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BETTI NUMBER ESTIMATES IN p-ADIC COHOMOLOGY

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Abstract In the framework of Berthelot's theory of arithmetic $\mathcal{D}$-modules, we prove the p-adic analogue of Betti number estimates and we give some standard applications.

Keywords: algebraic geometry; p-adic cohomologies; Betti numbers; arithmetic $\mathcal{D}$-modules

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Contents

Introduction

Let k be a perfect field of characteristic p and l be a prime number different from p. When k is algebraically closed, in the framework of Grothendieck's l-adic etale cohomology of k-varieties, Bernstein, Beilinson and Deligne in their famous paper on perverse sheaves, more precisely in [5, 4.5.1] (or see the p-adic translation here in Theorem 2.2.6), established some Betti number estimates. The goal of this paper is to get the same estimates in the context of Berthelot's arithmetic $\mathcal{D}$-modules. We recall that this theory of Berthelot gives a p-adic cohomology stable under six operations (see [15]) and admitting a theory of weights (see [3]) analogous to that of Deligne in the l-adic side (see [21]). This allows us to consider Berthelot's theory as a right p-adic analogue of Grothendieck's

l-adic etale cohomology. By trying to translate the proof of Betti number estimates in [5, 4.5.1] in the framework of arithmetic $\mathcal{D}$-modules, two specific problems appear. The first one is that we do not have a notion of local acyclicity in the theory of arithmetic $\mathcal{D}$-modules. We replace the use of this notion by another one that we might call 'relative generic $\mathcal{O}$-coherence'. The goal of the first chapter is to prove this property. The proof of this relative generic $\mathcal{O}$-coherence uses the precise description of the characteristic variety of a unipotent overconvergent F-isocrystal (see [11]). Berthelot's characteristic variety of a holonomic arithmetic $\mathcal{D}$-module endowed with a Frobenius structure. The second emerging problem when we follow the original l-adic proof of Betti number estimates is that we still do not have vanishing cycles theory as nice as in the l-adic framework (so far, following [2] we only have a p-adic analogue of Beilinson's unipotent nearby cycles and vanishing cycles). Here, we replace successfully in the original proof on Betti number estimates the use of vanishing cycles by that of some Fourier transform and of Abe–Marmora formula [4, 4.1.6(i)] relating the irregularity of an isocrystal with the rank of its Fourier transform. We conclude this paper by the remark that these Betti number estimates allow us to state that the results of [5, Chapters 4 and 5] are still valid (except [5, 5.4.7–8] because the translation is not clear so far).

Convention, notation of the paper

Let $\mathcal{V}$ be a complete discrete valued ring of mixed characteristic $(0, p)$, K its field of fractions, k its residue field which is supposed to be perfect, π be a uniformizer of $\mathcal{V}$. Let $F_k : k \to k$ be the Frobenius map given by $x \mapsto x^p$. When we deal with Frobenius structures, we suppose that there exists a lifting $\sigma_0 : \mathcal{V} \to \mathcal{V}$ of the Frobenius map F_k that we fix. A k-variety is a separated reduced scheme of finite type over k. We say that a k-variety X is realizable if there exists an immersion of the form $X \hookrightarrow \mathcal{P}$, where $\mathcal{P}$ is a proper smooth formal scheme over $\mathcal{V}$. In this paper, k-varieties will always be supposed realizable. For any k-variety X, we denote by $p_X : X \to \operatorname{Spec} k$ the canonical morphism. We denote formal schemes by curly or gothic letters and the corresponding straight roman letter will mean the special fibre (e.g., if $\mathfrak{X}$ is a formal scheme over $\mathcal{V}$, then X is the k-variety equal to the special fibre of $\mathfrak{X}$). The underlying topological space of a k-variety X is denoted by $|X|$. When M is a $\mathcal{V}$-module, we denote by $\widehat{M}$ its p-adic completion and we set $M_{\mathbb{Q}} := M \otimes_{\mathcal{V}} K$. By default, a module will mean a left module. Moreover, if $f : \mathcal{P}' \to \mathcal{P}$ is a morphism of formal schemes over $\mathcal{V}$, we denote by $\mathbb{L} f^*$ the functor defined by putting $\mathbb{L} f^*(\mathcal{M}) = \mathcal{O}_{\mathcal{P}',\mathbb{Q}} \otimes^{\mathbb{L}}_{f^{-1}\mathcal{O}_{\mathcal{P},\mathbb{Q}}} f^{-1}\mathcal{M}$, for any bounded below complex $\mathcal{M}$ of $\mathcal{O}_{\mathcal{P},\mathbb{Q}}$-modules. When f is flat, we remove $\mathbb{L}$ in the notation.

If $T \to S$ is a morphism of schemes and $f : X \to Y$ is an S-morphism, then we denote by $f_T : X_T \to Y_T$ or simply by $f : X_T \to Y_T$ the base change of f by $T \to S$.

Concerning the cohomological operations of the theory of arithmetic $\mathcal{D}$-modules of Berthelot, we follow the usual notation (for instance, see the beginning of [3]). More precisely, let S be a noetherian scheme such that p is nilpotent in $\mathcal{O}_S$. Let $f : X \to Y$ be morphism of quasi-compact smooth S-schemes. If f is smooth, then the extraordinary pull-back of level m by f has the factorization $f^{!(m)} : D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{(m)}_{Y/S}) \to D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{(m)}_{X/S})$ (see [10, 2.2.4]). If f is proper, then the push-forward of level m by f has the factorization

$f_{+^{(m)}}\colon D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{(m)}_{X/S}) \to D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{(m)}_{Y/S})$ (see [10, 2.4.4]). When there is no ambiguity with the basis S, we remove '$/S$' is the notation.

