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A double obstacle model for pricing bi-leg defaultable interest rate swaps†

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Two mathematical models under so-called intensity and structure frameworks to pricing a double defaultable interest rate swap are established. The default could happen or jump to a high probability in both fixed and floating parties on the predetermined boundaries. The models lead to a new and interesting mathematical problem. As the intensity approaches infinity in designated regions, the solutions of the intensity models converge to a solution of a structure-type model which is an initial value problem of a partial differential equation coupled with two obstacles problem in their restricted regions. According to the value of the fixed rate, three cases are discussed. The free boundary that determines the swap rate and the free boundaries that determine the earlier termination of the contract (due to counterparty's default) are analysed.

Key words: Free boundary, double obstacle variational inequality, interest rate swap, bi-leg defaultable, structure and intensity model

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1 Introduction

An interest rate swap (IRS) is a popular and highly liquid financial derivative instrument [9, 20]. In a swap contract, two parties agree to exchange interest rate cash flows, based on a specified notional amount of particular currency from a fixed rate to a floating rate, or vice versa. In details, a common IRS involves two counterparts: **A**, called 'payer', and **B**, called 'receiver'. In the contract life, **A** pays a fixed rate (the swap rate) to **B**, while **A** receives from **B** a floating rate indexed to a reference rate like London Interbank Offered Rate (LIBOR), Euro Interbank Offered Rate (EURIBOR) or Shanghai Interbank Offered Rate (SHIBOR). At the point of initiation of the swap, the swap rate is chosen so that the swap has a net present value of zero. In financial market, IRSs are used for both hedging and speculating, which also can be traded as an index through the FTSE MTIRS Index. By a December 2014 statistics release, the Bank for International Settlements reported that IRSs occupied 80% of the global Over the Counter (OTC) derivative market, with the notional amount outstanding \$630 trillion and the gross market value \$21 trillion, [1].

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A contract of IRS involves the risks from not only the uncertainty of the float rate, but also the credit default [5]. An IRS exposes a party to credit risk when it is in the money and faces a possible default by the counterparty. Even at the situation of out of the money, the counterparty also has probability to default for some other reason, which will cause the end of the contract. With the introduction of credit default swaps which act as insurance in practice, the IRS is negotiated through an intermediary financial institution who usually assumes the default risk in exchange for a fixed percentage of the transaction (the bid–ask spread); in an intermediated swap, the two parties are not typically even aware of the identity of the second party to the transaction, so that if the counterparty default happened out of money, the contract can be continued by transferring to another counterparty by the intermediary. In this case, the default did not 'real' happen to the counterparty. It is equivalent to the case that the default party could 'choose' if the contract would be default when it met the default boundary. Thus, it is necessary to know how credit risks affect the value of the swap.

Pricing an IRS boils down to discounting the series of future cash flows for each leg [4]. A clear way of representing the term structure of interest rate is via a discount factor curve that reflects the present values of future payments. It is called zero-coupon bond price (ZCBP), or discount curve function. Under further assumption that the interest rate follows some stochastic process, the ZCBP can be valued; see [6, 20]. There are many researches on pricing and managing on IRS; for example, see [15, 13, 19]

To deal with the default risks, there are primarily two types of models, the structural and intensity ones (it is also called reduced form one), respectively. A structural model came from Merton's work ([18], 1974), which set predetermined barriers between asset and debt to define the time of a default. The model was developed by Black–Cox ([3], 1976) and Longstaff–Schwartz ([17], 1995), leading to so-called first-time passage model. For an intensity model, the default time is governed by a default hazard rate with parameters inferred from market data and macroeconomic variables. Duffie–Singleton ([6], 1999) and Lando ([16], 1998) provide examples of research following this approach. With a structural style's intensity rate, these two models are linked; see ([11], 2012).

To price a defaultable IRS, an intensity model is usually applied; see [2, 12, 10]. However, so far as we know, there is no references relative to using the structure-type model to price a defaultable IRS. In this paper, we propose a structure model to price an IRS defaultable by both parties. It turns to a variational inequality problem restricted in a predetermined region. This is a new mathematical problem. We use the solution of an intensity model with a structure-type intensity to approximate the solution and prove the existence and uniqueness of the solution. Also by partial differential equation (PDE) techniques, the solution is analysed. Especially discussion is on the free boundary [7], or the parity curve, from which the swap rates are determined. So far as we know, this is the first time to deal with a defaultable IRS with default probabilities in both legs under structure framework.

We say 'in the money' if the value of a contract for the underlying party is positive, otherwise 'out of the money'. It is important to evaluate a contract when a party is at default and the defaulted party is 'in the money'. Thus, for proper transfer of the underlying IRS to a new party by the intermediary, we need a reliable mechanism to evaluate defaultable IRS. For defaultable IRS, the underlying problem is not linear – i.e. the sum of two contracts with swap rates c_1 and c_2 is not the same as two contracts of mean swap rate $(c_1 + c_2)/2$. This paper proposes a model to help to evaluate the IRS when one of the parties in contract is at default. We also consider a game theory where the payer (who receives floating rate) can choose to terminate contract if the floating rate is below a predetermined floor level and the receiver (who pays floating rate) can choose to terminate the contract if the floating rate is higher than a predetermined ceiling level. The framework proposed here can be easily adapted to deal with a variety of situations. This is a new way to study IRSs, and it established a bridge between intensity and structure models. It also turns to an interesting mathematical problem. We hope that the mathematical theory developed here can offer insight in handling the existing IRSs and designing new IRSs.

In the next section, we present an intensity model and a structure-type one, respectively, which can be transferred to PDE problems [14]; by choosing a special family of intensities, called 'structure type', the intensity models approximate the structure one. In the rest sections, we present rigorous mathematical analysis.

2 Modelling

In this section, we first present a basic IRS model, based on the celebrated Cox–Ingersoll–Ross (CIR) term structure model for interest rate dynamics. Then for defaultable swaps, we present an intensity model and a structure model, and make a connection between these two models.

2.1 A framework for non-defaultable swaps

Consider an IRS of exchanging a fixed rate h to a floating rate $\{r_t\}_{t\geq 0}$ for a time period $[0, T]$ without default risk. The present value of the swap for the payer (who receives floating rate and pays the fixed rate) can be calculated by evaluating

$$V(r, T) = \mathbb{E}\left[\int_0^T [r_t - h]e^{-\int_0^t r_s ds} dt \,\middle|\, r_0 = r\right],$$

where $\mathbb{E}$ is the expectation.

Note that T here stands for the remaining time to expiry; hence when $T = 0$ we have $V(\cdot, 0) = 0$.

For definiteness, assume that the floating rate obeys the CIR model

$$dr_t = (\kappa - \beta r_t)dt + \sigma\sqrt{\max\{r_t, 0\}}\, dW_t \qquad \forall\, t > 0, \tag{2.1}$$

where κ, β and σ are positive constants and $\{W_t\}_{t\geqslant 0}$ is the standard Brownian motion. Since risk-neutral measures can hardly be used for defaults, here in this paper, all expectations are measured under the natural probability.

Then V is the solution of the initial value problem

$$\begin{cases} \mathcal{L}V(r, T) = r - h & \forall\, r > 0, T > 0, \\ V(r, 0) = 0 & \forall\, r > 0, \end{cases} \tag{2.2}$$

where $\mathcal{L}$ is the Black–Scholes operator

$$\mathcal{L} = \frac{\partial}{\partial T} + L, \qquad L = -\frac{\sigma^2 r}{2}\frac{\partial^2}{\partial r^2} - (\kappa - \beta r)\frac{\partial}{\partial r} + r\,. \tag{2.3}$$

Note that here T is time to expiry. Mathematically, for uniqueness, the solution is restricted to a certain function class. Typically, one assumes that $\sigma^2 < 2\kappa$, which implies that $\mathbb{P}(\min_{0<s<t}\{r_s\} > 0) = 1$ for every $t > 0$, where $\mathbb{P}$ stands for the probability; hence one can use the function class $L^\infty((0, , \infty)^2)$ for uniqueness. In this paper, we drop this condition. Hence, throughout this paper, we only assume that

$$\kappa > 0, \quad \sigma > 0, \quad \beta > 0.$$

It is well known that when $\sigma^2 > 2\kappa$, $\mathbb{P}(\min_{0<s<t}\{r_s\} = 0) > 0$ for every $t > 0$. Note that (2.1) corresponds to the reflection when $r_t = 0$ so

$$\mathbb{P}(\{r_t > 0\}) = 1 \quad \forall\, t > 0.$$

To accommodate this, we use the function space

$$\mathbf{X} = \{\phi \in C(\bar{Q}) \mid \partial\phi/\partial r \in L^\infty(Q)\}; \quad Q := (0,\infty)^2. \tag{2.4}$$

Although a more accurate boundary condition is

$$\lim_{r\to 0} r^{2\kappa/\sigma^2}\frac{\partial\phi(r,T)}{\partial r} = 0 \qquad \forall\, T > 0,$$

for the problem at hand, the condition $\phi_r \in L^\infty$ will suffice our need. It is shown in [4] that problem (2.2), supplemented by $\phi_r \in L^\infty$, which we call the boundary condition at the origin, is well-posed: there exists a unique solution and the solution depends continuously on the parameters $(\kappa, \beta, \sigma) \in (0,\infty)^3$.

2.2 A framework for defaultable swaps

Suppose that both parties, the payer and the receiver, are subject to default. Assume that at the time of default, the right of the defaulted party in the swap can be transferred, if he is 'in the money'. Denote by τ_1 and τ_2 the default times of the payer and the receiver, respectively. Assume that at time τ_1 ($\leqslant \tau_2 \wedge T$), the payer's right and obligation in the swap is auctioned to a new party with a certain price $\phi_1(r_{\tau_1}, T-\tau_1)$, where $T-\tau_1$ is the remaining time to expiry; at time τ_2 ($\leqslant \tau_1 \wedge T$), the receiver's right and obligation is auctioned to a new receiver resulting an obligation cost of $\phi_2(r_{\tau_2}, T-\tau_2)$ to the payer. Then, from its cash flow, a model for the value of the swap for the payer can be expressed by

$$\begin{aligned} V(r,T) = \mathbb{E}\Big[\int_0^{\tau_1\wedge\tau_2\wedge T} &(r_t - h)e^{-\int_0^t r_s ds}dt + \phi_1(r_{\tau_1}, T-\tau_1)e^{-\int_0^{\tau_1} r_s ds}\mathbf{1}_{\{\tau_1<\tau_2\wedge T\}} \\ &- \phi_2(r_{\tau_2}, T-\tau_2)e^{-\int_0^{\tau_2} r_s ds}\mathbf{1}_{\{\tau_2<\tau_1\wedge T\}}\Big| r_0 = r, \tau_1\wedge\tau_2 > 0\Big]. \end{aligned} \tag{2.5}$$

If $\tau_1 \wedge \tau_2 > t$, the value of the swap at time t for the payer is $V(r_t, T-t)$.

There are variety of ways to model ϕ_1 and ϕ_2, for example,

$$\phi_1 = (1-\varepsilon_1)V^+, \qquad \phi_2 = (1-\varepsilon_2)V^-, \tag{2.6}$$

where $V^+ := \max\{V, 0\}$ and $V^- := \max\{-V, 0\}$; here $0 \leqslant \varepsilon_1, \varepsilon_2 < 1$ are discounter proportions when a default party sells the contract to a third party.

To complete the model, it remains to specify the default times τ_1 and τ_2. There are two types of specification, an intensity one and a structure one.

2.3 An intensity model for defaultable swaps

Assume that τ_1 and τ_2 are first arrival times of Poisson processes with variable intensities $\{\lambda_{1t}\}$ and $\{\lambda_{2t}\}$, respectively. Let $\{\mathcal{F}_t\}_{t\geqslant 0}$ be the information filtration generated by $\{r_t, \lambda_{1t}, \lambda_{2t}\}_{t\geqslant 0}$. Assume that the Poisson processes are conditionally independent in the sense that, for $t > s \geqslant 0$,

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$$\mathbb{P}(\tau_1 \wedge \tau_2 > t \mid \mathcal{F}_t, \tau_1 \wedge \tau_2 > s) = e^{-\int_s^t (\lambda_{1s}+\lambda_{2s})ds},$$
$$\mathbb{P}(\tau_1 \in (t-dt, t], \tau_2 > t \mid \mathcal{F}_t, \tau_1 \wedge \tau_2 > s) = \lambda_{1t} e^{-\int_s^t (\lambda_{1s}+\lambda_{2s})ds} dt,$$
$$\mathbb{P}(\tau_2 \in (t-dt, t], \tau_1 > t \mid \mathcal{F}_t, \tau_1 \wedge \tau_2 > s) = \lambda_{2t} e^{-\int_s^t (\lambda_{1s}+\lambda_{2s})ds} dt.$$

Denote by $\mathbb{E}^{0,r}$ the expectation conditioned on $r_0 = r$ and $\tau_1 \wedge \tau_2 > 0$. We can evaluate (2.5) by

$$\begin{aligned} V(r,T) &= \mathbb{E}^{0,r}\Big[\int_0^\infty \Big(\int_0^{\theta\wedge T} (r_t - h) e^{-\int_0^t r_s ds} dt\Big) d\Big(-e^{-\int_0^\theta (\lambda_{1s}+\lambda_{2s})ds}\Big) \\ &\quad + \int_0^T \Big(\phi_1(r_t, T-t)\lambda_{1t} - \phi_2(r_t, T-t)\lambda_{2t}\Big) e^{-\int_0^t (\lambda_{1s}+\lambda_{2s}+r_s)ds} dt\Big] \\ &= \mathbb{E}^{0,r}\Big[\int_0^T \Big(r_t - h + \phi_1(r_t, T-t)\lambda_{1t} - \phi_2(r_t, T-t)\lambda_{2t}\Big) e^{-\int_0^t (\lambda_{1s}+\lambda_{2s}+r_s)ds} dt\Big], \end{aligned}$$

where in the second equation, we have used integration by parts and an assumption that $\tau_1 \wedge \tau_2 < \infty$ almost surely.

Assume for simplicity that the intensities depend only on the variable interest rate via

$$\lambda_{it} = \Lambda_i(r_t), \quad i = 1, 2, \tag{2.7}$$

where $\Lambda_i(\cdot)$ are known non-negative functions. Also assume that $\{r_t\}$ obeys the CIR model (2.1). Then by the Feynman–Kac Formula, V is the solution of the initial value problem

$$(\mathcal{L} + \Lambda_1 + \Lambda_2)V = f + \Lambda_1\phi_1 - \Lambda_2\phi_2 \ \text{ on } \ (0,\infty)^2, \quad V(\cdot, 0) = 0,$$

where $f = r - h$ and $\mathcal{L}$ is as in (2.3). We solve the problem in the class $V \in \mathbf{X}$ defined in (2.4).

Finally, for definiteness, we take ϕ_1, ϕ_2 as in (2.6) with $\varepsilon_1 = 0, \varepsilon_2 = 0$. Then we obtain the following mathematical formulation for an evaluation of the defaultable swap for the payer:

$$\begin{cases} \mathcal{L}V - \Lambda_1 V^- + \Lambda_2 V^+ = f \ \text{ on } \ Q := (0,\infty)^2, \\ V(\cdot, 0) = 0, \quad V \in C(\bar{Q}), \quad V_r \in L^\infty(Q). \end{cases} \tag{2.8}$$

Remark 2.1 *Note that when $\varepsilon_1 = \varepsilon_2 = 0$ in (2.6), one cannot really distinguish a party defaulting or the same party simply selling his side of contract to a third party. Thus, talking positive ε_1 and ε_2 are more realistic. Our analysis for the intensity model works for general choices of ϕ_1 and ϕ_2. However, for the connection of structure and intensity models, the current analysis does not apply to the case of positive ε_1 and ε_2.*

We shall use the method introduced in [4] to show the well-posedness of (2.8) in Section 3.

