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Gauge freedom and objective rates in fluid deformable surfaces

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Morphodynamic descriptions of fluid deformable surfaces are relevant for a range of biological and soft matter phenomena, spanning materials that can be passive or active, as well as ordered or topological. However, a principled, geometric formulation of the correct hydrodynamic equations has remained opaque, with objective rates proving a central, contentious issue. We argue that this is due to a conflation of several important notions that must be disambiguated when describing fluid deformable surfaces. These are the Eulerian and Lagrangian perspectives on fluid motion, and three different types of gauge freedom: in the ambient space; in the parameterisation of the surface; and in the choice of frame field on the surface. We clarify these ideas, and show that objective rates in fluid deformable surfaces are time derivatives that are invariant under the first of these gauge freedoms, and which also preserve the structure of the ambient metric. The latter condition reduces a potentially infinite number of possible objective rates to only two: the material derivative and the Jaumann rate. The material derivative is invariant under the Galilean group, and therefore applies to velocities, whose rate captures the conservation of momentum. The Jaumann derivative is invariant under all time-dependent isometries, and therefore applies to local order parameters, or symmetry-broken variables, such as the nematic Q-tensor. We provide examples of material and Jaumann rates in two different frame fields that are pertinent to the current applications of the fluid mechanics of deformable surfaces.

Key word: membranes

1. Introduction

Morphodynamic descriptions of fluid and elastic deformable surfaces have relevance across a wide variety of biological and soft matter systems, including lipid membranes (Rangamani *et al.* 2013; Morris & Turner 2015; Sahu *et al.* 2017, 2020*a*; Tchoufag *et al.* 2022) and vesicles (Boedec *et al.* 2014; Keber *et al.* 2014; Al-Izzi *et al.* 2020*a*,*b*), thin layers of cortical cytoskeleton (Kruse *et al.* 2005; Bächer *et al.* 2021; Da Rocha *et al.* 2022), monolayers of epithelial tissue (Morris & Rao 2019; Al-Izzi & Morris 2021; Julicher *et al.* 2018; Blanch-Mercader *et al.* 2021*a*,*b*; Hoffmann *et al.* 2022; Khoromskaia & Salbreux 2023), liquid crystal shells (Napoli & Vergori 2012; Khoromskaia & Alexander 2017; Nestler & Voigt 2022; Nestler & Voigt 2022), actuating nematic elastomers and glasses (Modes & Warner 2011; Modes *et al.* 2011; Mostajeran *et al.* 2017; Feng *et al.* 2024), composites of microtubules and kinesin motors (Sanchez *et al.* 2012; Ellis *et al.* 2018; Pearce *et al.* 2019, 2021) and actively polymerising actin filaments (Simon *et al.* 2019). These examples encompass a broad range of media which can be active or passive (Salbreux & Juhlicher 2017; Al-Izzi & Alexander 2023), and which often possess additional order in the form of a nematic director field. Active nematic surfaces have been increasingly studied in the context of tissue mechanics and morphogenesis (Bächer *et al.* 2021; Rank & Voigt 2021; Alert 2022; Bell *et al.* 2022; Nestler & Voigt 2022; Salbreux *et al.* 2022; Vafa & Mahadevan 2022; Khoromskaia & Salbreux 2023), where the nematic ordering – and in particular the topological defects – are known to play an essential role in the development of protrusions and extrusions (Julicher *et al.* 2018; Metselaar *et al.* 2019; Al-Izzi & Morris 2021; Hoffmann *et al.* 2022; Pearce *et al.* 2022; Vafa & Mahadevan 2022, 2023; Hoffmann *et al.* 2023).

The underlying formulation of morphodynamics in this context has been explored in recent years (Torres-Sánchez *et al.* 2019; Al-Izzi & Morris 2023), and while there is a growing appreciation for the importance of a principled, geometric approach to the correct hydrodynamic equations, certain aspects have remained opaque. Most notably, and independent of any particular model or material under consideration, there is still confusion about how notions that have long been well understood in a fixed flat space translate to a moving and deforming surface, with the correct choice of objective rates being a persistently contentious issue (Marsden & Hughes 1994; Nitschke & Voigt 2022; Al-Izzi & Morris, 2023).

Partly, this difficulty arises from the large number of choices for objective rates. In Marsden & Hughes (1994), it is shown that any linear combination of Lie derivatives of the different covariant/contravariant/mixed forms of a tensor is objective. Furthermore, Kolev & Desmorat (2021) show that there are yet more classes of objective rates that do not arise from Lie derivatives. The choice of which rate is appropriate for a given application is not obvious, and various authors have made different choices. Oldroyd (1950) discussed the difference between upper- and lower-convected derivatives, introducing the “Oldroyd A” and “Oldroyd B” models which are widely used in the literature on viscoelastic materials. These have been used to model complex fluids and deforming surfaces Stone *et al.* (2023); de Kinkelder *et al.* (2021). Al-Izzi & Morris (2023) employ the Jaumann rate, an average of Oldroyd’s upper- and lower-convected rates, in their study of morphodynamics. The same rate is widely employed in the literature on liquid crystals, for example in deGennes & Prost (2013). Nitschke & Voigt (2022, 2023) have discussed the differences between various rates – including the upper and lower convected, material and Jaumann rates – in the context of deformable surfaces, and have computed them in various specific cases. They use these empirical differences to make suggestions regarding which rate is correct for which scenario. However, there remains no clear and principled method of choosing which is appropriate for a given application, and often there is little physical intuition underlying the choices made. We therefore argue that there remains no clear consensus about which rate is correct for a formulation of morphodynamics.

In this paper, we address the issue of objective rates in morphodynamics via three key contributions.

The first is split across §§ 2 and 3. In § 2 we provide a formal mathematical description of the motion of a deformable surface which clarifies the concepts of an 'Eulerian' and a 'Lagrangian' description. The way these terms are used in the literature on deformable surfaces is imprecise and at odds with the way they are defined formally (e.g. by Arnold & Khesin 2021) in the mathematical literature, which can lead to confusion. For example, in the computational studies of Torres-Sánchez *et al.* (2019) and Sahu *et al.* (2020*b*), the terms 'Eulerian' and 'Lagrangian' do not correspond to the use of the terms by Arnold & Khesin (2021), but to the difference between 'fixed-surface coordinate systems' and 'convected coordinate systems', a rather different concept. These are really manifestations of an inherent gauge freedom that exists in the mathematical description of a deforming surface, and the conflicting use of terminology has led to a muddling between the Eulerian/Lagrangian dichotomy and the notions of objectivity and gauge freedoms in morphodynamics, which we attempt to disentangle. In § 3, we precisely describe three group actions that relate to different gauge freedoms in the description of a deforming surface: a freedom in the ambient space which relates to objectivity, a freedom in the coordinate parameterisation of the surface itself and the freedom to choose a frame field along the surface. This last gauge freedom captures the distinction between 'fixed-surface coordinate systems' and 'convected coordinate systems': a 'Lagrangian' frame (really a convected coordinate system) is one that moves with the fluid, while an 'Eulerian' frame (really a fixed-surface coordinate system) is one which 'stays still'. A careful disambiguation of these distinct concepts provides both mathematical and physical insights.

Secondly, in § 4 we carefully examine the concept of an objective rate for a deformable surface. We clarify the formulation of objectivity by presenting it in terms of invariance under certain gauge transformations – this is a principled approach making use of the symmetries inherent in physics, and therefore our notions do not depend on the context of a specific material. We also stress that it is important to impose another physically motivated requirement on our rates, besides just objectivity: that the rate does not advect the Euclidean metric in the ambient space. This constraint has not, to our knowledge, been examined in the context of fluid dynamics in a fixed space, but when we consider a deformable surface it becomes especially relevant. With this additional requirement, the material and Jaumann rates emerge as the correct choices for the objective rate. These are shown to be invariant under different group actions. The former applies to velocities, whose rate is intimately linked with momentum conservation, and thus is synonymous with the Galilean group of transformations. The latter applies to local, symmetry-broken variables, such as the nematic Q-tensor, that are manifestly invariant under all time-dependent isometries. More concretely, this comes down to whether the rate of a given quantity should depend on (global) angular momentum, and therefore whether an observer is spinning: the material derivative does not account for this, whereas the Jaumann rate is "corotational".

Our third contribution is to provide formulas for the material and Jaumann rates of various quantities involved in surface dynamics. In § 5 we give expressions for these quantities in two different choices of frame. The first is a local coordinate chart which is not advected by the flow – an "Eulerian" parameterisation (Torres-Sánchez *et al.* 2019, 2020; Sahu *et al.* 2020*b*) – and the second is an orthonormal frame. Detailed computations of this formula using two different approaches – the Riemann–Cartan method of moving frames, as well as the more familiar Ricci calculus that employs a coordinate frame and Christoffel symbols – are presented in Appendices A and B. We also include Appendix C, which overviews the geometric notions of pushforward and pullback that we make use of throughout the text.

Sections 2 and 3 contain technical material regarding the distinction between Eulerian and Lagrangian formulations of deformable surface physics and the three gauge freedoms inherent in the formulation. In our discussion we employ various notions from differential manifold theory, such as group actions, Lie derivatives and fibre bundles, to bring some mathematical clarity to the subject. To aid readers who are unfamiliar with this material we have given informal descriptions of these concepts as they are introduced, and more technical details can be found in Frankel (2011). We strongly advocate a wider use of this machinery, especially when discussing deformable surfaces, where geometry plays such an essential role, and hope our discussion here helps promote this.

Readers who are less interested in these technical issues and more so in the practical question of which objective rate to use and how to compute it may skip §§ 2 and 3 and focus on §§ 4 and 5. We provide an overview of all the results in this paper in the discussion, § 6, suitable for readers of all backgrounds.

2. Eulerian vs Lagrangian perspectives

Consider the standard distinction between the Eulerian and Lagrangian perspectives on fluid flow inside a fixed manifold S. In the Eulerian perspective, we imagine standing still at a fixed point in S and watching the fluid flow past us. The Eulerian specification of the flow field at a point $p \in S$ and at time t is therefore a vector $\mathbf{v}_t(p)$ in the tangent space $\mathrm{T}_p S$ to the point p that gives the direction in which matter is flowing past us. Globally, this results in a time-dependent vector field $\mathbf{v}_t$ on S, a path in the space $\mathfrak{X}(S)$ of vector fields on S. By contrast, the Lagrangian perspective instead considers the medium to be made up of fluid particles, whose paths we follow through time. At time t, the fluid particle that was initially at point $p \in S$ has moved to some new point $\psi_t(p) \in S$. For a smooth fluid motion, this gives rise to a time-dependent diffeomorphism ψ_t of S, a path in the diffeomorphism group $\mathrm{Diff}(S)$ –recall that a diffeomorphism is nothing more than a smooth and invertible relabelling of the points of S, and that two diffeomorphisms can be composed to give another diffeomorphism, which gives the collection $\mathrm{Diff}(S)$ of all diffeomorphisms the mathematical structure of a group. Crucially, these two perspectives are completely equivalent and can be mapped onto one another: the Eulerian flow field $\mathbf{v}_t$ and the Lagrangian diffeomorphism ψ_t are related by

$$\mathbf{v}_t \circ \psi_t = \partial_t \psi_t, \tag{2.1}$$

where ∂_t denotes partial differentiation with respect to time.

The deeper mathematical relationship between Eulerian and Lagrangian perspectives has been studied using the language of differential geometry and Lie group theory, a set of ideas originally developed by Arnold (2014) and described in detail in the textbook of Arnold & Khesin (2021). We will not give a detailed discussion of this theory here, and instead only concern ourselves with aspects salient to the differences between ordinary fluid dynamics and the motion of deformable surfaces. Chief amongst these is the use of (2.1) to interpret $\partial_t \psi_t$ as a vector field on S: while this is fine in the setting of ordinary fluid dynamics, trying to carry this reasoning over to the setting of deformable surfaces requires care.

Properly, a Lagrangian motion is a path through the diffeomorphism group Diff(S) (or the group SDiff(S) of volume-preserving diffeomorphisms for an incompressible flow). At each time t the tangent direction $\partial_t \psi_t$ to this path lies in the tangent space $\mathrm{T}_{\psi_t}\mathrm{Diff}(S)$ to the diffeomorphism group at the point ψ_t. At any point the tangent space to the diffeomorphism group can be identified with its Lie algebra, and this is nothing more than the space $\mathfrak{X}(S)$ of vector fields on S – thus, for any fixed t, $\partial_t \psi_t$ can be understood as a vector field on S. When considering the motion and deformation of fluid surfaces, however, this identification is more complicated, and correspondingly the relationship between the Eulerian and Lagrangian perspectives is more complicated.