Let $f\colon \mathcal{P} \to \mathcal{Q}$ be a morphism of quasi-compact smooth formal $\mathcal{V}$-schemes. If f is smooth, then we have the extraordinary pull-back of level m by f of the form $f^{!(m)}\colon D^{\mathrm{b}}_{\mathrm{coh}}(\widehat{\mathcal{D}}^{(m)}_{\mathcal{Q}}) \to D^{\mathrm{b}}_{\mathrm{coh}}(\widehat{\mathcal{D}}^{(m)}_{\mathcal{P}})$ and $f^{!(m)}\colon D^{\mathrm{b}}_{\mathrm{coh}}(\widehat{\mathcal{D}}^{(m)}_{\mathcal{Q},\mathbb{Q}}) \to D^{\mathrm{b}}_{\mathrm{coh}}(\widehat{\mathcal{D}}^{(m)}_{\mathcal{P},\mathbb{Q}})$ (see [10, 3.4.6]), and we have the extraordinary pull-back by f of the form $f^{!}\colon D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathcal{Q},\mathbb{Q}}) \to D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathcal{P},\mathbb{Q}})$ (see [10, 4.3.4]). If f is proper, then we have the push-forward of level m of the form $f_{+^{(m)}}\colon D^{\mathrm{b}}_{\mathrm{coh}}(\widehat{\mathcal{D}}^{(m)}_{\mathcal{P}}) \to D^{\mathrm{b}}_{\mathrm{coh}}(\widehat{\mathcal{D}}^{(m)}_{\mathcal{Q}})$ and $f_{+^{(m)}}\colon D^{\mathrm{b}}_{\mathrm{coh}}(\widehat{\mathcal{D}}^{(m)}_{\mathcal{P},\mathbb{Q}}) \to D^{\mathrm{b}}_{\mathrm{coh}}(\widehat{\mathcal{D}}^{(m)}_{\mathcal{Q},\mathbb{Q}})$ (see [10, 3.5.3]), and the push-forward by f of the form $f_{+}\colon D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_{\mathcal{P},\mathbb{Q}}) \to D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_{\mathcal{Q},\mathbb{Q}})$ (see [10, 4.3.8]).

Let $a\colon X \to Y$ be a morphism of (realizable) k-varieties. By definition, there exist immersions $\iota\colon X \hookrightarrow \mathcal{P}$ and $\iota'\colon Y \hookrightarrow \mathcal{Q}$ where $\mathcal{P}$ and $\mathcal{Q}$ are proper smooth formal schemes over $\mathcal{V}$. Replacing $\mathcal{P}$ by $\mathcal{P} \times \mathcal{Q}$, we can suppose that there exist a (proper) smooth morphism of formal $\mathcal{V}$-schemes of the form $f\colon \mathcal{P} \to \mathcal{Q}$ such that $f \circ \iota = \iota' \circ a$. By definition, $D^{\mathrm{b}}_{\mathrm{ovhol}}(X, \mathcal{P}/K)$ is the full subcategory of $D^{\mathrm{b}}_{\mathrm{ovhol}}(\mathcal{D}^{\dagger}_{\mathcal{P},\mathbb{Q}})$ (the derived category of overholonomic complexes of $\mathcal{D}^{\dagger}_{\mathcal{P},\mathbb{Q}}$-modules) of the objects $\mathcal{E}$ such that there exists an isomorphism of the form $\mathbb{R}\underline{\Gamma}^{\dagger}_{X}(\mathcal{E}) \xrightarrow{\sim} \mathcal{E}$ (see [3, 1.1.6]). Since this category $D^{\mathrm{b}}_{\mathrm{ovhol}}(X, \mathcal{P}/K)$ does not depend on the choice of ι, we simply denote it by $D^{\mathrm{b}}_{\mathrm{ovhol}}(X/K)$ (see Definition [3, 1.1.5]). The extraordinary pull-back by a is by definition $\mathbb{R}\underline{\Gamma}^{\dagger}_{X} \circ f^{!}\colon D^{\mathrm{b}}_{\mathrm{ovhol}}(Y, \mathcal{Q}/K) \to D^{\mathrm{b}}_{\mathrm{ovhol}}(X, \mathcal{P}/K)$, which is simply denoted by $a^{!}\colon D^{\mathrm{b}}_{\mathrm{ovhol}}(Y/K) \to D^{\mathrm{b}}_{\mathrm{ovhol}}(X/K)$ (again, we check that this does not depend on ι, ι' and f). The push-forward by a is by definition $f_{+}\colon D^{\mathrm{b}}_{\mathrm{ovhol}}(X, \mathcal{P}/K) \to D^{\mathrm{b}}_{\mathrm{ovhol}}(Y, \mathcal{Q}/K)$, which is simply denoted by $a_{+}\colon D^{\mathrm{b}}_{\mathrm{ovhol}}(X/K) \to D^{\mathrm{b}}_{\mathrm{ovhol}}(Y/K)$. We have also the dual functor $\mathbb{D}_X := \mathbb{R}\underline{\Gamma}^{\dagger}_{X} \circ \mathbb{D}_{\mathcal{P}}\colon D^{\mathrm{b}}_{\mathrm{ovhol}}(X/K) \to D^{\mathrm{b}}_{\mathrm{ovhol}}(X/K)$. Then we get $a_{!} := \mathbb{D}_Y \circ a_{+} \circ \mathbb{D}_X$ and $a^{+} := \mathbb{D}_X \circ a^{!} \circ \mathbb{D}_Y$. There is a canonical t-structure on $D^{\mathrm{b}}_{\mathrm{ovhol}}(X/K)$ defined as follows: if $\mathfrak{U}$ is an open set of $\mathcal{P}$ so that X is closed in $\mathfrak{U}$ then $D^{\leqslant n}_{\mathrm{ovhol}}(X/K)$ (respectively $D^{\geqslant n}_{\mathrm{ovhol}}(X/K)$) is the subcategory of $D^{\mathrm{b}}_{\mathrm{ovhol}}(X/K)$ of complexes $\mathcal{E}$ such $\mathcal{E}|\mathfrak{U} \in D^{\leqslant n}_{\mathrm{ovhol}}(\mathcal{D}^{\dagger}_{\mathfrak{U},\mathbb{Q}})$ (respectively $D^{\geqslant n}_{\mathrm{ovhol}}(\mathcal{D}^{\dagger}_{\mathfrak{U},\mathbb{Q}})$), where the t-structure on $D^{\mathrm{b}}_{\mathrm{ovhol}}(\mathcal{D}^{\dagger}_{\mathfrak{U},\mathbb{Q}})$ is the obvious one. The heart of this t-structure is denoted by $\mathrm{Ovhol}(X/K)$ (see Definition [3, 1.2.6]).

