A REMARK ON THE SHIMURA CORRESPONDENCE

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(Received 25 April, 1987)

Introduction. In [**4**] an identity is given which relates the product of two Fourier coefficients of a Hecke eigenform g of half-integral weight and level $4N$ with N odd and squarefree to the integral of a Hecke eigenform f of even integral weight associated to g under the Shimura correspondence along a geodesic period on the modular curve $X_0(N)$. This formula contains as a special case a refinement of a result of Waldspurger [**6**] about special values of L-series attached to f at the central point.

The purpose of this note is to show that the restriction to N squarefree can be lifted and that our identity in the more general case is essentially a consequence of results of Waldspurger ([**8**, **9**, **10**]).

Notation. We let $\Gamma(1) = SL_2(\mathbb{Z})$ operate on integral binary quadratic forms $[a, b, c](x, y) = ax^2 + bxy + cy^2$ by

$$[a, b, c] \circ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}(x, y) = [a, b, c](\alpha x + \beta y, \gamma x + \delta y).$$

The symbol $\mathcal{H}$ denotes the upper half-plane.

The letters k and N denote positive integers, N is always assumed to be odd.

We write $N' \parallel N$ if $N' \mid N$ and $\left(N', \dfrac{N}{N'}\right) = 1$.

We let $M_{2k}(N)(S_{2k}(N))$ be the space of modular forms (cusp forms) of weight $2k$ on the group $\Gamma_0(N)$ and $S_{2k}^{\text{new}}(N) \subset S_{2k}(N)$ be the subspace of cuspidal newforms.

For a prime p we denote by $T_{2k}(p)$ the Hecke operator acting on $S_{2k}(N)$ by

$$T_{2k}(p) \sum_{n \geq 1} a(n) q^n = \sum_{n \geq 1} \left(a(pn) + \left(\frac{N^2}{p}\right) p^{2k-1} a\left(\frac{n}{p}\right) \right) q^n \qquad (q = e^{2\pi i z},\ z \in \mathcal{H})$$

(with the convention $a\left(\dfrac{n}{p}\right) = 0$ if $p \nmid n$). The Hecke operators leave $S_{2k}^{\text{new}}(N)$ stable.

For $f, f' \in S_{2k}(N)$ we write

$$\langle f, f' \rangle = \frac{1}{[\Gamma(1) : \Gamma_0(N)]} \int_{\Gamma_0(N) \backslash \mathcal{H}} f(z) \overline{f'(z)} y^{2k-2}\, dx\, dy \qquad (z = x + iy)$$

for the Petersson product of f and f'.

We let $S_{k+1/2}(N)$ be the subspace of cusp forms of weight $k + 1/2$ on $\Gamma_0(4N)$ whose nth Fourier coefficients at infinity vanish for $(-1)^k n \equiv 2, 3 \bmod 4$ ([**5**, **3**]).

For a prime p we write $T_{k+1/2}(p)$ for the Hecke operator acting on $S_{k+1/2}(N)$ given by

$$T_{k+1/2}(p) \sum_{n\geqq 1,(-1)^k n\equiv 0,1(4)} c(n)q^n$$
$$= \sum_{n\geqq 1,(-1)^k n\equiv 0,1(4)} \left(c(p^2n) + \left(\frac{N^2}{p}\right)\left(\frac{(-1)^k n}{p}\right)p^{k-1}c(n) + \left(\frac{N}{p}\right)^2 p^{2k-1}c(n/p^2)\right)q^n$$

(cf. [5], §1; [3], §3, Proposition and Remark, p. 46).

For $g, g' \in S_{k+1/2}(N)$ we denote by

$$\langle g, g'\rangle = \frac{1}{[\Gamma(1):\Gamma_0(4N)]}\int_{\Gamma_0(4N)\backslash\mathscr{H}} g(z)\overline{g'(z)}y^{k-3/2}\,dx\,dy \qquad (z = x+iy)$$

the Petersson product of g and g'.