Notably, when dealing with deformable surfaces there is not one manifold, but three: an abstract body B, which in our case is a two-dimensional (2-D) manifold because we are considering a deformable surface; an ambient space in which the motion happens, which for us will always be 3-D Euclidean space $\mathbb{R}^3$ equipped with the Euclidean metric e; and the image M of B in $\mathbb{R}^3$, which is the physical surface which we observe. The ambient space is the natural analogue of the fixed manifold in ordinary fluid dynamics, and must remain invariant under any dynamics. The body is not usually equipped with a metric, but often has a volume form μ which is interpreted as a density measure for an unstrained configuration, and is necessary to describe conservation of mass and incompressibility.

To clarify, we consider a configuration of the system as an embedding $r : B \to \mathbb{R}^3$ of the body into the ambient space, whose image is the material, a smooth submanifold M of $\mathbb{R}^3$. The collection of all such embeddings is a space Emb which has the structure of an infinite-dimensional manifold – the details of infinite-dimensional spaces are not relevant to our discussion here, what is relevant is that intuitive notions of smooth paths and variations make sense in this context. A motion of the system is then a path $r_t : B \to \mathbb{R}^3$ of embeddings with images M_t, that is, a path in Emb – the role played by this space is therefore analogous to the role played by the diffeomorphism group Diff(S) in the case of ordinary fluid dynamics in a fixed space S.

The derivative $\partial_t r_t$ defines a tangent vector to this path in the space Emb. Define $\mathbf{U}_t$ to be the value of this derivative at time t, an element of $\mathrm{T}_{r_t}\mathrm{Emb}$: this is a map $\mathbf{U}_t : B \to \mathrm{T}\mathbb{R}^3$. For a deformable surface this map is the analogue of the derivative $\partial_t \psi_t$ of the diffeomorphism giving the Lagrangian description of the motion. However, note that this map is not a vector field, either on B or in $\mathbb{R}^3$, and it cannot be identified with one via the relationship (2.1) used in ordinary fluid dynamics because the tangent space to Emb is not isomorphic to a space of vector fields. This is one of the key differences when considering the motion of a deformable surface.

We can derive two vector fields from the map $\mathbf{U}_t$ according to the diagram shown in figure 1. The first of the two vector fields is the "right Eulerian velocity", a map $\mathbf{v}_t : M_t \to \mathrm{T}\mathbb{R}^3$ defined by

$$\mathbf{v}_t := \mathbf{U}_t \circ r_t^{-1}. \tag{2.2}$$

This is a vector field in $\mathbb{R}^3$ defined along M_t. As such, it decomposes as $\mathbf{v}_t = \mathbf{v}_t^{\parallel} + v_t^n \mathbf{n}_t$, where $\mathbf{v}_t^{\parallel}$ is tangent to M_t and $\mathbf{n}_t$ is the normal. There is also the 'left Eulerian velocity', a vector field $\mathbf{V}_t : B \to \mathrm{T}B$ on B defined by

$$\mathbf{V}_t := (\mathrm{T}r_t)^{-1} \circ \mathbf{U}_t. \tag{2.3}$$

Here, $\mathrm{T}r_t$ denotes the tangent map induced by the embedding; see Appendix C for the definition. The right Eulerian velocity pulls back to the left Eulerian velocity, $r_t^* \mathbf{v}_t = \mathbf{V}_t$, but for purely dimensional reasons the normal component is lost and the pushforward of the left Eulerian velocity is accordingly $r_{t*}\mathbf{V}_t = \mathbf{v}_t^{\parallel}$, the tangent part $\mathbf{v}_t$.

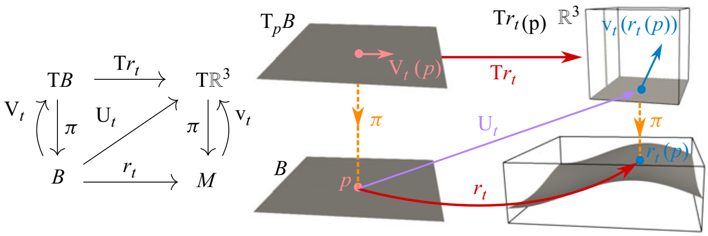

Figure 1. An embedding r_t of an abstract space B into $\mathbb{R}^3$ induces a vector field $\mathbf{V}_t$ on B called the left Eulerian velocity, as well as a map $\mathbf{v}_t : M_t \to \mathrm{T}\mathbb{R}^3$ on the image M_t of the embedding. We interpret that latter as a vector field along M_t which has a part tangent to M_t but may also have a part normal to M_t, and refer to it as the right Eulerian velocity. Pulling back this vector field along the embedding r_t 'forgets' the normal part, yielding the left Eulerian velocity $\mathbf{V}_t$. The mathematical relationships between these maps are shown in the diagram on the left, while a more visual representation is shown on the right. Here, Tr_t denotes the tangent map (matrix of partial derivatives) induced by r_t, π is the projection from the tangent bundle to the underlying manifold and $\mathbf{U}_t = \partial_t r_t$ as described in the text.

Neither of these vector fields quite corresponds to our intuitive idea of Eulerian motion: the right Eulerian velocity $\mathbf{v}_t$ is defined on a surface that is intrinsically moving; the left Eulerian velocity does not encode the normal motion of the surface. To properly specify the Eulerian and Lagrangian descriptions of surface motion, we need to introduce a small amount of additional structure which formally captures some intuitive notions about surface dynamics that would be unnecessary if B were itself a 3-D body.

The manifold M_t has two natural vector bundles associated with it. The first is TM_t, its usual 2-D tangent space. However, we also need to consider directions that contain a component normal to M_t, and these lie in a 3-D bundle that we denote by E_t, the restriction of T$\mathbb{R}^3$ to M_t. The Euclidean metric defines an orthogonal splitting $E_t = \mathrm{T}M_t \oplus N_t$, where N_t is the 1-D normal bundle. The orthogonal projection of the Euclidean metric onto TM_t is then exactly the induced metric on M_t (the first fundamental form). The right Eulerian velocity $\mathbf{v}_t$ is a section of the bundle E_t; to be explicit, a vector field defined along M_t that has both tangent and normal components. This bundle is visualised in figure 2.

We fix an initial embedding r_0 with image M_0. The motion r_t then induces diffeomorphisms $\lambda_t : M_0 \to M_t$ with $\lambda_t = r_t \circ r_0^{-1}$. The path $\lambda_t(p)$ of some point $p \in M_0$ is then naturally seen as the motion of a fluid particle in the ambient space $\mathbb{R}^3$, as thus is the a Lagrangian description of the motion. We have the relationship

$$\mathbf{v}_t \circ \lambda_t = \partial_t \lambda_t, \tag{2.4}$$

and hence the right Eulerian velocity $\mathbf{v}_t$ is the vector field naturally associated with the Lagrangian motion λ_t.

The Eulerian description must involve standing at a fixed point on the initial manifold M_0. If M_t were a 3-D manifold, there we would be no problem with pulling back the velocity field $\mathbf{v}_t$ on M_t along λ_t to give a vector field on M_0 that naturally corresponds to the Eulerian picture of the motion Casey & Papadopoulos (2002). Because we consider a 2-D surface this does not work out for dimensional reasons, but this is a purely technical issue that can easily be resolved with a simple definition.

The map Tλ_t : T$M_0 \to$ TM_t associates a tangent vector on M_0 with a tangent vector on M_t. Let us define a map $A_t : E_0 \to E_t$ that acts on the spaces of 3-D vector fields along M_0 and associates them with 3-D vector fields along M_t. To define this map, we use the splitting $E_t = \mathrm{T}M_t \oplus N_t$ into a tangent space and a normal bundle. By linearity, it suffices for us to define that A_t acts on the TM_0 factor exactly as Tλ_t, and acts on the N_0 factor

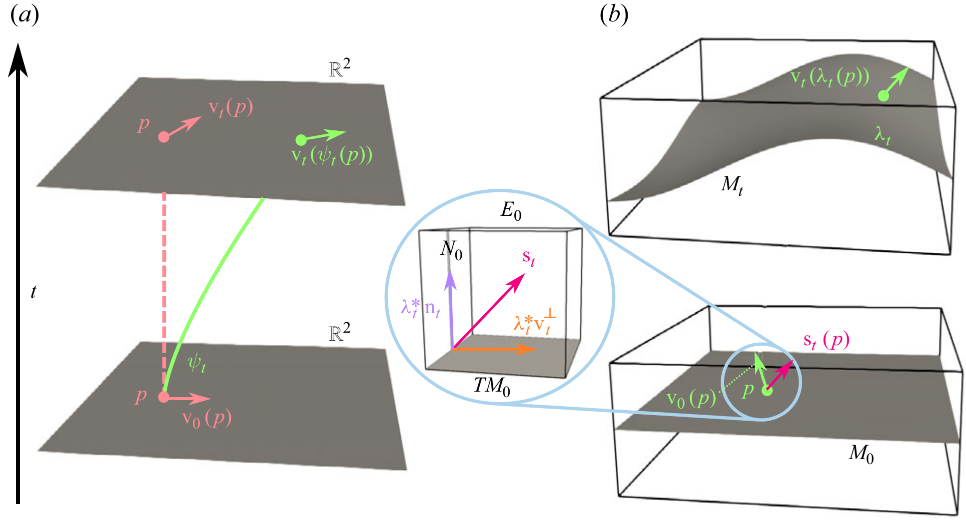

Figure 2. Comparison of the Eulerian and Lagrangian pictures for motion in a fixed surface (*a*) and for a deformable surface (*b*). In a fixed plane $\mathbb{R}^2$, the Lagrangian picture involves a diffeomorphism ψ_t of $\mathbb{R}^2$ that moves a fluid particle initially the point p to the point $\psi_t(p)$. We may also consider an Eulerian picture, where we stand still at the point p and watch the fluid flow past us, its direction at time t being given by the vector $\mathbf{v}_t(p)$. A deformable surface M_t embedded in $\mathbb{R}^3$ undergoes a Lagrangian motion described by a diffeomorphism $\lambda_t : M_0 \to M_t$ that maps the initial surface onto the time t surface. The associated right Eulerian velocity field $\mathbf{v}_t$ lies in the extended tangent space E_t to M_t, as described in the text. By pulling back the entire tangent space via λ_t (inset) as described in the text we can define a time-varying vector field $\mathbf{s}_t$ along M_0 which plays the role of the Eulerian velocity at a fixed point $p \in M_0$.

by mapping the unit normal $\mathbf{n}_0$ to M_0 to the unit normal $\mathbf{n}_t$ to M_t. Now, we may naturally define a section $\mathbf{s}_t$ of E_0 by

$$\mathbf{s}_t(p) := A_t^{-1} \circ \mathbf{v}_t \circ \lambda_t(p). \tag{2.5}$$

Concretely, if $\mathbf{v}_t \circ \lambda_t = \mathbf{v}_t^{\parallel} + v_t^n \mathbf{n}_t$, then

$$\mathbf{s}_t(p) = \lambda_t^* \mathbf{v}_t^{\parallel} + (\lambda_t^* v_t^n)\mathbf{n}_0. \tag{2.6}$$

Then $\mathbf{s}_t$ is always a 3-D vector field along M_0, as we may consider it to be the Eulerian flow field of the motion. It of course satisfies $\mathbf{s}_0 = \mathbf{v}_0$, because the map A_0 is just the identity map. We visualise this pullback process in figure 2. We could also describe this in a convective representation on the manifold B, by pulling the whole bundle E_t back along the embedding r_t, and taking the pullback $r_t^* \mathbf{s}_t$ as the left Eulerian velocity, a section of $r_t^* E_t$.

The ways in which the terms ‘Eulerian’ and ‘Lagrangian’ are often used in the literature – especially in computational studies (Torres-Sánchez *et al.* 2019, 2020; Sahu *et al.* 2020*b*) – refers to something quite different from the Eulerian–Lagrangian split we have just outlined. Rather, they describe different ways of parameterising quantities on the surface in terms of a surface frame field: this may be convected with the fluid (‘Lagrangian’) or not (‘Eulerian’). The freedom to choose the frame is a kind of ‘gauge freedom’, and the physics is agnostic to the particular choice of gauge. There are, in fact, several gauge freedoms that arise in the mathematical formulation of morphodynamics, some of which leave the physics invariant and some of which do not, and these choices play an essential role in the formulation of objectivity and observer motion. We describe this in detail in the next section.

3. Gauge freedom

We describe gauge freedoms in the usual language of gauge theory in physics, in terms of the action of a symmetry group and the principal bundle which it defines (Frankel 2011; Naber 2011*a*,*b*). We introduce this technical language to help make contact with other areas of physics and also for precision, but a familiarity with it is not essential to follow the key arguments of this section, as long as the reader grasps the intuitive notion of a symmetry group acting on a space, which we describe now.

Informally, a group action describes the way that transformations move points around in a physical space. For example, the circle group acts on a plane by rotations: the rotation by an angle α maps a point specified by (r, θ) in polar coordinates to the point $(r, \theta + \alpha)$. The group act divides the space up into pieces that it leaves invariant –in our example, these pieces are circles of constant radius in the plane, since any rotation will keep the points on one of these circles on that circle. This partition of the space up into a parameterised family of pieces is, loosely, the concept of a bundle.

