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The large-time asymptotic solution of the mKdV equation

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In this paper, an initial-value problem for the modified Korteweg-de Vries (mKdV) equation is addressed. Previous numerical simulations of the solution of

$$u_t - 6u^2u_x + u_{xxx} = 0, \quad -\infty < x < \infty, \quad t > 0,$$

where x and t represent dimensionless distance and time respectively, have considered the evolution when the initial data is given by

$$u(x, 0) = \tanh(Cx), \quad -\infty < x < \infty,$$

for C constant. These computations suggest that kink and soliton structures develop from this initial profile and here the method of matched asymptotic coordinate expansions is used to obtain the complete large-time structure of the solution in the particular case $C = 1/3$. The technique is able to confirm some of the numerical predictions, but also forms a basis that could be easily extended to account for other initial conditions and other physically significant equations. Not only can the details of the relevant long-time structure be determined but rates of convergence of the solution of the initial-value problem be predicted.

Key words: Asymptotic methods, large time solution, solitons and kinks

1 Introduction

The modified Korteweg-de Vries (mKdV) equation is a completely integrable equation which has been used to model a plethora of physically significant problems including large amplitude internal waves in the ocean and the study of negative ions in plasmas. It is known to admit a variety of exact solutions including self-similar solutions related to the second Painlevé transcendent together with travelling and soliton forms. In common with many other integrable partial differential systems, the mKdV equation has been the subject of many computational studies of which here we highlight [1] and [2]. Among other results, these calculations showed that various initial profiles tend to evolve towards their corresponding long-time structure rather quickly and it this aspect of the problem that is the central theme of the present paper. Here we demonstrate how the method of

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matched asymptotics can yield the complete large-time evolution and provide estimates as to the nature and convergence rates of the solution.

It is important to emphasise at the outset that although our technique is applied in detail to just one of the problems considered in [1] and [2], our approach generalises easily to other initial conditions and to other evolution equations. Nevertheless, we shall be studying the mKdV equation with initial conditions that are typical of those that apply in the development of undular bores discussed in [3] and [9] and references therein. Kotlyarov & Minakov [4] used rigorous methods using a Riemann–Hilbert formulation to infer the long-time dynamics of the solution of the mKdV equation from an initially step-like profile. Here we show how arguments based on matched asymptotics can also address similar issues; indeed, we note the methodology has previously been used to examine the long-time behaviour of solutions of the Korteweg-de Vries equation [6], the nonlinear hyperbolic Fisher equation [8] and reaction–diffusion equations of the Fisher–Kolmogorov type [7].

1.1 The initial value problem

Consider the normalised mKdV problem

$$u_t - 6u^2 u_x + u_{xxx} = 0, \quad -\infty < x < \infty, \quad t > 0, \tag{1.1}$$

which is to be solved subject to the initial condition

$$u(x,0) = \tanh\left(x/3\right), \quad -\infty < x < \infty \tag{1.2}$$

and the large-$|x|$ behaviours

$$u(x,t) \to \pm 1 \quad \text{as} \quad x \to \pm\infty. \tag{1.3}$$

Subsequently, we shall refer to problem (1.1)–(1.3) as our **IVP**. Before we outline our methods, it is worth noting that equation (1.1) admits the following solutions of interest:

(i) A travelling wave (or kink) solution given, up to an arbitrary phase shift, by

$$u(x,t) = \delta \tanh(\delta[x + 2\delta^2 t]), \tag{1.4}$$

where δ is a constant. This wave propagates in the $-x$ direction with speed $2\delta^2$ and connects $u = +\delta$ to $u = -\delta$.

(ii) A solitary wave solution with positive polarity given, up to an arbitrary phase shift, by

$$u(x,t) = -u_b + \frac{A^2}{2u_b + a_0 \cosh\left\{A(x - \left[A^2 - 6u_b^2\right] t)\right\}}, \tag{1.5}$$

where $-u_b$ is the constant background level, A is a constant ($0 < A^2 < 4u_b^2$), the wave speed $v = A^2 - 6u_b^2$ and

$$a_0 = \sqrt{4u_b^2 - A^2}.$$

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It was shown in [1] that the relevant Schrödinger potentials are reflectionless with an associated discrete set of eigenvalues given by $\lambda_0 = 0$, $\lambda_1 = 5/9$ and $\lambda_2 = 8/9$. Moreover, the corresponding wave speeds are given by

$$v_0 = -2, \quad v_1 = -\frac{38}{9} \quad \text{and} \quad v_2 = -\frac{50}{9} \tag{1.6}$$

respectively. We remark that $\lambda_0 = 0$ corresponds to the kink solution with speed $v_0 = -2$ whereas the eigenvalues λ_1 and λ_2 relate to soliton solutions with wave speeds v_1 and v_2 respectively. The upshot is that it is likely that at large times $t \gg 1$ the attractor of the solution of **IVP** will consist of two solitary waves with speeds v_1 and v_2 on a background $u = -1$ together with a kink solution of speed v_0 that connects $u = -1$ to $u = 1$. Excellent numerical verifications illustrating the development of this large-t attractor are given in [1] and [2] and our objective is to obtain a corresponding analytical description of the process. We will see that our procedure identifies the solitary wave and kink solutions without need to reference the Schrödinger potentials mentioned above.

