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On the coupled time-harmonic motion of water and a body freely floating in it

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We consider a spectral problem that describes the time-harmonic small-amplitude motion of the mechanical system that consists of a three-dimensional water layer of constant depth and a body (either surface-piercing or totally submerged), freely floating in it. This coupled boundary-value problem contains a spectral parameter – the frequency of oscillations – in the boundary conditions as well as in the equations governing the body motion. It is proved that the total energy of the water motion is finite and the equipartition of energy of the whole system is established. Under certain restrictions on body's geometry the problem is proved to have only a trivial solution for sufficiently large values of the frequency. The uniqueness frequencies are estimated from below.

Key words: surface gravity waves, waves/free-surface flows, wave–structure interactions

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1. Introduction

This paper deals with the coupled problem that describes the time-harmonic motion of the following mechanical system. It consists of an inviscid, incompressible, heavy fluid (water) and a bounded body (either totally submerged or surface-piercing), which floats freely. The latter means that there are no external forces acting on the body, for example due to constraints on the body motion. A totally submerged body is deemed to be freely floating when it is neutrally buoyant with the centre of buoyancy above the centre of mass. Water extends to infinity in horizontal directions, but has a finite depth being bounded from below by a horizontal rigid bottom and from above by a free surface. Thus, our model describes a vessel freely floating in an open sea of constant depth. The water motion is assumed to be irrotational and the surface tension is neglected on the free surface; moreover, the motion of the system is supposed to be of small amplitude near equilibrium, which allows us to use a linear model.

The time-dependent model of this mechanical system was developed by John (1949) in his pioneering work. Recently, the corresponding equations of the body motion were represented in a convenient matrix form by Nazarov & Videman (2011) (this notation was also used by Nazarov 2011), and this form is applied in the present work. Assuming that the water motion is simple harmonic in time, we apply an ansatz that introduces complex-valued unknowns and reduces the time-dependent

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problem to a coupled spectral problem. The frequency of oscillations, which plays the role of spectral parameter, appears in the boundary conditions as well as in the equations of the body motion. It is possible to reformulate this spectral problem as an operator equation in a Hilbert space (cf. Nazarov & Videman 2011, where this approach is developed for a similar problem in a channel symmetric with respect to the centreplane), but such a formulation is superfluous for our purpose.

One has to keep in mind that in the vast majority of published papers, the questions of uniqueness and existence of solutions and trapped modes are studied for the scattering and radiation problems under the assumption that an immersed body or several such bodies are fixed (see Kuznetsov, Maz'ya & Vainberg 2002 and Linton & McIver 2007 for surveys, and Kuznetsov 2008 for the most recent uniqueness theorem). Only John (1950) proved a uniqueness theorem for the problem of a freely floating body (without formulating the problem explicitly), and no other rigorous result about this problem had been obtained during the past six decades. In a number of papers (McIver & McIver 2006, 2007; Evans & Porter 2007; Newman 2008; Porter & Evans 2008, 2009; Fitzgerald & McIver 2010), the question of trapped modes was considered at the heuristic level; various two-dimensional and axisymmetric trapping structures are proposed in these papers. In most of these works, the authors deal with a simplified model of a freely floating body constrained to the heave motion only; however, the article by Porter & Evans (2009) involves an example of trapping structure whose motion is combined (heave and sway) and mentions the roll motion. Subsequently, Kuznetsov (2010) constructed a non-trivial solution for the two-dimensional problem analogous to that considered here. His solution proves the existence of time-harmonic waves trapped by each one of a family of symmetric, surface-piercing structures that have two immersed parts; these structures remain motionless despite the fact that they float freely. However, none of the above papers contain any uniqueness theorem for the problem of coupled motion.

In the present note, we deal with the case when a body is totally submerged as well as with the case when it is surface-piercing (unlike the most of previous works, in which only the latter case was considered), and our aim is twofold. First, we prove that the total energy is finite for the mechanical system in question; moreover, the equipartition equality is obtained for the kinetic and potential energy. Second, we prove a new uniqueness theorem for the case when a body is totally submerged. In this theorem, frequencies of oscillations are assumed to be sufficiently large, which is the same restriction as in the uniqueness theorem of John (1950). We prove the latter theorem in a slightly more general form, but what is more important is that the new proof is just a few lines instead of two pages in the original work.

2. Statement of the problem

Let us begin with geometrical assumptions. We choose a Cartesian coordinate system (x_1, x_2, y) so that the y-axis is directed upwards and the free surface at rest coincides with the horizontal (x_1, x_2)-plane. Let

$$\Pi = \{(\boldsymbol{x}, y) : \boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2, \ -h < y < 0\},$$

and let $\widehat{B}$ denote the domain whose closure is the floating body in its equilibrium position (see figures 1 and 2). The submerged part of $\widehat{B}$ is denoted by B, that is $B = \widehat{B} \cap \Pi$; note that $B = \widehat{B}$ for a totally submerged body. We suppose that $\overline{B} \subset \Pi$ in the latter case, whereas $\widehat{B} \setminus \overline{\Pi}$ is assumed to be non-empty for a surface-piercing body.

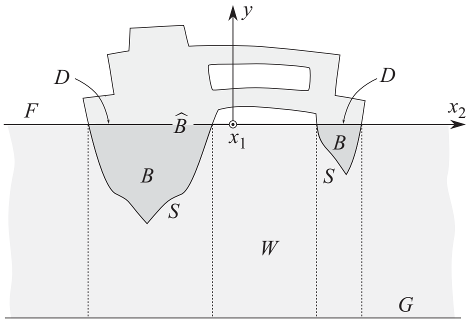

FIGURE 1. Definition sketch of a surface-piercing body.