2.4 A structure model for defaultable swaps

As we claim in the introduction, a defaultable swap would not be terminated when it was in money for the payer's at her default boundary and vice versa. It is equivalent that the both parities could choose to default, just like a game. Based on the limit of intensity models to be presented later, here we consider a game theory model. Assume that the payer does not really default but instead can choose to terminate the contract at any time $t \in [0, T]$, as long as $r_t \leqslant b_1$. Similarly, assume that the receiver can choose to terminate the contract at any time $t \in [0, T]$, as long as $r_t \geqslant b_2$. Hence, we define the admissible strategies of the payer and the receiver by

$$\mathcal{A}_1 = \{\tau \mid \tau \text{ is a non-negative stopping time}, r_\tau \leqslant b_1\},$$
$$\mathcal{A}_2 = \{\tau \mid \tau \text{ is a non-negative stopping time}, r_\tau \geqslant b_2\}.$$

Then we can evaluate the 'defaultable' swap for the payer by

$$u(r, T) = \inf_{\tau_2 \in \mathcal{A}_2} \sup_{\tau_1 \in \mathcal{A}_1} \mathbb{E}\left[\int_0^{\tau_1 \wedge \tau_2 \wedge T} (r_t - h) e^{-\int_0^t r_\theta d\theta} dt \,\middle|\, r_0 = r \right]. \tag{2.9}$$

We can explain our problem in terms of a game theory. For two parties **A** and **B**, if **A** knows that **B** will take a strategy τ_2, he can find and choose the best counter-strategy $\tau_1 = F(\tau_2)$ that maximises the expectation. Analogously, if **B** knows that **A** will take a strategy τ_1, he can find an optimal counter-strategy $\tau_2 = G(\tau_1)$. Assume that the two curves $\tau_1 = F(\tau_2)$ and $\tau_2 = G(\tau_1)$ admit a unique intersection, say (τ_1^*, τ_2^*), then (τ_1^*, τ_2^*) is the well-known Nash equilibrium and the sup & inf in (2.9) are interchangeable. As an equilibrium, **A** takes strategy τ_1^* and **B** takes strategy τ_2^*.

Assume for simplicity that $b_1 < b_2$. Using dynamical programming, one can formally derive that

$$\mathcal{F}[u] = 0 \quad \text{on } (0, \infty)^2, \qquad u(\cdot, 0) = 0, \qquad u \in \mathbf{X}, \tag{2.10}$$

where

$$\mathcal{F}[u] := (\mathcal{L}u - f)\mathbf{1}_{(b_1, b_2)}(r) + \min\{\mathcal{L}u - f, u\}\mathbf{1}_{(0, b_1]}(r) + \max\{\mathcal{L}u - f, u\}\mathbf{1}_{[b_2, \infty)}(r);$$

here $\mathbf{1}_\Omega$ is the characteristic function of the set Ω: $\mathbf{1}_\Omega(r) = 1$ if $r \in \Omega$ and $\mathbf{1}_\Omega(r) = 0$ if $r \notin \Omega$. This is an initial value problem of a PDE coupled with two obstacle problems in their restricted regions. For such a swap with expiry T, the optimal strategy is

$$\tau_1^* = \min\{t \in [0, T] \mid t = T \text{ or } r_t \leqslant b_1 \text{ and } u(r_t, T - t) = 0\},$$
$$\tau_2^* = \min\{t \in [0, T] \mid t = T \text{ or } r_t \geqslant b_2 \text{ and } u(r_t, T - t) = 0\}.$$

The corresponding infinite ($T = \infty$) horizon problem is denoted as follows:

$$\mathfrak{F}[v] := \min\{Lv - f, v\}\mathbf{1}_{(0, b_1]} + (Lv - f)\mathbf{1}_{(b_1, b_2)} + \max\{Lv - f, v\}\mathbf{1}_{[b_2, \infty)} = 0, \tag{2.11}$$

where L is defined in (2.3). The optimal strategy is

$$\tau_1^{**} = \inf\{t \geq 0 \mid r_t \leqslant b_1 \text{ and } v(r_t) = 0\},$$
$$\tau_2^{**} = \inf\{t \geq 0 \mid r_t \geqslant b_2 \text{ and } v(r_t) = 0\}.$$

Once we know the optimal strategy, we can use Ito Lemma to verify rigorously that u in (2.9) is the unique solution of (2.10); we omit the details. We shall study the infinite horizon problem ($T = \infty$) in Section 4 and the finite horizon problem in Section 5.

For rigorous mathematics, we solve (2.10) using viscosity solutions defined in the following.

Definition 2.1 *A viscosity solution of (2.10) is a Lipschitz continuous function u defined on* $[0, \infty)$ *that satisfies the following:*

1. *if* $r > 0$ *and* $\zeta \in C^2((0, \infty))$ *satisfies* $\zeta(r) = u(r)$ *and* $\zeta \geqslant u$ *in* $(0, \infty)$*, then* $\mathfrak{F}[\zeta](r) \leqslant 0$*;*
2. *if* $r > 0$ *and* $\zeta \in C^2((0, \infty))$ *satisfies* $\zeta(r) = u(r)$ *and* $\zeta \leqslant u$ *in* $(0, \infty)$*, then* $\mathfrak{F}[\zeta](r) \geqslant 0$.

2.5 A link from the intensity model to the structure one

Among the intensity models, there are special types, called 'structure types', whose limits are structure models. To demonstrate this, we consider a special family of structure-type intensity models characterised by

$$\Lambda_1(r) = p\mathbf{1}_{[0,b_1]}(r) = p\,\mathrm{H}(b_1 - r), \qquad \Lambda_2(r) = q\mathbf{1}_{[b_2,\infty]}(r) = q\,\mathrm{H}(r - b_2), \tag{2.12}$$

where p, q, b_1, b_2 are positive constants and H is the Heaviside function: $\mathrm{H}(x) = 0$ if $x < 0$ and $\mathrm{H}(x) = 1$ if $x > 0$.

Denote by τ_{1p} and τ_{2q} the corresponding stopping times. Since an intensity is the number of occurrence per unit time, we find that

$$\lim_{p\to\infty} \tau_{1p} = \hat{\tau}_1 := \inf\{t > 0 \mid r_t < b_1\},$$
$$\lim_{q\to\infty} \tau_{2q} = \hat{\tau}_2 := \inf\{t > 0 \mid r_t > b_2\}.$$

Since each party has his choice of transfer (auction sale) at time of his default, direct valuation of the structure model with the payer's default time $\hat{\tau}_1$ and receiver's default time $\hat{\tau}_2$ is very difficult. Nevertheless, we shall show that as $p, q \to \infty$, the underlying structure-type intensity model (2.8) approaches the structure model (2.10). Detailed analysis will be given in Section 3.

The rest of the paper is organised as follows. In Section 3, we first establish the well-posedness of the intensity model (2.8). Then we take the 'structure-type' intensity (2.12) and show that as $p, q \to \infty$, the solutions of the underlying intensity model (2.8) approach the solution of the structure model (2.10). In Section 4, we study the infinite horizon problem for the structure model, i.e. evaluate the function defined in (2.9) with $T = \infty$. We provide qualitative description of the value of the IRS in terms of the swap rate h, as well as the optimal strategies for the payer and the receiver. We study the structure model (2.10) in Section 5. Section 6 is our conclusion and remarks.

3 Intensity models and its connection to structure model

This section is a pure mathematical analysis aiming at the well-posedness of the intensity model and its connection to the structure model. We first as a preparation study the operator L in (2.3). Then we study the semi-linear equation (2.8), using the technique introduced in [4] for general non-negative Λ_1 and Λ_2. Then we take the special choice $\Lambda_1(r) = p\,\mathrm{H}(b_1 - r)$ and $\Lambda_2 = q\mathrm{H}(r - b_2)$ to investigate the asymptotic behaviour of the solution as $p, q \to \infty$.

Using the conventional notation, in the sequel we may write $u(x, T)$ as $u(x, t)$ and ∂_T as ∂_t. Also we use u_r and u_t for partial derivatives with respect to r and t.

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3.1 The operator L

The operator L defined in (2.3), generated from the Black–Scholes theory for the CIR model, is important in our analysis. Here we first construct a few auxiliary functions and then study the ordinary differential equation (ODE) $Lu = F$ on a bounded interval with homogeneous Neumann boundary conditions, where F is a function with at most a linear growth.

We begin with finding a fundamental solution set of the homogeneous linear equation $L\varphi = 0$ on $(0, \infty)$. The solutions are obtained by using the Laplace Transform. With suitable normalisation, we have the following.

Lemma 3.1 *Let L be defined in (2.3) where κ, β and σ are positive constants. Set*

$$s_1 = \frac{-\beta + \sqrt{\beta^2 + 2\sigma^2}}{\sigma^2}, \quad s_2 = \frac{-\beta - \sqrt{\beta^2 + 2\sigma^2}}{\sigma^2}, \quad \alpha_i = \frac{|s_i|}{s_1 - s_2}\,\frac{2\kappa}{\sigma^2}.$$

The followings are two linearly independent solutions of $L\varphi = 0$ on $(0, \infty)$:

$$\varphi_1(r) = \int_{s_1}^{\infty} (s - s_1)^{\alpha_1 - 1}(s - s_2)^{\alpha_2 - 1} e^{-rs} ds \Big/ \int_0^{\infty} t^{\alpha_1 + \alpha_2 - 1} e^{-t} dt,$$
$$\varphi_2(r) = \int_{s_2}^{s_1} (s_1 - s)^{\alpha_1 - 1}(s - s_2)^{\alpha_2 - 1} e^{-rs} ds \Big/ \int_{s_2}^{s_1} (s_1 - s)^{\alpha_1 - 1}(s - s_2)^{\alpha_2 - 1} ds.$$

In addition, φ_1 is smooth on $(0, \infty)$, φ_2 is analytic on $\mathbb{C}$ and for some positive constants c_i,

$$\varphi_1 > 0, \quad \varphi_1' < 0, \quad \varphi_2 > 0, \quad \varphi_2' > 0 \text{ in } (0, \infty),$$
$$|\varphi_i'(r)| = r^{-\alpha_i} e^{-s_i r}[c_i + O(r^{-1})] \text{ as } r \to \infty,$$
$$\varphi_1'(r) = [1 + O(r)] r^{-2\kappa/\sigma^2}, \quad \varphi_2(r) = 1 + O(r^2) \text{ as } r \searrow 0.$$

Proof The integrals in the definitions of φ_1 and φ_2 are uniformly convergent. As integrations and differentiations are interchangeable, for $r > 0$, we have

$$L\varphi_1(r) = \int_{s_1}^{\infty} \frac{(s - s_1)^{\alpha_1 - 1}(s - s_2)^{\alpha_2 - 1} e^{-rs}}{\Gamma(\alpha_1 + \alpha_2)} \Big[-\frac{\sigma^2}{2} rs^2 + (k - \beta r)s + r \Big] ds$$
$$= \frac{1}{\Gamma(\alpha_1 + \alpha_2)} \int_{s_1}^{\infty} \frac{d}{ds}\Big(\frac{\sigma^2}{2} (s - s_1)^{\alpha_1}(s - s_2)^{\alpha_2} e^{-rs} \Big) ds = 0,$$

by the special choices of s_1, s_2, α_1 and α_2. Similarly, we obtain $L\varphi_2(r) = 0$. The particular choice of the divisors $\int_0^\infty t^{\alpha_1+\alpha_2-1}e^{-t}dt$ and $\int_{s_1}^{s_2} (s - s_1)^{\alpha_1 - 1}(s - s_2)^{\alpha_2 - 1} ds$ provides the properties of (φ_1, φ_2) listed in the Lemma. We call it a standard fundamental solution set. □

Lemma 3.2 (i) *For each $\varepsilon \geqslant 0$, $\psi_1(r) := \sqrt{\kappa + \sigma^2/2 + (r - \varepsilon)^2}$ satisfies $L\psi_1 \geqslant 0$ in $[\varepsilon, \infty)$.*

(ii) *The inequality $L\psi_2(r) \geqslant 2b \max\{1, r\}$ and $\mathcal{L}\psi_2 > b\psi_2$ are satisfied by the following:*

$$a = \min\Big\{1, \frac{1}{\sigma^2 + 2\beta + 2\kappa}\Big\}, \quad b = a\kappa, \quad \psi_2(r) = \begin{cases} 1 & \textit{if } r \geqslant 1, \\ 1 + a(1 - r)^2 & \textit{if } r \in [0, 1]. \end{cases} \tag{3.1}$$

Proof **(i)** Set $A = \kappa + \sigma^2/2$. Direct calculation gives, for $r \geqslant \varepsilon$,

$$L\psi_1(r) = \frac{1}{\psi_1}\Big\{ r\Big[-\frac{\sigma^2}{2}\frac{A}{A + (r - \varepsilon)^2} - \kappa + A + (r - \varepsilon)^2 \Big] + \beta[r - \varepsilon] + \kappa\varepsilon \Big\} > 0.$$

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(ii) When $r > 1$, $L\psi_2 = r \geqslant 2br$. When $r \in [0, 1]$,

$$L\psi = 2a\kappa + r[1 - a(\sigma^2 + 2\kappa + 2\beta) + 2a\beta r + a(1-r)^2] \geqslant 2a\kappa = 2b.$$

Thus, $L\psi \geqslant 2b \max\{1, r\}$. Since $1 \leqslant \psi \leqslant 2$, we also have $L\psi > b\psi$. □

Next we study the equation $Lu = F$, where F may have a linear growth. For this, we introduce

$$\|F\| := \sup_{r>0} \frac{|F(r)|}{\max\{1, r\}}.$$

We say that F is at most linear growth if $\|F\| < \infty$.

Lemma 3.3 *Let c be a non-negative function on $[0, \infty)$, bounded near the origin. Let F be a function on $[0, \infty)$ with at most a linear growth. Fix $\varepsilon \in (0, 1/2]$ and let u be the solution of*

$$Lu + cu = F \ \text{ in } \ (\varepsilon, \varepsilon^{-1}), \qquad u'(\varepsilon) = 0, \quad u'(\varepsilon^{-1}) = 0. \tag{3.2}$$

Extend u to $[0, \infty)$ by setting $u(\cdot) = u(\varepsilon)$ on $[0, \varepsilon)$ and $u(\cdot) = u(\varepsilon^{-1})$ on $[\varepsilon^{-1}, \infty)$. Then there exists a positive constant K that depends only on κ, β and σ such that

$$\|u\|_{L^\infty((0,\infty))} \leqslant K\, \|F\|,$$
$$|u'(r)| \leqslant K \, \min\left\{1 + r^{-2\kappa/\sigma^2}, \ 1 + \|c\|_{L^\infty((0,r))}\right\} \|F\| \quad \forall\, r > 0.$$

The key here is that for any $B > 0$, the bounds of u' on $[B, \infty)$ does not depend on c.