The remainder of the paper is structured as follows. In Sections 2 and 3, we carefully trace the solution of **IVP** from the initial state through a sequence of crucial time-scales that lead to the complete description of the long-time structure. In particular in Section 3.2, we consider the soliton parts of the solution, while the structure of the kink solution is discussed in Section 4.1. We close in Section 5 with a few final remarks as to the main outcomes of the analysis and how it might find wider applicability in future work.

2 The short-time solution

Our considerations start in the obvious manner by seeking the form of the solution at early times. In view of the initial condition (1.2), we expand in ascending powers of t so that

$$u(x,t) = u_0(x) + t u_1(x) + O\left(t^2\right), \tag{2.1}$$

where $u_0(x) = \tanh(x/3)$ which has the property that

$$u_0(x) \sim \begin{cases} 1 - 2\,e^{-2x/3} + 2\,e^{-4x/3} - 2\,e^{-2x} + \cdots & \text{as} \quad x \to \infty, \\ -1 + 2\,e^{2x/3} - 2\,e^{4x/3} + 2\,e^{2x} + \cdots & \text{as} \quad x \to -\infty. \end{cases} \tag{2.2}$$

On substituting (2.1) into equation (1.1), we readily obtain

$$u(x,t) = u_0(x) + t\,\left[6\,u_0^2(x)u_0'(x) - u_0'''(x)\right] + O\left(t^2\right) \tag{2.3}$$

and for $x \gg 1$, expansion (2.3) with $(2.2)_1$ takes the form

$$u(x,t) \sim \left[1 - 2\,e^{-2x/3} + \cdots\right] + t\left[\frac{200}{27}\,e^{-2x/3} - \frac{1168}{27}\,e^{-4x/3} + 136\,e^{-2x}\right] + \cdots; \tag{2.4}$$

we conclude that (2.3) remains uniform for $x \gg 1$ as $t \to 0$. Similarly, for $(-x) \gg 1$ we obtain

$$u(x,t) \sim [-1 + 2\,e^{2x/3} + \cdots] + t\left[\frac{200}{27}\,e^{2x/3} - \frac{1168}{27}\,e^{4x/3} + 136\,e^{2x}\right] + \cdots \tag{2.5}$$

which remains uniform for $(-x) \gg 1$ as $t \to 0$.

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3 The asymptotic structure for large times

We can use the straightforward results derived immediately above to infer the details of the structure of the solution for $t = O(1)$, at least for $|x|$ sufficiently large. The form of expansion (2.4) for $x \gg 1$ as $t \to 0$ suggests that in this region and for $t = O(1)$ then

$$u(x,t) = 1 + g_0(t)\, e^{-2x/3} + g_1(t)\, e^{-4x/3} + g_2(t)\, e^{-2x} + o(e^{-2x}) \tag{3.1}$$

for some functions $g_j(t)$, $j = 0, 1, \ldots$ to be determined. On substituting (3.1) into equation (1.1) we can derive a sequence of equations for the unknown functions $g_j(t)$ and, in particular, it follows that

$$\frac{dg_0}{dt} = -\frac{100}{27} g_0, \qquad \frac{dg_1}{dt} + \frac{152}{27} g_1 = -8g_0^2 \quad \text{and} \quad \frac{dg_2}{dt} + 4g_2 = -4g_0^3 - 24 g_0 g_1. \tag{3.2}$$

These need to be solved subject to matching with with (2.3) as $t \to 0$ which forces $g_j(0) = 2(-1)^{j+1}$ for all integers j. It follows that

$$g_0(t) = -2e^{-\frac{100}{27}t}, \qquad g_1(t) = 18e^{-\frac{200}{27}t} - 16e^{-\frac{152}{27}t}, \qquad g_2(t) = 144e^{-\frac{252}{27}t} - 126e^{-\frac{100}{9}t} + 20e^{-4t}, \tag{3.3}$$

so

$$u(x,t) = 1 - 2\, e^{-\frac{2}{3}\left(\frac{50}{9}\right)t}\, e^{-\frac{2x}{3}} + \left[18\, e^{-\frac{4}{3}\left(\frac{50}{9}\right)t} - 16\, e^{-\frac{4}{3}\left(\frac{38}{9}\right)t}\right] e^{-\frac{4x}{3}}$$
$$+ \left[-20\, e^{-2(2t)} - 126\, e^{-2\left(\frac{50}{9}\right)t} + 144\, e^{-2\left(\frac{42}{9}\right)t}\right] e^{-2x} + o\left(e^{-2x}\right), \tag{3.4}$$

as $x \to \infty$ with $t = O(1)$. Expansion (3.4) remains uniform for $t \gg 1$ provided $x \gg t$, but becomes non-uniform when $x = O(t)$.