Suppose X smooth. Following [3, 1.2.14], we have a full subcategory $D^{\mathrm{b}}_{\mathrm{isoc}}(X/K)$ of $D^{\mathrm{b}}_{\mathrm{ovhol}}(X/K)$ whose cohomological spaces (for the above t-structure) belong to $\mathrm{Isoc}^{\dagger\dagger}(X/K)$ (the category of overconvergent isocrystals on X/K). Recall that $\mathrm{Isoc}^{\dagger\dagger}(X/K)$ is equivalent to the category of overconvergent isocrystals on X/K denoted by $\mathrm{Isoc}^{\dagger}(X/K)$.

If $j\colon U \hookrightarrow X$ is an open immersion of (realizable) varieties, the functor $j^{!}\colon D^{\mathrm{b}}_{\mathrm{ovhol}}(X/K) \to D^{\mathrm{b}}_{\mathrm{ovhol}}(U/K)$ (or the functor $j^{!}\colon \mathrm{Ovhol}(X/K) \to \mathrm{Ovhol}(U/K)$) will simply be denoted by $|U$ (in other papers, to avoid confusion, it was sometimes denoted by $\|U$ but, here, there is no such risk since we do not work 'partially').

Let s be a positive integer and $\sigma = \sigma_0^s\colon \mathcal{V} \to \mathcal{V}$ the corresponding lifting of the sth power of the Frobenius map $F^s_k\colon k \to k$. If X is a k-variety (respectively $\mathcal{P}$ is a smooth formal $\mathcal{V}$-scheme) then we denote by X^{σ} (respectively $\mathcal{P}^{\sigma}$) the corresponding k-scheme of finite type (respectively smooth formal $\mathcal{V}$-scheme) induced by the base change by F^s_k (respectively σ). We denote by $F^s_{X/k}\colon X \to X^{\sigma}$ the corresponding relative Frobenius

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which is a morphism of k-schemes. Notice, when X is k-smooth, $F^s_{X/k}$ is a morphism of smooth k-varieties. When it exists (e.g., when $\mathcal{P}$ is affine), we denote by $F^s_{\mathcal{P}/\mathcal{V}}\colon \mathcal{P} \to \mathcal{P}^\sigma$ a morphism of smooth formal $\mathcal{V}$-schemes which is a lifting of $F^s_{X/k}\colon X \to X^\sigma$. The functor $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathcal{P},\mathbb{Q}}) \to D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathcal{P}^\sigma,\mathbb{Q}})$ induced by the isomorphism $\mathcal{P}^\sigma \xrightarrow{\sim} \mathcal{P}$ is denoted by $\mathcal{E} \mapsto \mathcal{E}^\sigma$. We have the functor $(F^s_{P/k})^!\colon D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathcal{P}^\sigma,\mathbb{Q}}) \to D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathcal{P},\mathbb{Q}})$. Recall that when $F^s_{P/k}$ has a lifting $F^s_{\mathcal{P}/\mathcal{V}}\colon \mathcal{P} \to \mathcal{P}^\sigma$ then we have $(F^s_{P/k})^! = (F^s_{\mathcal{P}/\mathcal{V}})^!$ (in general, even if the lifting $F^s_{\mathcal{P}/\mathcal{V}}$ is not unique we can glue these functors: e.g., see [9, 2.1]). Finally, we get the functor $F^*\colon D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathcal{P},\mathbb{Q}}) \to D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathcal{P},\mathbb{Q}})$ which is defined for any $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathcal{P},\mathbb{Q}})$ by setting $F^*(\mathcal{E}) := (F^s_{P/k})^!(\mathcal{E}^\sigma)$. The derived category of overholonomic F-complexes of $\mathcal{D}^\dagger_{\mathcal{P},\mathbb{Q}}$-modules, denoted by $F\text{-}D^{\mathrm{b}}_{\mathrm{ovhol}}(\mathcal{D}^\dagger_{\mathcal{P},\mathbb{Q}})$, is the category whose objects are the data of an object $\mathcal{E}$ of $D^{\mathrm{b}}_{\mathrm{ovhol}}(\mathcal{D}^\dagger_{\mathcal{P},\mathbb{Q}})$ endowed with a Frobenius structure, i.e., an isomorphism ϕ of $D^{\mathrm{b}}_{\mathrm{ovhol}}(\mathcal{D}^\dagger_{\mathcal{P},\mathbb{Q}})$ of the form $\phi\colon F^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}$. We get similarly the category $F\text{-}D^{\mathrm{b}}_{\mathrm{ovhol}}(X/K)$ of overholonomic F-complexes on X/K. When X is smooth, we define similarly the categories of F-objects $F\text{-}D^{\mathrm{b}}_{\mathrm{isoc}}(X/K)$ and $F\text{-}\mathrm{Isoc}^{\dagger\dagger}(X/K)$ (see [3, 1.2.14]).

1. Relative generic $\mathcal{O}$-coherence

1.1. Preliminaries on cotangent spaces

Notation 1.1.1. Let X be a smooth k-variety. For any quasi-coherent $\mathcal{O}_X$-module $\mathcal{E}$, we denote by $\mathrm{Sym}(\mathcal{E})$ the symmetric algebra of $\mathcal{E}$ and by $\mathbb{V}(\mathcal{E}) := \mathrm{Spec}\,(\mathrm{Sym}(\mathcal{E}))$ endowed with its canonical projection $\mathbb{V}(\mathcal{E}) \to \mathrm{Spec}\,(\mathcal{O}_X) = X$. We denote by Ω^1_X the sheaf of differential form of $X/\mathrm{Spec}\,(k)$ (we skip k in the notation), and $\mathcal{T}_X$ the tangent space of $X/\mathrm{Spec}\,(k)$, i.e., the $\mathcal{O}_X$-dual of Ω^1_X. We denote by $T^*X := \mathbb{V}(\mathcal{T}_X)$ the cotangent space of X and $\pi_X\colon T^*X \to X$ the canonical projection. Recall that from [24, 1.7.9], there is a canonical bijection between sections of π_X and $\Gamma(X, \Omega^1_X)$. We denote by $T^*_X X$ the section corresponding to the zero section of $\Gamma(X, \Omega^1_X)$. If $t_1, \dots, t_d$ are local coordinates of X, we get local coordinates $t_1, \dots, t_d, \xi_1, \dots, \xi_d$ of T^*X, where ξ_i is the element associated with ∂_i, the derivation with respect to t_i. Is this case, $T^*_X X = V(\xi_1, \dots, \xi_d)$ is the closed subvariety of T^*X defined by $\xi_1 = 0, \dots, \xi_d = 0$.