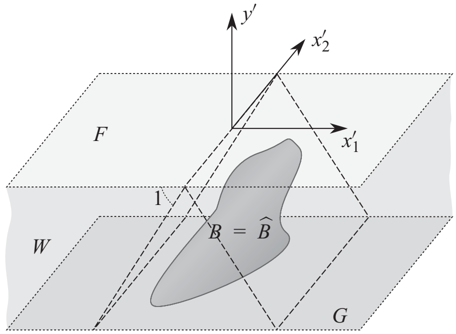

FIGURE 2. Definition sketch of a totally submerged body.

Anyway, the water domain at rest is $W = \Pi \setminus \overline{B}$ and $S = \partial \widehat{B} \cap \Pi$ is the wetted part of body's boundary. Let $G = \{\boldsymbol{x} \in \mathbb{R}^2,\ y = -h\}$ and $F = \partial W \setminus \overline{S}$ denote the bottom and the free surface at rest, respectively; moreover, if the body is surface-piercing, then we put $D = \{\boldsymbol{x} \in \mathbb{R}^2,\ y = 0\} \setminus \overline{F}$. We suppose that W is a simply connected Lipschitz domain, and so the unit normal is defined almost everywhere on ∂W; by $\boldsymbol{n}$ we denote the normal pointing to the exterior of W. Since W is simply connected, $\widehat{B} = B$ must be simply connected when the body is totally submerged; however, a surface-piercing body $\widehat{B}$ can have holes in its part above the water level, whereas B can consist of several connected components (D has the same number of components as B in this case; see figure 1).

Since the water motion is supposed to be of small amplitude, linear equations of motion are applicable. According to the standard linearisation procedure (see John 1949, § 2, and Kuznetsov *et al.* 2002, Introduction), all unknowns are proportional to a small non-dimensional parameter ϵ (in fact, expansions of unknowns in powers of ϵ are used), and the boundary conditions are imposed on the water boundary at rest ∂W. We begin with formulating (at the heuristic level) the first-order time-dependent problem in the form proposed by Nazarov & Videman (2011), and then apply a time-harmonic ansatz which introduces the spectral parameter instead of the dependence on time.

The assumption that the water motion is irrotational implies that there exists a velocity potential because W is simply connected. Let $\epsilon\, \Phi(\boldsymbol{x}, y; t)$ be the first-order velocity potential, which means that the velocity field is equal to $\epsilon\, \nabla \Phi(\boldsymbol{x}, y; t)$ at every instant t ($\nabla = (\partial_{x_1}, \partial_{x_2}, \partial_y)$ is the spatial gradient), then the continuity equation yields

$$\nabla^2 \Phi = 0 \quad \text{in } W \quad \text{for all } t. \tag{2.1}$$

In order to specify the behaviour of Φ near ∂W, we require that this function belongs to the Sobolev space $H^1_{loc}(W)$.

The standard linear boundary condition on the free surface has the following form:

$$\partial_t^2 \Phi + g \partial_y \Phi = 0 \quad \text{on } F \quad \text{for all } t. \tag{2.2}$$

Here $g > 0$ is the acceleration due to gravity that acts in the direction opposite to the y-axis. Equality (2.2) is a consequence of Bernoulli's equation and the kinematic condition both taken linearised; the first of these expresses the fact that the pressure is constant on F, whilst the second one means that there is no transfer of matter across F. Furthermore, we have the homogeneous Neumann condition on the bottom (no normal flow):

$$\partial_y \Phi(\boldsymbol{x}, y, t) = 0 \quad \text{on } G \quad \text{for all } t. \tag{2.3}$$

Now we turn to relations that describe the body motion coupled with the motion of water; one of them is the kinematic boundary condition on S, whereas the others are the equations of motion of the body whose position is characterised as follows. Let $(\boldsymbol{x}^{(0)}, y^{(0)})$ be the location of body's centre of mass in the equilibrium position; by $\boldsymbol{q}(t) \in \mathbb{R}^6$ we denote a vector column with the following components:

(i) ϵq_1, ϵq_2 and ϵq_4 are the first-order displacements of the centre of mass from its rest position in the horizontal and vertical directions, respectively;

(ii) ϵq_3, and ϵq_5, ϵq_6 are the first-order angles of rotation about the axes y, and x_1, x_2, respectively, through the centre of mass.

According to the linear theory (see, for example, John 1949) we have the following kinematic condition:

$$\partial_{\boldsymbol{n}} \Phi(\boldsymbol{x}, y, t) = \boldsymbol{n}^{\mathrm{T}}(\boldsymbol{x}, y)\, \boldsymbol{D}\big(\boldsymbol{x} - \boldsymbol{x}^{(0)}, y - y^{(0)}\big)\, \dot{\boldsymbol{q}}(t) \quad \text{on } S \quad \text{for all } t. \tag{2.4}$$

This condition and (2.6) below are written in the form proposed by Nazarov & Videman (2011). By superscript T we denote the operation of matrix transposition; when this operation is applied to a vector, the latter is considered as a one-column matrix. The vector $\dot{\boldsymbol{q}}$ characterises the body motion in the following manner: $\epsilon\,(\dot{q}_1, \dot{q}_2, \dot{q}_4)^{\mathrm{T}}$ is the velocity vector of the translational motion and $\epsilon\,(\dot{q}_3, \dot{q}_5, \dot{q}_6)^{\mathrm{T}}$ is the vector of angular velocities. The 3×6 matrix

$$\boldsymbol{D}(\boldsymbol{x}, y) = \begin{bmatrix} 1 & 0 & x_2 & 0 & 0 & -y \\ 0 & 1 & -x_1 & 0 & y & 0 \\ 0 & 0 & 0 & 1 & -x_2 & x_1 \end{bmatrix} \tag{2.5}$$

describes the motion of a rigid body so that its elements are in conformity with the order of components of the vector $\boldsymbol{q}$.