In an entirely analogous manner, we can determine the form of the solution as $x \to -\infty$ with $t = O(1)$. It is found that

$$u(x,t) = -1 + 2\, e^{\frac{2}{3}\left(\frac{50}{9}\right)t}\, e^{\frac{2x}{3}} + \left[-18\, e^{\frac{4}{3}\left(\frac{50}{9}\right)t} + 16\, e^{\frac{4}{3}\left(\frac{38}{9}\right)t}\right] e^{\frac{4x}{3}}$$
$$+ \left[20\, e^{2(2t)} + 126\, e^{2\left(\frac{50}{9}\right)t} - 144\, e^{2\left(\frac{42}{9}\right)t}\right] e^{2x} + o\left(e^{2x}\right) \tag{3.5}$$

which remains uniform for $t \gg 1$ provided $(-x) \gg t$, but becomes non-uniform when $(-x) = O(t)$.

3.1 The modified solution for $|x| = O(t)$ and $t \to \infty$

The solutions noted above point to the need of a modified structure for the solution once $|x| = O(t)$ to account for the breakdown of (3.4) and (3.5). First we deal with the problem for $x < 0$ and proceed by introducing the new scaled coordinate

$$y = \frac{x}{t},$$

where $y = O(1)$ as $t \to \infty$. The form of the solution (3.5) implies that

$$u(y,t) = -1 + e^{-t\theta(y,t)}, \tag{3.6}$$

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where

$$\theta(y,t) = \theta_0(y) + \theta_1(y)\frac{1}{t} + o\left(\frac{1}{t}\right) \tag{3.7}$$

as $t \to \infty$ with $y < 0$ and $\theta_0(y) > 0$. On substituting the ansatz (3.6) and (3.7) into equation (1.1) we derive the leading-order problem

$$\left(\theta_0'\right)^3 - (y+6)\theta_0' + \theta_0 = 0, \tag{3.8}$$

$$\theta_0(y) \sim -\frac{2}{3}(y - v_2) \quad \text{as} \quad y \to -\infty, \tag{3.9}$$

where, in accordance with (1.6), $v_2 = -50/9$. The final condition (3.9) arises from matching expansion (3.7) for $y \to -\infty$ with the form (3.5).

Equation (3.8) admits the one-parameter family of linear solutions

$$\theta_0(y) = Ay + A\left(6 - A^2\right) \tag{3.10}$$

which shows that the required solution to (3.8) subject to (3.9) is simply

$$\theta_0(y) = -\frac{2}{3}(y - v_2) \quad \text{for} \quad y < v_2. \tag{3.11}$$

If expansion (3.6)–(3.7) is taken to further terms it transpires that $\theta_1 = -\ln 2$ and, continuing further, then

$$\begin{aligned} u(y,t) = -1 &+ 2\,e^{\frac{2}{3}(y-v_2)t} - 18\,e^{\frac{4}{3}(y-v_2)t} + 16\,e^{\frac{4}{3}(y-v_1)t} + 126\,e^{2(y-v_2)t} \\ &- 144\,e^{2\left(y+42/9\right)t} + 20\,e^{2(y-v_0)t} + o\left(e^{2(y-v_0)t}\right) \end{aligned} \tag{3.12}$$

with y sufficiently negative and where the wave speeds v_0–v_2 are as given in (1.6).

3.1.1 *The breakdown of solution (3.12)*

The expansion (3.12) as $y \to -\infty$ matches directly onto the form of (3.5) but, unfortunately, it contains a non-uniformity in the vicinity of $y = 3v_2 - 2v_1 = -74/9$. This then requires us to monitor the nature of the appropriate solution near this point in order to identify the appropriate solution in $y > -74/9$.