Let $f\colon X \to Y$ be a morphism of smooth k-varieties. Using the equality [24, 1.7.11(iv)] we get the last one $X \times_Y T^*Y = X \times_Y \mathbb{V}(\mathcal{T}_Y) = \mathbb{V}(f^*\mathcal{T}_Y)$. The morphism $f^*\Omega^1_Y \to \Omega^1_X$ induced by f yields by duality $\mathcal{T}_X \to f^*\mathcal{T}_Y$ and then by functoriality $\mathbb{V}(f^*\mathcal{T}_X) \to \mathbb{V}(\mathcal{T}_Y) = T^*Y$. By composition, we get the morphism denoted by $\rho_f\colon X \times_Y T^*Y \to T^*X$. We write by $\varpi_f\colon X \times_Y T^*Y \to T^*Y$ the base change of f under π_Y (instead of $f_{T^*_Y}\colon X \times_Y T^*Y \to T^*Y$ which seems too heavy).

We denote by $\mathscr{T}_f$ the function from the set of subvarieties of T^*X to the set of subvarieties of T^*Y defined by posing, for any subvariety V of T^*X, $\mathscr{T}_f(V) := \varpi_f(\rho_f^{-1}(V))$. If f is an open immersion, then ρ_f is an isomorphism. In that case, $\mathscr{T}_f := \varpi_f \circ \rho_f^{-1}\colon T^*X \to T^*Y$ is an open immersion and this is compatible with the above definition of $\mathscr{T}_f$. The application $\mathscr{T}\colon f \mapsto \mathscr{T}_f$ is transitive (with respect to the composition), i.e., we have the equality $\mathscr{T}_g \circ \mathscr{T}_f = \mathscr{T}_{g\circ f}$ for any $g\colon Y \to Z$ (e.g., look at the bottom of the diagram (1.1.2.1) where f and u are replaced respectively by g and f).

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We define the k-variety $T^*_X Y$ (recall a k-variety is a separated reduced scheme of finite type over k from our convention) by setting $T^*_X Y := \rho_f^{-1}(T^*_X X)$. When f is an immersion, $T^*_X Y$ is viewed as a subvariety of T^*Y via $T^*_X Y \subset X \times_Y T^*Y \overset{\varpi_f}{\hookrightarrow} T^*Y$, i.e., we simply denote $\varpi_f(T^*_X Y)$ by $T^*_X Y$.

Lemma 1.1.2. *Let $u\colon Z \to X$ and $f\colon X \to P$ be two morphisms of smooth k-varieties.*

(1) *We have the equality $(Z \times_X \rho_f)^{-1}(T^*_Z X) = T^*_Z P$. When u is an immersion, this might be written of the form $\rho_f^{-1}(T^*_Z X) = T^*_Z P$ or $\mathscr{T}_f(T^*_Z X) = \varpi_f(T^*_Z P)$. If u and f are immersions, this might be written $\mathscr{T}_f(T^*_Z X) = T^*_Z P$. Finally, when u is an immersion and f is an open immersion, we might identify $T^*_Z X$ and $T^*_Z P$.*

(2) *When u is an immersion (respectively an open immersion), we have the inclusion $Z \times_X T^*_X P \subset T^*_Z P$ (respectively the equality $Z \times_X T^*_X P = T^*_Z P$) in $Z \times_P T^*P$.*

Proof. (1) First, let us prove part (1) of the lemma. The composition $(f \circ u)^*\Omega_P \xrightarrow{\sim} u^* \circ f^*\Omega_P \to u^*\Omega_X \to \Omega_Z$ is the canonical one. Indeed, since this is local, then we reduce to the case where varieties are affine and then this is checked by an easy computation. This implies that the composition $Z \times_P T^*P \xrightarrow{Z \times_X \rho_f} Z \times_X T^*X \xrightarrow{\rho_u} T^*Z$ is equal to $\rho_{f \circ u}$. Consider the following diagram

$$\begin{tikzcd}
T^*_Z Z \arrow[d, hook] \arrow[dr, phantom, "\square"] & T^*_Z X \arrow[l] \arrow[d, hook] \arrow[dr, phantom, "\square"] & T^*_Z P \arrow[l] \arrow[d, hook] & \\
T^*Z & Z \times_X T^*X \arrow[l, "\rho_u"'] \arrow[d, "\varpi_u"] \arrow[dr, phantom, "\square"] & Z \times_P T^*P \arrow[l, "Z \times_X \rho_f"'] \arrow[r, "\varpi_f"] \arrow[d] \arrow[dr, phantom, "\square"] & Z \times_X T^*P \arrow[d, "u"] \\
 & T^*X & X \times_P T^*P \arrow[l, "\rho_f"'] \arrow[r, "\varpi_f"] & T^*P,
\end{tikzcd} \tag{1.1.2.1}$$

where the upper left square and the composition of both upper squares are by definition Cartesian (for the second case, use $\rho_{f \circ u} = \rho_u \circ (Z \times_X \rho_f)$). This yields the cartesianity of the upper right square. Hence, we get the equality $(Z \times_X \rho_f)^{-1}(T^*_Z X) = T^*_Z P$. When u is an immersion, this yields $\mathscr{T}_f(T^*_Z X) = \varpi_f(T^*_Z P)$. The other assertions of (1) are obvious.