1. Statement of results. In [4] for every fundamental discriminant D with $(-1)^k D > 0$ we defined a Shimura lifting ζ_D mapping $S_{k+1/2}(N)$ to $M_{2k}(N)$ (to $S_{2k}(N)$ if $k \geqq 2$ or if N is cubefree) and a Shintani lifting ζ_D^* mapping $S_{2k}(N)$ to $S_{k+1/2}(N)$, and ζ_D and ζ_D^* were shown to be adjoint maps with respect to the Petersson products. Explicitly, for

$$g = \sum_{n\geqq 1,(-1)^k n\equiv 0,1(4)} c(n)q^n \in S_{k+1/2}(N)$$

one has

$$\zeta_D g = \sum_{n\geqq 1}\left(\sum_{d\mid n,(d,N)=1}\left(\frac{D}{d}\right)d^{k-1}c(|D|\,n^2/d^2)\right)q^n$$

and for $f \in S_{2k}^{\text{new}}(N)$ a newform (the case we will be interested in) one has

$$\zeta_D^* f = (-1)^{[k/2]}2^k \sum_{m\geqq 1,(-1)^k m\equiv 0,1(4)} r_{k,N,D}(f;|D|\,m)q^m.$$

Here for any positive integer Δ satisfying $\Delta \equiv 0, 1 \bmod 4$ and $D \mid \Delta$ we have put

$$r_{k,N,D}(f;\Delta) = \sum_{Q\in\mathfrak{q}_{N,\Delta}/\Gamma_0(N)} \omega_D(Q)\int_{C_Q} f(z)Q(z,1)^{k-1}\,dz, \tag{1}$$

where $\mathfrak{q}_{N,\Delta}/\Gamma_0(N)$ is the set of $\Gamma_0(N)$-classes of integral binary quadratic forms $Q = [a, b, c]$ with $b^2 - 4ac = \Delta$ and $N \mid a$, and where C_Q is the image in $X_0(N)(\mathbb{C}) = \Gamma_0(N) \mid \mathscr{H} \cup \mathbb{P}^1(\mathbb{Q})$ of the semicircle $a\,|z|^2 + b\,\mathrm{Re}\,z + c = 0$ oriented from $\frac{-b-\sqrt{\Delta}}{2a}$ to $\frac{-b+\sqrt{\Delta}}{2a}$, if $a \neq 0$ or of the vertical line $b\,\mathrm{Re}\,z + c = 0$, oriented from $\frac{-c}{b}$ to $i\infty$ if $b > 0$ and from $i\infty$ to $\frac{-c}{b}$ if $b < 0$, if $a = 0$. Furthermore, ω_D is the genus character given by

$$\omega_D(Q) = \begin{cases} 0 & \text{if } (a,b,c,D) \neq 1 \\ \left(\frac{D}{n}\right) & \text{if } (a,b,c,D) = 1, \text{ where } Q \text{ represents } n,\ (n,D) = 1. \end{cases}$$

https://doi.org/10.1017/S0017089500007370 Published online by Cambridge University Press

REMARK. In [**4**], p. 240 the factor $(-1)^{[k/2]}2^k$ is missing in the definition of ζ_D^*, and the orientation of C_Q should be the opposite one for $a=0$, $b<0$.

Recall that for any positive integer N' with $N' \parallel N$ we have an Atkin–Lehner involution $W_{N'}$ on $S_{2k}(N)$ leaving $S_{2k}^{\text{new}}(N)$ stable and defined by

$$W_{N'}f = N'^k(Nz+N'\delta)^{-2k}f\left(\frac{N'z+\beta}{Nz+N'\delta}\right) \quad (\beta,\ \delta \in \mathbb{Z},\ N'^2\delta - N\beta = N').$$

Now let f be a normalized Hecke eigenform in $S_{2k}^{\text{new}}(N)$ with $T_{2k}(p)f = \lambda_p f$ for all primes p, and let ε_p be the eigenvalues of f under W_{p^ν} (p prime, $p^\nu \parallel N$). We shall impose the assumption that $\varepsilon_p = 1$ whenever ν is even.

Let n_0 be an integer mod N such that $\left(\frac{(-1)^k n_0}{p^\nu}\right) = \varepsilon_p$ for all primes p with $p^\nu \parallel N$. We set

$$S_{k+1/2}(N;f,n_0) = \{g \in S_{k+1/2}(N) \mid T_{k+1/2}(p)g = \lambda_p f \text{ for all primes } p\text{; the } n\text{th Fourier coefficients of } g \text{ at infinity vanish unless } nn_0 \text{ is a square mod } N\}$$

(cf. [**2**, **6**, **8**, **3**]). Note that if $g = \sum c(n)q^n \in S_{k+1/2}(N;f,n_0)$ and $p^2 \mid N$, then $c(n)=0$ whenever $p \mid n$. In fact, since $p^2 \mid N$ it is well-known that $\lambda_p = 0$([**1**]), hence $c(n)=0$ for $p^2 \mid n$. However, if $p \parallel n$, then nn_0 cannot be a square mod p^2, since $p \nmid n_0$ by assumption, so $c(n)=0$ also in this case.