The linearised system of equations of the body motion expresses the conservation of linear and angular momentum. In the absence of external forces, this system is as follows:

$$\boldsymbol{E}\big(\boldsymbol{x}^{(0)}, y^{(0)}\big)\ddot{\boldsymbol{q}}(t) = -\int_S \partial_t \Phi(\boldsymbol{x}, y, t)\, \boldsymbol{D}^{\mathrm{T}}\big(\boldsymbol{x} - \boldsymbol{x}^{(0)}, y - y^{(0)}\big)\, \boldsymbol{n}(\boldsymbol{x}, y)\, \mathrm{d}s - g\, \boldsymbol{K}\big(\boldsymbol{x}^{(0)}, y^{(0)}\big)\, \boldsymbol{q}(t) \tag{2.6}$$

for all t. Here $\ddot{\boldsymbol{q}}(t)$ is the acceleration vector, and $\boldsymbol{E}(\boldsymbol{x}^{(0)}, y^{(0)})$ is the mass matrix defined as follows:

$$\boldsymbol{E}\left(\boldsymbol{x}^{(0)}, y^{(0)}\right) = \rho_0^{-1} \int_{\widehat{B}} \rho(\boldsymbol{x}, y)\, \boldsymbol{D}^{\mathrm{T}}\left(\boldsymbol{x} - \boldsymbol{x}^{(0)}, y - y^{(0)}\right) \boldsymbol{D}\left(\boldsymbol{x} - \boldsymbol{x}^{(0)}, y - y^{(0)}\right) \mathrm{d}\boldsymbol{x}\, \mathrm{d}y, \quad (2.7)$$

where $\rho(\boldsymbol{x}, y) \geqslant 0$ is the density distribution within the body and $\rho_0 > 0$ is the constant density of water. On the right-hand side of (2.6), we have forces and their moments, namely the first term is of the hydrodynamic origin and the second term is related to the buoyancy (see, for example, John 1949; Mei, Stiassnie & Yue 2005). The matrix in the latter term has the following blockwise form:

$$\boldsymbol{K} = \begin{pmatrix} \mathbb{O}_3 & \mathbb{O}_3 \\ \mathbb{O}_3 & \boldsymbol{K}' \end{pmatrix}, \quad \text{where } \boldsymbol{K}' = \begin{pmatrix} I^D & I_2^D & -I_1^D \\ I_2^D & I_{22}^D + I_y^B & -I_{12}^D \\ -I_1^D & -I_{12}^D & I_{11}^D + I_y^B \end{pmatrix} \quad (2.8)$$

and $\mathbb{O}_3$ is the null 3×3 matrix. The elements of $\boldsymbol{K}'$ involve the following moments:

$$\left.\begin{aligned} I^D &= \int_D \mathrm{d}\boldsymbol{x}, \quad I_y^B = \int_B \left(y - y^{(0)}\right) \mathrm{d}\boldsymbol{x}\, \mathrm{d}y, \\ I_i^D &= \int_D \left(x_i - x_i^{(0)}\right) \mathrm{d}\boldsymbol{x}, \quad I_{ij}^D = \int_D \left(x_i - x_i^{(0)}\right)\left(x_j - x_j^{(0)}\right) \mathrm{d}\boldsymbol{x}, \quad i, j = 1, 2. \end{aligned}\right\} \quad (2.9)$$

The matrix $\boldsymbol{K}$ is symmetric; moreover, we have that $\boldsymbol{K} = \mathrm{diag}\left(0, 0, 0, 0, I_y^B, I_y^B\right)$ in the case when $B = \widehat{B}$, that is, the body is totally submerged.

The elements of the mass matrix $\boldsymbol{E}$ are also composed of various moments, but of the whole body $\widehat{B}$. It is a direct calculation to obtain the explicit form of the matrix (2.7) which is as follows:

$$\boldsymbol{E} = \begin{pmatrix} I^{\widehat{B}} & 0 & 0 & 0 & 0 & 0 \\ 0 & I^{\widehat{B}} & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{x_1x_1}^{\widehat{B}} + I_{x_2x_2}^{\widehat{B}} & 0 & -I_{x_1y}^{\widehat{B}} & -I_{x_2y}^{\widehat{B}} \\ 0 & 0 & 0 & I^{\widehat{B}} & 0 & 0 \\ 0 & 0 & -I_{x_1y}^{\widehat{B}} & 0 & I_{x_2x_2}^{\widehat{B}} + I_{yy}^{\widehat{B}} & -I_{x_1x_2}^{\widehat{B}} \\ 0 & 0 & -I_{x_2y}^{\widehat{B}} & 0 & -I_{x_1x_2}^{\widehat{B}} & I_{x_1x_1}^{\widehat{B}} + I_{yy}^{\widehat{B}} \end{pmatrix}. \quad (2.10)$$