In order to help fix ideas, we briefly comment on how gauge freedoms manifest in the motion of a fluid in a fixed space S. There is a natural action of the group $\mathrm{Diff}(S)$ on this space which captures the notion of a Lagrangian flow – a diffeomorphism ψ moves the point p to the point $\psi(p)$ in S. It has a subgroup $\mathrm{Isom}(S)$ of isometries, those diffeomorphisms ϕ that fix the metric g on S, $\phi^* g = g$. In n-dimensional Euclidean space this is the Euclidean group $\mathrm{E}(n)$ of rotations and translations. If we add in time-dependent isometries with a constant velocity, so-called Galilean boosts, then we obtain the Galilean group, Gal. Hydrodynamics is manifestly not invariant under the action of $\mathrm{Diff}(S)$, but it is required to be invariant under Gal. This then corresponds to a gauge freedom in how we specify our equations of motion, as we are free to describe the physics itself in any inertial reference frame. Informally, objectivity is the invariance of our equations of motion under the action of the Galilean group, which can be interpreted as their invariance under a motion of a hypothetical observer in an inertial reference frame.

There is a second gauge freedom which is not concerned with the physics, but simply the representation of physical quantities. A frame field on S is a choice of basis for the tangent space $\mathrm{T}_p S$ at every point p, which varies smoothly on space. We should be careful to disambiguate this from the notion of an inertial reference frame or a hypothetical observer, and hence from the notion of objectivity – it is a fundamentally different concept. To describe a physical quantity that is a vector or a tensor – for example the velocity field $\mathbf{v}$ – we pick some frame field $\mathbf{e}_j$ – for example a coordinate frame – which spans the tangent space to S, and then we may write $\mathbf{v} = v^j \mathbf{e}_j$ for some set of functions v^j. However, the frame field is entirely arbitrary and bears no relation to the physics. We are free to choose a different frame field $\bar{\mathbf{e}}_j$ and instead write $\mathbf{v} = \bar{v}^j \bar{\mathbf{e}}_j$. Of course, the vector field $\mathbf{v}$ does not change under a change of frame, and it does not matter whether the frame field is a coordinate frame, whether it is orthonormal, or whether it varies in time. This freedom to choose the frame field is then associated with the action of $\mathrm{Diff}(S)$ not on the manifold itself, but on the frame bundle $\mathrm{F}S \rightarrow S$, an action which sends a frame $\mathbf{e}_j$ to the new frame $\psi_* \mathbf{e}_j$.

Now we return to the setting of deformable surfaces. In this problem there are really three distinct gauge freedoms, and part of the confusion around objectivity and "Eulerian" vs "Lagrangian" motions has to do with a failure to properly disambiguate between them. The state of the deforming surface is captured by an element $r \in \mathrm{Emb}$. There are two group actions on this space: $\mathrm{E}(3)$ and $\mathrm{Diff}(B)$ act on Emb respectively from the right and the left.

The first group $\mathrm{E}(3)$ is the isometry group of the ambient space, and it reflects our ability to change reference frames in the ambient space (not on the surface). The second group

Diff(B) is the diffeomorphism group of B, and its action on Emb reflects our ability to make arbitrary changes of coordinate system in the base space. This is a gauge freedom related to the parameterisation of the motion of the surface itself, independently on any physics or quantities defined on the surface. The third gauge freedom again concerns an action of Diff(B), but this time on the frame bundle $\mathrm{F}B \to B$; equivalently, an action of the diffeomorphism group Diff(M_t) of the embedded surface at time t on its own frame bundle $\mathrm{F}M_t \to M_t$. Informally, a frame on M consists of a choice, for each point $p \in M$, of basis for the tangent space T_pM, and the frame bundle $FM \to M$ consists of all possible choices of frame on M – for example, any global choice of coordinate function defines a frame. As with motion in a fixed space, this group action describes our freedom in choosing a local frame field with which to represent quantities defined along the surface.

We concretely define each of these group actions and their associated gauge freedoms in the following subsections, and describe their physical interpretation in more detail.

3.1. *Gauge freedom in the ambient space*

First we describe the action of the isometry group. For an isometry $\phi \in \mathrm{E}(3)$ and embedding $r \in \mathrm{Emb}$, the group action sends r to the composition $\phi \circ r$ and sends the image M of r to a different submanifold $\phi(M)$. While these two submanifolds will in general be different, they are related by a rigid body motion and not by a deformation (that is, a "pure motion") and the first and second fundamental forms induced on $\phi(M)$ are the same as those on M – more precisely, the pullbacks of these quantities to B are equal.

Readers familiar with bundle theory may appreciate an alternative perspective on this: it is possible to view this gauge freedom as introducing a fibre bundle structure on Emb, parameterising the choices of "observer reference frame" in the ambient space in the following way. Group actions on a space define quotient spaces and principal bundles over those spaces (Frankel 2011; Naber 2011*a,b*), by collapsing any submanifold left invariant by the group action down to a single point in the quotient. We define Iso to be the quotient space defined by the action of E(3) on Emb. This is the space of deformable surfaces up to isometry, which can be interpreted as surfaces with a fixed centre of mass (we can always use a translation to move this to the origin) and a fixed orientation at the centre of mass (a global rotation ensures this can always point along the z-axis). Alternatively, we may view it as the space of deformable surfaces with a single fixed point p at which the normal direction never changes. Associated with Iso is the fibre bundle

$$\mathrm{E}(3) \to \mathrm{Emb} \to \mathrm{Iso}. \tag{3.1}$$

Sitting above a point in Iso is the group E(3) which parameterises all possible placements and orientations for the centre of mass (equivalently a fixed point p) of the surface – this is the gauge freedom, and fixing a given isometry returns us to a point in the full space Emb. A path of distinct embeddings which correspond to the same point in Iso can be distinguished by an observer stood at a fixed point in space, but if the observer is allowed to move with the surface they can change their position so that their view of it never changes. More succinctly, in Iso we see only deformation and not the motion.

The action of $E(3)$ is then associated with a gauge freedom – the equations of motion are invariant under the action of $E(3)$, and so the physics does not really "see" motion in Emb, it only sees motion in the quotient space Iso.

As we saw in the case of a fixed space, the definition of objectivity requires extending this group action to the larger space of time-dependent isometries, of which the Galilean group *Gal* is a subgroup. We discuss this, including some of the subtleties, in more detail in the next section.

3.2. *Gauge freedom on the deforming surface*

The group Diff(B) acts on Emb as follows. Let $r \in$ Emb be an embedding and ψ a diffeomorphism (coordinate change) of B. The action of Diff(B) sends r to $r \circ \psi$. The image of B under $r \circ \psi$ is the same as its image under r, i.e. these two different embeddings define exactly the same submanifold M of $\mathbb{R}^3$, but they correspond to distinct points in Emb. This coordinate change therefore induces no motion and no deformation of the surface. Instead of considering motion relative to B we may instead consider it relative to M_0, the initial material surface, in which case we consider the action of the group Diff(M_0) – clearly isomorphic to the group Diff(B) – acting to change coordinates in the initial surface, with corresponding action on the diffeomorphism $\lambda_t : M_0 \to M_t$.

Again, it may help readers familiar with bundle theory to view this gauge freedom as parameterising choices of coordinates on the surface via a fibre bundle structure on Emb. The quotient space defined by the action of Diff(B) on Emb is the 'space of membranes' Memb, which can alternatively be described as the collection of all images of embeddings

$$\text{Memb} = \{r(B) \mid r \in \text{Emb}\}. \tag{3.2}$$

From the perspective of an outside observer it is impossible to distinguish points in Emb that correspond to the same point of Memb, even if the observer moves around – all that distinguishes them is the patameterisation of the surface, which has no physical meaning. We then have an associated fibre bundle,

$$\text{Diff}(B) \to \text{Emb} \to \text{Memb}. \tag{3.3}$$

Sitting above each point in Memb – which is nothing more than some submanifold M of Euclidean space with the same topology as B – is a copy of the group Diff(B), which can be thought of as parameterising all possible coordinate systems on M.

We want to consider the effects of changing this particular gauge on our description of the motion. Let $r_t : B \to \mathbb{R}^3$ be any path of embeddings describing a motion, with image M_t, and let ψ_t be an arbitrary time-dependent diffeomorphism of B. We associate with ψ_t its "drive velocity" $\mathbf{w}_t^{\psi} = (\partial_t \psi_t) \circ \psi_t^{-1}$, which is a vector field on B which can be seen as the velocity of an observer moving around on B (not in the ambient space) starting at an initial point p whose own (Lagrangian) motion is $\psi_t(p)$. A time-dependent drive velocity generates a "fictitious flow" $\partial_t \mathbf{w}_t^{\psi}$ on the surface, and thus the hydrodynamics of any physical quantity defined on the surface is not invariant under this group action even though the motion of the surface itself is.

Concretely, if we define a new embedding $r_t^{\psi} := r_t \circ \psi_t$, then this embedding has the same image M_t as r_t and the associated right Eulerian velocity field is

$$\mathbf{v}_t^{\psi} = \mathbf{v}_t + r_{t*}\mathbf{w}_t^{\psi}, \tag{3.4}$$

where $\mathbf{v}_t = (\partial_t r_t) \circ r_t^{-1}$ is the Eulerian velocity field associated with r_t. We note that, since $\mathbf{w}_t^{\psi}$ is a vector field on B, its pushforward $r_{t*}\mathbf{w}_t^{\psi}$ is a tangent vector field on M_t with no component along the normal direction. In particular, this suggests a natural choice of gauge transformation in which ψ_t is chosen so that $-r_{t*}\mathbf{w}_t^{\psi}$ is exactly the tangential part of $\mathbf{v}_t$, and hence the velocity field in this gauge is directed along the normal direction to M_t. Indeed, this gauge can be defined as one in which the pullback of $\mathbf{v}_t^{\psi}$ vanishes, which obviously requires that

$$0 = r_t^* \mathbf{v}_t^{\psi} = \mathbf{V}_t + \mathbf{w}_t^{\psi}. \tag{3.5}$$

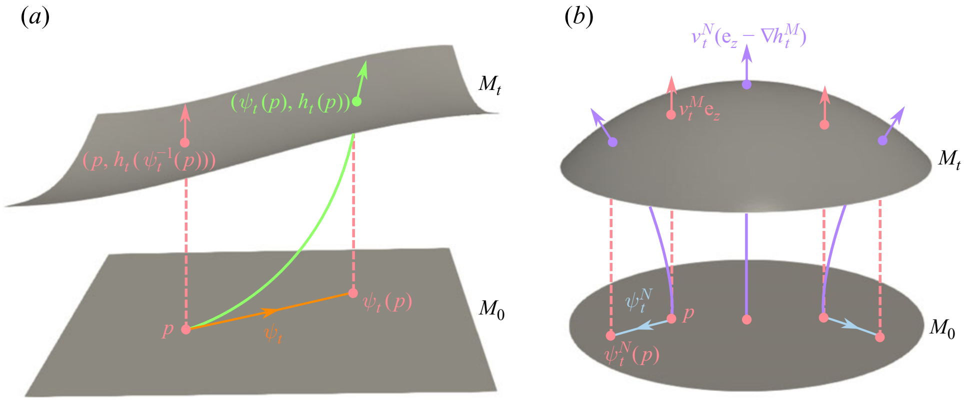

Figure 3. (*a*) Construction of a Monge gauge. The evolution of an initially flat surface (green) can be decomposed as a pair $(\psi_t(p), h_t(p))$ where ψ_t (orange) is a diffeomorphism is the initial surface M_0 and the h_t is a height function. By making a time-dependent change of gauge using the inverse ψ_t^{-1} as described in the text, we can ensure the evolution is determined purely by a gauge-transformed height function $h_t \circ \psi_t^{-1}$, which ensures a fluid particle initially at the point p evolves purely in the vertical direction (pink). (*b*) Transition from the Monge gauge (pink) to the normal gauge (purple). This involves a diffeomorphism ψ_t^N (blue) whose drive velocity is $-\nabla h_t^M$, where $h_t^M = h_t \circ \psi_t^{-1}$ is the height function in the Monge gauge. In the Monge gauge the velocity vector of the surface is $\mathbf{v}^M = v^M \mathbf{e}_z$, while in the normal gauge it is $\mathbf{v}^N = v^N(\mathbf{e}_z - \nabla h^M)$.

We call this the 'normal gauge'. This tells us that the space Memb only 'sees' the normal part of the motion and never the tangential part, which can always be cancelled out by a relabelling of the fluid particles. Computationally, when describing the evolution of the deforming surface itself (but not quantities on the surface) it is convenient to work in the space Memb, moving the mesh points purely along the normal direction Torres-Sánchez *et al.* (2019). Because the surface is invariant under these gauge transformations this presents no issue. However, Memb ignores all evolution on the surface, and therefore we cannot describe the evolution of other quantities on the surface in this gauge, only the motion of the surface itself.