(2) Now, let us check part (2) of the lemma. Since $(Z \times_X \rho_f)^{-1}(Z \times_X T^*_X X) = Z \times_X \rho_f^{-1}(T^*_X X) = Z \times_X T^*_X P$ and $(Z \times_X \rho_f)^{-1}(T^*_Z X) = T^*_Z P$ (this is the first part of the Lemma), we reduce to check the inclusion $Z \times_X T^*_X X \subset T^*_Z X$ (respectively equality $Z \times_X T^*_X X = T^*_Z X$) of subvarieties of T^*X.

(i) First, suppose that u is an open immersion. In that case $\rho_u\colon Z \times_X T^*X \to T^*Z$ is an isomorphism and we check easily the desired equality $\rho_u^{-1}(T^*_Z Z) = Z \times_X T^*_X X$ by coming back to the definition of $T^*_X X$ and $T^*_Z Z$.

(ii) Suppose now that u is only an immersion. Let $\overline{Z}$ be the closure of Z in X. From the part 2(i) of the proof, the respective case of the part (2) of the lemma is satisfied. Hence, since $Z \to \overline{Z}$ is an open immersion, we get $Z \times_{\overline{Z}} T^*_{\overline{Z}} X = T^*_Z X$. Using this latter equality,

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we reduce to check the inclusion $\overline{Z} \times_X T^*_X X \subset T^*_{\overline{Z}} X$. In other words, we can suppose $Z = \overline{Z}$. Moreover, from the part (1) and the respective case of the part (2) of the Lemma, the check is local in X. Hence, we can suppose that X has local coordinates $t_1, \dots, t_d$ such that $\bar{t}_1, \dots, \bar{t}_r$, the global section of $\mathcal{O}_Z$ induced by $t_1, \dots, t_r$, are local coordinates of Z. We get local coordinates $t_1, \dots, t_d, \xi_1, \dots, \xi_d$ of T^*X, where ξ_i is the element associated with ∂_i, the derivation with respect to t_i. We get also local coordinates $\bar{t}_1, \dots, \bar{t}_r, \overline{\xi}_1, \dots, \overline{\xi}_r$ of T^*Z, where $\overline{\xi}_i$ is the element associated with $\overline{\partial}_i$, the derivation with respect to $\bar{t}_i$. Then, $\rho_u \colon Z \otimes_X T^*X \to T^*Z$ is smooth and $1 \otimes \xi_{r+1}, \dots, 1 \otimes \xi_d$ are local coordinates relatively to ρ_u (in fact they induce an isomorphism of the form $Z \otimes_X T^*X \xrightarrow{\sim} \mathbb{A}^{d-r}_{T^*Z}$) and the image of $\overline{\xi}_1 \dots, \overline{\xi}_r$ via ρ_u are $1 \otimes \xi_1, \dots, 1 \otimes \xi_r$. Hence we get $\rho_u^{-1}(T^*_Z Z) = T^*_Z X = V(1 \otimes \xi_1, \dots, 1 \otimes \xi_r)$ and $Z \times_X T^*_X X = V(1 \otimes \xi_1, \dots, 1 \otimes \xi_d)$ as subvarieties of $Z \times_X T^*X$. Hence, $Z \times_X T^*X \subset T^*_Z X$. □

Lemma 1.1.3. *Let $f \colon X \to P$ be a morphism of smooth k-varieties and $(X_i)_{1 \leqslant i \leqslant r}$ be a family of smooth subvarieties of X (respectively open subvarieties of X) such that $X \subset \bigcup_{i=1}^r X_i$. Then, $T^*_X P \subset \bigcup_{i=1}^r T^*_{X_i} P$ (respectively $T^*_X P = \bigcup_{i=1}^r T^*_{X_i} P$).*

Proof. From the first equality of Lemma 1.1.2, we reduce to the case $X = P$. Using the second part of Lemma 1.1.2, we get the inclusions $X_i \times_X T^*_X X \subset T^*_{X_i} X$ (respectively the equalities $X_i \times_X T^*_X X = T^*_{X_i} X$), which yields the desired result when $X = P$. □

Proposition 1.1.4. *Let $f \colon X \to Y$ be a smooth morphism of smooth varieties. Let B be a smooth subvariety of Y and $A := f^{-1}(B)$.*

(1) *The morphism ρ_f is a closed immersion. If f is étale then ρ_f is an isomorphism.*

(2) *We have $\rho_f^{-1}(T^*_A X) \overset{1.1.2(1)}{=} T^*_A Y = \varpi_f^{-1}(T^*_B Y)$, $\rho_f(\varpi_f^{-1}(T^*_B Y)) \subset T^*_A X$ and $\mathscr{T}_f(T^*_A X) \subset T^*_B Y$.*

(3) *When f is surjective, we have the equality $\mathscr{T}_f(T^*_A X) = T^*_B Y$.*

Proof. (1) From [28, 17.11.1], the canonical morphism $f^*\Omega_Y \to \Omega_X$ is injective. By duality we get the surjection $\mathcal{T}_X \to f^*\mathcal{T}_Y$. Hence, from [24, 1.7.11(iv) and (v)], the morphism $\rho_f \colon X \times_Y T^*Y \to T^*X$ is a closed immersion. When f is etale, from [28, 17.11.2], the canonical morphism $f^*\Omega_Y \to \Omega_X$ is an isomorphism and then so is ρ_f.

(2) (a) In this step, we check the equality $T^*_X Y = \varpi_f^{-1}(T^*_Y Y)$ $(= X \times_Y T^*_Y Y)$.

(i) From the respective case of Lemma 1.1.2(2), this is local in X. Moreover, this is local in Y. Indeed, let V be an open set of Y. We put $U := f^{-1}(V)$ and $f_V \colon U \to V$ the induced morphism. On the one hand $(U \times_Y T^*Y) \cap \varpi_f^{-1}(T^*_Y Y) = U \times_Y T^*_Y Y = U \times_V (V \times_Y T^*_Y Y) \overset{1.1.2(2)}{=} U \times_V T^*_V Y \overset{1.1.2(1)}{=} U \times_V T^*_V V = \varpi_{f_V}^{-1}(T^*_V V)$ and on the other hand $(U \times_Y T^*Y) \cap T^*_X Y = U \times_X T^*_X Y \overset{1.1.2(2)}{=} T^*_U Y \overset{1.1.2(1)}{=} T^*_U V$, which give the localness in Y.