Here

$$\left.\begin{aligned} I^{\widehat{B}} &= \rho_0^{-1} \int_{\widehat{B}} \rho(\boldsymbol{x}, y)\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}y, \quad I_\sigma^{\widehat{B}} = \rho_0^{-1} \int_{\widehat{B}} \rho(\boldsymbol{x}, y)\left(\sigma - \sigma^{(0)}\right) \mathrm{d}\boldsymbol{x}\, \mathrm{d}y, \\ I_{\sigma\tau}^{\widehat{B}} &= \rho_0^{-1} \int_{\widehat{B}} \rho(\boldsymbol{x}, y)\left(\sigma - \sigma^{(0)}\right)\left(\tau - \tau^{(0)}\right) \mathrm{d}\boldsymbol{x}\, \mathrm{d}y, \end{aligned}\right\} \quad (2.11)$$

where σ and τ attain their values in the set of the coordinate notation, which is $\{x_1, x_2, y\}$; in the same way, we have that $\sigma^{(0)}$ and $\tau^{(0)}$ belong to $\{x_1^{(0)}, x_2^{(0)}, y^{(0)}\}$. In formula (2.10), it is also taken into account that all first-order moments vanish by the definition of $(x_1^{(0)}, x_2^{(0)}, y^{(0)})$. The matrix $\boldsymbol{E}$ is obviously symmetric, and it is straightforward to verify that $\boldsymbol{E}$ is always positive definite.

The relations listed above must be augmented by the following conditions concerning the equilibrium position.

(i) The mass of the displaced liquid is equal to that of the body (Archimedes' law):

$$I^{\widehat{B}} = \int_B \mathrm{d}\boldsymbol{x}\,\mathrm{d}y. \tag{2.12}$$

(ii) The centre of buoyancy lies on the same vertical line as the centre of mass (see, for example, Mei *et al.* 2005, § 8.2.3):

$$\int_B \left(x_i - x_i^{(0)}\right) \mathrm{d}\boldsymbol{x}\,\mathrm{d}y = 0, \quad i = 1, 2. \tag{2.13}$$

(iii) The matrix $\boldsymbol{K}$ is positive semi-definite; moreover, $\boldsymbol{K}'$ is positive definite for a surface-piercing body and the inequality $I_y^B > 0$ holds for a totally submerged body (see John 1949).

The last requirement is the classical condition yielding the stability of the equilibrium position of a body. This follows from the results formulated, for example, by John (1949, § 2.4). (The stability is understood in the usual sense that an instantaneous, infinitesimal disturbance causes changes in the position which remain infinitesimal, except for purely horizontal, for all subsequent times.) In the case of a totally submerged body the condition has simple meaning: the centre of buoyancy must be above the centre of mass.

Of course, relations (2.1)–(2.6) must be complemented by proper initial conditions in order to obtain a well-posed initial-value problem (see Beale 1977, § 3, where this question was considered). However, our aim is to study the motion simple harmonic in time, and so not depending on the initial conditions. An heuristic explanation as to how the phenomenon of time-harmonic waves is to be conceived is given by John (1950), who discussed it in detail on page 46.

Now we are in a position to formulate the problem describing the coupled time-harmonic motion of the mechanical system under consideration. Let $\omega > 0$ be the radian frequency of oscillations, then we represent the velocity potential and the displacement vector in the form

$$(\Phi(\boldsymbol{x}, y, t), \boldsymbol{q}(t)) = \mathrm{Re}\{\mathrm{e}^{-\mathrm{i}\omega t}(\varphi(\boldsymbol{x}, y), \boldsymbol{\chi})\}, \tag{2.14}$$

where φ is a complex-valued function and $\boldsymbol{\chi} \in \mathbb{C}^6$. Substituting the latter expression into relations (2.1)–(2.6), we immediately get

$$\nabla^2\varphi = 0 \quad \text{in } W, \tag{2.15}$$

$$\partial_y\varphi - \frac{\omega^2}{g}\varphi = 0 \quad \text{on } F, \tag{2.16}$$

$$\partial_y\varphi = 0 \quad \text{on } G, \tag{2.17}$$

$$\partial_{\boldsymbol{n}}\varphi = -\mathrm{i}\omega\, \boldsymbol{n}^{\mathrm{T}}\boldsymbol{D_0}\,\boldsymbol{\chi} \quad \text{on } S, \tag{2.18}$$

$$\omega^2\boldsymbol{E}\boldsymbol{\chi} = -\mathrm{i}\omega \int_S \varphi \boldsymbol{D_0}^{\mathrm{T}}\boldsymbol{n}\,\mathrm{d}s + g\,\boldsymbol{K}\boldsymbol{\chi}, \tag{2.19}$$

where $\boldsymbol{D_0} = \boldsymbol{D}(\boldsymbol{x} - \boldsymbol{x}^{(0)}, y - y^{(0)})$. It is clear that ω is a spectral parameter in this boundary-value problem, and so ω is sought together with the eigenvector $(\varphi, \boldsymbol{\chi})$. Besides, we have to explain how the boundary condition (2.18) is understood because we assumed that W is a Lipschitz domain and $\varphi \in H^1_{loc}(W)$. As usual, it is sufficient to understand the whole set of relations (2.15)–(2.18) in the sense of the following integral identity:

$$\int_W \nabla\varphi\nabla\psi \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}y = \frac{\omega^2}{g}\int_F \varphi\,\psi\,\mathrm{d}\boldsymbol{x} - \mathrm{i}\omega\int_S \psi\,\boldsymbol{n}^{\mathrm{T}}\boldsymbol{D}_0\,\boldsymbol{\chi}\,\mathrm{d}s, \tag{2.20}$$

which must hold for an arbitrary smooth ψ having a compact support in $\overline{W}$. (For smooth S, equality (2.20) is a consequence of relations (2.15)–(2.18) and the first Green's identity applied to φ and ψ.)