The neighbourhood of $y = -74/9$ turns out to play little more than a passive role and here we are led to the introduction of the scaled coordinate $\eta = (y + 74/9)t$, where $\eta = O(1)$ as $t \to \infty$. Then the structure of (3.12) suggests that

$$u(\eta,t) = -1 + G_0(\eta)e^{-\frac{16}{9}t} + G_1(\eta)e^{-\frac{32}{9}t} + G_2(\eta)e^{-\frac{48}{9}t} + G_3(\eta)e^{-\frac{64}{9}t} + O\left(e^{-\frac{112}{9}t}\right) \tag{3.13}$$

for functions $G_j(\eta)$ to be determined by the routine substitution of expansion (3.13) into equation (1.1) and solving sequentially. If this is done, and the constants fixed by ensuring that the solution matches to (3.12) as $\eta \to -\infty$, then

$$\begin{aligned} u(\eta,t) = &-1 + 2e^{\frac{2\eta}{3}}e^{-\frac{16}{9}t} - 18e^{\frac{4\eta}{3}}e^{-\frac{32}{9}t} + \left[16e^{\frac{4\eta}{3}} + 126e^{2\eta}\right]e^{-\frac{48}{9}t} \\ &- \left[144e^{2\eta} + 810e^{\frac{8\eta}{3}}\right]e^{-\frac{64}{9}t} + O\left(e^{-\frac{112}{9}t}\right). \end{aligned} \tag{3.14}$$

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The $\eta \to \infty$ limit of (3.14) tells us the structure of the solution in the $y = O(1)$ regime on the other side of $y = -74/9$ and then standard manipulations lead to

$$\begin{aligned} u(y,t) = &-1 + 2\,e^{\frac{2}{3}(y-v_2)t} - 18\,e^{\frac{4}{3}(y-v_2)t} + 126\,e^{2(y-v_2)t} + 16\,e^{\frac{4}{3}(y-v_1)t} \\ &- 810\,e^{\frac{8}{3}(y-v_2)t} - 144\,e^{2\left(y+42/9\right)t} + o\left(e^{2\left(y+42/9\right)t}\right). \end{aligned} \tag{3.15}$$

It might be hoped that this solution holds for all $y > -74/9$ but it too contains another non-uniformity in the vicinity of $y = -62/9$. To resolve this difficulty, it is again necessary to re-expand the solution in what turns out to be another passive layer very reminiscent of the one around $y = -74/9$. In the interests of brevity, we do not detail the precise forms of the expansion and solutions in this zone as they closely mimic (3.14), (3.15) above with only minor changes. All that needs to be noted is that the analysis leads to the solution for $-62/9 < y < v_2$ in which

$$\begin{aligned} u(y,t) = &-1 + 2\,e^{\frac{2}{3}(y-v_2)t} - 18\,e^{\frac{4}{3}(y-v_2)t} + 126\,e^{2(y-v_2)t} - 810\,e^{\frac{8}{3}(y-v_2)t} \\ &+ 16\,e^{\frac{4}{3}(y-v_1)t} - 144\,e^{2\left(y+42/9\right)t} + o\left(e^{2\left(y+42/9\right)t}\right). \end{aligned} \tag{3.16}$$

This solution contains no further non-uniformities and holds for all $y \in (-62/9, v_2)$ but as $y \to v_2^-$ it is plain that the many of the small exponential terms in (3.16) become $O(1)$. This suggests that this location marks the position of a significant jump in the leading-order form of the solution and indeed this is precisely where the first soliton component of the large time solution appears. (It has been pointed out to us that while many of the details above may seem routine, extreme care needs to be exercised when series become disordered in this way; for a seminal exposition of this phenomenon the interested reader is referred to [5].)

3.2 The soliton zone

Within this soliton region $x \sim s(t)$ and

$$u(z,t) = U_0(z) + o(1), \tag{3.17}$$

where the travelling wave coordinate

$$z \equiv x - s(t) = O(1) \tag{3.18}$$

and

$$s(t) = v_2 t + \phi_0 + \phi_1(t) + o(\phi_1(t)),$$

with ϕ_0 a constant and $\phi_1(t) = o(1)$ an as yet undetermined gauge function. This function can be determined by rewriting (3.16) in terms of z; routine manipulations show that this form can be expressed as

$$\begin{aligned} u = &-1 + 2\,e^{\frac{2}{3}(z+\phi_0)}e^{\frac{2}{3}\phi_1} - 18\,e^{\frac{4}{3}(z+\phi_0)}e^{\frac{4}{3}\phi_1} + 126\,e^{2(z+\phi_0)}e^{2\phi_1} - 810\,e^{\frac{8}{3}(z+\phi_0)}e^{\frac{8}{3}\phi_1} \\ &+ 16\,e^{\frac{4}{3}(z+\phi_0)}\,e^{-\frac{16}{9}t} - 144\,e^{2(z+\phi_0)}e^{-\frac{16}{9}t} + \cdots; \end{aligned} \tag{3.19}$$

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the terms in the second line of this result imply that (3.17) should be refined to

$$u(z,t) = U_0(z) + e^{-\frac{16}{9}t}U_1(z) + \cdots \tag{3.20}$$

and the fact that for $\phi_1(t) = o(1)$ then $e^{k\phi_1} = 1 + k\phi_1 + \cdots$ implies also that $\phi_1(t) = e^{-\frac{16}{9}t}$. On substituting expansion (3.20) into equation (1.1), we find that

$$U_0''' - 6U_0^2U_0' - v_2U_0' = 0 \quad \text{for} \quad -\infty < z < \infty, \tag{3.21}$$

which is to be solved subject to matching to (3.19) as $z \to -\infty$.