THEOREM. *Let $k \geqq 1$ and let N be odd. Let f be a normalized Hecke eigenform in $S_{2k}^{\text{new}}(N)$, and let n_0 and $S_{k+1/2}(N;f,n_0)$ be as above. Then:*

(i) $\dim_{\mathbb{C}} S_{k+1/2}(N;f,n_0) = 1$.

(ii) *For any generator* $g = \sum_{n \geqq 1,\, (-1)^k n \equiv 0,1(4)} c(n)q^n$ of $S_{k+1/2}(N;f,n_0)$ *the formula*

$$\frac{c(m)\overline{c(n)}}{\langle g, g\rangle} = \frac{(-1)^{[k/2]}2^k}{\langle f, f\rangle} r_{k,N,(-1)^k n}(f;mn) \tag{2}$$

($m, n \in \mathbb{N}$; $(-1)^k m$, $(-1)^k n \equiv 0, 1 \bmod 4$; $(-1)^k n$ *a fundamental discriminant*; $(-1)^k nn_0$ *congruent to a square* mod N)

holds, where $r_{k,N,(-1)^k n}(f;mn)$ is the cycle integral defined by (1).

As already mentioned above the proof of (i) strongly depends on results of Waldspurger ([**8**, **9**, **10**]). In certain cases (i) can also be derived from [**6**, Chap. 4] by using an isomorphism between modular forms of half-integral weight and Jacobi modular forms and then applying the "multiplicity 1" theorem proved in [**7**]. Assertion (ii) can be deduced as in [**4**] using (i) and the fact that the Shimura liftings and the Shintani liftings are adjoint maps with respect to the Petersson products.

For a fundamental discriminant D with $(D, N) = 1$ let

$$L(f, D, S) = \sum_{n \geqq 1} \left(\frac{D}{n}\right) a(n) n^{-s} (\operatorname{Re} s \gg 0) \tag{3}$$

be the L-series of $F(z)=\sum_{n\geq 1} a(n)q^n$ twisted with the quadratic character $\left(\frac{D}{n}\right)$. Recall that $L(f, D, s)$ has a holomorphic continuation to $\mathbb{C}$ and that

$$L^*(f, D, s)=(2\pi)^{-s}(ND^2)^{s/2}\Gamma(s)L(f, D, s)$$

satisfies the functional equation

$$L^*(f, D, s)=(-1)^k\left(\frac{D}{-N}\right)L^*(W_Nf, D, 2k-s).$$

As in [**4**], letting $m=n$ in (2) we can deduce a refined version of a result of Waldspurger ([**8**]).

COROLLARY. *Suppose that the hypotheses of the Theorem are satisfied. Let D be a fundamental discriminant and suppose that $(-1)^kD>0$, $(D, N)=1$ and that Dn_0 is a square modulo N. Then*

$$\frac{c(|D|)^2}{\langle g, g\rangle}=2^{\nu(N)}\frac{(k-1)!}{\pi^k}|D|^{k-1/2}\frac{L(f, D, k)}{\langle f, f\rangle},$$

where $\nu(N)$ denotes the number of different prime divisors of N.

Of course, Corollaries 2–6 in [**4**] also have natural generalizations to the more general situation here. However, we leave their explicit formulation to the reader.

2. Proofs.

PROPOSITION. *For all primes p and all fundamental discriminants D with $(-1)^kD>0$ one has*

$$\zeta_D^*T_{2k}(p)f=T_{k+1/2}(p)\zeta_D^*f \qquad (\forall f\in S_{2k}^{\text{new}}(N)). \tag{4}$$

Proof of Proposition. We shall give the proof only for $(D, N)=1$, leaving the more general case which is similar but more tedious to the reader. We may assume that f is a normalized Hecke eigenform.

If $p\nmid N$, then $T_{2k}(p)$ and $T_{k+1/2}(p)$ are hermitian, and since ζ_D and ζ_D^* are adjoint maps and ζ_D commutes with the action of Hecke operators (immediate verification), identity (4) is obvious in this case.