Proof Since $c(r) + r \geqslant \varepsilon$ for $r \in [\varepsilon, \varepsilon^{-1}]$, the second-order elliptic operator $L + c$ on $(\varepsilon, \varepsilon^{-1})$, together with the homogeneous Neumann boundary condition is invertible (cf. [8]), the boundary value problem (3.2) admits a unique solution. Next, set $\bar{u} = \|F\|\, \psi_2(r)/2b$, where (b, ψ_2) is defined in (3.1). Note that $\bar{u}'(\varepsilon) < 0$, $\bar{u}'(\varepsilon^{-1}) = 0$ and $(L + c)\bar{u} \geqslant L\bar{u} \geqslant \|F\| \, \max\{1, r\} \geqslant |F|$. Hence, by comparison, $\pm u \leqslant \bar{u}$. Thus,

$$\|u\|_{L^\infty((0,\infty))} \leqslant \|\bar{u}\|_{L^\infty((0,\infty))} = b^{-1}\, \|F\|\,.$$

Next we estimate u'. First we consider the case $F \geqslant 0$. By the maximum principle, $u \geqslant 0$ in $[0, \infty)$. Set

$$W(r) = r^{\mu} e^{-\nu r}, \qquad \mu = \frac{2\kappa}{\sigma^2}, \qquad \nu = \frac{2\beta}{\sigma^2}.$$

We derive from (3.2) that

$$-\left(Wu'\right)' = \frac{2WF}{\sigma^2 r} - \left(1 + \frac{c(r)}{r}\right)\frac{2Wu}{\sigma^2} \leqslant \frac{2WF}{\sigma^2 r}. \tag{3.3}$$

For $r \in [1, \varepsilon^{-1})$, integrating the inequality in (3.3) over $[r, \varepsilon^{-1}]$, we then obtain

$$u'(r) \leqslant \frac{2}{\sigma^2} \int_r^{\varepsilon^{-1}} \frac{W(\rho)}{W(r)} \frac{F(\rho)}{\rho} d\rho \leqslant \frac{2\|F\|}{\sigma^2} \int_r^{\infty} \frac{W(\rho)}{W(r)} d\rho \leqslant \frac{2\|F\|}{\sigma^2} \int_0^{\infty} \left(1 + z\right)^{\mu} e^{-\nu z} dz.$$

Next, for each $r \in [2, \varepsilon^{-1}]$, there exists $\xi \in [r-1, r]$ such that, by the mean value theorem,

$$u'(\xi) = u(r) - u(r-1) \geqslant -\|u\|_{L^\infty} \geqslant -\|F\|/b.$$

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Thus, integrating the inequality in (3.3) over $[\xi, r]$, we obtain

$$u'(r) \geqslant \frac{W(\xi)}{W(r)}u'(\xi) - \frac{2}{\sigma^2}\int_\xi^r \frac{W(\rho)}{W(r)}\frac{F(\rho)}{\rho}d\rho \geqslant -\Big(\frac{e^\nu}{b} + \frac{2e^\nu}{\sigma^2}\Big)\|F\|.$$

Thus, there exists a positive constant K_1 that depends only on κ, β and σ such that

$$|u'(r)| \leqslant K_1\|F\| \quad \forall\, r \in [2, \varepsilon^{-1}].$$

Next we estimate u' on $[\varepsilon, 2]$. We begin with integrating the equation (3.3) over $[\varepsilon, \varepsilon^{-1}]$ to obtain the identity

$$\int_\varepsilon^{\varepsilon^{-1}} \frac{W(\rho)F(\rho)}{\rho}d\rho = \int_\varepsilon^{\varepsilon^{-1}} \Big(1 + \frac{c(\rho)}{\rho}\Big)W(\rho)u(\rho)d\rho.$$

Now integrating the equation in (3.3) over $r \in [\varepsilon, r]$, we obtain

$$-\int_\varepsilon^r \Big(1 + \frac{c(\rho)}{\rho}\Big)u(\rho)\}d\rho \leqslant -\frac{\sigma^2}{2}W(r)u'(r) \leqslant \int_\varepsilon^r \frac{W(\rho)F(\rho)}{\rho}d\rho.$$

Thus, for each $r \in [\varepsilon, \varepsilon^{-1}]$,

$$\frac{\sigma^2}{2}W(r)|u'(r)| \leqslant \int_\varepsilon^{\varepsilon^{-1}} \frac{W(\rho)F(\rho)}{\rho}d\rho \leqslant \|F\| \int_0^\infty \max\{\rho^\mu, \rho^{\mu-1}\}e^{-\nu\rho}d\rho.$$

Thus, there exists a constant K_2 that depends only on κ, β and σ such that

$$|u'(r)| \leqslant K_2\|F\|r^{-\mu}e^{\nu r} \quad \forall\, r \in [\varepsilon, \varepsilon^{-1}].$$

Finally, we estimate $u'(r)$ using bonds of c. Integrating the equation in (3.3) over $[\varepsilon, r]$, we obtain

$$\begin{aligned}\frac{\sigma^2}{2}W(r)|u'(r)| &\leqslant \int_\varepsilon^r \frac{W(\rho)F(\rho)}{\rho}d\rho + \int_\varepsilon^r \Big(1 + \frac{c(\rho)}{\rho}\Big)W(\rho)u(\rho)d\rho \\ &\leqslant \|F\| \int_0^r \rho^{\mu-1}d\rho + \|u\|_{L^\infty} \int_0^\rho \Big[\rho^\mu + c(\rho)\rho^{\mu-1}\Big]d\rho \\ &\leqslant \|F\|\, r^\mu \Big\{\frac{1}{\mu} + \frac{1}{b}\Big[\frac{1}{\mu+1} + \frac{\|c\|_{L^\infty((0,r)}}{\mu}\Big]\Big\}.\end{aligned}$$

Thus, there exists a constant K_3 depending only κ, β and σ such that

$$|u'(r)| \leqslant K_3\big(1 + \|c\|_{L^\infty((0,r))}\big)\|F\| \qquad \forall\, r \in [\varepsilon, 1]. \tag{3.4}$$

This completes the proof of the lemma for the case $F \geqslant 0$ (recalling $u' = 0$ on $[0, \varepsilon] \cup [\varepsilon^{-1}, \infty)$).

For general F, we write $F = F^+ - F^-$ and $u = u_+ - u_-$, where $u_\pm$ are solutions of

$$Lu_\pm + cu_\pm = F^\pm \text{ in } (\varepsilon, \varepsilon^{-1}), \quad u'_\pm(\varepsilon) = 0, u'_\pm(\varepsilon^{-1}) = 0.$$

Since $F^\pm \geqslant 0$ and $\|F^\pm\| \leqslant \|F\|$, applying the previous estimate for $u'_\pm$ we then obtain the assertion of Lemma 3.3. □

3.2 Well-posedness of intensity model

Now we are ready to study the intensity model (2.8).

Theorem 1 *Let κ, β and σ be positive constants, $\mathcal{L}$ be defined in (2.3), $\Lambda_1(\cdot)$ and $\Lambda_2(\cdot)$ be non-negative and locally bounded functions defined on $[0, \infty)$ and f be a function on $[0, \infty)$*

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that has at most a linear growth. Then problem (2.8) admits a unique solution. In addition, there exists a positive constant K that depends only on κ, β and σ such that

$$\|V\|_{L^\infty((0,\infty)^2)} + \|V_t\|_{L^\infty((0,\infty)^2)} \leqslant K\|\!|f|\!\|,$$
$$|V_r(r,t)| \leqslant K \min\{1 + r^{-2\kappa/\sigma^2}, 1 + \|\Lambda_1 + \Lambda_2\|_{L^\infty((0,r))}\} \|\!|f|\!\| \quad \forall\, r>0, t>0.$$

Furthermore, there exists a positive constant b that depends only on κ, β and σ such that

$$\|V(\cdot,t) - V_*(\cdot)\|_{L^\infty((0,\infty))} \leqslant 2K\,\|\!|f|\!\|e^{-bt} \quad \forall\, t>0;$$

here V_* is the unique solution of the corresponding infinite horizon problem:

$$LV_* - \Lambda_1 V_*^- + \Lambda_2 V_*^+ = f \quad \text{in } (0,\infty), \qquad V_*' \in L^\infty((0,\infty)). \tag{3.5}$$

Proof 1. We begin with the uniqueness proof, using a contradiction argument. Suppose V_1 and V_2 are solutions of (2.8) and there exist positive r_0 and t_0 such that $V_1(r_0,t_0) > V_2(r_0,t_0)$. Let φ_1 and φ_2 be those defined in Lemma 3.1. Set $\varepsilon = [V_1(r_0,t_0) - V_2(r_0,t_0)]/[2\varphi_1(r_0) + \varphi_2(r_0)]$ and

$$w(r,t) = V_1(r,t) - V_2(r,t) - \varepsilon[\varphi_1(r) + \varphi_2(r)] \quad \forall\, r>0, t\in[0,t_0].$$

Since V_{1r} and V_{2r} are bounded and $\lim_{r\to\infty}[\varphi_1'(r) + \varphi_2'(r)] = \infty$ and $\lim_{r\searrow 0}[\varphi_1'(r) + \varphi_2'(r)] = -\infty$, there exist positive constants A and B such that

$$w_r > 0 \text{ in } (0,A]\times[0,t_0], \qquad w_r < 0 \text{ in } [B,\infty)\times[0,t_0].$$

Consequently, there exist $\hat{r} \in (A,B)$ and $\hat{t} \in (0,t_0]$ such that

$$w(\hat{r},\hat{t}) = \max_{[A,B]\times[0,t_0]} w = \max_{(0,\infty)\times[0,t_0]} w \geqslant w(r_0,t_0) > 0.$$

Now set $\Omega = (A,B)\times(0,t_0]$ and consider the function $\zeta(r,t) = w(r,t) - w(\hat{r},\hat{t})$ for $(r,t)\in\bar{\Omega}$.

First we have $\zeta < 0$ on the parabolic boundary of Ω. Next,

$$\mathcal{L}\zeta = \mathcal{L}V_1 - \mathcal{L}V_2 - rw(\hat{r},\hat{t}) = \Lambda_1[V_1^- - V_2^-] - \Lambda_2[V_1^+ - V_2^+] - rw(\hat{r},\hat{t}).$$

By the mean value theorem, there exist functions C_1 and C_2 such that $V_1^- - V_2^- = -C_1[V_1 - V_2]$ and $V_1^+ - V_2^+ = C_2[V_1 - V_2]$. In addition, $0 \leqslant C_1 \leqslant 1$ and $0 \leqslant C_2 \leqslant 1$. It then follows that

$$\mathcal{L}\zeta + [C_1\Lambda_1 + C_2\Lambda_2]\zeta = -[C_1\Lambda_1 + C_2\Lambda_2][\varepsilon\varphi_1 + \varepsilon\varphi_2 + w(\hat{r},\hat{t})] - rw(\hat{r},\hat{t}) < 0,$$

for all $(r,t)\in\bar{\Omega}$. Hence, applying the maximum principle for ζ on $\bar{\Omega}$ we find then $\zeta < 0$ in Ω. But this contradicts $\zeta(\hat{r},\hat{t}) = 0$. This contradiction proves the uniqueness of the solution of (2.8).

2. For existence, we first consider approximation problems. Denote $g(r,v) = \Lambda_2(r)v^+ - \Lambda_1(r)v^-$. For each $\varepsilon\in(0,1/2]$, let v^ε be the solution of the semi-linear initial boundary value problem

$$\begin{cases} \mathcal{L}v^\varepsilon + g(r,v^\varepsilon) = f & \text{on } (\varepsilon,\varepsilon^{-1})\times(0,\infty), \\ v_r^\varepsilon = 0 & \text{on } ([0,\varepsilon]\cup[\varepsilon^{-1},\infty))\times[0,\infty), \\ v_\varepsilon(\cdot,0) = 0 & \text{on } [0,\infty)\times\{0\}. \end{cases}$$

By a classical PDE theory, there exists a unique solution. We now provide *a priori* estimates.

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Let ψ_2 be as in Lemma 3.2. Set $\bar{v} = \psi_2 \|f\|/2b$. Then $\bar{v}_r(\varepsilon) < 0 = \bar{v}_r(\varepsilon^{-1})$, and $\mathcal{L}\bar{v} + g(r, \bar{v}) \geqslant \mathcal{L}\bar{v} \geqslant \|f\| \max\{1, r\} \geqslant |f(r)|$. Thus by comparison, $v^\varepsilon \leqslant \bar{v}$. Similarly comparing v^ε with $-\bar{v}$, we have $v^\varepsilon \geqslant -\bar{v}$. Thus,

$$\|v^\varepsilon\|_{L^\infty((0,\infty)^2)} \leqslant \|\bar{v}\|_{L^\infty((0,\infty))} \leqslant b^{-1} \|f\|.$$

Next, set $w^\varepsilon = v_t^\varepsilon$. Differentiating the equations for v^ε with respect to t, we obtain

$$\begin{cases} \mathcal{L}w^\varepsilon + g_v(r, v^\varepsilon) w^\varepsilon = 0 & \text{on } (\varepsilon, \varepsilon^{-1}) \times (0, \infty), \\ w_r^\varepsilon = 0 & \text{on } ([0, \varepsilon] \cup [\varepsilon^{-1}, \infty)) \times (0, \infty), \\ w_\varepsilon(\cdot, 0) = f & \text{on } (\varepsilon, \varepsilon^{-1}) \times \{0\}. \end{cases}$$

Let $\bar{w}^\varepsilon := b_1 \|f\| \psi_1(r)$, where ψ_1 is as in Lemma 3.2 and $b_1 = \max\{2, (\kappa + \sigma^2/2)^{-1/2}\}$. Since $g_v \geqslant 0$, comparing $\pm w^\varepsilon$ with $\bar{w}^\varepsilon$ we obtain $\pm w^\varepsilon \leqslant \bar{w}^\varepsilon$ on $[\varepsilon, \varepsilon^{-1}] \times [0, \infty)$. Thus, $|w^\varepsilon| \leqslant \bar{w}^\varepsilon \leqslant K_3 \|f\| \max\{1, r\}$, where $K_3 = b_1 \sqrt{1 + \kappa + \sigma^2/2}$. Hence, we have

$$\|v_t^\varepsilon(\cdot, t)\| \leqslant K_3 \|f\| \quad \forall\, t > 0.$$

Finally, we estimate v_r^ε. Fix $t > 0$. Set $u(r) = v^\varepsilon(r, t)$, $c(r) = \Lambda_1(r) \mathbf{1}_{v_\varepsilon(r,t)<0} + \Lambda_2(r) \mathbf{1}_{v_\varepsilon(r,t)>0}$ and $F(r) = f(r) - v_t^\varepsilon(r, t)$. Then u is the unique solution of (3.2). Applying Lemma 3.3, we obtain

$$|v_r^\varepsilon(r, t)| \leqslant K \min\{1 + r^{-2\kappa/\sigma^2}, 1 + \|\Lambda_1 + \Lambda_2\|_{L^\infty((0,r))}\} \|F\| \quad \forall\, r \in (0, \infty), t > 0.$$

Note that $\|F\| \leqslant [1 + K_3] \|f\|$. Thus, there exists a positive constant K_4 that depends only on κ, β and σ such that for each $B \in (0, 1]$,

$$\|v_r^\varepsilon\|_{L^\infty((B,\infty)\times(0,\infty))} \leqslant K_4 \big[1 + B^{-2\kappa/\sigma^2}\big] \|f\|, \tag{3.6}$$

$$\|v_r^\varepsilon\|_{L^\infty((0,B)\times(0,\infty))} \leqslant K_4 \big[1 + \|\Lambda_1 + \Lambda_2\|_{L^\infty((0,B))}\big] \|f\|. \tag{3.7}$$

3. With the above *a priori* estimates, we can find a sequence of $\varepsilon \searrow 0$ such that along the sequence, v^ε approaches a limit V which is a solution of (2.8). The *a priori* estimates for v^ε carry over to the limit V. Since V is unique, the whole family $\{v^\varepsilon\}$ converges to V as $\varepsilon \searrow 0$.

4. Following a similar analysis as above with $\mathcal{L}$ replaced by L, we can obtain a solution V_* of the infinite horizon problem (3.5).

Now let V_* be any solution of (3.5). Set $\zeta = V - V_*$. There are functions C_1 and C_2 such that $V^- - V_*^- = -C_1(V - V_*)$, $V^+ - V_*^+ = C_2(V - V_*)$. In addition, $0 \leqslant C_1 \leqslant 1$ and $0 \leqslant C_2 \leqslant 1$. Thus, setting $C = C_1 \Lambda_1 + C_2 \Lambda_2$, we have

$$(\mathcal{L} + C)\zeta = 0 \text{ on } (0, \infty)^2, \qquad \zeta_r \in L^\infty((0, \infty)^2).$$

Let (b, ψ_2) be as in Lemma 3.2. Now compare with $\pm\zeta$ with $\psi_2(r) \|V_*\|_{L^\infty((0,\infty))} e^{-bt}$, we find $\pm\zeta \leqslant \psi_2(r) \|V_*\|_{L^\infty(0,\infty))} e^{-bt}$. Hence, $\|\zeta(\cdot, t)\|_{L^\infty((0,\infty))} \leqslant 2 \|V_*\|_{L^\infty(0,\infty))} e^{-bt}$. This implies that $V_* = \lim_{t\to\infty} V(\cdot, t)$ so V_* is unique, $\|V_*\|_{L^\infty((0,\infty))} \leqslant K \|f\|$ and $\|\zeta(\cdot, t)\|_{L^\infty((0,\infty))} \leqslant 2K \|f\| e^{-bt}$. This completes the proof of Theorem 1. □

3.3 Convergent process from intensity model to structure model

Theorem 2 *Let $\kappa, \beta, \sigma, h, b_1$ and b_2 be fixed positive constants, $\mathcal{L}$ be defined in (2.3), $f(r) = r - h$ and $Q = (0, \infty)^2$. For each pair of positive constants p and q, let $u_{pq} = V$ be the unique solution of*

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$$\begin{cases} \mathcal{L}V - p\,\mathrm{H}(b_1 - r)\,V^- + q\,\mathrm{H}(r - b_2)\,V^+ = f \text{ in } Q, \\ V(\cdot, 0) = 0, \quad V \in C(\bar{Q}), \quad V_r \in L^\infty(Q). \end{cases} \tag{3.8}$$

Then as $p, q \longrightarrow \infty$, $u_{pq} \longrightarrow u$ locally uniformly on $\bar{Q}$, where u is the unique solution of the following variational inequalities:

1. *If $0 < b_1 < b_2$,*

$$\begin{cases} \max\{\mathcal{L}u - f, u\} = 0 & \text{on } [b_2, \infty) \times (0, \infty), \\ \mathcal{L}u - f = 0 & \text{in } (b_1, b_2) \times (0, \infty), \\ \min\{\mathcal{L}u - f, u\} = 0 & \text{in } (0, b_1] \times (0, \infty), \\ u(\cdot, 0) = 0, \quad u \in C(\bar{Q}), & u_r \in L^\infty(Q); \end{cases} \tag{3.9}$$

2. *If $0 < b_2 \leqslant b_1$,*

$$\begin{cases} \max\{\mathcal{L}u - f, u\} = 0 & \text{in } (b_1, \infty) \times (0, \infty), \\ u = 0 & \text{on } [b_2, b_1] \times (0, \infty), \\ \min\{\mathcal{L}u - f, u\} = 0 & \text{in } (0, b_2) \times (0, \infty), \\ u(\cdot, 0) = 0, \quad u \in C(\bar{Q}), & u_r \in L^\infty(Q). \end{cases} \tag{3.10}$$

Proof We divide the proof into several steps. Set $\Lambda_1(r) = pH(b_1 - r)$ and $\Lambda_2(r) = qH(r - b_2)$.