We can immediately integrate equation (3.21) once to deduce that

$$U_0'' - 2U_0^3 - v_2U_0 - (2 + v_2) = 0 \tag{3.22}$$

and a phase–plane analysis of this second-order autonomous non-linear second-order differential equation establishes the existence of a homoclinic connection with

$$U_0(z) = -1 + \frac{2}{9 + 6\sqrt{2}\cosh\left(2z/3\right)}. \tag{3.23}$$

The translational invariance in z has been chosen (without loss of generality) so that matching expansion (3.17) with (3.23) requires that

$$\phi_0 = -\frac{3}{2}\ln\left(3\sqrt{2}\right). \tag{3.24}$$

We conclude that

$$u(z,t) = -1 + \frac{2}{9 + 6\sqrt{2}\cosh\left(2z/3\right)} + O\left(e^{-\frac{16}{9}t}\right) \tag{3.25}$$

with $z = O(1)$, where $z = x - s(t)$ and

$$s(t) = v_2t - \frac{3}{2}\ln\left(3\sqrt{2}\right) + O\left(e^{-\frac{16}{9}t}\right). \tag{3.26}$$

At leading order, we have the expected soliton which has an amplitude $2/(9 + 6\sqrt{2})$ and travels with speed $v_2\,(= -50/9)$ on the background level $u = -1$ (see (1.5) with $u_b = 1$ and $A = 2/3$). We also note that the rate of convergence of the solution of **IVP** to the soliton is exponential and of size $O(e^{-\frac{16}{9}t})$.

As $z \to \infty$ we move back towards the region where $y = O(1)$ but now $y > v_2$. In order to assist with the subsequent calculation, it is instructive to examine the form of expansion (3.25) for $z \gg 1$. In this limit

$$\begin{aligned} u(z,t) \sim &\left(-1 + \frac{2}{3\sqrt{2}}e^{-\frac{2z}{3}} - \frac{18}{(3\sqrt{2})^2}e^{-\frac{4z}{3}} + \frac{126}{(3\sqrt{2})^3}e^{-2z} + \cdots\right) \\ &+ \left(A_0e^{\frac{4z}{3}} + \cdots\right)e^{-\frac{16}{9}t} + \cdots \end{aligned} \tag{3.27}$$

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for some constant A_0, which, when rewritten in terms of $y = x/t = (z + s(t))/t$ gives that

$$\begin{aligned} u(y,t) = & -1 + \frac{2}{(3\sqrt{2})^2} e^{-\frac{2}{3}(y-v_2)t} - \frac{18}{(3\sqrt{2})^4} e^{-\frac{4}{3}(y-v_2)t} + \frac{126}{(3\sqrt{2})^6} e^{-2(y-v_2)t} + \cdots \\ & + \frac{A_0}{(3\sqrt{2})^2} e^{\frac{4}{3}(y-v_1)t} + \cdots . \end{aligned} \tag{3.28}$$

It turns that the solution $u(y,t)$ is uniformly valid for $v_2 < y < -226/45$. Now we recall the behaviour noted earlier for (3.16) where it was clear that one of the exponentially small terms in that solution became $O(1)$ as $y \to v_2$ pointing the way to a soliton zone. Exactly the same phenomenon occurs here. The last term in (3.28) becomes as large as the leading order form as $y \to v_1^-$. Across the region $v_2 < y < v_1$, there will be a number of locations where various of the terms within (3.28) become disordered and thus strictly speaking require further analysis. However, again as before, the re-ordering of the terms occurs in an entirely passive way and there is little need to spell out all the details. Rather, we just note that after the last of the re-orderings but prior to the second soliton region wherein $-42/9 < y < v_1$ we have

$$u(y,t) \sim -1 + \left(\frac{A_0}{(3\sqrt{2})^2} e^{\frac{4}{3}(y-v_1)t} + \cdots \right) + O\left(e^{-\frac{2}{3}(y-v_2)t} \right). \tag{3.29}$$