For a concrete example of this gauge transformation, choose $B = \mathbb{R}^2$ to be a plane. Let us fix the usual x, y, z on $\mathbb{R}^3$, and identify B with the plane M_0 with coordinates $(x, y, 0)$. Thus the deforming surface M_t at time t can be related to its initial value by the Lagrangian motion λ_t, which can be written in the form

$$\lambda_t(x, y) = (\psi_t(x, y), h_t(x, y)). \tag{3.6}$$

In this expression $\psi_t : \mathbb{R}^2 \to \mathbb{R}^2$ is a diffeomorphism of M_0 and $h_t : \mathbb{R}^2 \to \mathbb{R}$ is a 'height' function. At a point $p = (x, y, z)$ that lies on M_t the velocity field is

$$\mathbf{v}_t(p) = (\partial_t \psi_t) \circ \psi_t^{-1}(x, y) + h_t(\psi_t^{-1}(x, y))\mathbf{e}_z, \tag{3.7}$$

where $\mathbf{e}_z$ is the unit vector along the Cartesian z direction in $\mathbb{R}^3$. Now we make a change of gauge using the diffeomorphism $\psi_t^{-1} \in \mathrm{Diff}(M_0)$. The Lagrangian motion in this new gauge is $\lambda_t^M = \lambda_t \circ \psi_t^{-1}$, or in coordinates

$$\lambda_t^M(x, y) = (x, y, h_t(\psi_t^{-1}(x, y))). \tag{3.8}$$

Defining $h_t^M = h_t \circ \psi_t^{-1}$, we see that that the motion in this gauge can be described entirely in terms of a height function. We call this the 'Monge gauge' (Monge 1850) because it is related to a local parameterisation of the surface in terms of Monge patches, and we illustrate this transition in the leftmost panel of figure 3*a*.

It is easy to see the transformation from the Monge gauge to the normal gauge. The normal direction on the surface is $\mathbf{e}_z - \nabla h_t^M$, and so we see that the transformation to this gauge involves a diffeomorphism $\psi_t^N \in \mathrm{Diff}(M_0)$ with drive velocity $\mathbf{w}_t^N = -\nabla h_t^M$. This describes the motion of a surface in the quotient space Memb, where the velocity is only ever in the normal direction. This is illustrated in the rightmost panel of figure 3*b*.

3.3. *Frame fields*

To express vector and tensor quantities on the deforming surface M_t we choose a frame field $\mathbf{e}_1, \mathbf{e}_2, \mathbf{n}$ spanning the bundle E_t along M_t where $\mathbf{n}$ is the unit normal and $\mathbf{e}_j$ is a tangential frame field on M_t – that is, for each point $p \in M_t$ we choose a basis for E_t at p, such that this basis varies smoothly as we move around on M. Given that our deformable surface is moving in time, our frame will also vary in time. The most natural way to define the tangential frame is to pick a fixed set of coordinates x_1, x_2 on the base space B and push the coordinate directions $\mathbf{e}_{x_j}$, $j = 1, 2$, forwards along the embedding r_t to give a frame for the tangent space to M_t

$$\mathbf{e}_j := r_{t*}\mathbf{e}_{x_j}. \tag{3.9}$$

Conversely, any tangential frame field specified on M_t can be pulled back to give a frame field on B. The gauge freedom is then associated with the freedom to pass to a different frame field on B (or equivalently M_t).

A frame field obtained by pushing forward a constant coordinate basis is what Torres-Sánchez *et al.* (2019) refer to as a “Lagrangian parameterisation”. This is because the frame freely moves along with the flow field, and therefore it is conceptually “Lagrangian”, although we emphasise that it has nothing to do with the Eulerian–Lagrangian split in the formulation of hydrodynamics we outlined in § 2. The tangent part of this frame field is advected along with the velocity field of the surface according to

$$\partial_t \mathbf{e}_j = -L_{\mathbf{v}}\mathbf{e}_j. \tag{3.10}$$

We are of course free to choose a different time-dependent frame $\bar{\mathbf{e}}_j(t)$ on B (which may or may not be associated with time-dependent coordinates $x_j(t)$) and push it forward to give a frame $\mathbf{e}_j := r_{t*}\bar{\mathbf{e}}_j$ on the surface. The choice is completely arbitrary; this is what Torres-Sánchez *et al.* (2019) and Sahu *et al.* (2020*b*) refer to as an ‘arbitrary mixed Eulerian–Lagrangian parameterisation’. The time evolution is then

$$\partial_t \mathbf{e}_j = -L_{\mathbf{v}}\mathbf{e}_j + r_{t*}\partial_t\bar{\mathbf{e}}_j. \tag{3.11}$$

There is a special case where the frame on B is chosen so that $r_{t*}\partial_t\bar{\mathbf{e}}_j = -\nabla_i v^j \mathbf{e}_j$; that is, the frame field is not advected by the tangential part of the flow. This is the standard choice for hydrodynamics in a fixed space. For a deforming surface, this gives rise to the concept of a frame that is advected only by the normal part of the flow, which conceptually corresponds to an observer that is ‘standing still’ on the deforming surface. Torres-Sánchez *et al.* (2019) refer to this as an ‘Eulerian parameterisation’.

In the language of group actions, this freedom to choose the tangential part of the frame field is associated with the action of $\mathrm{Diff}(B)$ on the frame bundle $\mathrm{F}B \to B$, where the diffeomorphism $\psi \in \mathrm{Diff}(B)$ acts on a frame by pushforward. Because our surface is evolving in time we consider time-dependent paths of frame fields defined by the pushforward along a time-dependent diffomorphism ψ_t. The gauge transformations associated with parameterisations of the surface in the “Monge gauge” and “normal gauge” that we introduced in the previous subsection act on the frame bundle $\mathrm{F}B$, selecting a frame field that moves only in the z direction and only in the normal direction respectively; the latter is of course the “Eulerian parameterisation” of Torres-Sánchez *et al.* (2019). An arbitrary time-dependent change of gauge then gives a "mixed Eulerian–Lagrangian parameterisation" (Torres-Sánchez *et al.* 2019; Sahu *et al.* 2020*b*), of which the Monge gauge is a particular example.

4. Objective rates

We now consider questions of objectivity and objective rates in fluid dynamics. Physical quantities are not generally required to be invariant under general diffeomorphisms, as a change to a non-inertial reference frame will introduce 'fictitious forces' as we described in the previous section – a familiar example is the Coriolis force experienced by a rotating observer. However, proper formulation of physical laws requires them to be invariant under an appropriate subgroup of symmetry transformations, represented by diffeomorphisms, which gives rise to the concept of objectivity. Which group of transformations is required by objectivity is determined by the physics of the object under consideration.

For hydrodynamics in a fixed space S, a rate $D_{\mathbf{v}_t}$ is a way of taking the derivative of material quantities along the Eulerian flow field $\mathbf{v}_t$. To be a rate, the map D must satisfy the Liebniz rule: for any time-dependent tensor field $\boldsymbol{\sigma}_t$ and function f_t of time

$$D(f_t\boldsymbol{\sigma}_t) = (Df_t)\boldsymbol{\sigma}_t + f_t D\boldsymbol{\sigma}_t. \tag{4.1}$$

This condition implies that all rates will act the same on scalar functions. Intuitively, the distinction between the ordinary time derivative ∂_t and a rate D is that the former only captures the way in which a material quantity itself changes in time, while the latter captures this as well as accounting for its advection along the flow.

There is enormous freedom in defining rates. There are three notions of differentiation that make sense on a manifold – the Lie derivative, the exterior derivative, and covariant derivatives – and all of them may appear in a rate in various forms. The first two depend only on the manifold topology and are essentially unique, while there are many possible choices for the covariant derivative – requiring metric compatibility and torsion-freeness fixes the Levi-Civita connection ∇ of the metric, but *a priori* neither of these conditions is required to define a rate. Typical rates appearing in the literature include the Oldroyd rate (Oldroyd 1950), also called the upper-convected derivative

$$D^{\mathrm{O}}\boldsymbol{\sigma}_t = \partial_t\boldsymbol{\sigma}_t + L_{\mathbf{v}_t}\boldsymbol{\sigma}_t, \tag{4.2}$$

the material derivative

$$D^{\mathrm{M}}\boldsymbol{\sigma}_t = \partial_t\boldsymbol{\sigma}_t + \nabla_{\mathbf{v}_t}\boldsymbol{\sigma}_t, \tag{4.3}$$

and the Jaumann rate (Jaumann 1911; Prager 1961; Masur 1961), also called the corotational derivative,

$$D^{\mathrm{J}}\boldsymbol{\sigma}_t = \partial_t\boldsymbol{\sigma}_t + \frac{1}{2}\left(L_{\mathbf{v}_t}\boldsymbol{\sigma}_t + \left(L_{\mathbf{v}_t}\boldsymbol{\sigma}_t^{\flat}\right)^{\sharp}\right). \tag{4.4}$$

The Jaumann rate is the average of Oldroyd's upper-convected derivative and his lower-convected derivative Oldroyd (1950).

Invariance of a rate under the action of a group of diffeomorphisms G of the space S in which the fluid moves is expressed in the following condition: the rate D of any tensor σ_t has to satisfy the equation

$$\phi^* D_{\phi_*\mathbf{v}_t}\phi_*\boldsymbol{\sigma}_t = D_{\mathbf{v}_t}\boldsymbol{\sigma}_t, \tag{4.5}$$

for every element $\phi \in G$ of the group. Classically, objectivity of a rate is formulated as invariance under the group of Galilean transformations Gal, which consists of all time-independent isometries of S along with time-dependent isometries that have a constant velocity, Galilean boosts.

The Lie derivative satisfies (4.5) for any diffeomorphism ϕ, which means that the Oldroyd rate (4.2) is "general covariant". The material derivative (4.3) satisfies (4.5) for an isometry ϕ provided the covariant derivative is metric compatible, i.e. ∇ is the Levi-Civita connection. The Jaumann rate, being expressed in terms of Lie derivatives, also satisfies (4.5) for all isometries, but unlike the Oldroyd rate it does not satisfy (4.5) for a general diffeomorphism.

Now we return to the motion of a deformable surface. In this setting the rate is not a derivative on the ambient space $\mathbb{R}^3$, rather, we take the rate of quantities defined along a path r_t of embeddings. The rates depend on both the path and the velocity fields associated with the tangent direction $\mathbf{U}_t = \partial_t r_t$ to the path. In fluid dynamics on a fixed surface possible objective rates are induced by covariant derivatives on the diffeomorphism group; mathematically this is the correct description from the perspective of the geometric theory of fluid dynamics (Arnold 2014; Arnold & Khesin 2021). The natural extension of this theory to the motion of deformable surfaces is to consider the configuration space Met(B) of Riemannian metrics on B (Rougée 2006; Fiala 2011). This is equivalent to considering dynamics in the quotient space Memb, with the motion now viewed as the path $g_t = r_t^* e$ of metrics on B induced by the embeddings (note that distinct paths in Emb that descend to the same path in Memb induce the same metrics). The tangent and cotangent spaces of Met(B) play host to the strain rate and stress in this theory – as the corresponding tangent and cotangent spaces to Diff(B) do for fluid dynamics in a fixed space (Arnold & Khesin 2021) – and covariant derivatives on Met(B) correspond exactly to the possible objective rates (Kolev & Desmorat 2021). This correspondence implies that there are an enormous number of objective rates which are all equally valid from a purely mathematical point of view; physical intuition is required to choose the correct one.

We will not adopt this perspective here. Instead, we describe the dynamics in terms of the motion in the ambient space, and objectivity in terms of invariant under a subgroup $G \subset \mathrm{Diff}(\mathbb{R}^3)$ of the diffeomorphism group in the ambient space. Technically we should consider an appropriate subgroup of the group of diffeomorphisms of spacetime, but this is equivalent to considering time-dependent paths of diffeomorphisms $\phi_t \in \mathrm{Diff}(\mathbb{R}^3)$, and we will not labour the distinction. Under a change of coordinates corresponding to such a time-dependent paths of diffeomorphisms ϕ_t the velocity field $\mathbf{v}_t$ changes to $\phi_{t*}\mathbf{v}_t + \mathbf{w}_t$, where $\mathbf{w}_t = (\partial_t \phi_t) \circ \phi_t^{-1}$ is the drive velocity. The condition of invariance of a rate D under this change of coordinates is

$$\phi_t^* D_{\phi_{t*}\mathbf{v}_t + \mathbf{w}_t} \phi_{t*} \boldsymbol{\sigma}_t = D_{\mathbf{v}_t} \boldsymbol{\sigma}_t. \tag{4.6}$$

Satisfying this condition requires the rate to account for, and in some sense counteract, the drive velocity of the diffeomorphism.