(1) First we search for upper bounds of u_{pq} when $r \in (b_2, \infty]$. Since $f(r) = r - h \leqslant r$, by comparison, we have $u_{pq} \leqslant 1$. Next, fix an arbitrary $\rho > 0$ and consider the function

$$m(\rho) = [b_2 + 2\rho]\left\{1 + \frac{\sigma^2 + \beta^2 + \kappa^2/b_2^2}{\rho^2}\right\},$$
$$V(r) = \frac{(r - b_2 - \rho)^2}{\rho^2} + \frac{m(\rho)}{q} \quad \text{for } r \in [b_2, b_2 + 2\rho].$$

When $r = b_2$ or $r = b_2 + 2\rho$, we have $V > 1 > u_{pq}$. For $r \in [b_2, b_2 + 2\rho]$, we have

$$\begin{aligned} &\mathcal{L}V - \Lambda_1 V^- + \Lambda_2 V^+ - f \geqslant LV - f + m(\rho) \\ &\geqslant \frac{r}{\rho^2}\left\{-\sigma^2 - \left(\frac{\kappa}{r} - \beta\right)(r - b_2 - \rho) + (r - b_2 - \rho)^2\right\} - f + m(\rho) \\ &\geqslant m(\rho) - r - \frac{r}{\rho^2}\left(\sigma^2 + \beta^2 + \frac{\kappa^2}{r^2}\right) \geqslant 0. \end{aligned}$$

Thus, by comparison $u_{pq} \leqslant V$ for $r \in [b_2, b_2 + 2\rho]$. In particular,

$$u_{pq}(b_2 + \rho, t) \leqslant \frac{m(\rho)}{q} \quad \forall\, \rho > 0, t > 0.$$

Using the L^∞ bound of $u_{pq,r}$ (c.f. (3.6)), we then find that

$$\lim_{q\to\infty} \sup_{t\geq 0, r\in[b_2,R], p>0} u_{pq}^+(r, t) = 0 \quad \forall\, R > b_2.$$

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(2) Next, we search for lower bounds of u_{pq} when $r \in [0, b_1]$. First of all, we have $\|u_{pq}\|_{L^\infty(Q)} \leqslant M := K\|f\|$. Next fix $\rho \in [0, b_1)$ and set

$$m_1(\rho) = \frac{h}{M} + \frac{\sigma^2 b_1 + (\kappa + \beta b_1)b_1}{(b-\rho)^2}, \qquad v = -M\Big\{\frac{(r-\rho)^2}{(b_1-\rho)^2} + \frac{m_1(\rho)}{p}\Big\}.$$

When $r = b_1$, we have $MV < -M < u_{pq}$. For $r \in [0, b_1]$, we have

$$\begin{aligned}\mathcal{L}V - \Lambda_1 V^- + \Lambda_2 V^+ - f &\leqslant LV + h - Mm_1(\rho)\\ &= \frac{M}{(b_1-\rho)^2}\Big\{\sigma^2 r + (\kappa - \beta r)(r-\rho) - (r-\rho)^2\Big\} + h - Mm(\rho)\\ &\leqslant M\Big\{-m(\rho) + \frac{h}{M} + \frac{\sigma^2 b_1 + (\kappa + \beta b_1)b_1}{(b-\rho)^2}\Big\} \leqslant 0.\end{aligned}$$

Thus, by comparison $u_{pq} \geqslant V$ on $[0, b_1] \times [0, \infty)$. In particular,

$$u_{pq}(\rho, t) \geqslant -\frac{Mm_1(\rho)}{p}, \quad \forall\, t > 0, \rho \in [0, b_1). \tag{3.11}$$

Hence, using the L^∞ bound of $u_{pq,r}$ in $[b_1/2, \infty) \times [0, \infty)$ (c.f. (3.6)), we derive that

$$\lim_{p\to\infty} \sup_{t\geq 0, r\in[0,b_1], q>0} u^-_{pq}(r, t) = 0.$$

(3) We estimate $u_{pq,r}$ for $r \in [0, b_1 \wedge b_2/2]$. We write the equation for u_{pq} as

$$Lu_{pq} = F := f + pu^-_{pq} - u_{pq,t} \quad \text{in } \ (0, b_1 \wedge b_2/2].$$

Using the estimate (3.11), we have when $r \in [0, b_1 \wedge b_2/2]$, we have

$$0 \leqslant pu^-_{pq} \leqslant Mm(b_1 \wedge b_2/2).$$

Thus, by an estimate similar to that for (3.4) with $c = 0$ we find that $u_{pq,r}$ is bounded; here we use $\lim_{\varepsilon\searrow 0} W(\varepsilon)u_{pq,r}(\varepsilon, t) = 0$. Hence, there exists a positive constant $K(\kappa, \beta, \sigma, h, b_1 \wedge b_2)$ that depends only on $\kappa, \beta, \sigma, h, b_1 \wedge b_2$ such that, for each $p > 0$ and $q > 0$,

$$\Big\|u_{pq}\Big\|_{L^\infty(Q)} + \Big\|\frac{\partial u_{pq}}{\partial t}\Big\|_{L^\infty(Q)} + \Big\|\frac{\partial u_{pq}}{\partial r}\Big\|_{L^\infty(Q)} \leqslant K(\kappa, \beta, \sigma, h, b_1 \wedge b_2).$$

(4) Now, we can use the local compactness of the family $\{u_{pq}\}_{p>0.q>0}$ and a diagonal process to find a sequence $\{p_j, q_j\}_{j=1}^\infty$ in $(0, \infty)^2$ and a function v defined on $[0, \infty)^2$ such that

$$\lim_{j\to\infty} p_j = \infty, \lim_{j\to\infty} q_j = \infty, \quad \lim_{j\to\infty} \Big\|u_{p_j q_j} - v\Big\|_{C^{1/2}([0,R]^2)} = 0 \quad \forall\, R > 0.$$

In addition,

$$\|u\|_{L^\infty(Q)} + \|u_t\|_{L^\infty(Q)} + \|u_r\|_{L^\infty(Q)} \leqslant K(\kappa, \beta, \sigma, h, b_1 \wedge b_2).$$

(5) Now we derive the equation for u.

(i) First, we have

$$u \geqslant 0 \text{ on } [0, b_1] \times [0, \infty), \qquad u \leqslant 0 \text{ on } \ [b_2, \infty) \times [0, \infty).$$

Thus, if $b_1 \geqslant b_2$, then $u = 0$ in $[b_2, b_1] \times [0, \infty)$. If $b_1 < b_2$, then for $r \in (b_1, b_2)$, we obtain from $\mathcal{L}u_{pq} = f$ that $\mathcal{L}u - f = 0$ in $(b_1, b_2) \times [0, \infty)$.

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(ii) In $(0,b_2)\times(0,\infty)$, we have $\mathcal{L}u_{pq}-f=pH(b_1-r)u^-_{pq}\geqslant 0$. Hence, in the distribution sense,

$$\mathcal{L}u-f\geqslant 0 \quad \text{in } (0,b_2)\times(0,\infty).$$

Similarly, in the distribution sense we have

$$\mathcal{L}u-f\leqslant 0 \quad \text{in } (b_1,\infty)\times(0,\infty).$$

(iii) Now suppose $u(r_0,t_0)=3\delta>0$ for some $r_0\in(0,b_1]\cap(0,b_2)$ and $t_0>0$. Then in a small neighbourhood (r_0,t_0), $u\geqslant 2\delta$. By uniform convergence, we have $u_{p_jq_j}\geqslant\delta$ in this small neighbourhood for all $j\gg 1$. Hence, for $i\gg 1$, we have $\mathcal{L}u_{p_iq_i}-f=0$ in this small neighbourhood. Sending $i\to\infty$, we obtain $\mathcal{L}u-f=0$. Thus, we have

$$\min\{\mathcal{L}u-f,u\}=0, \quad \text{in } \Big((0,b_1]\cap(0,b_2)\Big)\times(0,\infty).$$

Similarly, we can show that

$$\max\{\mathcal{L}u-f,u\}=0, \quad \text{in } \Big((b_1,\infty)\cap[b_2,\infty)\Big)\times(0,\infty).$$

Thus, u is the solution of (3.9) when $b_1<b_2$, and is the solution of (3.10) when $b_1\geqslant b_2$.

Since (3.10) consists of two standard variational inequalities, there exists a unique solution. The uniqueness of (3.9) will be proved in Section 5. Once we have the uniqueness of the limit, we conclude that as $p,q\to\infty$, $u_{pq}\longrightarrow u$ locally uniformly in $[0,\infty)^2$. This completes the proof of Theorem 2. □

We end this section by providing a few more estimates for the case $b_1<b_2$. Since $\mathcal{L}u-f=0$ in $(b_1,b_2)\times[0,\infty)$, using the bounds of u, u_r and u_t, we find that there exists a positive constant K depending only on $\kappa,\beta,\sigma,h,b_1,b_2$ such that

$$\Big\||u|+|u_r|+|u_t|+|u_{rr}|\Big\|_{L^\infty((b_1,b_2)\times(0,\infty))}\leqslant K.$$

By interpolation, there exists a positive constant K_5 depending only on $\kappa,\beta,\sigma,h,b_1,b_2$ such that

$$\|u_r(r,\cdot)\|_{C^{1/2}([0,\infty))}\leqslant K_5, \quad \forall\, r\in[b_1^+,b_2^+].$$

In particular, both $u_r(b_1^+,t)$ and $u_r(b_2^-,t)$ are well defined for all $t\geqslant 0$.

Now we consider the equation $\min\{\mathcal{L}u-f,u\}(b_1,t)=0$.

(i) Suppose $u(b_1,t)>0$. Then $u>0$ in a neighbourhood of (b_1,t) and we can derive that $\mathcal{L}u-f=0$ in a neighbourhood of (b_1,t).

(ii) Suppose $u(b_1,t)=0$. Since $\mathcal{L}u_{pq}-f\geqslant 0$ for $r\in(0,b_2)$, we derive that there exists a positive constant K depending only on $\kappa,\beta,\sigma,b_1,b_2,h$ such that

$$ru_{pq,rr}\leqslant K \quad \text{in } (0,b_2)\times(0,\infty), \forall\, p>0, q>0.$$

Thus, for small positive $\varepsilon>0$,

$$\begin{aligned}\frac{1}{\varepsilon}\int_0^\varepsilon u_{pq,r}(b_1+\rho,t)d\rho &= \frac{1}{\varepsilon}\int_0^\varepsilon\Big(u_{pq,r}(b_1-\rho,t)+\int_{b_1-\rho}^{b_1+\rho}u_{pp,rr}(\theta,t)d\theta\Big)d\rho\\ &\leqslant \frac{u_{pq}(b_1,t)-u_{pq}(b_1-\varepsilon,t)}{\varepsilon}+K\varepsilon.\end{aligned}$$

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Set $(p,q)=(p_j,q_j)$, and sending $j\to\infty$, we find that

$$\frac{1}{\varepsilon}\int_0^{\varepsilon} u_r(b_1+\rho,t)d\rho \leqslant \frac{u(b_1,t)-u(b_1-\varepsilon,t_1)}{\varepsilon}+K\varepsilon=-\frac{u(b_1-\varepsilon,t)}{\varepsilon}+K\varepsilon\leqslant K\varepsilon.$$

Sending $\varepsilon\searrow 0$, we derive that $u_r(b_1^+,t)\leqslant 0$.

Similarly we can show that if $u(b_2,t)=0$, then $u_r(b_2^-,t)\leqslant 0$.

Remark 3.1 *In* (3.9)*, the equation* $\min\{\mathcal{L}u-f,u\}(b_1,t)=0$ *can be interpreted as follows:*

1. *if* $u(b_1,t_1)>0$*, then* $\mathcal{L}u-f=0$ *in a neighbourhood of* (b_1,t)*;*
2. *if* $u(b_1,t_1)=0$*, then* $u_r(b_1^+,t)\leqslant 0$*.*

Similarly, the equation $\max\{\mathcal{L}u-f,u\}(b_2,t)=0$ *can be defined as follows:*

1. *if* $u(b_2,t)<0$*, then* $\mathcal{L}u-f=0$ *in a neighbourhood of* (b_2,t)*;*
2. *if* $u(b_2,t)=0$*, then* $u_r(b_2^-,t)\geqslant 0$*.*

In the rest of this paper, we study the variational inequality (3.9).

4 The infinite horizon problem from the structure-type model

As a preparation for the study of the mathematical formulation of the structure-type model (3.9), here we consider the corresponding infinite horizon (i.e. $T=\infty$) problem, being the limit as $p,q\to\infty$ of the infinite horizon problem of the intensity model (3.5) with $\Lambda_1=pH(r-b_1)$ and $\Lambda_2=qH(r-b_2)$. The case $b_2\leqslant b_1$ is relatively easy since we have the boundary condition $u=0$ for $r\in[b_2,b_1]$. Hence, for definiteness, we consider only the case $0<b_1<b_2$. We focus on the qualitative behaviour of the solution. We recall that the model is formally derived according to the following assumption:

> *the payer can select to terminate the contract whenever* $r_t<b_1$ *and the receiver can select to terminate the contract whenever* $r_t>b_2$*.*

We shall use τ_1^{**} the optimal termination time for payer and τ_2^{**} the optimal termination time for the receiver. More detailed properties, such as the relationship among the solution and the swap rate h and default bounds b_1 and b_2, for the solution of this infinite horizon structure model will be analysed.

4.1 The double obstacle problem and one of the main results

The infinite horizon problem is to find a solution $u=u(r)$ of the double variational inequality

$$\begin{cases} \min\{Lu-f,u\}=0 \quad \text{in} \quad (0,b_1], \\ Lu-f=0 \quad \text{in} \quad (b_1,b_2), \\ \max\{Lu-f,u\}=0 \quad \text{in} \quad [b_2,\infty), \\ u_r\in L^{\infty}((0,\infty)). \end{cases} \tag{4.1}$$

Here for a parameter $h\in(0,\infty)$, $f(r)=r-h$. Also L is as in (2.3) with fixed positive constants κ,β and σ. Due to the degenerate nature of the elliptic operator at $r=0$, it is not appropriate

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to supply boundary conditions at $r=0$. Hence, we use $u_r \in L^\infty$ to ensure the uniqueness. Note that $u_r \in L^\infty$ implies that $u(0) = \lim_{r\searrow 0} u(r)$ is well defined. As we shall see in the following, the solution is piecewise smooth. Hence, we can use the convention that $u_{rr}(r) = \infty$ if $u_r(r^+) - u_r(r^-) > 0$ and $u_{rr}(r) = -\infty$ if $u_r(r^+) - u_r(r^-) < 0$. Consequently, the condition $Lu - f \geqslant 0$ at $r = b_1$ and $Lu - f \leqslant 0$ at $r = b_2$ imply that

$$u_r(b_1^+) \leqslant u_r(b_1^-), \qquad u_r(b_2^-) \leqslant u_r(b_2^+).$$

From a mathematical finance point of view, it is natural to define a solution of problem (4.1) by the viscosity solution $F[u] = 0$ on $(0, \infty)$, where $F(\cdot)$ is defined in (2.11).

Here ζ is called a test function. It is well known that in the definition the test function can be replaced a local one: $\zeta \in C^2((r-\varepsilon, r+\varepsilon))$ for some $\varepsilon > 0$. Also, a viscosity solution of $Lu = f$ in an open interval (in $(0,\infty)$) is also a smooth and classical solution of $Lu = f$ in that open interval. Hence, if u is a viscosity solution and $u(r) > 0$ with $r \in (0, b_2)$ or $u(r) < 0$ with $r \in (b_1, \infty)$, then u is smooth and $Lu = f$ in an neighbourhood of r.