Furthermore, it is necessary to specify the behaviour of φ at infinity so that the velocity potential given by formula (2.14) describes outgoing waves. Therefore, we complement relations (2.15)–(2.19) by the following radiation condition:

$$\int_{W\cap\{|\boldsymbol{x}|=a\}} |\partial_{|\boldsymbol{x}|}\varphi - \mathrm{i}\nu_0\varphi|^2\,\mathrm{d}s = o(1) \quad \text{as } a\to\infty, \tag{2.21}$$

where ν_0 is the unique positive root of $\nu_0\tanh(\nu_0 h) = \omega^2/g$. It is natural that (2.21) is the same radiation condition as in the water-wave problem for a fixed obstacle (see, for example, John 1950).

Nazarov & Videman (2011) introduced and investigated problem (2.15)–(2.19) in the case when a body floats freely in an infinitely long channel of uniform cross-section. They studied only solutions that have finite energy in which case no radiation condition is required.

Problem (2.15)–(2.19), (2.21) is related to various important applications. Here we restrict ourselves by mentioning only one of them, namely, the scattering problem which is as follows. In the coupling conditions (2.18) and (2.19), φ is split into the sum $\varphi_{sc}+\varphi_{in}$, where the latter term describes a given wave (say, a plane wave), and so φ_{in} satisfies the Laplace equation throughout Π and the boundary conditions (2.16) and (2.17) on the whole upper and lower parts of $\partial\Pi$, respectively, but the radiation condition (2.21) usually does not hold for φ_{in}. The function φ_{sc} is sought from problem (2.15)–(2.19) and (2.21) which is inhomogeneous now because of the presence of φ_{in} in the coupling conditions.

In §3, we show that all solutions of problem (2.15)–(2.19) and (2.21) have finite energy. Some conditions, which guarantee that this problem has only a trivial solution, are given in §4, that is, uniqueness theorems for the scattering problem are proved there.

3. On the energy of the coupled time-harmonic motion

It is known (see, for example, Kuznetsov *et al.* 2002, §2.2.1) that a potential, satisfying relations (2.15)–(2.17) and (2.21), has the asymptotic representation at infinity of the same type as Green's function, namely

$$\varphi(\boldsymbol{x},y) = A(\theta,y)|\boldsymbol{x}|^{-1/2}\,\mathrm{e}^{\mathrm{i}\nu_0|\boldsymbol{x}|} + r(\boldsymbol{x},y), \quad \text{where } |r|+|\nabla r| = O\left(|\boldsymbol{x}|^{-3/2}\right) \quad \text{as } |\boldsymbol{x}|\to\infty, \tag{3.1}$$

and the following equality holds

$$\nu_0\int_0^{2\pi}\mathrm{d}\theta\int_{-h}^0 |A(\theta,y)|^2\,\mathrm{d}y = -\,\mathrm{Im}\int_S \overline{\varphi}\,\partial_{\boldsymbol{n}}\varphi\,\mathrm{d}s. \tag{3.2}$$

Here θ is the polar angle in the (x_1,x_2)-plane measured anticlockwise.

Assuming that $(\varphi,\boldsymbol{\chi})$ is a solution of problem (2.15)–(2.19) and (2.21), let us rearrange the latter formula using the coupling conditions (2.18) and (2.19).

Transposing the complex conjugate of (2.19), we get

$$\omega^2 \overline{\boldsymbol{\chi}}^{\mathrm{T}} \boldsymbol{E} - g\overline{\boldsymbol{\chi}}^{\mathrm{T}} \boldsymbol{K} = \mathrm{i}\omega \int_S \overline{\varphi}\, \boldsymbol{n}^{\mathrm{T}} \boldsymbol{D_0}\, \mathrm{d}s. \tag{3.3}$$

This relation and condition (2.18) yield that the inner product of both sides with $\boldsymbol{\chi}$ can be written in the form

$$\omega^2 \overline{\boldsymbol{\chi}}^{\mathrm{T}} \boldsymbol{E}\boldsymbol{\chi} - g\overline{\boldsymbol{\chi}}^{\mathrm{T}} \boldsymbol{K}\boldsymbol{\chi} = -\int_S \overline{\varphi}\, \partial_{\boldsymbol{n}}\varphi\, \mathrm{d}s. \tag{3.4}$$

Using this equality in (3.2), we obtain

$$\nu_0 \int_0^{2\pi} \mathrm{d}\theta \int_{-h}^{0} |A(\theta, y)|^2\, \mathrm{d}y = \mathrm{Im}\{\omega^2 \overline{\boldsymbol{\chi}}^{\mathrm{T}} \boldsymbol{E}\boldsymbol{\chi} - g\overline{\boldsymbol{\chi}}^{\mathrm{T}} \boldsymbol{K}\boldsymbol{\chi}\}. \tag{3.5}$$

Now, we are in a position to prove the following assertion.