To understand how this constrains the form of the rate, we explain how to construct rates starting from the most simple candidate, the partial derivative ∂_t with respect to time, and describe the groups of diffeomorphisms that they are invariant under. Under the action of a diffeomorphism ϕ_t the rate of a tensor $\boldsymbol{\sigma}_t$ changes to

$$\begin{aligned} \partial_t \phi_{t*} \boldsymbol{\sigma}_t &= \phi_{t*} \partial_t \boldsymbol{\sigma}_t - L_{\mathbf{w}_t} \phi_{t*} \boldsymbol{\sigma}_t, \\ &= \phi_{t*} \left(\partial_t \boldsymbol{\sigma}_t - L_{\phi_t^* \mathbf{w}_t} \boldsymbol{\sigma}_t \right). \end{aligned} \tag{4.7}$$

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In the second line we have used the facts that $\phi^*\phi_* = \phi_*\phi^*$ is the identity that that $\phi_* L_{\mathbf{u}}\sigma = L_{\phi_*\mathbf{u}}\phi_*\boldsymbol{\sigma}$ for any diffeomorphism ϕ, vector field $\mathbf{u}$ and tensor $\boldsymbol{\sigma}$. This calculation shows that the partial derivative is only invariant under time-independent diffeomorphisms. This calculation motivates the definition of the Oldroyd rate (4.2), as the addition of a Lie derivative causes the extra terms that involve the drive velocity to cancel

$$\begin{aligned}\partial_t(\phi_{t*}\boldsymbol{\sigma}_t) + L_{\phi_{t*}\mathbf{v}_t+\mathbf{w}_t}(\phi_{t*}\boldsymbol{\sigma}_t) &= \phi_{t*}\left(\partial_t\boldsymbol{\sigma}_t - L_{\phi_t^*\mathbf{w}_t}\boldsymbol{\sigma}_t + L_{\mathbf{v}_t}\boldsymbol{\sigma}_t + L_{\phi_t^*\mathbf{w}_t}\boldsymbol{\sigma}_t\right), \\ &= \phi_{t*}\left(\partial_t\boldsymbol{\sigma}_t + L_{\mathbf{v}_t}\boldsymbol{\sigma}_t\right),\end{aligned} \tag{4.8}$$

and thus the Oldroyd rate is invariant under all time-dependent diffeomorphisms. By contrast, the material derivative (4.3) is not invariant under general diffeomorphisms, or even general time-dependent isometries. Indeed, we compute that

$$\partial_t(\phi_{t*}\boldsymbol{\sigma}_t) + \nabla_{\phi_{t*}\mathbf{v}_t+\mathbf{w}_t}(\phi_{t*}\boldsymbol{\sigma}_t) = \phi_{t*}\left(\partial_t\boldsymbol{\sigma}_t - L_{\phi_t^*\mathbf{w}_t}\boldsymbol{\sigma}_t + \nabla_{\mathbf{v}_t}\boldsymbol{\sigma}_t + \nabla_{\phi_t^*\mathbf{w}}\boldsymbol{\sigma}_t\right). \tag{4.9}$$

When ϕ_t is a translation with constant velocity, then covariant derivatives equal Lie derivatives $\nabla_{\phi_t^*\mathbf{w}}\boldsymbol{\sigma}_t = L_{\phi_t^*\mathbf{w}}\boldsymbol{\sigma}_t$. Using this relation in (4.9) causes the extra terms to cancel and illustrates that the material derivative is objective when we restrict our transformations to constant rotations and time-dependent translations with constant velocity. The latter are Galilean boosts, and hence this calculation shows that the material derivative is invariant under exactly the Galilean group Gal.

We argue that it is a physical necessity that rates satisfy an additional condition beyond objectivity: the Euclidean metric e in the ambient space should not be advected by the flow

$$D_{\mathbf{v}_t} e = 0. \tag{4.10}$$

This asserts that the motion of the fluid does not change the fundamental structure of space. Of course, there are examples where advection of the metric is desired – such as in general relativity, where the movement of masses changes the curvature of space–time – but we argue that this is not one of those cases. There is no physical reason why the metric of the embedding space knows anything about non-relativistic flows, either on a 3-D submanifold or on a 2-D curved surface. Additionally, this condition is important to avoid the appearance of changes in the induced metric on the surface that are not related to a real deformation. Consider the following simple example. The induced metric (first fundamental form) g on the surface is defined by taking the Euclidean metric and projecting it down onto the surface. The rate of the surface metric is therefore

$$Dg = De - \mathbf{n}^\flat \otimes D\mathbf{n}^\flat - D\mathbf{n}^\flat \otimes \mathbf{n}^\flat, \tag{4.11}$$

where $\mathbf{n}$ is the surface normal. Suppose we have a flat surface $z = 0$ undergoing a purely tangential flow. The surface does not change, and hence the metric induced on the surface is constant in both space and time, as is the normal – thus, there should be no advection with the flow at all. This would also be the case for a flow normal to the surface but spatially constant, meaning the surface simply moves without deforming. If the rate advects the Euclidean metric then (4.11) implies an advection of either the metric or the normal. Plainly this should not be the case for the examples just given where there is no deformation.

Lie derivatives are general covariant but do not preserve the metric except in the special case when the velocity field is a Killing field (i.e. it is the velocity field of a time-dependent isometry). Typically this will not be the case, and hence the Oldroyd rate is out if we require this additional condition. The partial derivative clearly does preserve the ambient metric, but lacks the appropriate objectivity properties. The material derivative obviously

Rate	Objective with respect to	Preserves ambient metric?
Partial derivative	Time-independent $\phi \in \mathrm{Diff}(\mathbb{R}^3)$	Yes
Oldroyd	Time-dependent $\phi_t \in \mathrm{Diff}(\mathbb{R}^3)$	No
Material	Galilean transforms $\phi_t \in \mathrm{Gal}$	Yes
Jaumann	Time-dependent $\phi_t \in \mathrm{E}(3)$	Yes

Table 1. For the four rates considered in the main text we show the largest group of diffeomorphisms under which the given rate is invariant. We also indicate whether the rate preserves the Euclidean metric in the ambient space.

preserves the metric – provided it involves a metric-compatible connection – but is not invariant under general isometries.

The requirement of preserving the metric as well as being invariant under all time-dependent isometries motivates the addition of a corotational part to the Oldroyd rate, which leads us to the Jaumann rate (5.4). The corotational terms must themselves be invariant under a time-dependent isometry. Equation (4.9) illustrates that these terms cannot involve covariant derivatives, only combinations of Lie derivatives (Marsden & Hughes 1994). The simplest possible expression with this property is then

$$\frac{1}{2}\left(\left(L_{\mathbf{v}_t}\boldsymbol{\sigma}_t^{\flat}\right)^{\sharp} - L_{\mathbf{v}_t}\boldsymbol{\sigma}_t\right). \tag{4.12}$$

Adding this term to the Oldroyd rate indeed yields the Jaumann rate (5.4). Because these terms involve the metric connection (through the raising and lowering operators) the Jaumann rate is not invariant under a general diffeomorphism, but it is invariant under all time-dependent paths of isometries. Furthermore, the Jaumann rate satisfies the condition (4.10) and does not advect the ambient metric. The result of these calculations for the different rates is summarised in table 1.

It follows that the plethora of rates in the literature can be reduced to two, the material and the Jaumann. Which of these two rates is the correct one? The essential difference between them is that the material derivative "feels" constant speed rotations while the Jaumann does not. The correct choice of rate for a given material quantity then boils down to whether that quantity should be sensitive to angular velocities on the deforming surface. In the gauge transformation picture outlined in the previous section this can be interpreted as a question about which of the spaces Emb, Memb, and Iso the quantity lives in.

A quantity (by which we mean physical fields as well as differential operators like rates) defined on Emb can be "pushed down" to one of the quotient spaces Memb and Iso by removing the part of it that changes under the action of the appropriate gauge group. A quantity can be defined on Iso if it only depends on the first and second fundamental forms of the surface and is insensitive to a spatially constant but time-dependent angular momentum, which has no meaning when we quotient out by the action of the isometry group. Concretely, if $\mathbf{v}$ is the velocity field of our surface, we can decompose it as $\mathbf{v} = \mathbf{v}^{ess} + \mathbf{w}^{iso}$, where $\mathbf{w}^{iso}$ is the drive velocity of some time-dependent isometry and $\mathbf{v}^{ess}$ is the "essential" part. The Jaumann rate depends only on this essential part and ignores the isometry part – this is equivalent to the rate descending to a well-defined derivative on the quotient space Iso. The key example of a quantity that is invariant under the effect of a time-dependent isometry is a nematic director or Q-tensor in the standard Frank–Oseen or Beris–Edwards formulations, respectively (deGennes & Prost 2013; Stewart 2019). In both cases, the free energy functionals in these models only capture local gradients, and as such a nematic director will not feel a translation nor a continuous rotation of the surface about an axis. The Jaumann rate is the appropriate choice for any physical quantity which has this property, and indeed corotational derivatives are used to formulate the equations of nematodynamics in a flat space (deGennes & Prost 2013; Stewart 2019).

A physical quantity which does not have this property is the surface velocity. For example, a spherical surface experiencing a constant speed rotation about an axis (a time-dependent isometry, but not an element of the Galilean group) will experience a centripetal force inwards that will act to deform it. Velocities do care about ambient space motions, and hence we should choose a rate that involves invariance under the Galilean group but not the full group of time-dependent isometries. The correct rate for a velocity field is therefore the material rate, not the Jaumann. The appropriate rate for a general physical quantity in morphodynamics can be deduced analogously by determining its behaviour under rotations with constant speed.

This analysis leads to a natural question: What does this mean for models that use the Oldroyd rate, or some other rate entirely? Are such models inherently incorrect? It appears that our analysis implies this, but there is a subtlety. Suppose a model is of the form $D_t^O\mathbf{u} = \mathbf{F}$, where $\mathbf{u}$ is a broken symmetry variable and $\mathbf{F}$ a force that may depend on both $\mathbf{u}$ and the velocity field $\mathbf{v}$. It may well be the case there are terms in the force $\mathbf{F}$ which can be moved over to the left-hand side of the equations, allowing us to rewrite the model as $D_t^J\mathbf{u} = \mathbf{F}'$. This would then be an example of a model which appears to violate the objectivity constraints we have discussed, but only because it is written in a way that is slightly misleading from the perspective of our analysis.

For a concrete example of how this could occur, consider the Eriksen–Leslie equation for the evolution of a nematic director field $\mathbf{n}$ moving with velocity $\mathbf{v}$

$$D_t^{\mathrm{J}}\mathbf{n} = \frac{1}{\gamma_1}\mathbf{h} + \lambda\mathbf{n} - \frac{\gamma_2}{\gamma_1}\left(\nabla\mathbf{v} + (\nabla\mathbf{v})^T\right)\cdot\mathbf{n}. \tag{4.13}$$

Here, γ_1, γ_2 are viscosities, λ is a Lagrange multiplier and $\mathbf{h}$ is the molecular field arising from the variation of the nematic free energy (deGennes & Prost 2013). This is written in terms of the Jaumann rate, as we advocate. However, one can split the Jaumann rate into the material rate plus the corotational part (A16), and then move the latter term over to the right-hand side of the equation with the forces. Then the Eriksen–Leslie equation will be written, apparently incorrectly, in terms of a material rate – of course, it is exactly the same model. We can perform a similar shuffling of terms to write it using the Oldroyd rate. Even more confusingly, we can actually move the term involving the viscosity γ_2 to the left-hand side and incorporate it into the rate. After some rescaling of the coefficients, the equation will then be written in terms of a rate of the form

$$D_t\mathbf{n} = \partial_t\mathbf{n} + cL_{\mathbf{v}}\mathbf{n} + (1-c)\left(L_{\mathbf{v}}\mathbf{n}^\flat\right)^\sharp. \tag{4.14}$$

This is an interpolation between the upper- and lower-convected rates, with a dimensionless parameter $0 \leqslant c \leqslant 1$– in this example it is a ratio of viscosities. This rate is invariant under the same group of transformations as the Jaumann rate, but preserves the metric only when $c = 1/2$ (in which case it is the Jaumann rate). It is possible that some models making use of more esoteric objective rates have implicitly performed such a process, bringing a term into the rate that should properly be included among the forces in the model.

We suggest that writing models in this way obscures the fundamental physics, and ought to be avoided. To avoid such confusions we advocate for always making use of either the material or Jaumann rate, as appropriate for the variable under consideration, when formulating a model. The others terms in the model should be derived in a principled way, for example from an Onsager expansion.