Next assume $p\mid N$. Then by definition of ζ_D^* and $T_{k+1/2}(p)$ we must show that

$$r_{k,N,D}(f;|D|mp^2)=r_{k,N,D}(T_{2k}(p)f;|D|m) \qquad (\forall m\geqq 1 \text{ with } (-1)^km\equiv 0,1 \ (\mathrm{mod}\ 4)).$$

According to Proposition 7 in [**4**]

$$r_{k,N,D}(f;|D|m)=\alpha_k(|D|m)^{k-1/2}\langle f, f_{k,N,D,(-1)^km}\rangle,$$

where α_k is a constant depending only on N and k and $f_{k,N,d,(-1)^km}(z)$ $(m\geqq 1$, $(-1)^km\equiv 0, 1 \bmod 4$; $z\in\mathcal{H})$ is the modular form (cusp form if N is cubefree or if $k\geqq 2)$

https://doi.org/10.1017/S0017089500007370 Published online by Cambridge University Press

in $M_{2k}(N)$ defined by

$$f_{k,N,D,(-1)^k m}(z)=\begin{cases}\sum_{Q\in \mathcal{Q}_{N,|D|m}} \omega_D(Q)Q(z,1)^{-k} & \text{if } k>1\\ \lim_{s\downarrow 0}\sum_{Q\in \mathcal{Q}_{N,|D|m}} \omega_D(Q)Q(z,1)^{-1}|Q(z,1)|^{-s} & \text{if } k=1\end{cases}$$

(cf. [**4**, §1]). In the following we will assume $k\geqq 2$ (the case $k=1$ is entirely similar). We will distinguish two cases.

Case (i): $p^2\mid N$. Then $T_{2k}(p)f=0$([**1**]), hence we must show

$$\langle f, f_{k,N,D,(-1)^k mp^2}\rangle = 0. \tag{5}$$

Since $p^2\mid N$, the conditions $b^2-4ac=|D|\,mp^2$, $N\mid a$ imply $p^2\mid a$, $p\mid b$, hence

$$f_{k,N,D,(-1)^k mp^2}(z)=\sum_{[a,b,c]\in \mathcal{Q}_{N',|D|M}} \omega_D(p^2a,pb,c)(p^2az^2+pbz+c)^{-k}$$

where $N'=N/p^2$.

Since $p\nmid D$, we have $\omega_D(p^2a,pb,c)=\omega_D(a,b,c)$; in fact, $(a,b,c,D)=1$ is equivalent to $(p^2a,pb,c,D)=1$, and if $[a,b,c]$ represents n, then $[p^2a,pb,c]$ represents p^2n.

Hence

$$f_{k,N,D,(-1)^k mp^2}(z)=f_{k,N',D,(-1)^k m}(pz),$$

and since f is in $s_{2k}^{\text{new}}(N)$, (5) follows.

Case (ii): $p\parallel N$. By [1] we have $T_{2k}(p)f=-p^{k-1}W_pf$, hence we have to show that

$$\langle f, f_{k,N,D,(-1)^k mp^2}\rangle = -p^{-k}\langle W_pf, f_{k,N,D,(-1)^k m}\rangle. \tag{6}$$

If $a,b,c\in\mathbb{Z}$ with $b^2-4ac=|D|\,mp^2$ and $N\mid a$, then it follows that $p\mid b$ and either $p^2\mid a$ or $p\parallel a$ and $p\mid c$. Hence setting $N'=N/p$ we have

$$f_{k,N,D,(-1)^k mp^2}(z)=\sum_{[a,b,c]\in \mathcal{Q}_{N',|D|m}} \omega_D(p^2a,pb,c)(p^2az^2+pbz+c)^{-k}$$
$$+p^{-k}\sum_{\substack{a,b,c\in\mathbb{Z},N'\mid a,p\nmid a\\ b^2-4ac=|D|m}} \omega_D(pa,pb,pc)(az^2+bz+c)^{-k}.$$

As in case (i) the first sum equals $f_{k,N',D,(-1)^k m}(pz)$. Since $\omega_D(pa,pb,pc)=\left(\frac{D}{p}\right)\omega_D(a,b,c)$ the second sum equals

$$p^{-k}\left(\frac{D}{p}\right)(f_{k,N',D,(-1)^k m}(z)-f_{k,N,D,(-1)^k m}(z)),$$

hence since $f\in S_{2k}^{\text{new}}(N)$ it follows that

$$\langle f, f_{k,N,D,(-1)^k mp^2}\rangle = -p^{-k}\left(\frac{D}{p}\right)\langle f, f_{k,N,D,(-1)^k m}\rangle.$$