For easy reference, we define

$$\omega(r) = r^{2\kappa/\sigma^2} e^{-2\kappa r/\sigma^2} \qquad \forall\, r \geqslant 0, \tag{4.2}$$

$$h_0 = \frac{\int_0^{b_2} \frac{1}{\omega(r)\varphi_2^2(r)} \int_0^r \omega(s)\varphi_2(s)ds}{\int_0^{b_2} \frac{1}{\omega(r)\varphi_2^2(r)} \int_0^r \frac{1}{s}\omega(s)\varphi_2(s)ds}, \tag{4.3}$$

$$h_1 = \frac{\int_{b_1}^{b_2} \frac{1}{\omega(r)\varphi_2^2(r)} \int_{b_1}^r \omega(s)\varphi_2(s)ds}{\int_{b_1}^{b_2} \frac{1}{\omega(r)\varphi_2^2(r)} \int_{b_1}^r \frac{1}{s}\omega(s)\varphi_2(s)ds}, \tag{4.4}$$

$$h_2 = \frac{\int_{b_1}^{b_2} \frac{1}{\omega(r)\varphi_2^2(r)} \int_r^{b_2} \omega(s)\varphi_2(s)ds}{\int_{b_1}^{b_2} \frac{1}{\omega(r)\varphi_2^2(r)} \int_r^{b_2} \frac{1}{s}\omega(s)\varphi_2(s)ds}. \tag{4.5}$$

Using mean value theorem, one can derive that $0 < h_0 < h_1 < h_2$ and $b_1 < h_1 < h_2 < b_2$.

In the rest of this section, we solve problem (4.1) and describe its qualitative behaviour in term of the swap rate h by proving the following.

Theorem 3 *Assume that $0 < b_1 < b_2$. For each $h \in \mathbb{R}$, (4.1) with $f = r - h$ admits a unique solution $u = u(h;\cdot)$. The solution is decreasing in h. Define h_0, h_1 and h_2 by (4.2)–(4.5). Then $0 < h_0 < h_1 < h_2$, $b_1 < h_1 < h_2 < b_2$ and the following holds:*

(1) *If $h \in (-\infty, h_0]$, u does not depend on b_1 and is the unique solution of the following:*

$$Lu - f = 0 \;\; in \;\; (0, b_2), \qquad u = 0 \;\; in \;\; [b_2, \infty), \qquad u_r \in L^\infty((0,\infty)).$$

In addition, $u > 0$ in $(0, b_2)$ and $u(h;0)$ is strictly decreasing in $h \in (-\infty, h_0]$ with $u(h_0;0) = 0$.

'Here the receiver is always out of money, so he has to terminate the contract whenever he is allowed. Thus, $\tau_1^{**} = \infty$, $\tau_2^{**} = \inf\{t > 0 \mid r_t \geqslant b_2\}$.'

(2) *If $h \in (h_0, h_1]$, there exists $z = z(h) \in (0, h) \cap (0, b_1]$ such that (z, u) is the unique solution of*

$$Lu - f = 0 \;\&\; u > 0 \; in \;\; (z, b_2), \quad u = 0 \;\; in \;\; [0, z] \cup [b_2, \infty), \quad u'(z) = 0.$$

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In addition, $z(\cdot)$ *is continuous and strictly increasing on* $[h_0, h_1]$ *with* $z(h_0) = 0, z(h_1) = b_1$.
'Here the receiver is out of money when $r_t > z(h)$ and the payer is out of money when $r_t < z(h)$. Hence, $\tau_1^{**} = \inf\{t > 0 \mid r_t \leqslant z(h)\}$ and $\tau_2^{**} = \inf\{t > 0 \mid r_t \geqslant b_2\}$.'

(3) *If* $h \in (h_1, h_2)$*, then* u *is the unique continuous solution of*

$$Lu - f = 0 \ in \ (b_1, b_2), \quad u = 0 \ in \ [0, b_1] \times [b_2, \infty).$$

In addition, $u'(b_1^+) < 0, u'(b_2^-) < 0$ *and there exists a unique* $z = z(h) \in (b_1, b_2)$*, such that*

$$u < 0 \ in \ (b_1, z), \quad u(h;\, z(h)) = 0, \quad u > 0 \ in \ (z, b_2).$$

Also $z(\cdot)$ *is continuous and strictly increasing on* $[h_1^+, h_2^-]$ *with* $z(h_1^+) = b_1$ *and* $z(h_2^-) = b_2$.
'Here the payer is out of money when $r_t < z(h)$ and the receiver is out of money when $r_t > z(h)$. Hence, $\tau_1^{**} = \inf\{t > 0 \mid r_t \leqslant b_1\}$, $\tau_2^{**} = \inf\{t > 0 \mid r_t \geqslant b_2\}$.'

(4) *If* $h \in [h_2, \infty)$*, there exists* $z(h) \in (h, \infty) \cap [b_2, \infty)$ *such that* (z, u) *is the unique solution of*

$$Lu - f = 0 \ \&\ u < 0 \ in \ (b_1, z), \quad u = 0 \ in \ [0, b_1] \cup [z, \infty), \quad u'(z) = 0. \tag{4.6}$$

In addition $z(\cdot)$ *is continuous and strictly increasing on* $[h_2, \infty)$ *with* $z(h_2) = b_2$.
'Here the payer is out of money when $r_t < z(h)$ and the receiver is out of money when $r_t > z(h)$. Hence, $\tau_1^{**} = \inf\{t > 0 \mid r_t \leqslant b_1\}$, $\tau_2^{**} = \inf\{t > 0 \mid r_t \geqslant z(h)\}$.'

The proof will be given in the subsequent subsections.

Note that the piecewisely defined function $z(\cdot)$ is continuous and strictly increasing on $[h_0, \infty)$. Define $z(h) = 0^-$ for $h < h_0$. We have the following characterisations: for every $h \in \mathbb{R}$,

$$u(h;\cdot) \leqslant 0 \text{ on } (0, z(h)], \quad u(h,\cdot) \geqslant 0 \text{ on } [z(h), \infty),$$

We call

$$r = z(h), \tag{4.7}$$

the *parity level*. If the current floating interest rate is at the parity level, the value of the swap is zero; above the parity level, the swap favours the payer; and below the parity level, the swap favours the receiver.

When $h \leqslant h_0$, there is no parity level since the contract always favours the payer.

When $h \in (-h_0, h_1]$, the swap always favours the payer, provided that he terminates the contract when floating rate is too low.

When $h \in (h_1, h_2)$, we have $z(h) \in (b_1, b_2)$.

When $h \in [h_2, \infty)$, the swap always favours the receiver provided that he chooses to terminate the contract whenever interest is too high.

Thus, at initiation the swap rate h should be set in the range of $(h_1, h_2) \subset (b_1, b_2)$. If the current interest rate is $r_0 \in (b_1, b_2)$, then the swap rate should be the unique h satisfying $u(h; r_0) = 0$, i.e. $r_0 = z(h)$ or $h = h(r_0)$, where $h(\cdot)$ is defined in (4.9).

Remark 4.1 *As the domain* $[0, \infty)$ *of the independent variable* r *is bounded from below, there is no parity-level free boundary when* $h \in (0, h_0]$*; on the other side, no matter how big* h *is, the parity-level free boundary,* $z(h)$*, always exists.*

In the rest of this section, we perform pure mathematical analysis to prove Theorem 3.

4.2 Uniqueness and monotonicity in the swap rate *h*

Lemma 4.1 *Problem* (4.1) *admits at most one solution.*

Proof Suppose, on the contrary, that there are two different solutions, say u_1 and u_2. Exchanging the roles of u_1 and u_2 if necessary, we can find $r_0 > 0$ such that $u_1(r_0) > u_2(r_0)$. Let φ_1 and φ_2 be as in Lemma 3.1. Set $\varepsilon = [u_1(r_0) - u_2(r_0)]/[2\varphi_1(r_0) + \varphi_2(r_0)]$ and consider the function

$$w(r) = u_1(r) - u_2(r) - \varepsilon[\varphi_1(r) + \varphi_2(r)], \quad \forall\, r > 0.$$

Since u_1' and u_2' are bounded, we have $\lim_{r\searrow 0} w'(r) = \infty$ and $\lim_{r\to\infty} w(r) = -\infty$. Hence, there exists $\hat{r} \in (0, \infty)$ such that

$$w(\hat{r}) = \max_{(0,\infty)} w(\cdot) \geqslant w(r_0) > 0.$$

We now derive a contradiction by considering three cases:

$$(1)\ \hat{r} \in (b_1, b_2), \quad (2)\ \hat{r} \in (0, b_1], \quad (3)\ \hat{r} \in [b_2, \infty).$$

(1) Suppose $\hat{r} \in (b_1, b_2)$. Then $Lu_1 = f$ and $Lu_2 = f$ in a neighbourhood of $\hat{r}$ and $w'(\hat{r}) = 0 \geqslant w''(\hat{r})$. This gives the following contradiction:

$$0 < \hat{r}w(\hat{r}) \leqslant Lw(\hat{r}) = Lu_1 - Lu_2 - \varepsilon L\varphi_1 - \varepsilon L\varphi_2\Big|_{r=\hat{r}} = 0.$$

(2) Suppose $\hat{r} \in (0, b_1]$. Then $u_2(\hat{r}) \geqslant 0$ and $u_1(\hat{r}) = w(\hat{r}) + u_2(\hat{r}) + \varepsilon[\varphi_1(\hat{r}) + \varphi_2(\hat{r})] > 0$. Hence, u_1 is smooth and $Lu_1 = f$ in a neighbourhood of $\hat{r}$. Consequently, $\zeta(r) := u_1(r) - \varepsilon[\varphi_1(r) + \varphi_2(r)] - w(\hat{r})$ can be used as a test function for the viscosity solution u_2 at $r = \hat{r}$, leading to the following contradiction:

$$0 \leqslant \min\{L\zeta - f, \zeta\}|_{r=\hat{r}} \leqslant L\zeta - f|_{r=\hat{r}} = -\hat{r}w(\hat{r}) < 0.$$

(3) Suppose $\hat{r} \in [b_2, \infty)$. Then $u_1(\hat{r}) \leqslant 0$ and $u_2(\hat{r}) = u_1(\hat{r}) - w(\hat{r}) - \varepsilon[\varphi_1(\hat{r}) + \varphi_2(\hat{r})] < 0$. Hence, u_2 is smooth and $Lu_2 - f = 0$ in a neighbourhood of $\hat{r}$. Consequently, $\zeta(r) := u_2(r) + \varepsilon[\varphi_1(r) + \varphi_2(r)] + w(\hat{r})$ is a test function for the viscosity solution u_1 at $r = \hat{r}$, leading to the following contradiction:

$$0 \geqslant \max\{L\zeta - f, \zeta\}|_{r=\hat{r}} \geqslant L\zeta - f|_{r=\hat{r}} = \hat{r}w(\hat{r}) > 0.$$

All of the above contradictions imply that (4.1) admits at most one solution. □

Following the same proof as above, we can show the following.

Lemma 4.2 *Suppose $h < \ell$ and $u_1(\cdot)$ and $u_2(\cdot)$ are solutions of* (4.1) *with $f(r) = r - h$ and $f(r) = r - \ell$, respectively. Then $u_1 \gneqq u_2$.*

Once we have uniqueness and monotonicity, what we need now is to construct solutions. This will be done in the following subsections.

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4.3 The case $h \in (-\infty, h_0]$

Lemma 4.3 *Let h_0 be defined in (4.3), φ_2 be defined as in Lemma 3.1 and ω as in (4.2). For each $h \in (-\infty, h_0]$, the solution of (4.1) with $f = r - h$ is given by*

$$u(h;r) = \varphi_2(r) \int_{r\wedge b_2}^{b_2} \int_0^{\rho} \frac{2(s-h)}{\sigma^2 s} \frac{\omega(s)\varphi_2(s)}{\omega(\rho)\varphi_2^2(\rho)} ds d\rho. \tag{4.8}$$

It has the property that $Lu - f = 0 < u$ in $[0, b_2)$ and $u = 0$ on $[b_2, \infty)$. Also, $u(h_0; 0) = 0$.

Proof Using variation of constant, we can solve $Lu - f = 0$ in $(0, b_2]$ together with the boundary condition $u(b_2) = 0$ and $u_r \in L^\infty$ to obtain the formula (4.8). Since $u = 0$ and $f > 0$ on $[b_2, \infty)$, we have $\max\{Lu - f, u\} = 0$ in (b_2, ∞). It remains to show that $u > 0$ in $(0, b_2)$ so that $\min\{Lu - f, u\} = 0$ in $(0, b_1]$, $Lu - f = 0$ in (b_1, b_2) and $\max\{Lu - f, u\} = 0$ at $r = b_2$ (since $u_r|_{r=b_2-} < 0$ and $u_{rr}|_{r=b_2} = \infty$.)

For $r \in [0, b_2]$, write (4.8) as $u(h;r) = \varphi_2(r)c(h;r)$. Note that $c(h;0)$ is a strictly decreasing function of h and by the definition of h_0, we have $c(h_0;0) = 0$. Thus, $c(h;0) \geqslant 0 = c(h;b_2)$ for $h \leqslant h_0$. Also, direct differentiation gives, for $r \in (0, b_2]$,

$$\omega(r)\varphi_2^2(r)\, c_r(h;r) = \int_0^r \frac{2(h-s)}{\sigma^2 s}\omega(s)\varphi_2(s)ds.$$

Note that the integrand changes sign at moist once, so $c_r(h;\cdot)$ in $[0, b_2]$ changes sign at most once. It follows that $c(h;\cdot)$ in $[0, b_2]$ is either strictly decreasing or first increasing and then decreasing. Consequently, $c(h;r) > 0$ for $r \in (0, b_2)$ and $h \leqslant h_0$. This completes the proof. □

4.4 The case $h \in [h_0, h_1]$

Lemma 4.4 *Define*

$$h(z) = \frac{\int_z^{b_2} \frac{1}{\omega(r)\varphi_2^2(r)} \int_z^r \omega(s)\varphi_2(s)ds}{\int_z^{b_2} \frac{1}{\omega(r)\varphi_2^2(r)} \int_z^r \frac{1}{s}\omega(s)\varphi_2(s)ds} \quad \forall\, z \in [0, b_1]. \tag{4.9}$$

Then $h(\cdot)$ is continuous and strictly increasing on $[0, b_1]$, $h(0) = h_0$ and $h(b_1) = h_1$. For $z \in [0, b_1]$ and $h = h(z)$, the unique solution of (4.1) is given by

$$u(h;r) = \varphi_2(r) \int_z^{[r\vee z]\wedge b_2} \int_z^{\rho} \frac{2(h-s)}{\sigma^2 s} \frac{\omega(s)\varphi_2(s)}{\omega(\rho)\varphi_2^2(\rho)} ds\, d\rho; \tag{4.10}$$

the solution satisfies $u = 0$ in $[0, z] \cup [b_2, \infty)$, $Lu - f = 0 < u$ in (z, b_2) and $u_r|_{r=z} = 0$.

Proof Suppose $z \in (0, b_1]$ and $h = h(z)$. Using variation of constant to solve $Lu = f$ in $[z, b_2]$ supplemented with the initial conditions $u(z) = 0$ and $u'(z) = 0$, we obtain the formula (4.10).

Note that $h = h(z) > z$ so $f < 0$, $u = 0$ and $\min\{Lu - f, u\} = 0$ in $[0, z)$. In addition, since $u'(z) = 0$, we have $\min\{Lu - f, u\} = 0$ in at $r = z$. Furthermore, since $h = h(z) < b_2$, we have $\max\{Lu - f, u\} = 0$ in (b_2, ∞). Hence, to show that u given by (4.10) is a solution, it remains to show that $u > 0$ in (z, b_2).

Write (4.10) as $u = \varphi_2(r)c(h;r)$. We see that $c(h;z) = 0$ and, by the definition of $h(z)$, $c(h;b_2) = 0$. Also,

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$$\omega(r)\varphi_2^2(r)\, c_r(h;r) = \int_z^r \frac{2(h-s)}{\sigma^2 s}\omega(s)\varphi_2(s)ds \quad \forall\, r \in [z, b_2].$$

It follows that c_r changes sign at most once. Thus, c is first increasing from 0 at $r = z$ and then decreasing to 0 at $r = b_2$. This implies that $c > 0$ for $r \in (z, b)$.