3.2.1 *The second soliton*

As $y \to v_1^-$ a rescaling is required to account for the second soliton structure. In the interest of conciseness, we restrict ourselves to a statement of the main results. Here

$$u(\hat{z},t) = -1 + \frac{8}{9 + 3\sqrt{5}\cosh\left(4\hat{z}/3\right)} + O\left(e^{-\frac{8}{9}t}\right), \tag{3.30}$$

where $\hat{z} = x - \hat{s}(t)$ and

$$\hat{s}(t) = v_1 t + \frac{3}{4}\ln\left(\frac{96}{A_0\sqrt{5}}\right) + O\left(e^{-\frac{8}{9}t}\right). \tag{3.31}$$

At leading order, we thus have a soliton of amplitude $8/(9 + 3\sqrt{5})$ which travels at speed v_1 on the background level $u = -1$. Once more the convergence of the solution of **IVP** is exponential in t and occurs at an $O(e^{-\frac{8}{9}t})$ rate. We note that as $z \to \infty$ we move back into the zone where $y = x/t = O(1)$ but now $y > v_1$ $(= -38/9)$.

We have now developed the whole of the large time solution in the region $x < -38t/9$ which includes two soliton zones. To complete our description, it is necessary to examine the region $y > v_1$ and this is tackled next.

4 The solution structure for $x > -38t/9$

We commence our study of the region as $x \to +\infty$ in much the same manner as for large negative values of x. For $y = x/t$ satisfying $y < -74/9$, we proceeded via steps (3.6)–(3.11)

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to derive the form (3.12); in a completely analogous manner we find that for $y > 46/9$

$$u(y,t) = 1 - 2e^{-\frac{2}{3}(y-v_2)t} - 16e^{-\frac{4}{3}(y-v_1)t} + 18e^{-\frac{4}{3}(y-v_2)t} - 20e^{-2(y-v_0)t} + O\left(e^{-2(y+42/9)t}\right). \quad (4.1)$$

As previously, a number of non-uniformities occur as y is reduced corresponding to various of the terms in (4.1) becoming disordered. Again these technical issues can be resolved by consideration of a succession of entirely passive regions; for the record these occur at $y = 46/9$, $y = 22/9$ and $y = 2/9$ but we do not dwell on the details. Rather we just note that to the left of the last of these passive regions where $y < 2/9$ the solution is

$$u(y,t) = 1 - 20e^{-2(y-v_0)t} - 2e^{-\frac{2}{3}(y-v_2)t} - 16e^{-\frac{4}{3}(y-v_1)t} + o\left(e^{-\frac{4}{3}(y-v_1)t}\right). \quad (4.2)$$

It is clear that this form must be modified as $y \to v_0^+$, that is to say, as $y \to -2^+$. It is to be expected that in this limit the solution will be altered at leading order and, to anticipate the below, we shall see this where the kink solution appears.

4.1 The solution in the kink region

Within the kink zone, we expect that $u = O(1)$ and $y = -2 + O(t^{-1})$. Written in terms of the travelling co-ordinate

$$u(\tilde{z},t) = U_K(\tilde{z}) + o(1), \quad (4.3)$$

where $\tilde{z} = x - \tilde{s}(t)$ with

$$\tilde{s}(t) = v_0 t + \tilde{\phi}_0 + \tilde{\phi}_1(t) + o(\tilde{\phi}_1(t)).$$

In order to identify the appropriate size of $\tilde{\phi}_1 = o(1)$, it is instructive to recast (4.2) in terms of these new co-ordinates. It follows that as $y \to -2^+$ so

$$u \sim 1 - 20e^{-2z}e^{2\tilde{\phi}_0}e^{2\tilde{\phi}_1} - 2e^{-\frac{2}{3}z}e^{-\frac{2}{3}\tilde{\phi}_0}e^{-\frac{2}{3}\tilde{\phi}_1}e^{-\frac{64}{27}t} - 16e^{-\frac{4}{3}z}e^{-\frac{4}{3}\tilde{\phi}_0}e^{-\frac{4}{3}\tilde{\phi}_1}e^{-\frac{80}{27}t} + \cdots \quad (4.4)$$

which suggests that (4.3) be refined to

$$u(\tilde{z},t) = U_K(\tilde{z}) + U_1(\tilde{z})e^{-\frac{64}{27}t} + O\left(e^{-\frac{80}{27}t}\right) \quad (4.5)$$

and that $\tilde{\phi}_1 = O(e^{-\frac{64}{27}t})$. The leading order problem can then be cast as

$$U_K''' - 6U_K^2U_K' - v_0U_K' = 0, \quad -\infty < \tilde{z} < \infty, \quad (4.6)$$

$$U_K(\tilde{z}) \to 1^- \quad \text{as} \quad \tilde{z} \to \infty, \quad (4.7)$$

$$U_K(\tilde{z}) \quad \text{bounded as} \quad \tilde{z} \to -\infty. \quad (4.8)$$