5. Computations of the material and Jaumann rate

We now give expressions for the rates of various quantities that are important for membrane dynamics and morphology. Throughout this section we denote the Jaumann rate along the Eulerian velocity $\mathbf{v}_t$ by D_t^J and the material derivative by D_t^M. As described in the previous section, the material derivative is the appropriate rate for velocities, while the Jaumann is the appropriate rate for quantities that are insensitive to angular momentum, such as a nematic Q-tensor or polarisation field. We also give two expressions, one in a coordinate basis and and a second where the quantities are computed relative to an orthonormal frame.

In Appendix A we give a clear derivation of these formulas using Cartan's method of moving frames in the ambient space. In Appendix B we give an alternative derivation using the surface covariant derivative. Both approaches are equivalent and give the same result.

5.1. *Rates in a coordinate frame*

We begin with the formulas in a coordinate frame. Let x_j, $j = 1, 2$, denote coordinates on B. We push the associated coordinate basis $\mathbf{e}_{x_j}$ on B forward along the embeddings r_t to obtain a coordinate basis on M_t

$$\mathbf{e}_j := r_{t*}\mathbf{e}_{x_j}. \tag{5.1}$$

These will not in general be orthogonal or normalised. We follow Al-Izzi & Morris (2023) and choose a time-varying coordinate frame on B that is not advected by the left Eulerian velocity $\mathbf{V}_t$. We make this choice to ensure that our formulas are consistent with those derived for ordinary hydrodynamics in a flat space, and also because this is the most practical choice for computations.

Write $\mathbf{n}$ for the surface normal. The velocity field is $\mathbf{v} = v^j\mathbf{e}_j + v^n\mathbf{n}$. Let $\mathbf{u} = u^j\mathbf{e}_j + u^n\mathbf{n}$ be some other vector field. Its material derivative is

$$\begin{aligned} D_t^{\mathrm{M}}\mathbf{u} = &\left(\partial_t u^j + v^i\nabla_i u^j + u^i v^k\Gamma^j_{ki} - u^i v^n b^j_i - v^i u^n b^j_i - u^n\nabla^j v^n\right)\mathbf{e}_j \\ &+ \left(\partial_t u^n + v^i\nabla_i u^n + v^i u^j b_{ij} + u^i\nabla_i v^n\right)\mathbf{n}. \end{aligned} \tag{5.2}$$

This agrees with the formula obtained by Nitschke & Voigt (2023). This leads to a formula for the acceleration

$$\begin{aligned} D_t^{\mathrm{M}}\mathbf{v} = &\left(\partial_t v^j + v^i\nabla_i v^j + v^i v^k\Gamma^j_{ki} - v^i v^n b^j_i - v^i v^n b^j_i - v^n\nabla^j v^n\right)\mathbf{e}_j \\ &+ \left(\partial_t v^n + v^i\nabla_i v^n + v^i v^j b_{ij} + v^i\nabla_i v^n\right)\mathbf{n}. \end{aligned} \tag{5.3}$$

The details of these calculations are shown in Appendices A and B. In these and all other formulas in this section, ∇ denotes the Euclidean metric connection in the ambient space, not the connection on the surface, and Γ^k_{ij} are the Christoffel symbols. A discussion of the relationship between the ambient connection and the surface is given in Appendix B. This formula for the acceleration is identical to the one derived in Waxman (1984) by less formal means, as well the formulas derived by Sahu (2022), Nitschke & Voigt (2022) and Yavari *et al.* (2016).

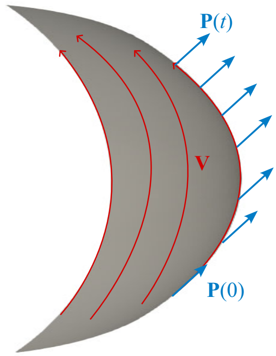

Figure 4. Lie dragging a tangent vector field **P** along a flow **v** on the surface results in a new vector field which has a component out of the surface – we illustrate **P** at a single point being dragged along an integral curve of **v**. In order to keep the field **P** in the surface, a counteracting force must push back against this effect.

We also compute the Jaumann rate of **u**

$$D_t^{\mathrm{J}}\mathbf{u} = \left(\partial_t u^j + v^i\nabla_i u^j - u^i v^n b_i^j - \frac{1}{2}v^i u^n b_i^j - \frac{1}{2}u^n\nabla_j v^n - \frac{u^i}{2}(\nabla_i v^j - \nabla^j v_i)\right)\mathbf{e}_j + \left(\partial_t u^n + \frac{1}{2}u^i\nabla_i v^n + v^i\nabla_i u^n + \frac{1}{2}v^i u^j b_{ij}\right)\mathbf{n}. \tag{5.4}$$

By setting the normal component of **u** to zero in (5.4), we obtain the Jaumann rate of a nematic polarisation field $\mathbf{P} = P^1\mathbf{e}_1 + P^2\mathbf{e}_2$ tangent to the surface

$$D_t^{\mathrm{J}}\mathbf{P} = \left(\partial_t P^j + v^i\nabla_i P^j - P^i v^n b_i^j - \frac{P^i}{2}(\nabla_i v^j - \nabla^j v_i)\right)\mathbf{e}_j + \frac{1}{2}\left(P^i\nabla_i v^n + v^i P^j b_{ij}\right)\mathbf{n}. \tag{5.5}$$

Note this formula has a normal component, which may seem surprising. However, we can see this as the predictable result of Lie dragging a vector field along a curved surface. Consider a flow that is tangent to the surface, and drag a vector $\mathbf{P}(0)$ at a given point along that flow as in figure 4 to get a new vector $\mathbf{P}(t)$. We see that this results in the vector lifting off the surface.

Keeping a vector field confined to the tangent plane of a surface therefore requires additional forces. These might arise from an embedding fluid, or from some other, unspecified interfacial physics, but a mathematical understanding of the precise nature of such forces typically requires a consideration of how finite thickness surfaces formally limit to manifolds (e.g. Lomholt & Miao 2006; Lomholt 2006*a*,*b*). Either way, we can imagine a counteracting force whose strength is sufficient to ensure that the nematic director being tangential is a hard constraint. This can be implemented using a Lagrange multiplier, or simply by projecting (5.5) into the surface, which results in the formula derived elsewhere (deGennes & Prost 2013; Stewart 2019; Salbreux *et al.* 2022; Al-Izzi & Morris 2023).

5.2. *Rates in an orthonormal frame*

An alternative approach to the use of a coordinate frame is to work with an orthonormal basis $\mathbf{e}_1$, $\mathbf{e}_2$ for the deforming surface. This approach may be a convenient choice when dealing with a nematic polarisation field on the surface, as we can take this to be the basis vector $\mathbf{e}_1$, and it is also more appropriate for an arbitrary Eulerian–Lagrangian approach to surface dynamics Torres-Sánchez *et al.* (2019).

When dealing with an arbitrary frame as opposed to a coordinate frame it is more convenient to employ the Riemann–Cartan ("moving frame") formalism of differential geometry, in which the gradients of the frame field are encoded by the connection 1-form ω_{ij}^k rather than the Christoffel symbols Γ_{ij}^k. For readers who are unfamiliar with this framework, we define the connection form and describe its relationship to the intrinsic and extrinsic geometry of the surface in Appendix A, where we also give the full derivation of each formula in this section. The reader may also see Frankel (2011) for a detailed introduction to Riemann–Cartan geometry.

For a general basis, the time derivatives are

$$\begin{aligned} \partial_t \mathbf{e}_j &= R_t J \mathbf{e}_j + \nabla_j v^n \mathbf{n}, \\ \partial_t \mathbf{n} &= -\nabla^j v^n \mathbf{e}_j. \end{aligned} \tag{5.6}$$

Here, we have introduced the map $J = \mathbf{n}\times$ which affects a 90 degree rotation about the unit normal, $J\mathbf{e}_1 = \mathbf{e}_2$ and $J\mathbf{e}_2 = -\mathbf{e}_1$. The function R_t is a time-dependent function which characterises the in-plane rotation of the basis. The choice of this function is a gauge freedom, and it may be set to zero by a particular choice of the orthonormal (ON) frame.

The equation for the material derivative is

$$\begin{aligned} D_t^{\mathrm{M}} \mathbf{u} = {}& \left(\partial_t u^k + v^i \nabla_i u^k + v^i u^j \omega_{ij}^k - u^n \nabla^k v^n - v^i u^n b_i^k \right) \mathbf{e}_k + u^k R_t J \mathbf{e}_k \\ & + \left(\partial_t u^n + v^i \nabla_i u^n + u^j \nabla_j v^n + v^i u^j b_{ij} \right) \mathbf{n}. \end{aligned} \tag{5.7}$$

The Jaumann rate of a vector field is

$$\begin{aligned} D_t^{\mathrm{J}} \mathbf{u} = {}& u^j R_t J \mathbf{e}_j + \left(\partial_t u^k + v^i \nabla_i u^k + \frac{u^j}{2} \left(\nabla^k v_j - \nabla_j v^k \right) + \frac{v^i u^j}{2} \left(2\omega_{ij}^k + \omega_{jk}^i - \omega_{kj}^i \right) \right. \\ & \left. - \frac{u^n}{2} \left(\nabla^k v^n + v^j b_j^k \right) \right) \mathbf{e}_k + \left(\partial_t u^n + v^i \nabla_i u^n + \frac{1}{2} v^i u^j b_{ij} + \frac{1}{2} u^j \nabla_j v^n \right) \mathbf{n}. \end{aligned} \tag{5.8}$$

Again, these equations use the ambient space connection ∇ and not the surface connection. Written out in this choice of frame, the rate now involve the connection coefficients ω_{ij}^k, which are not be symmetric in the lower two indices as they are for a coordinate frame. If we do choose the component $\mathbf{e}_1$ of the frame to be a nematic polarisation direction, then these coefficients have the physical interpretation as the surface splay and bend distortions of the polarisation field. Write κ for the (in-plane) bend magnitude and s for the splay of the polarisation field. Then the connection coefficients are (Pollard & Alexander 2021; Da Silva & Efrati 2021)

$$\omega_{11}^2 = -\omega_{12}^1 = \kappa, \qquad \omega_{21}^2 = -\omega_{22}^1 = s. \tag{5.9}$$

We can use these to express (5.8) in terms of the nematic distortion modes.

6. Discussion

In this paper, we have brought some mathematical clarity to the formulation of morphodynamics. We have described the Eulerian and Lagrangian perspectives on the motion of a fluid deformable surface and shown how these map on to equivalent formulations for the motion of a fluid in a fixed space. Using a symmetry group analysis, we have shown that morphodynamics has three inherent (and independent) gauge freedoms, which are associated with isometries of the ambient space, the parametrisation of the surface and the choice of the frame field. The latter two gauge freedoms are especially important when developing numerical simulations of the motion of a fluid deformable surface, and neatly capture the distinction between 'fixed-surface coordinate systems' and 'convected coordinate systems'.

Ultimately, once we have disambiguated between these notions, we argue that objective rates in morphodynamics should be treated precisely as with their fixed space counterparts: as invariant under time-dependent isometries of the ambient (full) space. As shown elsewhere, however, this yields an infinite number of possibilities (Marsden & Hughes 1994; Kolev & Desmorat 2021). We therefore require that these rates should also conserve the ambient space metric, as they would in a fixed space. Under this additional constraint, we show that the material and Jaumann rates are the only possible choices.

The material derivative leads to rates that are invariant under the action of the Galilean group – fixed isometries and constant velocity translations, or "boosts". Since Noether's theorem identifies the Galilean group with conservation of momentum, this therefore applies to the velocity field; the resultant rate (i.e. acceleration) is not invariant under non-inertial gauges, which induce fictitious forces. The Jaumann derivative, by contrast, is invariant under all time-dependent isometries. It is the correct rate for hydrodynamic variables whose long wavelength behaviour arises not from conservation laws, but due to symmetry breaking, and the related presence of Goldstone modes (Hohenberg & Halperin 1977): a pertinent example is the Q-tensor that captures nematic degrees of freedom (deGennes & Prost 2013).

Taken together, this protocol essentially ensures that the largest group of time-dependent isometries that the whole system is invariant under is just the Galilean group, as required by the informal understanding of objectivity. We provide several examples of how to calculate such rates, both in a coordinate system which is convected by the flow and also for an arbitrary frame field. The latter formulas are needed for numerical simulations making use of an "arbitrary mixed Eulerian–Lagrangian" parameterisation of the surface 2020, Torres-Sánchez *et al.* (2019); Sahu *et al.* (2020*b*). For this calculation we have employed the Riemann–Cartan method of moving frames, a formulation which is especially convenient should one wish to develop simulations making use of discrete differential geometry and discrete exterior calculus techniques.

One point of contention that our work raises is with the use of the Oldroyd rate in constitutive relations of certain non-Newtonian fluids (Oldroyd 1950; Edwards & Beris 2023). It is not clear to us how this practice fits into the framework that we have outlined. Our results call into question the use of upper- and lower-convected rates, and suggest that one should instead use their average, the Jaumann rate. It remains unclear exactly what consequences this has for the predictions of models using the Oldroyd or some other rate. It is possible that many such models are fine, and are subject to the issue we described at the end of § 4: there is some other term appearing in the equations that can be folded into the rate to give the Jaumann rate.