(ii) Let $g \colon Y \to Z$ be another smooth morphism of varieties. If this equality is satisfied for f and g, i.e., if $T^*_X Y = \varpi_f^{-1}(T^*_Y Y)$ and $T^*_Y Z = \varpi_g^{-1}(T^*_Z Z)$, then we get that the squares of the following diagram are Cartesian:

$$\begin{tikzcd}
T^*_X X \arrow[d, hook] & X \times_Y T^*_Y Y \arrow[l] \arrow[d, hook] & X \times_Z T^*_Z Z \arrow[l] \arrow[d, hook] \\
T^* X \arrow[ur, phantom, "\square"] & X \times_Y T^* Y \arrow[l, "\rho_g"'] \arrow[ur, phantom, "\square"] & X \times_Z T^* Z \arrow[l, "\rho_f"'] \arrow[ll, bend left, "\rho_{g\circ f}"]
\end{tikzcd}$$

Hence, the rectangle is also Cartesian, i.e., $T^*_X Z = X \times_Z T^*_Z Z$, which is the desired equality.

(iii) Since from the step (i) the check of equality $T^*_X Y = \varpi_f^{-1}(T^*_Y Y) = X \times_Y T^*_Y Y$ is local in X and Y, then we can suppose that Y has local coordinates $t_1, \ldots, t_d$ and that there exists an etale morphism of the form $X \to \mathbb{A}^n_Y$ whose composition with the projection $\mathbb{A}^n_Y \to Y$ gives f. Since from the step (ii) the equality $T^*_X Y = \varpi_f^{-1}(T^*_Y Y) = X \times_Y T^*_Y Y$ is transitive with respect to the composition, we reduce to the case where $n = 0$ or $X = \mathbb{A}^n_Y$. Suppose $n = 0$, i.e., f is etale. Let $t'_1, \ldots, t'_d$ be the local coordinates of X induced by $t_1, \ldots, t_d$. We get local coordinates $t_1, \ldots, t_d, \xi_1, \ldots, \xi_d$ of T^*Y, where ξ_i is the element associated with the derivation with respect to t_i and local coordinates $t'_1, \ldots, t'_d, \xi'_1, \ldots, \xi'_d$ of T^*X, where ξ'_i is the element associated with the derivation with respect to t'_i. The isomorphism $\rho_f \colon X \times_Y T^*Y \xrightarrow{\sim} T^*X$ sends ξ'_i to $1 \otimes \xi_i$. Since $T^*_X X = V(\xi'_1, \ldots, \xi'_d)$ then $T^*_X Y := \rho_f^{-1}(T^*_X X) = V(1 \otimes \xi_1, \ldots, 1 \otimes \xi_d) = X \times_Y V(\xi_1, \ldots, \xi_d) = X \times_Y T^*_Y Y$. Suppose now that $X = \mathbb{A}^n_Y$. Let $t_{d+1}, \ldots, t_{d+n}$ be the coordinates of $\mathbb{A}^n_k$ and let $\xi_{d+1}, \ldots, \xi_{d+n}$ be the element of $T^*\mathbb{A}^n_k$ associated respectively with the derivation with respect to $t_{d+1}, \ldots, t_{d+n}$. We get local coordinates $t_1, \ldots, t_{d+n}, \xi_1, \ldots, \xi_{d+n}$ of T^*X. In that case the morphism ρ_f is a closed immersion of the form $\rho_f \colon \mathbb{A}^n \times T^*Y \hookrightarrow T^*X$ so that $\mathbb{A}^n \times T^*Y = V(\xi_{d+1}, \ldots, \xi_{d+n})$ in T^*X. Since $T^*_X X = V(\xi_1, \ldots, \xi_{d+n})$ in T^*X then $\rho_f^{-1}(T^*_X X)$ is the closed subvariety of $\mathbb{A}^n \times T^*Y$ defined by $\xi_1 = 0, \ldots, \xi_d = 0$, i.e., $\rho_f^{-1}(T^*_X X) = \mathbb{A}^n \times T^*_Y Y$ (recall $T^*_Y Y = V(\xi_1, \ldots, \xi_d)$ in T^*Y), which is the desired equality.

(b) Let $i \colon B \hookrightarrow Y$ be the structural immersion. By applying Lemma 1.1.2(1) to the case of $A \to B \to Y$, we get $(A \times_B \rho_i)^{-1}(T^*_A B) = T^*_A Y$. Moreover, $(A \times_B \rho_i)^{-1}(A \times_B T^*_B B) = A \times_B T^*_B Y$. From part (a), we have $T^*_A B = A \times_B T^*_B B$. This implies the first equality $T^*_A Y = A \times_B T^*_B Y = X \times_Y T^*_B Y = \varpi_f^{-1}(T^*_B Y)$. Hence, we have checked the equality $\rho_f^{-1}(T^*_A X) = \varpi_f^{-1}(T^*_B Y)$. By applying ρ_f to this equality, we get $\rho_f(\varpi_f^{-1}(T^*_B Y)) = \rho_f(\rho_f^{-1}(T^*_A X)) \subset T^*_A X$. By applying this time ϖ_f to this equality, we get $\mathscr{T}_f(T^*_A X) = \varpi_f(\rho_f^{-1}(T^*_A X)) = \varpi_f(\varpi_f^{-1}(T^*_B Y)) \subset T^*_B Y$. When f is surjective, then so is ϖ_f (see [23, 3.5.2(ii)]). Hence, the latter inclusion is in fact an equality. $\square$

1.2. Inverse and direct images of complexes of arithmetic $\mathcal{D}$-modules and characteristic varieties

1.2.1 (Characteristic variety and characteristic cycle (of level 0)). Let $\mathfrak{X}$ be a smooth $\mathcal{V}$-formal scheme, X be the reduction of $\mathfrak{X}$ modulo π (recall π is a uniformizer of $\mathcal{V}$). Let $m \in \mathbb{N}$ be an integer. Let us recall Berthelot's definition of characteristic varieties (of level m) as explained in [10, 5.2].