Proposition 1. *Let* $(\varphi, \boldsymbol{\chi})$ *be a solution of problem* (2.15)–(2.19) *and* (2.21), *then*

$$\int_W |\nabla\varphi|^2\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}y < \infty \quad \textit{and} \quad \int_F |\varphi|^2\, \mathrm{d}\boldsymbol{x} < \infty, \tag{3.6}$$

that is, the kinetic energy of the water motion is finite as well as the potential one. Moreover, the following equality holds:

$$\int_W |\nabla\varphi|^2\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}y + \omega^2 \overline{\boldsymbol{\chi}}^{\mathrm{T}} \boldsymbol{E}\boldsymbol{\chi} = \frac{\omega^2}{g} \int_F |\varphi|^2\, \mathrm{d}\boldsymbol{x} + g\, \overline{\boldsymbol{\chi}}^{\mathrm{T}} \boldsymbol{K}\boldsymbol{\chi}, \tag{3.7}$$

where the kinetic energy of the water/body system is on the left-hand side, whereas on the right-hand side we have the potential energy of this coupled motion.

Proof. In view of formula (3.1), relations (3.6) are true provided the right-hand side in (3.5) vanishes. Since the matrix $\omega^2\boldsymbol{E} - g\boldsymbol{K}$ is real and symmetric, we have

$$\overline{\boldsymbol{\chi}}^{\mathrm{T}}(\omega^2\boldsymbol{E} - g\boldsymbol{K})\boldsymbol{\chi} = \left[\overline{\boldsymbol{\chi}}^{\mathrm{T}}(\omega^2\boldsymbol{E} - g\boldsymbol{K})\boldsymbol{\chi}\right]^{\mathrm{T}} = \boldsymbol{\chi}^{\mathrm{T}}(\omega^2\boldsymbol{E} - g\boldsymbol{K})\overline{\boldsymbol{\chi}} = \overline{\overline{\boldsymbol{\chi}}^{\mathrm{T}}(\omega^2\boldsymbol{E} - g\boldsymbol{K})\boldsymbol{\chi}}, \tag{3.8}$$

which is true for any $\boldsymbol{\chi}$. Taking into account that $(\varphi, \boldsymbol{\chi})$ is a solution, we can use the latter fact in equality (3.5), thus arriving at the required result.

Let us turn to proving equality (3.7). By $\zeta_a(|\boldsymbol{x}|)$ we denote an infinitely differentiable cutoff function, which is equal to one when $0 \leqslant |\boldsymbol{x}| \leqslant a$ and to zero for $|\boldsymbol{x}| \geqslant a+1$. Let a be so large that S lies within the cylinder $\{|\boldsymbol{x}| < a\}$, then substituting $\psi = \overline{\varphi}\, \zeta_a(|\boldsymbol{x}|)$ into (2.20), we see that relations (3.6) allow us to let $a \to \infty$, which combined with (2.19) gives equality (3.7). □

Remark 1. Formula (3.7) generalises the equality of energy equipartition valid for a fixed body. Indeed, $\boldsymbol{\chi} = 0$ for such a body, and (3.7) turns into the well-known equality (see, for example, formula (4.99) in Kuznetsov *et al.* 2002, p. 208).

Corollary 1. *For a solution* $(\varphi, \boldsymbol{\chi})$ *of problem* (2.15)–(2.19) *and* (2.21), *the first component* φ *belongs to the space* $H^1(W)$.

Proof. It follows from relations (3.6) that

$$\left(\int_W |\nabla\varphi|^2\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}y + h^{-1} \int_F |\varphi|^2\, \mathrm{d}\boldsymbol{x}\right)^{1/2} < \infty, \tag{3.9}$$

which is a norm in $H^1(W)$ equivalent to the usual one. □

Proposition 1 allows us to consider separate problems for the real/imaginary and imaginary/real parts of the complex-valued unknowns $(\varphi, \boldsymbol{\chi})$. Thus, if $(\phi, \boldsymbol{\xi}) = (\mathrm{Re}\,\varphi, \mathrm{Im}\,\boldsymbol{\chi})$, then problem (2.15)–(2.19) and (2.21) reduces to the following one:

$$\nabla^2 \phi = 0 \quad \text{in } W, \tag{3.10}$$

$$\partial_y \phi - \frac{\omega^2}{g} \phi = 0 \quad \text{on } F, \tag{3.11}$$

$$\partial_y \phi = 0 \quad \text{on } G, \tag{3.12}$$

$$\partial_n \phi = \omega\, \boldsymbol{n}^{\mathrm{T}} \boldsymbol{D}_0\, \boldsymbol{\xi} \quad \text{on } S, \tag{3.13}$$

$$\omega^2 \boldsymbol{E\xi} = -\omega \int_S \phi\, \boldsymbol{D}_0^{\mathrm{T}} \boldsymbol{n}\, \mathrm{d}s + g\, \boldsymbol{K\xi}, \tag{3.14}$$

where $(\phi, \boldsymbol{\xi}) \in H^1(W) \times \mathbb{R}^6$.

For $(\mathrm{Im}\,\varphi, \mathrm{Re}\,\boldsymbol{\chi})$, the same problem arises, but with $\omega < 0$. Therefore, we consider problem (3.10)–(3.14), in which $\omega \in \mathbb{R} \setminus \{0\}$, in what follows. The two-dimensional problem analogous to (3.10)–(3.14) was investigated by Kuznetsov (2010).