Nonetheless, there is still a question of what the fundamental difference is between the predictions of a model using the Oldroyd rate and those of one involving the Jaumann. We have not been able to find a simple example which is especially illuminating in this regard. One example of a study examining these issues is Hinch & Harlen (2021), who discuss in detail the difference between the predictions of models using upper- and lower-convected derivatives and whether they agree with experiment. While Hinch & Harlen (2021) argue very differently from us, considering the specific microstructure of materials and not the general symmetry constraints we examine here, it is interesting that they also conclude that "something in between is appropriate", i.e. that one should use the Jaumann rate. In the specific context of morphodynamics, Nitschke & Voigt (2022) have empirically evaluated the various rates we have considered here for several simple examples. They find scenarios in which the Oldroyd rates are equal to the Jaumann, and other scenarios in which they are different. Nonetheless, a clear understanding of the practical differences in the choice of rate *vis a vis* model predictions remains elusive, and we believe this is a deep question requiring further thought. We hope our results here draw greater attention to this issue in the modelling of complex fluids and stimulate additional studies, and we further emphasise the need for a mathematically precise approach to building models so as to avoid introducing further confusion. Coordinate-free descriptions that rely on fundamental symmetry laws are especially helpful in this regard.

More generally, we remark that many of the ideas that we present borrow from, or are motivated by, Arnold's theory of hydrodynamics (Arnold & Khesin 2021), which has been greatly influential and has led to a deeper understanding of fluid dynamics, especially the role played by topology and geometry. To take the analogy further would involve a formulation of morphodynamics which focuses on the Riemannian metric and second fundamental form as the central dynamical objects, as opposed to the embedding. Such an approach has been discussed previously by Morris & Rao (2019), where both objects were treated as symmetry-broken variables, but to our knowledge the idea not been developed since. Aside from offering a fresh perspective, computer simulations in this formulation of morphodynamics seemingly sidestep computational issues which require the adoption of a normal gauge or a Monge gauge. Either way, we argue that geometrical tools, such as the pushforward and pullback, gauge symmetry analysis and the language of differential geometry and/or exterior calculus can help to highlight and clarify the role played by geometry in the morphodynamics of fluid deformable surfaces, and we welcome further work in the area.

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Appendix A. Calculation of rates in the Cartan formalism

We calculate the material and Jaumann rates using Cartan's moving frame approach to differential geometry (Cartan 1945; Frankel 2011). In what follows, Greek indices α, β, γ run from from 1 to 3, while Latin indices i, j, k run from 1 to 2. Let $\mathbf{e}_\alpha$ be any frame along the surface, with $\mathbf{e}_3 = \mathbf{n}$ the normal direction. Write e^α for the dual 1-form. Define the connection 1-forms associated with the Euclidean connection ∇ in the ambient space by

$$\begin{aligned} \nabla \mathbf{e}_\alpha &= \omega_\alpha^\beta \mathbf{e}_\beta, \\ \mathrm{d}e^\alpha &= -\omega_\beta^\alpha \wedge e^\beta, \end{aligned} \tag{A1}$$

with components

$$\omega_\beta^\gamma(\mathbf{e}_\alpha) = \omega_{\alpha\beta}^\gamma. \tag{A2}$$

In our problem it does not make sense to take derivatives along the normal $\mathbf{n}$, so the surface geometry is encoded by dropping all coefficients of e^3 from the connection form. The 1-forms ω_3^1, ω_3^2 then encode exactly the second fundamental form $b = -\nabla\mathbf{n}$, while ω_1^2 is the connection 1-form of the induced metric on the surface

$$\begin{aligned}\omega_3^1 &= -b_1^1 e^1 - b_2^1 e^2,\\ \omega_3^2 &= -b_1^2 e^1 - b_2^2 e^2,\\ \omega_1^2 &= \omega_{11}^2 e^1 + \omega_{21}^2 e^2.\end{aligned} \tag{A3}$$

We now separate out the normal component of the frame from the tangential components. Then the connection 1-form encodes the following relationships:

$$\begin{aligned}\nabla_i \mathbf{e}_j &= \omega_{ij}^k \mathbf{e}_k + b_{ij}\mathbf{n},\\ \nabla_i \mathbf{n} &= -b_i^j \mathbf{e}_j.\end{aligned} \tag{A4}$$

We are free to choose the tangential frame however we wish. Following Al-Izzi & Morris (2023) we take the components $\mathbf{e}_1, \mathbf{e}_2$ of the frame to be the pushforward of some coordinates x_1, x_2 on the base space B along the embedding r. Explicitly, we define coordinate vectors $\mathbf{e}_{x_j}$ on the base B, and then set

$$\mathbf{e}_j := r_* \mathbf{e}_{x_j}. \tag{A5}$$

Using the rules for the time derivative of a pushforward, see Appendix C, along with the fact that coordinates x_1, x_2 do not depend on time, we conclude that

$$\partial_t \mathbf{e}_j = -L_{\mathbf{v}} \mathbf{e}_j, \tag{A6}$$

where $\mathbf{v} = \partial_t r \circ r^{-1}$ is the velocity field. Moreover, since the x_j are coordinates we have

$$[\mathbf{e}_1, \mathbf{e}_2] = [r_* \mathbf{e}_{x_1}, r_* \mathbf{e}_{x_2}] = r_* [\mathbf{e}_{x_1}, \mathbf{e}_{x_2}] = 0. \tag{A7}$$

This implies $(\omega_{12}^k - \omega_{12}^k) = 0$, and so the coefficients in the components of the connection 1-forms are all symmetric. The time derivative of the normal component can be derived from the fact that it is orthogonal to the $\mathbf{e}_j$, which implies that $\mathbf{n} \cdot \partial_t \mathbf{e}_j = -\mathbf{e}_j \cdot \partial_t \mathbf{n}$.

Write the velocity as $\mathbf{v} = v^j \mathbf{e}_j + v^n \mathbf{n}$. To compute the rate of any vector field $\mathbf{u} = u^\alpha \mathbf{e}_\alpha$, we can use the Leibniz formula

$$D_t \mathbf{u} = \left(\partial_t u^\alpha + v^\beta \nabla_\beta u^\alpha\right) \mathbf{e}_\alpha + u^\alpha D_t \mathbf{e}_\alpha. \tag{A8}$$

In order to compute the material and Jaumann rates, we need only compute their action on the basis elements. Firstly, let us compute the time derivatives of the frame. By (A6), we obtain this from the Lie derivatives

$$\begin{aligned}L_{\mathbf{v}} \mathbf{e}_j &= \nabla_{\mathbf{v}} \mathbf{e}_j - \nabla_{\mathbf{e}_j} \mathbf{v},\\ &= v^i \left(\omega_{ij}^k - \omega_{ji}^k\right) \mathbf{e}_k - \nabla_j v^i \mathbf{e}_i - v^n \nabla_j \mathbf{n} - \nabla_j v^n \mathbf{n},\\ &= \left(v^n b_j^i - \nabla_j v^i\right) \mathbf{e}_i - \nabla_j v^n \mathbf{n}\\ L_{\mathbf{v}} \mathbf{n} &= \nabla_{\mathbf{v}} \mathbf{n},\\ &= -v^j b_j^i \mathbf{e}_i.\end{aligned} \tag{A9}$$

Here, we have used the fact that $\mathbf{e}_i$ is a coordinate basis, so that ω_{ij}^k is symmetric. These yield the time derivatives for a constant coordinate frame on the base space. In practice, it is preferable to choose a time-dependent coordinate basis for B, corresponding to performing a time-dependent gauge transformation. The formula for the change of the frame is then

$$\partial_t \mathbf{e}_j = -L_{\mathbf{v}} \mathbf{e}_j + r_* \partial_t \mathbf{e}_{x_j}. \tag{A10}$$

We choose the time-dependent change of gauge so that $r_*\partial_t \mathbf{e}_{x_j}$ cancels out the term $\nabla_j v^i \mathbf{e}_i$ that would appear in the Lie derivative of $\mathbf{v}$; informally, this involves moving to a coordinate frame that is not advected by the tangential part of the flow. This particular choice then results in the time derivatives obtained by Al-Izzi & Morris (2023) and Salbreux *et al.* (2022)

$$\begin{aligned} \partial_t \mathbf{e}_i &= -v^n b_i^j \mathbf{e}_j + \nabla_i v^n \mathbf{n}, \\ \partial_t \mathbf{n} &= -\nabla^j v^n \mathbf{e}_j. \end{aligned} \tag{A11}$$

We continue to work in this natural choice of time-dependent coordinate system in what follows.

Now we compute our rates. We begin with the material rate. The action of this rate on the basis vectors is

$$\begin{aligned} \partial_t \mathbf{e}_i + \nabla_{\mathbf{v}} \mathbf{e}_i &= \left(v^k \omega_{ki}^j - v^n b_i^j\right) \mathbf{e}_j + \left(v^j b_{ij} + \nabla_i v^n\right) \mathbf{n}, \\ \partial_t \mathbf{n} + \nabla_{\mathbf{v}} \mathbf{n} &= -\left(v^i b_i^j + \nabla^j v^n\right) \mathbf{e}_j. \end{aligned} \tag{A12}$$

Inserting these terms in the Leibniz formula (A8) for a general vector field $\mathbf{u}$ leads us to (5.2). We already have the Lie derivatives of the frame, Eq. A9, so to obtain the Jaumann rate it remains to compute the Lie derivatives of the coframe

$$\begin{aligned} L_{\mathbf{v}} e^j &= \iota_{\mathbf{v}} \mathrm{d}e^j + \mathrm{d}v^j, \\ &= \iota_v \left(-\omega_{ik}^j e^i \wedge e^k + b_i^j e^i \wedge \mathbf{n}^\flat\right) + \nabla_i v^j e^i, \\ &= \left(v^i b_i^j \mathbf{n}^\flat - v^n b_i^j e^i + v^k (\omega_{ik}^j - \omega_{ki}^j) e^i\right) + \nabla_i v^j e^i, \\ &= \left(\nabla_i v^j - v^n b_i^j\right) e^i + v^i b_i^j \mathbf{n}^\flat, \\ L_{\mathbf{v}} \mathbf{n}^\flat &= \iota_{\mathbf{v}} \mathrm{d}\mathbf{n}^\flat + \mathrm{d}v^n, \\ &= \nabla_i v^n e^i. \end{aligned} \tag{A13}$$

We have again used the fact that ω_{ij}^k is symmetric for the vanishing of the tangential part of the connection form, and the 2-form $d\mathbf{n}^\flat$ vanishes because the second fundamental form is also symmetric. These then lead us to Lie derivatives of a general vector field $\mathbf{u}$ and its dual 1-form $\mathbf{u}^\flat$

$$\begin{aligned} L_{\mathbf{v}} \mathbf{u} &= u^j L_{\mathbf{v}} \mathbf{e}_j + v^i \nabla_i u^j \mathbf{e}_j + u^n L_{\mathbf{v}} \mathbf{n} + v^i \nabla_i u^n \mathbf{n}, \\ &= \left(v^i \nabla_i u^j - u^i \nabla_i v^j - u^n v^i b_i^j + v^n u^i b_i^j\right) \mathbf{e}_j + \left(v^i \nabla_i u^n - u^i \nabla_i v^n\right) \mathbf{n}, \end{aligned} \tag{A14}$$

$$\begin{aligned} L_{\mathbf{v}} \mathbf{u}^\flat &= u_j L_{\mathbf{v}} \mathbf{e}^j + v^i \nabla_i u_j e^j + u^n L_{\mathbf{v}} \mathbf{n}^\flat + v^i \nabla_i u^n \mathbf{n}^\flat, \\ &= \left(v^i \nabla_i u_j + u^n \nabla_j v^n + u_i \nabla_j v^i - v^n u_i b_j^i\right) e^j + \left(v^i \nabla_i u^n + u_i v^j b_j^i\right) \mathbf{n}^\flat. \end{aligned} \tag{A15}$$

Putting these together leads to

$$\begin{aligned} \frac{1}{2}\left(L_{\mathbf{v}} \mathbf{u} + (L_{\mathbf{v}} \mathbf{u}^\flat)^\sharp\right) &= \frac{1}{2}\left(2v^i \nabla_i u^j + u^n \nabla^j v^n - v^i u^n b_i^j + u^i (\nabla^j v_i - \nabla_i v^j)\right) \mathbf{e}_j \\ &\quad + \frac{1}{2}\left(v^i u^j b_{ij} + 2v^i \nabla_i u^n - u^i \nabla_i v^n\right) \mathbf{n}. \end{aligned} \tag{A16}$$

Now we compute the time derivative of $\mathbf{u}$ using (A11),

$$\partial_t \mathbf{u} = \left(\partial_t u^j - u^i v^n b_i^j - u^n \nabla^j v^n\right) \mathbf{e}_j + \left(\partial_t u^n + u^i \nabla_i v^n\right) \mathbf{n}. \tag{A17}$$

Combining (A16) with (A17) then gives the final result, (5.4). We emphasise that this formula holds only when $\mathbf{e}_j$ is a specific time-dependent choice of coordinate basis of the base space B. For a time-independent coordinate basis on B we pick up the additional term $u^i \nabla_i v^j \mathbf{e}_j$ that accounts for the advection of the coordinates.