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(1) Let $\mathcal{G}$ be a coherent $\mathcal{D}_X^{(0)}$-module. Berthelot checked the equality $T^*X := \operatorname{Spec} \operatorname{gr}\mathcal{D}_X^{(0)}$, where $\mathcal{D}_X^{(0)}$ is filtered by the order. Since $\pi_X\colon T^*X \to X$ is affine, the functor π_{X*} induces an equivalence from the category of (quasi-)coherent $\mathcal{O}_{T^*X}$-modules to that of (quasi-)coherent $\operatorname{gr}\mathcal{D}_X^{(0)}$-modules. We denote by $\sim$ a quasi-inverse functor. Even if it is tempting to identify both categories, we try to distinguish them to avoid confusion. Choose a good filtration $(\mathcal{G}_n)_{n\in\mathbb{N}}$, i.e., a filtration such that $\operatorname{gr}\mathcal{G}$ is a coherent $\operatorname{gr}\mathcal{D}_X^{(0)}$-module (see the definition [10, 5.2.3]). We denote by $\widetilde{\operatorname{gr}}$ the composition of gr with $\sim$. So, $\widetilde{\operatorname{gr}}\mathcal{G}$ is a coherent $\mathcal{O}_{T^*X}$-module. The characteristic variety of level 0 of $\mathcal{G}$, denoted by $\operatorname{Car}^{(0)}(\mathcal{G})$ is by definition the support of $\widetilde{\operatorname{gr}}\mathcal{G}$ in T^*X which is viewed canonically as a subvariety of T^*X. Berthelot checked that this is well defined (i.e., that this is independent of the choice of the good filtration). Moreover, he defined the characteristic cycle associated with $\mathcal{G}$ that we denote (we add (0) to avoid confusion) by $Z\operatorname{Car}^{(0)}(\mathcal{G})$ (for a detailed definition see [10, 5.4.1]).

(2) Let $\mathcal{F}$ be a coherent $\widehat{\mathcal{D}}_{\mathfrak{X}}^{(0)}$-module. The characteristic variety $\operatorname{Car}^{(0)}(\mathcal{F})$ of level 0 of $\mathcal{F}$ is by definition the characteristic variety of level 0 of $\mathcal{F}/\pi\mathcal{F}$ as coherent $\mathcal{D}_X^{(0)}$-module, i.e., $\operatorname{Car}^{(0)}(\mathcal{F}) := \operatorname{Car}^{(0)}(\mathcal{F}/\pi\mathcal{F})$. Similarly, we define the characteristic cycle $Z\operatorname{Car}^{(0)}(\mathcal{F})$ of level 0 of $\mathcal{F}$ by setting $Z\operatorname{Car}^{(0)}(\mathcal{F}) := Z\operatorname{Car}^{(0)}(\mathcal{F}/\pi\mathcal{F})$.

(3) Let $\mathcal{E}$ be a coherent $\widehat{\mathcal{D}}_{\mathfrak{X},\mathbb{Q}}^{(0)}$-module. Choose a coherent $\widehat{\mathcal{D}}_{\mathfrak{X}}^{(0)}$-module $\mathring{\mathcal{E}}$ without p-torsion such that there exists an isomorphism of $\widehat{\mathcal{D}}_{\mathfrak{X},\mathbb{Q}}^{(0)}$-modules of the form $\mathring{\mathcal{E}}_{\mathbb{Q}} \xrightarrow{\sim} \mathcal{E}$. The characteristic variety of level 0 of $\mathcal{E}$ denoted by $\operatorname{Car}^{(0)}(\mathcal{E})$ is by definition that of $\mathring{\mathcal{E}}$ as coherent $\widehat{\mathcal{D}}_{\mathfrak{X}}^{(0)}$-module, i.e., $\operatorname{Car}^{(0)}(\mathcal{E}) := \operatorname{Car}^{(0)}(\mathring{\mathcal{E}}/\pi\mathring{\mathcal{E}})$. Berthelot checked that this is well defined. Similarly, we define the characteristic cycle $Z\operatorname{Car}^{(0)}(\mathcal{E})$ of level 0 of $\mathcal{E}$ by setting $Z\operatorname{Car}^{(0)}(\mathcal{E}) := Z\operatorname{Car}^{(0)}(\mathring{\mathcal{E}}/\pi\mathring{\mathcal{E}})$.

(4) Let $(\mathcal{N},\phi)$ be a coherent F-$\mathcal{D}^{\dagger}_{\mathfrak{X},\mathbb{Q}}$-module, i.e., a coherent $\mathcal{D}^{\dagger}_{\mathfrak{X},\mathbb{Q}}$-module $\mathcal{N}$ and an isomorphism of $\mathcal{D}^{\dagger}_{\mathfrak{X},\mathbb{Q}}$-modules ϕ of the form $\phi\colon F^*\mathcal{N} \xrightarrow{\sim} \mathcal{N}$. Then there exists a (unique up to isomorphism) coherent $\widehat{\mathcal{D}}_{\mathfrak{X},\mathbb{Q}}^{(0)}$-module $\mathcal{N}^{(0)}$ and an isomorphism $\phi^{(0)}\colon \widehat{\mathcal{D}}_{\mathfrak{X},\mathbb{Q}}^{(s)} \otimes_{\widehat{\mathcal{D}}_{\mathfrak{X},\mathbb{Q}}^{(0)}} \mathcal{N}^{(0)} \xrightarrow{\sim} F^*\mathcal{N}^{(0)}$ which induces canonically ϕ. Then, the characteristic variety of $\mathcal{N}$ denoted by $\operatorname{Car}(\mathcal{N})$ is by definition the characteristic variety of level 0 of $\mathcal{N}^{(0)}$, i.e., $\operatorname{Car}(\mathcal{N}) := \operatorname{Car}^{(0)}(\mathcal{N}^{(0)})$. Finally, the characteristic cycle of $\mathcal{N}$ denoted by $Z\operatorname{Car}(\mathcal{N})$ is by definition the characteristic variety of level 0 of $\mathcal{N}^{(0)}$.