Thus, u in (4.10) is a solution of (4.1). Since the solution is unique, we know that $h(\cdot)$ is a strictly increasing function. Finally, by the definition of h_0 and h_1, we see that $h_0 = h(0)$ and $h_1 = h(b_1)$. This completes the proof. □

As an inverse function of $h = h(z)$, $z = z(h)$ is typically referred to as the free boundary for the variational inequality $\min\{Lu - f, u\} = 0$ in $(0, b_2)$ with $u(b_2) = 0$ and $u_r \in L^\infty$. This also solves the infinite horizon problem for (3.10) in $[0, b_2]$.

4.5 The case $h \in [h_2, \infty)$

Lemma 4.5 *Define*

$$h(z) = \frac{\int_{b_1}^z \frac{1}{\omega(r)\varphi_2^2(r)} \int_r^z \omega(s)\varphi_2(s)ds}{\int_{b_1}^z \frac{1}{\omega(r)\varphi_2^2(r)} \int_r^z \frac{1}{s}\omega(s)\varphi_2(s)ds} \quad \forall\, z \in [b_2, \infty). \tag{4.11}$$

Then $h(b_2) = h_2$ and $h(\cdot)$ is continuous and strictly increasing in $[b_2, \infty)$. For $z \in [b_2, \infty)$ and $h = h(z)$, the unique solution of (4.1) is given by

$$u(h;r) = \varphi_2(r) \int_{[r\vee b_1]\wedge z}^z \int_\rho^z \frac{2(h-s)}{\sigma^2 s} \frac{\omega(s)\varphi_2(s)}{\omega(\rho)\varphi_2^2(\rho)} ds\, d\rho; \tag{4.12}$$

the solution satisfies $u = 0$ on $[0, b_1] \cup [z, \infty)$, $u_r|_{r=z} = 0$ and $Lu - f = 0 > u$ in (b_1, z).

The proof follows from an analysis similar to that in the previous section, so it is omitted.

We remark that the solution is also that for the infinite horizon problem of (3.10) in the interval $[b_1, \infty)$.

4.6 The case $h \in [h_1, h_2]$

Lemma 4.6 *Let $u(h_1; r)$ and $u(h_2; r)$ be the solutions of (4.1) with $f(r) = r - h_1$ and $f(r) = r - h_2$, respectively. Then for every $h \in [h_1, h_2]$, the solution of (4.1) is given by*

$$u(h;r) = \frac{h_2 - h}{h_2 - h_1} u(h_1; r) + \frac{h - h_1}{h_2 - h_1} u(h_2; r). \tag{4.13}$$

It satisfies $u = 0$ in $[0, b_1] \cup [b_2, \infty)$ and $Lu = f$ in (b_1, b_2). In addition, for each $h \in [h_1, h_2]$, there exists $z(h) \in [b_1, b_2]$ such that $u(h;\cdot) < 0$ in $(b_1, z(h))$ and $u(h;\cdot) > 0$ in $(z(h), b_2)$. Furthermore, the function $z(\cdot)$ is continuous and strictly increasing on $[h_1, h_2]$ with $z(h_1) = b_1$ and $z(h_2) = b_2$.

Proof Fix $h \in (h_1, h_2)$ and define $u(h;\cdot)$ as in (4.13). By the definitions of $u(h_1;\cdot)$ in Lemma 4.4 and $u(h_2;\cdot)$ in Lemma 4.5, we see that $u(h;\cdot) = 0$ in $[0, b_1] \cup [b_2, \infty)$ and that $Lu(h;\cdot) = f$ in (b_1, b_2). Furthermore, since

$$u_r(h_1; b_1) = 0 > u_r(h_2; b_1+), \qquad u_r(h_1; b_2-) < 0 = u_r(h_2; b_2),$$

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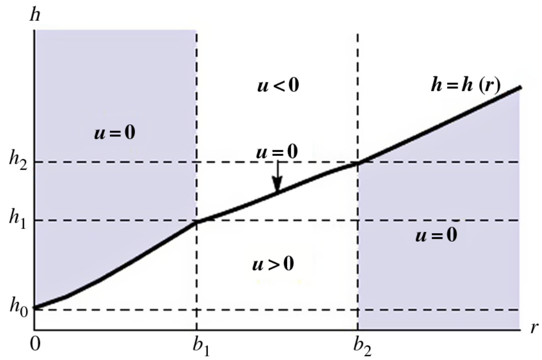

FIGURE 1. The function $h = h(r)$, and its inverse $r = z(h)$.

we see that $u_r(h; b_1+) < 0$ and $u_r(h; b_2-) < 0$; see Remark 3.1. Thus, u is the unique solution of (4.1). It remains to show the existence of $z(h)$.

First applying the maximum principle to $Lu = f$ in (b_1, b_2) with boundary conditions $u(h; b_1) = 0$ and $u(h; b_2) = 0$, we find that $u < 1$ in $[b_1, b_2]$. Next, differentiating $Lu = f$, we have

$$\left\{-\frac{\sigma^2 r}{2}\frac{\partial^2}{\partial r^2} - \left(\frac{\sigma^2}{2} + \kappa - \beta r\right)\frac{\partial}{\partial r} + \beta + r\right\}u_r = 1 - u > 0.$$

This implies that u_r does not have a local negative minimum in (b_1, b_2). Since $u_r(h; b_1+) < 0$, $u_r(h; b_2-) < 0$ and $\int_{b_1}^{b_2} u_r dr = 0$, the equation $u_r(h; \cdot) = 0$ has exactly two roots in (b_1, b_2). Thus, u first decreases from 0 at $r = b_1$ to its global negative minimum, then increases to its global positive maximum and finally decreases to 0 at $r = b_2$. Hence, there exists a unique $z(h) \in (b_1, b_2)$ such that $u < 0$ in $(b_1, z(h))$ and $u > 0$ in $(z(h), b_2)$. Since by Lemma 4.2, $u(h; r)$ is a decreasing function of h, we see that z is continuous and strictly increasing on (h_1, h_2), $z(h_1) = b_1$ and $z(h_2) = b_2$. This completes the proof. □

Proof of Theorem 3. The assertions of Theorem 3 follows from Lemmas 3.2–4.6.

Figure 1 uses the parameters

$$\kappa = 0.5, \;\; \beta = 1, \;\; \sigma = 0.5, \;\; b_1 = 1, b_2 = 2,$$

and the calculated results are

$$s_1 = 0.898979, \;\; s_2 = -8.89898, \;\; \alpha_1 = 0.0917517, \;\; \alpha_2 = 0.908248,$$

and

$$h_0 = 0.225406, \;\; h_1 = 1.07801, \;\; h_2 = 1.66554.$$

5 The finite horizon problem of the structure-type model

In this section, we study the parabolic variational inequality (3.9), being a structure model for the underlying IRS. For definiteness, we consider the case $0 < b_1 < b_2$. Also we define h_0, h_1, h_2 as in (4.3), (4.4), (4.5). We prove the following.

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Theorem 4 *Assume that* $0 < b_1 < b_2$. *For each* $h \in \mathbb{R}$, (3.9) *with* $f = r - h$ *admits a unique solution. The solution is decreasing in h. Also, the following holds:*

(1) *If* $h \in [h_1, h_2]$, *u is the solution of the initial boundary value problem*

$$\begin{cases} \mathcal{L}u - f = 0 & \text{in } D := (b_1, b_1) \times [0, \infty), \\ u = 0 & \text{on } [0, \infty)^2 \setminus D. \end{cases}$$

In addition, there exists $s \in C^\infty([0, \infty))$ *such that for each* $t > 0$, $s(t) \in (b_1, b_2)$ *and*

$$u(\cdot, t) < 0 \;\; in \;\; (b_1, s(t)), \quad u(\cdot, t) > 0 \;\; in \;\; (s(t), b_2).$$

Moreover, $s(0) = h$ *and* $s(\infty) = z(h)$, *where* $z(h)$ *is defined by* (4.7) *as in Theorem 3.*
'Here the parity level is $r_t = s(T - t)$, the payer is out of money if $r_t < s(T - t)$ and the receiver is out of money if $r_t > s(T - t)$. Hence, the optimal strategies are $\tau_1^* = \inf\{t > 0 \mid r_t < b_1\}$ and $\tau_2^* = \inf\{t > 0 \mid r_t > b_2\}$.'

(2) *If* $h \in (h_2, \infty)$, *there exist* $T_1 \in [0, \infty)$ *and* $s \in C([0, \infty)) \cap C^\infty([0, T_1) \cup (T_1, \infty))$ *such that* (u, s) *is the unique solution of the following free boundary problem:*

$$\begin{cases} \mathcal{L}u - f = 0, & \forall\, t > 0, r \in (b_1, s(t) \vee b_2), \\ u(r, t) = 0, & \forall\, t > 0, r \in [0, b_1] \cup [s(t) \vee b_2, \infty), \\ u(\cdot, 0) = 0, \quad s(0) = h & \text{at } t = 0, \\ s(t) \in (b_1, b_2), \quad u(s(t), t) = 0, \; \forall\, t \in (0, T_1), & \\ \quad u(\cdot, T_1) < 0 \; in \; (b_1, b_2), \quad u_r(b_2^-, T_1) = 0, \quad s(T_1) = b_2, & \\ s(t) > b_2, \quad u_r(s(t), t) = 0, & \forall\, t > T_1. \end{cases} \tag{5.1}$$

In addition, $T_1 > 0$ *if* $h \in (h_2, b_2)$ *and* $T_1 = 0$ *if* $h \in [b_2, \infty)$. *Also,* $s' > 0$ *in* (T_1, ∞) *and* $s(\infty) = z(h)$, *where* $z(h)$ *is as in Theorem 3.*
'Here the payer is out of money if $r_t < s(T - t)$ and the receiver is out of money if $r_t > s(T - t)$, so the optimal strategy is $\tau_1^* = \min\{t > 0 \mid s_t \leqslant b_1\}$, $\tau_2^* = \inf\{t > 0 \mid r_t \geqslant b_2 \vee s(T - t)\}$.'

(3) *If* $h \in (0, h_1)$, *there exist* $T_1 \in [0, \infty)$, $T_0 \in (T_1, \infty]$ *and* $s \in C([0, \infty)) \cap C^\infty([0, T_1) \cup (T_1, T_0))$ *such that* (u, s) *is the unique solution of the following free boundary problem:*

$$\begin{cases} \mathcal{L}u - f = 0, & \forall\, t > 0, r \in (s(t) \wedge b_1, b_2), \\ u(r, t) = 0, & \forall\, t > 0, r \in (0, s(t) \wedge b_1] \cup [b_2, \infty), \\ u(\cdot, 0) = 0, \quad s(0) = h & \text{at } t = 0, \\ s(t) \in (b_1, b_2), u(s(t), t) = 0, & \forall\, t \in (0, T_1), \\ \quad u(\cdot, T_1) > 0 \; in \; (b_1, b_2), \quad u_r(b_1^+, T_1) = 0, \quad s(T_1) = b_1, & \\ s(t) \in (0, b_1), u_r(s(t), t) = 0, & \forall\, t \in (T_1, T_0), \\ s(t) := 0^-, |u_r(0, t)| < \infty, & \forall\, t > T_0. \end{cases} \tag{5.2}$$

In addition, $T_1 = 0$ *if* $h \leqslant b_1$; $T_1 > 0$ *if* $h \in (b_1, h_1)$; $T_0 = \infty$ *and* $s(\infty) = z(h)$ *if* $h \in [h_0, h_1)$; $T_0 < \infty$ *if* $h \in (0, h_0)$. *Also* $s' < 0$ *on* (T_1, T_0).

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'The receiver is out of money when $r_t > s(T-t)$, and the payer is out of the money when $r_t < s(T-t)$. Thus $\tau_1^* = \inf\{t > | r_t \leqslant b_1 \wedge s(T-t)\}$ and $\tau_2^* = \inf\{t > 0 \mid r_t \geqslant b_2\}$.'

(4) *If* $h \in (-\infty, 0]$, *u is the solution of the following initial boundary value problem:*

$$\begin{cases} \mathcal{L}u - f = 0, & \forall\, t > 0, r \in (0, b_2), \\ u(r,t) = 0 & \text{if } \ t = 0 \text{ or } \ r \geqslant b_2, \\ u_r \in L^\infty((0,\infty)^2). \end{cases} \tag{5.3}$$

'The receiver is always out of money so $\tau_1^* = \infty$, $\tau_2^* = \inf\{t > 0 \mid r_t \geqslant b_2\}$.'

We call the curve $r = s(T)$ *the* **parity curve** *since it has the following properties:*

$$u(\cdot, T) \leqslant 0 \ on \ (0, s(T)], \qquad u(\cdot, T) \geqslant 0 \ on \ [s(T), \infty) \quad \forall\, T \geqslant 0.$$

The rest of this section is devoted to the pure mathematical proof of the above theorem.

5.1 Uniqueness and monotonicity in *h*

Lemma 5.1 *For each* $h \in \mathbb{R}$, *Problem* (3.9) *with* $f = r - h$ *admits at most one solution. In addition, the solution is decreasing in h.*

Proof Let h and ℓ be constants with $h \geqslant \ell$. Let $u = v_1$ and $u = v_2$ be (viscosity) solutions of (3.9) with $f(r) = r - h$ and $f(r) = r - \ell$, respectively. We want to show that $v_1 \leqslant v_2$. Suppose $v_1 \leqslant v_2$ is not true. Then there exist positive r_0 and t_0 such that $v_1(r_0, t_0) > v_2(r_0, t_0)$. Let φ_1 and φ_2 be as in Lemma 3.1. Set $\varepsilon = [v_1(r_0,t_0) - v_2(r_0,t_0)]/[2\varphi_1(r_0) + \varphi_2(r_0)]$ and

$$w(r,t) = v_1(r,t) - v_2(r,t) - \varepsilon[\varphi_1(r) + \varphi_2(r)] \quad \forall\, r > 0, t \in [0, t_0].$$

Since v_{1r} and v_{2r} are bounded, there exist positive constants A and B such that

$$w_r > 0 \text{ in } \ (0, A] \times [0, t_0], \qquad w_r < 0 \text{ in } \ [B, \infty) \times [0, t_0].$$

Consequently, there exist $\hat{r} \in (A, B)$ and $\hat{t} \in (0, t_0]$ such that

$$w(\hat{r}, \hat{t}) = \max_{[A,B]\times[0,t_0]} w = \max_{(0,\infty)\times[0,t_0]} w \geqslant w(r_0, t_0) > 0.$$

We now derive a contradiction by considering three cases:

$$(1)\ \hat{r} \in (b_1, b_2), \quad (2)\ \hat{r} \in (0, b_1], \quad (3)\ \hat{r} \in [b_2, \infty).$$

(1) Suppose $\hat{r} \in (b_1, b_2)$. Then $\mathcal{L}v_1 = r - h$ and $\mathcal{L}v_2 = r - \ell$ in a neighbourhood of $(\hat{r}, \hat{t})$, $w_r(\hat{r}, \hat{t}) = 0$, $w_{rr}(\hat{r}, \hat{t}) \leqslant 0$ and $w_t(\hat{r}, \hat{t}) \geqslant 0$. But this gives the contradiction

$$0 < \hat{r} w(\hat{r}, \hat{t}) \leqslant \mathcal{L}w(\hat{r}, \hat{t}) = \ell - h \leqslant 0.$$

(2) Suppose $\hat{r} \in (0, b_1]$. Then $v_2(\hat{r}, \hat{t}) \geqslant 0$ and $v_1(\hat{r}, \hat{t}) = w(\hat{r}, \hat{t}) + v_2(\hat{r}, \hat{t}) + \varepsilon[\varphi_1(\hat{r}) + \varphi_2(\hat{r})] > 0$. Hence, v_1 is smooth and $\mathcal{L}v_1 = r - h$ in a neighbourhood of $(\hat{r}, \hat{t})$. Thus, we can use $\zeta(r,t) := v_1(r,t) - \varepsilon[\varphi_1(r) + \varphi_2(r)] - w(\hat{r}, \hat{t})$ as a test function for the viscosity function v_2 at $(\hat{r}, \hat{t})$ to derive the following contradiction: since $v_2(\hat{r}, \hat{t}) = \zeta(\hat{r}, \hat{t})$ and $\zeta \leqslant v_2$,

$$0 \leqslant \min\{\mathcal{L}\zeta - r + \ell, \zeta\}|_{r=\hat{r},t=\hat{t}} \leqslant \mathcal{L}\zeta - r + \ell|_{r=\hat{r},t=\hat{t}} = \ell - h - \hat{r}w(\hat{r},\hat{t}) < 0.$$

(3) Suppose $\hat{r} \in [b_2, \infty)$. Then $v_1(\hat{r},\hat{t}) \leqslant 0$ and $v_2(\hat{r},\hat{t}) = v_1(\hat{r},\hat{t}) - w(\hat{r},\hat{t}) - \varepsilon[\varphi_1(\hat{r}) + \varphi_2(\hat{r})] < 0$. Hence, v_2 is smooth and $\mathcal{L}v_2 = r - \ell$ in a neighbourhood of $(\hat{r},\hat{t})$. Thus, following an analogous argument as above, we can use $\zeta(r,t) := v_2(r,t) + \varepsilon[\varphi_1(r) + \varphi_2(r)] + w(\hat{r},t)$ as a test function for the viscosity v_1 at $(\hat{r},\hat{t})$ to derive another contradiction.