By definition

$$W_p f_{k,N,D,(-1)^k m}(z) = \sum_{Q \in \mathfrak{q}_{N,\,|D|\,m}} \omega_D(Q \circ W_p)(Q \circ W_p)(z, 1)^{-k}.$$

Since acting by W_p is a permutation of $\mathfrak{q}_{N,\,|D|\,m}$ and $\omega_D(Q \circ W_p) = \left(\frac{D}{p}\right)\omega_D(Q)$ for $p \nmid D$ we conclude that

$$W_p f_{k,N,D,(-1)^k m} = \left(\frac{D}{p}\right) f_{k,N,D,(-1)^k m},$$

and (6) follows, since W_p is hermitian. This concludes the proof of the Proposition.

We now turn to the proof of the Theorem. Suppose that D is a fundamental discriminant such that $(-1)^k D > 0$ and $|D|\,n_0$ is congruent to a square mod N. The above Proposition and the definition of ζ_D^* then show that $\zeta_D^* f$ is in $S_{k+1/2}(N; f, n_0)$.

Let p be a prime with $p^\nu \| N$. If $(D, N) = 1$, then from $\left(\frac{(-1)^k n_0}{p^\nu}\right) = \varepsilon_p$ and $\left(\frac{(-1)^k D n_0}{p^\nu}\right) = 1$ it follows that $\left(\frac{D}{p^\nu}\right) = \varepsilon_p$, and the same computations as in [**4**, p. 243] give

$$r_{k,N,D}(f; D^2) = 2^{\nu(N)}(-1)^{[k/2]}(2\pi)^{-k}\Gamma(k)\,|D|^{k-1/2}L(f, D, k), \tag{7}$$

where $L(f, D, s)$ is the L-function defined for $\operatorname{Re} s > 0$ by (3) and $\nu(N)$ is the number of different prime factors of N. But according to [**9**, Theorem 4] there is a fundamental discriminant D_0 with $(-1)^k D_0 > 0$, $\varepsilon_p = \left(\frac{D_0}{p^\nu}\right)$ for all primes p with $p^\nu \| N$ and $L(f, D_0, k) \neq 0$. Hence $\zeta_{D_0}^* f$ is a non-zero function in $S_{k+1/2}(N; f, n_0)$, and it follows that $\dim_{\mathbb{C}} S_{k+1/2}(N; f, n_0) \geqq 1$. On the other hand, Waldspurger [**10**] kindly informs the author that for N odd it follows from his Theorem 1, (ii) in [**8**, p. 378] and the explicit description of the sets $U_p(e, f)$ ([**8**, p. 454]) that $\dim_{\mathbb{C}} S_{k+1/2}(N; f, n_0) \leqq 1$.

Let us now prove the second assertion of the Theorem. Since f and $\zeta_{(-1)^k n} g$ have the same eigenvalues under all Hecke operators, $\zeta_{(-1)^k n} g$ is a cusp form and is a multiple of f by the "multiplicity 1" theorem valid for $S_{2k}^{\text{new}}(N)$. Comparing the Fourier coefficients at q we find that

$$\zeta_{(-1)^k n} g = c(n) f. \tag{8}$$

From what was proved above,

$$\zeta_{(-1)^k n}^* f = \beta_n g$$

for some $\beta_n = \mathbb{C}$.

Now

$$\beta_n c(m) = \text{coefficient of } q^m \text{ in } \zeta_{(-1)^k n}^* f = (-1)^{[k/2]} 2^k r_{k,n,(-1)^k n}(f; mn).$$

On the other hand

$$\begin{aligned}\beta_n c(m)\langle g, g\rangle &= c(m)\langle \zeta^*_{(-1)^k n} f, g\rangle \\ &= c(m)\langle f, \zeta_{(-1)^k n} g\rangle \\ &= c(m)\overline{c(n)}\langle f, f\rangle,\end{aligned}$$

where in the last line we have used (8). Comparing these two formulas we obtain (2).

The Corollary to the Theorem, of course, follows from (7).

ACKNOWLEDGEMENT. I would like to thank J.-L. Waldspurger for a helpful discussion.

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