Remark 2. In the same way as in Nazarov & Videman (2011, § 3), problem (3.10)–(3.14) can be formulated as a variational problem for which one has to use the integral identity (2.20), where $\psi \in H^1(W)$, combined with the variational form of relation (3.4).

Definition 1. A non-trivial solution $(\phi, \boldsymbol{\xi})$ of problem (3.10)–(3.14) is called a *trapped mode*; the corresponding value of ω is referred to as a *trapping frequency*.

Conditions guaranteeing the absence of trapped modes are given in the next section.

4. Uniqueness theorems

Here we prove two uniqueness theorems for problem (3.10)–(3.14); the first theorem concerns a surface-piercing body and the second theorem deals with a totally submerged body.

4.1. *John's theorem*

We begin with the uniqueness theorem which slightly generalises that of John (1950), being valid for B consisting of more than one connected component (see figure 1). The form (3.13) and (3.14) of coupling conditions allows us to give a simple proof. We recall that this theorem was proved under the assumption that a body satisfies the following geometric restriction (nowadays, it is referred to as John's condition): no point of S lies on vertical lines going through F (see figure 1). This condition is essential for proving the following assertion (cf. Lemma II in John 1950).

Lemma 1. *Let body's submerged part B satisfy John's condition, and let the free surface F be connected. If relations* (3.10)–(3.12) *hold for $\phi \in H^1(W)$, then*

$$\int_S \phi\, \partial_n \phi\, \mathrm{d}s > 0 \tag{4.1}$$

unless ϕ vanishes identically in $\overline{W}$.

Remark 3. Implications that arise when F has a component separated by the body $\widehat{B}$ from infinity are discussed by Kuznetsov *et al.* (2002, see in particular §§ 4.2.1, 4.2.3 and 4.2.5 therein).

THEOREM 1. *Let the geometrical assumptions of Lemma 1 hold, then problem* (3.10)–(3.14) *has only a trivial solution in* $H^1(W) \times \mathbb{R}^6$ *provided* $|\omega|$ *is sufficiently large.*

Proof. Let us assume that $(\phi, \boldsymbol{\xi})$ is a non-trivial solution. It is clear that equality (3.4) holds for this solution. Since the right-hand side in this equality is negative in view of (4.1) and the assumption made, we get a contradiction provided the matrix $\omega^2\boldsymbol{E} - g\boldsymbol{K}$ is positive definite which is true for sufficiently large $|\omega|$. □

REMARK 4. It is clear from the proof of Theorem 1 that its assertion is valid for $\omega^2 > \lambda_*$, where $\lambda_* > 0$ is the largest λ such that $\det(\lambda\boldsymbol{E} - g\boldsymbol{K}) = 0$.

EXAMPLE 1. Let $\widehat{B}$ be a rectangular box with a square horizontal cross-section (John's condition is fulfilled for this body, and so Theorem 1 is valid for it). It is easy to calculate λ_* in the case when the density ρ is constant. Indeed, $\lambda_* g^{-1}$ is the largest eigenvalue of $\boldsymbol{E}^{-1}\boldsymbol{K}$, which is a diagonal matrix for such a box provided the frame of reference is chosen so that the y-axis goes through the centre of mass and the horizontal axes are parallel to the corresponding edges. Using formulae (2.8) and (2.10), one easily expresses the elements of this matrix in terms of ρ/ρ_0, b and ℓ, where the latter two values are the height of the box and the length of its horizontal edge, respectively.

A straightforward calculation gives that

$$\boldsymbol{E}^{-1}\boldsymbol{K} = d^{-1}\,\mathrm{diag}(0, 0, 0, 1, 1 - \Delta, 1 - \Delta), \quad \Delta = \frac{b^2 + 6d(b - d)}{b^2 + \ell^2}, \tag{4.2}$$

where d stands for the draft of the box; moreover, ρ/ρ_0, appearing in (2.10), is replaced by d/b by virtue of Archimedes' law. (Note that the second additional condition is an immediate consequence of box's symmetry, whereas the requirement that the matrix $\boldsymbol{K}'$ is positive definite reduces to the inequality $\ell^2 > 6d(b - d)$, which we assume to hold.) Now one immediately obtains that the largest eigenvalue of $\boldsymbol{E}^{-1}\boldsymbol{K}$ is equal to d^{-1}, which implies that $\lambda_* = g/d$.

4.2. *Uniqueness theorem for a totally submerged body*

We state that a totally submerged body satisfies the prism condition if there exists an appropriate Cartesian frame of reference (x_1', x_2', y'), obtained by a rotation about a vertical axis and a horizontal translation of the (x_1, x_2, y)-coordinates, and such that this body is a subset of

$$P = \{(r', \theta', x_2') : x_2' \in \mathbb{R},\ \theta' \in (-\pi + 1, -1),\ -h < r' \sin\theta' < 0\}, \tag{4.3}$$

where (r', θ', x_2') are the cylindrical coordinates such that $x_1' = r'\cos\theta'$, $y' = r'\sin\theta'$ and $\theta' \in (-\pi, 0)$ in W (see figure 2). This condition means that B is contained between the bottom and two planes inclined at the angle $(\pi/2) - 1$ to the vertical and crossing each other along an appropriately chosen line in the $\boldsymbol{x}$-plane.