All of the above contradictions imply that $v_1 \leqslant v_2$. This gives monotonicity of solutions in h. Setting $h = \ell$, we also obtain the uniqueness. □

5.2 Case 1: $h \in [h_1, h_2]$ and the parity curve

We solve (3.9) for $h \in [h_1, h_2]$ and partially solve (3.9) for $h \in (b_1, h_1) \cup (h_2, b_2)$.

Lemma 5.2 *Let $h \in (b_1, b_2)$, $f = r - h$ and v be the solution of the initial boundary value problem*

$$\mathcal{L}v = f \text{ in } D := (b_1, b_2) \times (0, \infty), \qquad v = 0 \text{ on } [0,\infty)^2 \setminus D. \tag{5.4}$$

There exists a continuous function s defined on $[0,\infty)$ such that $s(0) = h$ and for each $t > 0$,

$$v(\cdot,t) < 0 \text{ in } (b_1, s(t)), \qquad v(s(t),t) = 0, \qquad v(\cdot,t) > 0 \text{ in } (s(t), b_2).$$

In addition, there exists $T_1 \in (0,\infty]$ such that

$$s(t) \in (b_1, b_2), \quad v_r(b_1^+, t) < 0, \qquad v_r(b_2^-, t) < 0 \quad \forall\, t \in (0, T_1).$$

Consequently, $u := v$ solves (3.9) in $[0,\infty) \times [0, T_1]$. Furthermore, the following holds:

1. *If $h \in [h_1, h_2]$, then $T_1 = \infty$;*
2. *If $h \in (b_1, h_1)$, then $T_1 < \infty$, $v_r(b_1, T_1) = 0$ and $v > 0$ and $v_t > 0$ in $(b_1, b_2) \times [T_1, \infty)$;*
3. *If $h \in (h_2, b_2)$, then $T_1 < \infty$, $v_r(b_2, T_1) = 0$ and $v < 0$ and $v_t < 0$ in $(b_1, b_2) \times [T_1, \infty)$.*

When $h < h_1$, the receiver is out of money when $r_t > s(T-t)$, since $s(T-t) < b_2$ so $\tau_2^* = \inf\{t > 0 | \, r_t > b_2\}$. When $h \in (b_1, h_1)$ and time to expiration is T with $T < T_1$, the payer is out of money if $r_t > s(T-t)$, so $\tau_1^* = \inf\{t > 0 \,|\, r_t < b_1\}$. Thus, $v(r,T)$ is the solution of (5.4) when $T \leqslant T_1$. Analogous discussion holds for the case $h \in (h_2, b_2)$.

Proof We divide the proof into several steps.

(1) First we investigate $v_t = \frac{\partial v}{\partial t}$. Differentiating $\mathcal{L}v = f$ with respect to t, we find that

$$\mathcal{L}v_t = 0 \text{ in } D, \quad v_t = 0 \text{ on } \{b_1, b_2\} \times (0,\infty), \quad v_t(r,0) = r - h \text{ for } r \in (b_1, b_2). \tag{5.5}$$

The equation $\mathcal{L}v_t = 0$ in D and condition $v_t = 0$ on the lateral boundary imply that the number of roots of $v_t(\cdot,t) = 0$ in (b_1, b_2) does not increase as t increases. Hence, there exists $\hat{T} \in (0,\infty]$ such that $v_t(\cdot,t) = 0$ admits exactly one root in (b_1, b_2) when $t \in [0, \hat{T})$; if $\hat{T} < \infty$, either $v_t > 0$ in $(b_1, b_2) \times [\hat{T}_1, \infty)$ or $v_t < 0$ in $(b_1, b_2) \times [\hat{T}, \infty)$.

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(2) Next we consider the sign of v_r on the lateral boundary. We consider $v_{rt}(b_1^+, \cdot)$ and $v_{rt}(b_2^-, \cdot)$.

(a) For each $t \in (0, \hat{T})$, there exists a unique $\tilde{s}(t) \in (b_1, b_2)$ such that $v_t(\cdot, t) < 0$ in $(0, \tilde{s}(t))$ and $v_t(\cdot, t) > 0$ in $(\tilde{s}(t), b_2)$. Then by Hopf's Lemma, $v_{tr}(b_1^+, t) < 0$ and $v_{tr}(b_2^-, t) < 0$.

(b) Suppose $\hat{T} < \infty$ and $v_t > 0$ in $(b_1, b_2) \times [\hat{T}, \infty)$. Then by Hopf's lemma, $v_{tr}(b_1^+, t) > 0 > v_{tr}(b_2^-, t)$ for all $t > \hat{T}$. This implies that $v_{rt}(b_2^-, t) < 0$ and $v_r(b_2^-, t) < 0$ for all $t > 0$. Also $v_{rt}(b_1^+, \cdot) < 0$ in $(0, \hat{T})$ and $v_{rt}(b_1^+, \cdot) > 0$ in $(\hat{T}, \infty)$. Thus, there exists $T_1 \in (\hat{T}, \infty]$ such that $v_r(b_1^+, \cdot) < 0$ in $(0, T_1)$ and $v_r(b_1^+, \cdot) > 0$ in $(T_1, \infty]$.

(c) Suppose $\hat{T} < \infty$ and $v_t < 0$ in $(b_1, b_2) \times [\hat{T}, \infty)$. Then by Hopf's Lemma, $v_{tr}(b_1^+, t) < 0 < v_{tr}(b_2^-, t)$ for all $t > \hat{T}$. This implies that $v_r(b_1^+, t) < 0$ for all $t > 0$ and there exists $T_1 \in (\hat{T}, \infty]$ such that $v_r(b_2^-, \cdot) < 0$ in $(0, T_1)$ and $v_r(b_2^-, \cdot) > 0$ in $(T_1, \infty]$.

Set $T_1 = \infty$ in the case (a). Then in any case we have $v_r(b_1^+, \cdot) < 0$ and $v_r(b_2^+, \cdot) < 0$ in $(0, T_1)$ so $u := v$ is the solution of (3.9) on $[0, \infty) \times [0, T_1]$.

Observe that $v(\cdot, \infty) := \lim_{t\to\infty} v(\cdot, t)$ is the solution of $Lv(\cdot, \infty) = f$ on (b_1, b_2) with boundary condition $v = 0$ at $r = b_1$ and $r = b_2$. Using Theorem 3, we derive the following:

(a) If $h \in [h_1, h_2]$, then $T_1 = \infty$.

(b) If $h \in (b_1, h_1)$, then $T_1 < \infty$, $v_r(b_1^+, T_1) = 0$ and $v_t > 0$ in $(b_1, b_2) \times [T_1, \infty)$.

(c) If $h \in (h_2, b_2)$, then $T_1 < \infty$, $v_r(b_2^-, T_1) = 0$ and $v_t < 0$ in $(b_1, b_2) \times [T_1, \infty)$.

(3) Next we investigate the roots of $v = 0$ in $(b_1, b_2) \times (0, T_1]$.
For each $t \in (0, T_1)$, we have $v_r(b_1^-, t) < 0$ and $v_r(b^+, t) < 0$. We can define

$$\ell(t) = \sup\{r \in [b_1, b_2] \mid v_r(\cdot, t) < 0 \text{ in } (b_1, r)\},$$
$$\rho(t) = \inf\{r \in [b_1, b_2] \mid v_r(\cdot, t) < 0 \text{ in } (r, b_2)\}.$$

Since $\int_{b_1}^{b_2} v_r(r, t)dr = v(b_2, t) - v(b_1, t) = 0$, for each $t \in (0, T_1)$,

$$\ell(t) < \rho(t), \qquad v_r(\cdot, t) < 0 \text{ in } [b_1, \ell(t)) \cup (\rho(t), b_2], \qquad v_r(\ell(t), t) = 0, v_r(\rho(t), t) = 0.$$

Since when t is small $v_t \approx r - h$, $v_{rt} \approx 1$ and $v_r \approx t$, we have $\ell(0^+) = b_1$ and $\rho(0^+) = b_2$. We will show that $v_r(r, t) \geqslant 0$ for $r \in (\ell(t), \rho(t))$. For this, we observe the following:

(i) Since **1** is a supersolution, by comparison, $v < 1$ on $\bar{D}$.

(ii) Set $\mathcal{L}_1 = \mathcal{L} - \frac{\sigma^2}{2}\frac{\partial}{\partial r} + \beta$. Differentiating $\mathcal{L}v = 0$, we find that

$$\mathcal{L}_1 v_r = 1 - v > 0 \text{ in } [b_1, b_2] \times (0, \infty).$$

Thus, v_r cannot attain a negative local minimum. Hence, $v_r \geqslant 0$ on $\cup_{0<t<T_1}[\ell(t), \rho(t)] \times \{t\}$. Using strong maximum principle, we then conclude that for each $t \in (0, T_1)$,

$$v_r(\cdot, t) < 0 \text{ in } [b_1, \ell(t)), \quad v_r > 0 \text{ in } (\ell(t), \rho(t)), \quad v_r < 0 \text{ in } (\rho(t), b_2].$$

Since $v(b_1, t) = 0$ and $v(b_2, t) = 0$, there exists a unique $s(t) \in (\ell(t), \rho(t))$ such that

$$v(\cdot, t) < 0 \text{ in } (b_1, s(t)), \quad v(s(t), t) = 0, \quad v(\cdot, t) > 0 \text{ in } (s(t), b_2).$$

In addition, if $h \in (b_1, h_1)$, then $T_1 < \infty$, $s(T_1) = b_1$ and $v > 0$ on $(b_1, b_2) \times [T_1, \infty)$; if $h \in (h_2, b_2)$, then $T_1 < \infty$, $s(T_1) = b_2$ and $v < 0$ on $(b_1, b_2) \times [T_1, \infty)$.

This completes the proof of Lemma 5.2. □

Figure 2 uses the parameters as in Figure 1. and $T = 1$, $s(0) = 1.25$.

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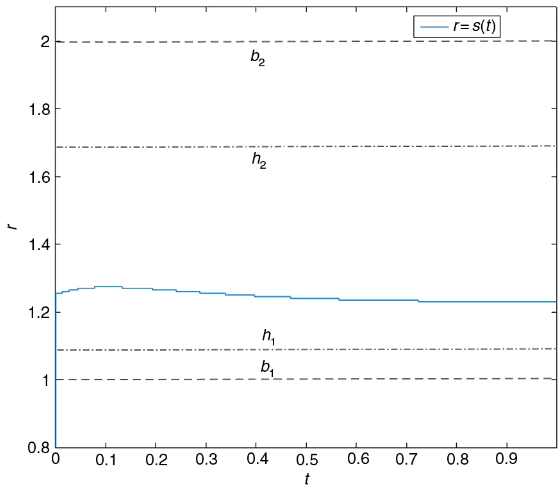

FIGURE 2. Parity-level $s(t)$ in Case 1.

5.3 Case 2: $h \in (h_2, \infty)$ and monotonicity of free boundary

For $h \in (h_2, b_2)$, we set T_1 as in Lemma 5.2 and set $u_0(\cdot) = v(\cdot, T_1)$, $w_0 = v_t(\cdot, T_1)$ and $s_0 = b_2$. For $h \in [b_2, \infty)$, we set $T_1 = 0$, $u_0(\cdot) = 0$, $w_0 = (r - h)\mathbf{1}_{(b_1,h]}(r)$ and $s_0 = h$. Let u on $[0, \infty) \times [T_1, \infty)$ be the solution of the following initial boundary value variational inequality:

$$\max\{\mathcal{L}u - f, u\} = 0 \;\text{ in } (b_1, \infty) \times [T_1, \infty), \quad u(\cdot, T_1) = u_0, \quad u = 0 \;\text{ on } [0, b_1] \times [T_1, \infty).$$

Note that this is exactly one part of the problems stated in (3.10). We shall show that there exists an increasing $s \in C([T_1, \infty))$ with $s(T_1) = s_0$ such that $u(\cdot, t) < 0$ in $(b_1, s(t))$ and $u(\cdot, t) = 0$ in $[s(t), \infty)$. Thus, $\mathcal{L}u - f = 0$ in $(b_1, b_2) \times [T_1, \infty)$. As $u_r(b_1^+, \cdot) < 0$ on (T_1, ∞), we also have $\min\{\mathcal{L}u - f, u\} = 0$ on $[0, b_1] \times [T_1, \infty)$. Thus, u is the solution of (3.9).

To solve the variational inequality and study the regularity of the free boundary $\{r = s(t)\}$, we convert the problem to the classical Stefan problem for $w = u_t$. We first provide a formal derivation of the free boundary condition at $r = s(t)$. Assume for the moment that $w := u_t$ is continuous near the free boundary $r = s(t)$. Then $u = u_r = u_t = 0$ at $r = s(t)$. From $\mathcal{L}u = f$ at $r = s(t)^-$, we find that $-\frac{\sigma^2 s(t)}{2} u_{rr}(s(t)^-, t) = s(t) - h$. Hence, differentiating the free boundary condition $u_r(s^-(t), t) = 0$, we find that

$$\frac{ds(t)}{dt} = -\frac{u_{rt}}{u_{rr}}\bigg|_{r=s(t)^-} = \frac{\sigma^2 s(t)}{2} \frac{w_r(s(t)^-, t)}{s(t) - h} \quad \forall\, t > 0.$$

Thus, formally, $w = u_t$ is the solution of the following free boundary problem: for (w, s),

$$\begin{cases} \mathcal{L}w = 0, \; w < 0, & \text{in } Q := \{(r, t) |\; t > T_1, r \in (b_1, s(t))\}, \\ w(\cdot, T_1) = w_0(\cdot), \quad s(T_2) = s_0, & \text{at } t = T_1, \\ w(r, t) = 0, & \forall\, t > T_1, r \in [0, b_1] \cup [s(t), \infty), \\ (s(t) - h)\dfrac{ds(t)}{dt} = \dfrac{\sigma^2}{2} s(t) w_r(s(t)^-, t), \; \forall\, t > T_1. \end{cases} \tag{5.6}$$

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Note that we have $w_0(\cdot) = u_t(\cdot, T_1) < 0$ in $(0, s_0)$ and $w_0(\cdot) = 0$ in $[s_0, \infty)$. This is a Stefan problem modelling a solidification process where w stands for the temperature, $(b_1, s(t))$ is the solid region, where $w < 0$, and $[s(t), \infty)$ is the liquid region, where $w = 0$; the latent heat at position r is $1 - h/r$. Once we solve (5.6), we can obtain the solution of (3.9) in $[0, \infty) \times (T_1, \infty)$ by defining u by

$$u(r,t) = u_0(r) + \int_{T_1}^{t} w(r,t)dt \quad \forall\, r \in (0,\infty), t \geqslant T_1. \tag{5.7}$$

Lemma 5.3 *Problem* (5.6) *admits a unique solution. In addition, $w < 0$ in Q and*

$$s \in C([T_1,\infty)) \cap C^\infty((T_1,\infty)), \quad s' > 0 \ \text{ in } \ (T_1,\infty), \quad s(\infty) = z(h).$$

Moreover, u defined by (5.7) *is the unique solution of* (3.9) *in* $[0,\infty) \times [T_1, \infty)$.

Proof When $h \in (h_2, b_2)$, we have a standard Stefan problem and the assertion of the theorem is quite standard; see, for example, Friedman [7]. When $h \geqslant b_2$, we have $s_0 = h$ so the equation for ds/dt in (5.6) at $t = 0$ is singular. For reader's convenience and also for the simplicity of presentation, we provide a proof here for the case $h \in [b_2, \infty)$ – i.e. $T_1 = 0$, $s_0 = h$, and $w_0(r) = (r - h)\mathbf{1}_{(b_1,h]}$.