We now choose to work with an ON basis $\mathbf{e}_1, \mathbf{e}_2$ for the tangent space to the the deforming surface, and introduce $\mathbf{e}_3 = \mathbf{n}$. Along the surface, the euclidean metric e does not change, and therefore we have $\partial_t \mathbf{e}_\alpha \cdot \mathbf{e}_\beta = 0$. The evolution of the normal is constrained by the fact that $\partial_t \mathbf{n} = -\nabla^j v^n \mathbf{e}_j$, and therefore we have

$$\begin{aligned} \partial_t \mathbf{e}_1 &= R_t \mathbf{e}_2 + \nabla^1 v^n \mathbf{n}, \\ \partial_t \mathbf{e}_2 &= -R_t \mathbf{e}_1 + \nabla^2 v^n \mathbf{n}, \\ \partial_t \mathbf{n} &= -\nabla^1 v^n \mathbf{e}_1 - \nabla^2 v^n \mathbf{e}_2. \end{aligned} \tag{A18}$$

Here, R_t is a function on M_t of both position and time. It describes an intrinsic rotation of the coordinate system which we are free to choose as we wish, including by setting it to be zero, as it is a gauge freedom. For reasons of generality we choose to leave it in our calculations. Introduce the operation $J = \mathbf{n}\times$. Then we may write more succinctly

$$\begin{aligned} \partial_t \mathbf{e}_j &= R_t J \mathbf{e}_j + \nabla^j v^n \mathbf{n}, \\ \partial_t \mathbf{n} &= -\nabla^j v^n \mathbf{e}_j. \end{aligned} \tag{A19}$$

The material rate is then readily seen to be (5.7). The Lie derivatives of $\mathbf{n}$, $\mathbf{n}^\flat$ in this ON frame are the same as in the coordinate frame. The Lie derivatives of $\mathbf{e}_j$ and e^j are then

$$\begin{aligned} L_{\mathbf{v}} \mathbf{e}_j &= \left(v^i (\omega_{ij}^k - \omega_{ji}^k) + v^n b_j^k - \nabla_j v^k\right) \mathbf{e}_k - \nabla_j v^n \mathbf{n} \\ L_{\mathbf{v}} e^j &= \left(v^i (\omega_{ki}^j - \omega_{ik}^j) + \nabla_k v^j - v^n b_k^j\right) e^k + v^i b_i^j \mathbf{n}^\flat. \end{aligned} \tag{A20}$$

Thus

$$\begin{aligned} D_t^{\mathrm{J}} \mathbf{e}_j &= R_t J \mathbf{e}_j + \frac{1}{2}\Big(\nabla^k v_j - \nabla_j v^k + v^i (2\omega_{ij}^k + \omega_{jk}^i - \omega_{kj}^i)\Big)\mathbf{e}_k + \frac{1}{2}\left(v^i b_{ij} + \nabla_j v^n\right) \mathbf{n}, \\ D_t^{\mathrm{J}} \mathbf{n} &= -\frac{1}{2}\left(\nabla^k v^n + v^j b_j^k\right) \mathbf{e}_k. \end{aligned} \tag{A21}$$

Using the Leibniz formula, we then derive the Jaumann rate of a general vector field in an ON frame, (5.8).

Appendix B. Calculation of rates using Christoffel symbols

Here, we give a computation of the objective and Jaumann rates using more traditional differential geometry notation based on a coordinate basis. The Gauss and Weingarten equations are

$$\partial_i \mathbf{e}_j = \Gamma_{ij}^k \mathbf{e}_k + b_{ij} \mathbf{n}, \tag{B1}$$

$$\partial_i \mathbf{n} = -b_i{}^k \mathbf{e}_k, \tag{B2}$$

where $\Gamma^k_{ij} = (1/2)g^{kl}(\partial_i g_{lj} + \partial_j g_{li} - \partial_l g_{ij})$ are the Christoffel symbols associated with the induced metric of the surface. The dynamics of the basis is given by

$$\partial_t \mathbf{e}_j = -v^n b_i{}^j \mathbf{e}_j + \bar{\nabla}_i v^n \mathbf{n}, \tag{B3}$$

$$\partial_t \mathbf{n} = -\bar{\nabla}_i v^n \mathbf{e}_i, \tag{B4}$$

where $\bar{\nabla}$ is the covariant derivative on the surface (not the ambient connection ∇). Note that we have made a choice of frame here where our coordinates move only with the normal velocity, such that for a fixed flat manifold we will recover the standard formulas of fluid mechanics.

In addition we note that the triad $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{n}\}$ is not a coordinate basis of $\mathbb{R}^3$, so the components of the dual basis are not closed. This means that we also need the following formulas:

$$de^i = b^i{}_j e^j \wedge \mathbf{n}^\flat, \tag{B5}$$

$$d\mathbf{n}^\flat = 0, \tag{B6}$$

as the second fundamental form measures the deviation from a closed coordinate frame of the embedding space. We now give the formulas for partial, covariant and Lie derivatives of the surface vector $\mathbf{u} = u^i \mathbf{e}_i + u^n \mathbf{n}$.

The partial derivative is

$$\partial_t \mathbf{u} = \left(\partial_t u^i - u^j v^n b_j{}^i - u^n \bar{\nabla}^i v^n\right) \mathbf{e}_i + \left(\partial_t u^n + u^i \bar{\nabla}_i v^n\right) \mathbf{n}, \tag{B7}$$

and the covariant derivative along the velocity $\mathbf{v} = v^i \mathbf{e}_i + v^n \mathbf{n}$ is given by

$$\mathbf{v}(\mathbf{u}) = v^i \partial_i \left(u^j \mathbf{e}_j + u^n \mathbf{n}\right) = \left(v^i \bar{\nabla}_i u^j - v^i b_i{}^j u^n\right) \mathbf{e}_j + \left(v^i u^j b_{ij} + v^i \bar{\nabla}_i u^n\right) \mathbf{n}. \tag{B8}$$

Summing (B7) and (B8) gives the main result (5.2). To obtain the exact formula (5.2) it is also necessary to replace the surface connections $\bar{\nabla}$ with the ambient space connection ∇, which accounts for the presence of the connection form in (5.2).

The Lie derivative of the vector $\mathbf{u}$ along the flow $\mathbf{v}$ is given by, =

$$L_\mathbf{v}\mathbf{u} = \mathbf{v}(\mathbf{u}) - \mathbf{u}(\mathbf{v}), \tag{B9}$$

$$= v^i \partial_i \left(u^j \mathbf{e}_j + u^n \mathbf{n}\right) - u^i \partial_i \left(v^j \mathbf{e}_j + v^n \mathbf{n}\right)$$

$$= \left[v^i \bar{\nabla}_i u^j - u^i \bar{\nabla}_i v^j + b_i{}^j \left(u^i v^n - v^i u^n\right)\right] \mathbf{e}_j + \left[v^i \bar{\nabla}_i u^n - u^i \bar{\nabla}_i v^n\right] \mathbf{n}. \tag{B10}$$

The computation of the Lie derivative of the 1-form $\mathbf{u}^\flat = u_i e^i + u^n \mathbf{n}^\flat$ is a little more involved. We make use of Cartan's formula $L_\mathbf{v}\mathbf{u}^\flat = \mathrm{d}\iota_\mathbf{v}\mathbf{u}^\flat + \iota_\mathbf{v} d\mathbf{u}^\flat$. The exterior derivative of $\mathbf{u}^\flat$ is, =

$$d\mathbf{u}^\flat = \partial_j u_i e^j \wedge e^i + u_i b^i{}_j e^j \wedge \mathbf{n}^\flat + \partial_j u^n e^j \wedge \mathbf{n}^\flat. \tag{B11}$$

We then find that, =

$$L_\mathbf{v}\mathbf{u}^\flat = \left[u_j \bar{\nabla}_i v^j + u^n \bar{\nabla}_i v^n + v^j \bar{\nabla}_j u_i - u_j b^j{}_i v^n\right] e^i + \left[u^i v^j b_{ij} + v^i \bar{\nabla}_i u^n\right] \mathbf{n}^\flat. \tag{B12}$$

The Lie derivative part of the Jaumann rate is therefore, =

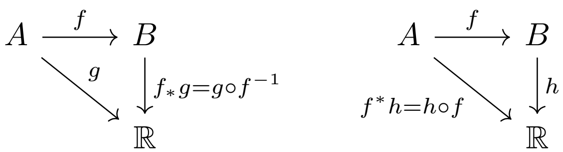
$$\frac{1}{2}\left[L_{\mathbf{v}}\mathbf{u}+\left(L_{\mathbf{v}}\mathbf{u}^{\flat}\right)^{\sharp}\right]=\left[\frac{1}{2}u_j\bar{\nabla}^i v^j+\frac{1}{2}u^n\bar{\nabla}^i v^n+v^j\bar{\nabla}_j u_i-\frac{1}{2}u_j b^{ji}v^n-\frac{1}{2}u^j\bar{\nabla}_j v^i\right]e_i + \left[\frac{1}{2}u^i v^j b_{ij}+v^i\bar{\nabla}_i u^n-\frac{1}{2}u^i\bar{\nabla}_i v^n\right]\mathbf{n}. \tag{B13}$$

Summing (B7) and (B13) gives the Jaumann rate given in (5.4). In this formula the extra terms that appear when replacing surface derivatives by ambient space derivatives cancel.

Appendix C. Pushforward and pullback

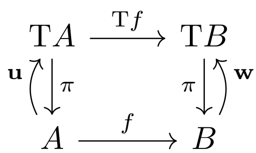
Let $f: A \to B$ be a diffeomorphism between manifolds A, B. Recall that f induces a linear map $\mathrm{T}f: \mathrm{T}A \to \mathrm{T}B$. In local coordinates x_i on A and X_j on B we may write f in terms of its components f^j as $X^j = f_j(x_1, x_2, \ldots,)$, and then the tangent map $\mathrm{T}f$ is just the matrix of partial derivatives $\partial f^j/\partial x_i$. It also defines a dual linear map $\mathrm{T}f^*: \mathrm{T}^*B \to \mathrm{T}^*A$ (the adjoint or transpose of $\mathrm{T}f$), a pullback f^*, and a pushforward f_*. We give the action of the pushforward and pullback on general tensors.

On functions



(C1)

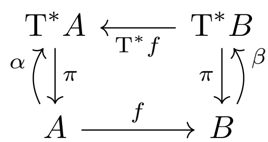
On vector fields



(C2)

$$f^*\mathbf{w} = \mathrm{T}f^{-1} \circ \mathbf{w} \circ f, \qquad f_*\mathbf{u} = \mathrm{T}f \circ \mathbf{u} \circ f^{-1}. \tag{C3}$$

On covector fields (1-forms)



(C4)

$$f^*\beta = \mathrm{T}^*f \circ \beta \circ f, \qquad f_*\alpha = (\mathrm{T}^*f)^{-1} \circ \alpha \circ f^{-1}. \tag{C5}$$

On general differential forms and tensors the action is determined by distributivity over the wedge product of forms and the tensor product of tensors. Also note that $f^* = f_*^{-1}$ and *vice versa*, so these operations are inverse to one another.

For a time-dependent family of maps $f_t: A \to B$ we also make use of the time derivatives of the pushforward and pullback of quantities along this map. Let

$\mathbf{w}_t = \partial_t f_t \circ f_t^{-1}$ be the drive velocity, a vector field on B. Given a time-independent tensor field $\boldsymbol{\sigma}$ on A, we have

$$\partial_t f_t^* \boldsymbol{\sigma} = f_t^* L_{\mathbf{w}} \boldsymbol{\sigma}, \qquad \partial_t f_{t*} \boldsymbol{\sigma} = -L_{\mathbf{w}} f_{t*} \boldsymbol{\sigma}. \tag{C6}$$

If $\boldsymbol{\sigma}_t$ also depends on time, then

$$\partial_t f_t^* \boldsymbol{\sigma}_t = f_t^* \left(\partial_t \boldsymbol{\sigma}_t + L_{\mathbf{w}} \boldsymbol{\sigma}_t \right), \qquad \partial_t f_{t*} \boldsymbol{\sigma}_t = f_{t*} \partial_t \boldsymbol{\sigma}_t - L_{\mathbf{w}} f_{t*} \boldsymbol{\sigma}_t. \tag{C7}$$

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