The requirement (4.7) arises from matching expansion (4.2) to (4.3) and equation (4.6) integrates once to obtain

$$U_K'' - 2U_K^3 - v_0U_K = 0. \quad (4.9)$$

It then follows that

$$U_K(\tilde{z}) = \tanh\left(\tilde{z} + \tilde{\phi}_c\right), \quad (4.10)$$

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for some constant $\tilde{\phi}_c$. On setting $\tilde{\phi}_c = 0$ the translational invariance in $\tilde{z}$ is fixed and it is now clear that (4.10) is the kink, or double layer, solution of (1.1). Matching the solution to (4.2) as $y \to v_0^+$ up to $O(1)$ fixes

$$\tilde{\phi}_0 = \frac{1}{2} \ln 10. \tag{4.11}$$

The correction term $U_1(\tilde{z})$ satisfies

$$U_1''' + U_1'\left(2 - 6U_k^2\right) - U_1\left(\frac{64}{27} + 12U_K U_K'\right) = -\frac{64}{27}U_K' \tag{4.12}$$

and has asymptotic properties

$$U_1(\tilde{z}) \sim A_2 e^{-\frac{2}{3}\tilde{z}} + 4e^{-2\tilde{z}} \quad \text{as} \quad \tilde{z} \to \infty, \tag{4.13}$$

$$U_1(\tilde{z}) \sim B_1 e^{-\frac{2}{3}\tilde{z}} + 4e^{2\tilde{z}} \quad \text{as} \quad \tilde{z} \to -\infty, \tag{4.14}$$

where A_2 and B_1 are constants. Matching expansion (4.5) with expansion (4.3) (as $y \to v_0^+$) up to $O(e^{-\frac{64}{27}t})$ determines

$$A_2 = -\frac{2}{10^{1/3}}.$$

We conclude therefore that in the kink solution zone

$$u(\tilde{z}, t) = \tanh \tilde{z} + O\left(e^{-\frac{64}{27}t}\right), \tag{4.15}$$

where $\tilde{z} = x - \tilde{s}(t)$ and

$$\tilde{s}(t) = v_0 t + \frac{1}{2} \ln 10 + O(e^{-\frac{64}{27}t})$$

as $t \to \infty$. Hence again the convergence of the solution of **IVP** to the kink is exponential in t.

The calculation is now almost complete but before describing the final step it is instructive to examine the next term in expansion (4.5). If we look for an expansion of the form

$$u(\tilde{z}, t) = U_K(\tilde{z}) + U_1(\tilde{z})e^{-\frac{64}{27}t} + U_2(\tilde{z})e^{-\frac{80}{27}t} + o\left(e^{-\frac{80}{27}t}\right), \tag{4.16}$$

it follows that $U_2(\tilde{z}) \sim B_2 e^{-\frac{4}{3}\tilde{z}}$ as $\tilde{z} \to -\infty$ for some B_2. The upshot is that as in this limit we move back towards the region where $y\,(= x/t) < -2$. If we write the solution (4.16) in terms of y we find that

$$u(y, t) \sim -1 + \frac{2}{10} e^{2(y-v_0)t} + B_1(10)^{\frac{1}{3}} e^{-\frac{2}{3}(y-v_2)t} + B_2(10)^{\frac{2}{3}} e^{-\frac{4}{3}(y-v_1)t}. \tag{4.17}$$

4.2 The solution in the region $v_1 < y < v_0$

In Section 3, we developed the form of the solution when $t \to \infty$ for $y < v_1$, and above we followed the solution through the kink zone located at $y = v_0 = -2$. All that is left to do

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is to examine the intermediate region $v_1 < y < v_0$. Here it is straightforward to establish that

$$u(y,t) = -1 + \frac{2}{10}e^{2(y-v_0)t} + B_1(10)^{\frac{1}{3}}e^{-\frac{2}{3}(y-v_2)t} + B_2(10)^{\frac{2}{3}}e^{-\frac{4}{3}(y-v_1)t} + o\left(e^{-\frac{4}{3}(y-v_1)t}\right). \quad (4.18)$$

Unfortunately, expansion (4.18) becomes non-uniform as $y \to -\frac{26}{9}^{+}$ which points to the need for one final expansion. It is interesting that in this limit all three exponential terms within (4.18) become comparable in size and we examine the solution here by introducing a suitable scaled co-ordinate $\xi = (y + \frac{26}{9})t$ so that

$$u(\xi,t) = -1 + S(\xi)e^{-\frac{48}{27}t} + o\left(e^{-\frac{48}{9}t}\right). \quad (4.19)$$