1. Suppose there is a solution for $t \in [0, T]$. By comparing the solution u with the infinite horizon problem, we obtain a precise upper bound: $h < s(t) < z(h)$ for all $t \in (0, T]$, where $z(h)$ is defined in Theorem 3.
2. Now we establish a local in time existence of (5.6). For this, let δ be a positive constant to be determined later. We define

$$\mathbf{X}_\delta = \{s \in C([0,\delta]) \mid (s-h)^2 \in C^1([0,\delta]), s' \geqslant 0, s \leqslant 2z(h) + 1\}. \tag{5.8}$$

For $s \in \mathbf{X}_\delta$, we define W as the solution of the initial boundary value problem

$$\begin{cases} \mathcal{L}W = 0, & \forall\, r \in (b_1, s(t)), t \in [0,\delta], \\ W(r,t) = 0, & \forall\, t \in [0,\delta], r \in [0,b_1] \cup [s(t),\infty), \\ W(r,0) = r - h, & \forall r \in [b_1, h]. \end{cases} \tag{5.9}$$

We now estimate $W_r(s(t),t)$. First of all, by the maximum principle, we have $W \leqslant 0$. Hence, $W_r(s(t)^-, s) > 0$ for all $t \in (0,\delta]$. Next, for each $R \in [h, z(h)+1]$, we denote by $\Phi(r,R)$ the solution of the initial value problem

$$L\Phi = 0 \ \text{ in } \ (0,R], \quad \Phi|_{r=R} = 0, \quad \Phi_r|_{r=R} = 1.$$

Then with $\mu = 2\kappa/\sigma^2$ and $\nu = 2\beta/\sigma^2$, we have $\sigma^2\left(r^\mu e^{-\nu r}\Phi_r\right)_r = 2r^\mu e^{-\rho r}\Phi$. Thus, $\Phi < 0 < \Phi_r$ for $r \in (0,R)$. We now define

$$N = \sup_{R \in [h, 2z(h)+1], r \in [b_1, R)} \frac{R - r}{|\Phi(r,R)|}, \qquad \delta = \frac{1}{\sigma^2 N}.$$

Then for $r \in [b_1, R]$, we have $N|\Phi(r,R)| \geqslant R - r \geqslant h - r$. Since $s' \geqslant 0$, for each $T \in (0,\delta]$, comparing $W(r,t)$ with $N\Phi(r, s(T))$ for $r \in [b_1, s(\tau)]$ and $t \in [0,T]$, we find that $0 \geqslant$

$W(r,t) \geqslant N\Phi(r, s(T))$ for $r \in [0, s(t)]$ and $t \in [0, T]$. This implies that

$$0 \leqslant W_r(s(T), T) \leqslant N\Phi_r(s(T), s(T)) = N \quad \forall\, T \in [0, \delta]. \tag{5.10}$$

Now, define $\hat{s} \geqslant h$ by

$$\hat{s}(T) = h + \sqrt{\sigma^2 \int_0^T s(t) W_r(s(t), t) dt} \quad \forall\, T \in [0, \delta].$$

Then we have, for each $t \in (0, \delta]$,

$$\hat{s}(t) > h, \quad 2\Big(\hat{s}(t) - h\Big)\frac{\hat{d}s(t)}{dt} = \sigma^2 s(t) w_r(s(t), t).$$

In addition, by Cauchy inequality, $\hat{s}(t) - h < s(t)/2 + \sigma^2 N\delta/2$ so $\hat{s}(t) \leqslant 2h + \sigma^2 N\delta \leqslant 2s_0 + 1$. Hence, the map $\mathcal{T}s := \hat{s}$ maps $\mathbf{X}_\delta$ to itself. It is easy to see that $\mathcal{T}$ is a compact and continuous operator. Thus, by Schauder's fixed points theorem, $\mathcal{T}$ admits a fixed point in $\mathbf{X}_\delta$. We denote (one of) the fixed point by s, and the corresponding solution of (5.9) by w. Then (w, s) is a solution of (5.6) for $t \in [0, \delta]$. By a bootstrap argument, we can show that $s \in C^\infty((0, \delta])$ and $(s - h)^2 \in C^1([0, \delta])$. From Step 4 below, we see that the corresponding u is the unique solution of (3.9) for $t \in [0, \delta]$. Hence, by comparing with the solution of the infinite horizon problem, we have $s(t) < z(h)$ for all $t \in [0, \delta]$.

3. Repeating the same argument, we can establish the solution of (5.6) for $t \in [0, \delta]$, $[\delta, 2\delta], \ldots$. Hence, we obtain a solution of (5.6). The solution satisfies $s \in C^\infty((0, \infty))$, $(s - h)^2 \in C^1([0, \infty))$, $0 < w_r(s(t)^-, t) < N$ and $h < s(t) \leqslant z(h)$ for all $t > 0$.
4. We now recover u from w to verify our formal derivation of the free boundary problem. We define (i) $\psi(r) = 0$ for $r \in (0, s_0]$, and (ii) $\psi(r) = t$ is the inverse function of $r = s(t)$ when $t \in (0, \infty)$. Then for $t > 0$ and $r \in (b_1, s(t))$, we have

$$\begin{aligned}
u(r,t) &= \int_0^t w(r,\tau)d\tau = \int_{\psi(r)}^t w(r,\tau)d\tau, \qquad u_r(r,t) = \int_{\psi(r)}^t w_r(r,\tau)d\tau,\\
u_t(r,t) &= w(r,t) = \int_{\psi(r)}^t w_t(r,\tau)d\tau + f\mathbf{1}_{r\in[b_1,h]},\\
u_{rr}(r,t) &= \int_{\psi(r)}^t w_{rr}(r,\tau)d\tau - \mathbf{1}_{r\in(h,s(t))}\frac{w_r(s(t),t)}{\frac{ds(t)}{dt}}\\
&= \int_{\psi(r)}^t w_{rr}(r,\tau)d\tau - \frac{\sigma^2}{2} f\mathbf{1}_{r\in(h,s(t))}.
\end{aligned}$$

It then follows that

$$\begin{gathered}
u < 0, \quad \mathcal{L}u - f = 0 \quad \forall\, t > 0, r \in (b_1, s(t)),\\
u(s(t), t) = 0, \quad u_r(s(t), t) = 0 \quad \forall\, t > 0,\\
u(r, 0) = 0 \quad \forall\, r \in [b_1, h], \quad s(0) = h, \quad s'(t) > 0 \quad \forall\, t > 0.
\end{gathered}$$

Thus, u is a solution of (5.1) on $(0, \infty) \times [0, \infty)$. Since $b_2 < s(t)$ and $u_{rr}(b_1, t) = -\infty$ for $t \in (0, \infty)$, u is also a solution of (3.9). By uniqueness of the solution of (3.9), we see that (w, s) is unique. This completes the proof of Lemma 5.1.

□

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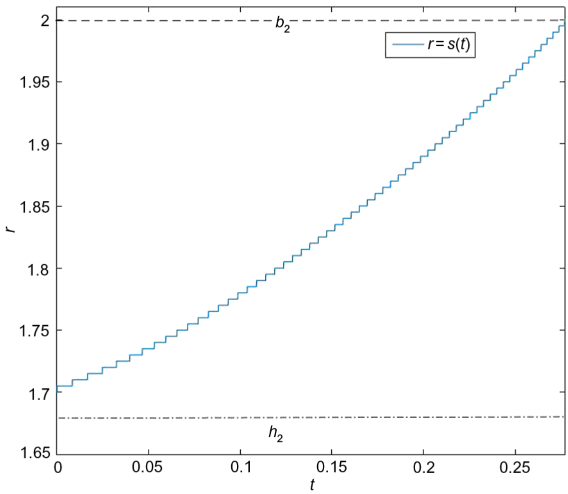

FIGURE 3. Parity-level $s(t)$ in Case 2.

Figure 3 uses the parameters as in Figure 2 but $s(0) = 1.70$, and it hit b_2 about $t = 0.27$.

5.4 Case 3: $h \in (0, h_1)$ and monotonicity of free boundary

For $h \in (b_1, h_1)$, we set T_1 as in Lemma 5.2 and set $u_0(\cdot) = v(\cdot, T_1)$, $w_0 = v_t(\cdot, T_1)$ and $s_0 = b_1$. For $h \in (0, b_1)$, we set $T_1 = 0$, $u_0(\cdot) = 0$, $w_0 = (r - h)\mathbf{1}_{[h,b_2)}(r)$ and $s_0 = h$. Let u on $(0, \infty) \times [T_1, \infty)$ be the solution of the following initial boundary value variational inequality:

$$\min\{\mathcal{L}u - f, u\} = 0 \text{ in } (0, b_2) \times (T_1, \infty),\ u(\cdot, T_1) = u_0,\ u = 0 \text{ on } [b_2, \infty) \times [T_1, \infty),\ u_r \in L^\infty.$$

Note that this is one part of the problem in (3.10). One can formally derive that $w := u_t$ is the solution of

$$\begin{cases} \mathcal{L}w = 0, \quad w > 0, & \forall\, t > T_1,\ r \in (s(t), b_2), \\ w(\cdot, T_1) = w_0(\cdot), s(T_1) = s_0 & \text{at } t = T_1, \\ w(r, t) = 0, & \forall\, t > T_1, r \in (0, s(t)] \cup [b_2, \infty) \\ s(t) \in (0, h), & \forall\, t \in (T_1, T_0), \\ \dfrac{ds(t)}{dt} = \dfrac{\sigma^2 s(t) w_r(s^+(t), t)}{2[s(t) - h]}, & \forall\, t \in (T_1, T_0), \\ s(t) := 0^-, |w_r(0, t)| < \infty & \forall\, t > T_0. \end{cases} \tag{5.11}$$

One can perform a similar analysis as in the previous section to show the following.

Lemma 5.4 *Assume that $h \in (0, h_1)$. Then problem (5.11) admits a unique solution satisfying*

$$s \in C([T_1, \infty)) \cap C^\infty((T_1, T_0)), \quad s' < 0 \text{ in } (T_1, T_0).$$

In addition, $T_0 = \infty$ and $s(\infty) = z(h)$ if $h \in [h_0, h_1)$; $T_0 < \infty$ and $u > 0$ in $[0, b_2) \times (T_0, \infty)$ if $h \in (0, h_0)$. Moreover, u defined by (5.7) is the unique solution of (3.9) on $[0, \infty) \times [T_1, \infty)$.

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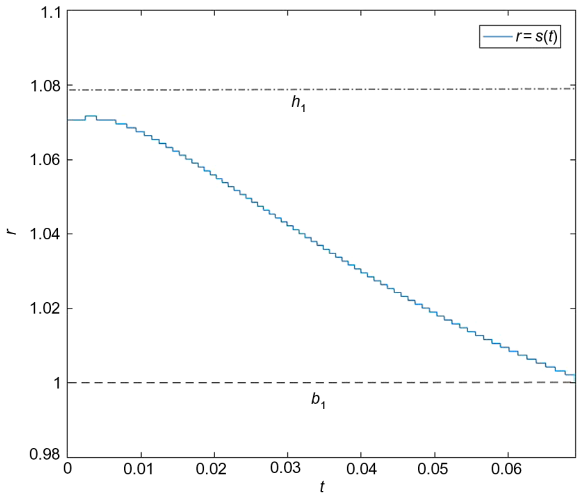

FIGURE 4. Parity-level $s(t)$ in Case 3.

It is never optimal for the payer to choose to terminate the contract if time to expiration is longer than T_0. Hence, when $h \in (h_0, h_1)$, the payer still needs to choose to terminate the contract if the floating interest rate is too low.

The proof follows from a similar idea as that in the previous section. We omit the details. We remark that when $t \in [0, T_0)$, we can use the boundary condition $u_r(s(t), t) = 0$ to show that $u_r \in L^\infty((0, \infty) \times [0, T_0])$. When $T > T_0$, u is the solution of

$$\mathcal{L}u = f \text{ in } [0, b_2) \times [T_0, \infty), \quad u = 0 \text{ on } [b_2, \infty) \times [T_0, \infty), \quad u_r \in L^\infty.$$

Here the condition $u_r \in L^\infty$ is needed to ensure the uniqueness.

Proof of Theorem 4. Assertions (1)–(3) of Theorem 4 follows from Lemmas 5.1–5.4. For the case $h \leqslant 0$, we have $f > 0$ in $(0, \infty)$ so the solution of (5.3) satisfies $u > 0$ in $(0, b_1) \times [0, \infty)$. This implies that u is the solution of (3.9). This completes the proof of Theorem 4.

Figure 4 uses the parameters as in Figure 2, but $s(0) = 1.074$. It hits b_1 at about $t = 0.061$.

6 Conclusion

In this paper, we establish intensity and structure-type models for pricing defaultable IRSs, where the float interest rate follows a CIR process. It is proved that the solution of the intensity model with structural-type intensities goes to the one of the structure-type models. Existence and uniqueness of the both solutions are proved.

For the predetermined default barriers b_1, b_2, the IRS has possibilities of ‘in’ or ‘out’ money on the barriers. If a proper credit default swap insures a party from the loss due to counterparty’s default, an IRS may be resumed by the intermediary at time when one party is at default. It is equivalent to the case that the party can choose to terminate the contract. As a result, the structural model turns to an optimal stopping problem; mathematically, it becomes a problem with double variational inequalities combined with an equation in their corresponding regions.

We have discussed both the time-dependent and time-independent problems for the structure-type models, whose solutions are able to be obtained by limits of the one of the intensity models.

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Also, the solutions are proved to be unique. The structures of the solutions are carefully analysed. Different cases of the swap rates are considered. We found the following facts:

For the infinite horizon problem from the structure-type model:

1. For a 'reasonable' swap rate h, there exists a so-called parity-level curve $r = z(h)$. If the current floating interest rate is at the parity level, the value of the swap is zero; above the parity level, the swap favours the payer; and below the parity level, the swap favours the receiver.
2. There exist constants h_0, h_1 and h_2, which can be easily evaluated and it satisfies $0 < h_0 < h_1 < h_2$ and $b_1 < h_1 < h_2 < b_2$, to divide the swap rate h into regions of distinct characteristic behaviour described as follows:
   (a) When $h \leqslant h_0$, there is no parity level since the contract always favours the payer.
   (b) When $h \in (h_0, h_1]$, there exists a parity-level free boundary $z(h) \in (0, h) \cap (0, b_1]$, such that the receiver is out of money when float interest rate is above $z(h)$ and the payer is out of money when float interest rate is below $z(h)$; thus, the payer terminates the contract when $r_t \leqslant z(h) \wedge b_1$.
   (c) When $h \in (h_1, h_2)$, which is properly contained in (b_1, b_2), there exists a parity-level free boundary $z(h) \in (b_1, b_2)$, such that the receiver is out of money when float interest rate is above $z(h)$ and the payer is out of money when float interest rate is below $z(h)$; thus, the payer terminates the contract when $r_t \leq b_1$ and the receiver terminates the contract when $r_t \geq b_2$.
   (d) When $h \in [h_2, \infty)$, there exists a parity-level free boundary $z(h) \in (h, \infty) \subset [b_2, \infty)$, such that the receiver is out of money when float interest rate is above $z(h)$ and the payer is out of money when float interest rate is below $z(h)$.

   A'reasonable' initiation swap rate should be set in the range of $(h_1, h_2) \subset (b_1, b_2)$.
3. The swap value is monotone with respect to the swap rate h.

For the time-depending problem from the structure-type model:

1. Above parity level, the free boundary becomes a function of time $r = s(h; T)$.
2. A 'reasonable' initiation swap rate should be set in the range of $(h_1, h_2) \subset (b_1, b_2)$:
   (a) when $h \in (h_1, h_2)$, the parity-level free boundary is $r = s(h; T) \in C^\infty(0, \infty)$;
   (b) when $h \in [h_2, \infty)$, the parity-level free boundary $s(h; T) \in C([0, \infty)) \cap C^\infty([0, T_1) \cup (T_1, \infty))$ – i.e. the free boundary will hit the default boundary b_2 at finite time T_1;
   (c) when $h \in (0, h_1]$, the parity-level free boundary $s(h; T) \in C([0, \infty)) \cap C^\infty([0, T_1) \cup (T_1, T_0))$ – i.e. the free boundary will hit the default boundary b_1 at finite time T_1, and the free boundary falls below zero after T_0.

There are IRSs having 'cap locks', i.e. if the receiver only pays a maximum ceiling rate, M, if the floating interest rate is higher than the ceiling rate, and will guarantee to pay a minimum floor rate, m, if the floating rate is below the floor rate. These kinds of IRSs are within the scope of our model if we modify f by setting

$$f(r) = \begin{cases} M - h & \text{if } r > M, \\ r - h & \text{if } r \in [m, M], \\ m - h & \text{if } r < m. \end{cases}$$

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Conflicts of Interest

None.

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