THEOREM 2. *Let the totally submerged body* B *satisfy the prism condition, then problem* (3.10)–(3.14) *has only a trivial solution in* $H^1(W) \times \mathbb{R}^6$ *provided* $|\omega|$ *is sufficiently large.*

Proof. Let relations (3.10)–(3.12) hold for $\phi \in H^1(W)$. Then the so-called Maz'ya's identity is valid (see Kuznetsov *et al.* 2002, § 2.2.3)

$$\int_W (\nabla\phi)^{\mathrm{T}} \boldsymbol{Q}\, \nabla\phi \,\mathrm{d}\boldsymbol{x}'\,\mathrm{d}y' - \int_{S\cup G} |\nabla\phi|^2\, \boldsymbol{V}\cdot\boldsymbol{n}\,\mathrm{d}s + \int_F \big(\nu^2 V_{y'} + \nu[2H - \nabla_{\boldsymbol{x}'}\cdot\boldsymbol{V}]\big)\phi^2\,\mathrm{d}\boldsymbol{x}' \\ - \int_F V_{y'}|\nabla\phi|^2\,\mathrm{d}\boldsymbol{x}' + 2\int_{S\cup G}\big[\boldsymbol{V}\cdot\nabla\phi + H\phi\big]\partial_{\boldsymbol{n}}\phi\,\mathrm{d}s = 0, \quad \text{where } \nu = \omega^2/g. \tag{4.4}$$

Here $\nabla_{\boldsymbol{x}'} = (\partial_{x_1'}, \partial_{x_2'}, 0)^{\mathrm{T}}$, H is a constant, and $\boldsymbol{V} = (V_1, V_2, V_{y'})^{\mathrm{T}}$ is a vector field, whose components are real and uniformly Lipschitz functions in $\overline{W}$. Furthermore, the 3×3 matrix $\boldsymbol{Q}$ has the following elements:

$$\boldsymbol{Q}_{ij} = (\nabla\cdot\boldsymbol{V} - 2H)\delta_{ij} - (\partial_{x_j'}V_i + \partial_{x_i'}V_j), \quad i, j = 1, 2, 3, \tag{4.5}$$

where $V_3 = V_{y'}$, $x_3' = y'$ and δ_{ij} is the Kronecker delta. It is worth mentioning that $\boldsymbol{n}$ is directed out of W in (4.4), whereas the oppositely directed normal is used in Kuznetsov *et al.* (2002).

Let us assume that $(\phi, \boldsymbol{\xi})$ is a non-trivial solution. Under the geometrical assumptions made, we apply Maz'ya's identity (4.4) for ϕ, choosing $H = -1/2$ and the vector field $\boldsymbol{V} = \alpha(\theta')(x_1', 0, y')^{\mathrm{T}}$, where α is the following function:

$$\alpha(\theta') = \begin{cases} -(1+\theta'), & \text{for } \theta' \in [-1, 0], \\ 0, & \text{for } \theta' \in (-\pi+1, -1), \\ \pi - 1 + \theta', & \text{for } \theta' \in [-\pi, -\pi+1]. \end{cases} \tag{4.6}$$

(It was Weck 1990 who proposed to use these H and $\boldsymbol{V}$ for proving the uniqueness in the water-wave problem with a fixed body.) The resulting identity is as follows:

$$\int_W (\nabla\phi)^{\mathrm{T}} \boldsymbol{Q}\, \nabla\phi\,\mathrm{d}\boldsymbol{x}'\,\mathrm{d}y' - \int_G |\nabla\phi|^2\,\boldsymbol{V}\cdot\boldsymbol{n}\,\mathrm{d}\boldsymbol{x}' = \int_S \phi\,\partial_{\boldsymbol{n}}\phi\,\mathrm{d}s. \tag{4.7}$$

Here $\boldsymbol{V}\cdot\boldsymbol{n} < 0$ on G (see figure 2.4*b* in Kuznetsov *et al.* 2002, p. 80), and the matrix $\boldsymbol{Q}$ is as follows. It is equal to the unit matrix in P, whereas

$$\boldsymbol{Q} = \begin{pmatrix} 1 \mp \sin 2\theta' & 0 & \pm\cos 2\theta' \\ 0 & 1 & 0 \\ \pm\cos 2\theta' & 0 & 1 \pm \sin 2\theta' \end{pmatrix} \quad \text{in } \varPi \setminus P, \tag{4.8}$$

the upper (lower) sign must be used for $\theta' \in [-1, 0]$ ($\theta' \in [-\pi, -\pi+1]$, respectively). The eigenvalues of the latter matrix are equal to 0, 1 and 2 irrespective of to which interval θ' belongs, and so this matrix is a non-negative definite.

Combining (4.7) and (3.4), we get

$$\int_W (\nabla\phi)^{\mathrm{T}} \boldsymbol{Q}\, \nabla\phi\,\mathrm{d}\boldsymbol{x}'\,\mathrm{d}y' - \int_G |\nabla\phi|^2\,\boldsymbol{V}\cdot\boldsymbol{n}\,\mathrm{d}\boldsymbol{x}' = -\omega^2\,\boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{E}\boldsymbol{\xi} + g\boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{\xi}. \tag{4.9}$$

Since the left-hand side is positive under the assumption made, we arrive at a contradiction provided $|\omega|$ is sufficiently large (cf. the proof of Theorem 1), which proves the assertion. □

REMARK 5. The same lower estimate for ω^2 as in Remark 4 is valid under the assumptions of Theorem 2.

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