The function $S(\xi)$ satisfies

$$S''' - \frac{28}{9}S' - \frac{48}{27}S = 0 \quad (4.20)$$

and this is to be solved subject to matching with form (4.18) as $\xi \to \infty$ which requires

$$S(\xi) \sim \frac{2}{10}e^{2\xi} + B_1(10)^{\frac{1}{3}}e^{-\frac{2}{3}\xi} + B_2(10)^{\frac{2}{3}}e^{-\frac{4}{3}\xi} \quad \text{as} \quad \xi \to \infty. \quad (4.21)$$

The solution of (4.20) subject to (4.21) is seen to be

$$S(\xi) = \frac{2}{10}e^{2\xi} + B_1(10)^{\frac{1}{3}}e^{-\frac{2}{3}\xi} + B_2(10)^{\frac{2}{3}}e^{-\frac{4}{3}\xi} \quad (4.22)$$

and as $\xi \to -\infty$ we move back into the region where $v_1 < y < -26/9$. Here

$$u(y,t) = -1 + B_2(10)^{\frac{2}{3}}e^{-\frac{4}{3}(y-v_1)t} + B_1(10)^{\frac{1}{3}}e^{-\frac{2}{3}(y-v_2)t} + \frac{2}{10}e^{2(y-v_0)t} + o\left(e^{2(y-v_0)t}\right); \quad (4.23)$$

a solution that again becomes disordered as $y \to v_1^{+}$. However, this is just the location of the second soliton solution that was discussed in Section 3.2.1. Therefore, all that remains to complete the large-t asymptotic structure of **IVP** is to match expansion (4.23) as $y \to v_1^{+}$ to expansion (3.30) as $z \to \infty$ which is ensured if

$$B_2 = \frac{512}{10^{\frac{2}{3}}5A_0} \quad \text{and} \quad B_1 = \frac{4\sqrt{6}}{10^{\frac{1}{3}}A_0^{\frac{1}{2}}5^{\frac{1}{4}}}.$$

5 Closing remarks and discussion

In this paper, we have presented the complete asymptotic structure of the large-t solution to **IVP** and have shown the emergence of the solution consisting of two solitary waves plus a kink solution. In particular, we have ascertained the existence of the following.

(i) A relatively fast solitary wave of amplitude $2/(9+6\sqrt{2})$ which travels with a speed $v_2 (= -\frac{50}{9})$ on the background level $u = -1$, see (3.25). The rate of convergence of the solution of **IVP** to this soliton is $O(e^{-\frac{16}{9}t})$.

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(ii) A slower solitary wave of amplitude $8/(9+3\sqrt{5})$ which travels at speed $v_1\,(=-\frac{38}{9})$, again on the background level $u=-1$, see (3.30). In this case, the rate of convergence of the solution to the soliton is $O(e^{-\frac{8}{9}t})$.

(iii) A kink solution which travels at speed v_0 $(=-2)$ and which connects the level $u=+1$ to the level $u=-1$. The rate of convergence to the kink is $O(e^{-\frac{64}{27}t})$.

All these results are in excellent accord with the numerical simulations described within [1] and [2]. While some of the features of the solution are well known, our matched asymptotic analysis furnishes new estimates of the convergence rates of the computations. It would be of interest to couple the results of our work to more accurate numerical solutions that are possible with modern methods and this is a topic of ongoing research.

What is most attractive about the techniques outlined here is the potential for application to a whole hierarchy of non-linear evolution equations. Although the analysis on the surface appears to be intricate and complicated, with many different regions of interest that require separate consideration, in reality most of the detail follows in a natural manner. Given the specified initial condition, in Section 2 we used standard methods to infer the structure of the solution at relatively early times. The results allowed us to develop solutions for large $|x|$ to moderate times and these in turn enabled us to identify the relevance of the scale $x\sim O(t)$. Once this scaling was highlighted, the remainder of the calculation was rendered little more than routine. As might be expected the two soliton zones and the kink solution required some more careful analysis to ensure that matching was accomplished properly but otherwise the methodology was standard. It should be emphasised that the overall procedure required little more than familiar methods but led to accurate predictions of the size and positions of the various features as well as new information concerning the rates of convergence of the solution.

Finally, we remark that the computations in [1] and [2] examined the evolution of the mKdV equation from a number of different initial conditions including profiles leading to many more soliton structures. As might be expected, a more complicated long-time structure of the solution requires a correspondingly involved analysis, but it can still be followed through using precisely the techniques described here. It is perhaps somewhat surprising that the suitable application of standard methods of matched asymptotics can yield complete details of the long-term structure of complicated non-linear evolution equations.

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