(5) Let $\overline{\mathcal{E}} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X^{(0)})$ and let $(\mathcal{E},\phi) \in F\text{-}D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{X},\mathbb{Q}})$. By definition, we define the characteristic variety of these complexes by setting $\operatorname{Car}^{(0)}(\overline{\mathcal{E}}) := \bigcup_r \operatorname{Car}^{(0)}(\mathcal{H}^r(\overline{\mathcal{E}}))$ and $\operatorname{Car}(\mathcal{E}) := \bigcup_r \operatorname{Car}^{(0)}(\mathcal{H}^r(\mathcal{E}))$.

Lemma 1.2.2. *Let $u\colon V \to W$ be a morphism of k-varieties.*

(1) *If u is flat then for any $\mathcal{O}_W$-module $\mathcal{N}$ we have* $\operatorname{Supp}(u^*\mathcal{N}) = u^{-1}(\operatorname{Supp}\mathcal{N})$.

(2) *If u is finite, then for any coherent $\mathcal{O}_V$-module $\mathcal{M}$ we have* $\operatorname{Supp}(u_*(\mathcal{M})) = u(\operatorname{Supp}(\mathcal{M}))$.

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Proof. Since a flat morphism of local rings is faithfully flat, we get the first assertion. Let $\mathcal{M}$ be a coherent $\mathcal{O}_V$-module. From [23, 5.2.2], $\operatorname{Supp}\mathcal{M}$ is a closed subset of V. Since u is closed, by using [23, 3.4.6], we get the inclusion $\operatorname{Supp}(u_*(\mathcal{M})) \subset u(\operatorname{Supp}(\mathcal{M}))$. Set $W' := W \setminus \operatorname{Supp}(u_*\mathcal{M})$, $V' := u^{-1}(W')$ and $u'\colon V' \to W'$ the morphism induced by u. Since u is finite, then $u_*\mathcal{M}$ is a coherent $\mathcal{O}_W$-module. Hence W' is an open subset of W. Since $u'_*(\mathcal{M}|V') = u_*(\mathcal{M})|W' = 0$, since u' is affine and $\mathcal{M}|V'$ is a quasi-coherent $\mathcal{O}_{V'}$-module, this yields $\mathcal{M}|V' = 0$. Hence, $\operatorname{Supp}\mathcal{M} \subset V \setminus V' = u^{-1}(\operatorname{Supp}(u_*\mathcal{M}))$, which is equivalent to the inclusion $u(\operatorname{Supp}(\mathcal{M})) \subset \operatorname{Supp}(u_*(\mathcal{M}))$. $\square$

Proposition 1.2.3. *Let $f\colon X \to Y$ be an étale morphism of integral smooth k-varieties. Let $\overline{\mathcal{E}} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{(0)}_X)$, $\overline{\mathcal{F}} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{(0)}_Y)$.*

(1) *We have the equality $|\mathrm{Car}^{(0)}(f^{!(0)}(\overline{\mathcal{F}}))| = \rho_f(\varpi_f^{-1}(|\mathrm{Car}^{(0)}(\overline{\mathcal{F}})|))$.*

(2) *If f is moreover finite, then $|\mathrm{Car}^{(0)}(f_{+^{(0)}}(\overline{\mathcal{E}}))| = \varpi_f \circ \rho_f^{-1}(|\mathrm{Car}^{(0)}(\overline{\mathcal{E}})|) =: \mathscr{T}_f(|\mathrm{Car}^{(0)}(\overline{\mathcal{E}})|)$.*

(3) *If f is moreover finite and surjective of degree d and if $\overline{\mathcal{F}}$ is a coherent $\mathcal{D}^{(0)}_Y$-module, then*

$$Z\mathrm{Car}^{(0)}(f_{+^{(0)}} f^{!(0)}(\overline{\mathcal{F}})) = d\, Z\mathrm{Car}^{(0)}(\overline{\mathcal{F}}). \tag{1.2.3.1}$$

Proof. Since f is etale, ρ_f is an isomorphism (this is equivalent to say that the canonical morphism $\operatorname{gr}\mathcal{D}^{(0)}_X \to f^*\operatorname{gr}\mathcal{D}^{(0)}_Y$ is an isomorphism). We get the etale morphism of k-varieties $\mathscr{T}_f := \varpi_f \circ \rho_f^{-1}\colon T^*X \to T^*Y$ which is included in the Cartesian square:

$$\begin{tikzcd} T^*X \arrow[r, "\mathscr{T}_f"] \arrow[d, "\pi_X"'] \arrow[dr, phantom, "\square"] & T^*Y \arrow[d, "\pi_Y"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

To check the first assertion we can suppose that $\overline{\mathcal{F}}$ is a coherent $\mathcal{D}^{(0)}_Y$-module. Let $(\overline{\mathcal{F}}_n)$ be a good filtration of $\overline{\mathcal{F}}$. Then $(f^*\overline{\mathcal{F}}_n)$ be a good filtration of $f^{!(0)}(\overline{\mathcal{F}})$ (which is equal as $\mathcal{O}_X$-module to $f^*\overline{\mathcal{F}}$). With this filtration, we check $\widetilde{gr}(f^{!(0)}(\overline{\mathcal{F}})) \xrightarrow{\sim} \mathscr{T}_f^*(\widetilde{gr}(\overline{\mathcal{F}}))$ (remark that $\pi_{T^*X,*} \circ \mathscr{T}_f^* \circ \sim$ is isomorphic to $\operatorname{gr}\mathcal{D}^{(0)}_X \otimes_{f^{-1}\operatorname{gr}\mathcal{D}^{(0)}_Y} f^{-1}(-)$ as functor from the category of quasi-coherent $\operatorname{gr}\mathcal{D}^{(0)}_Y$-modules to that of quasi-coherent $\operatorname{gr}\mathcal{D}^{(0)}_X$-modules). From Lemma 1.2.2(1), since $\mathscr{T}_f$ is flat we get $\operatorname{Supp}\mathscr{T}_f^*(\widetilde{gr}(\overline{\mathcal{F}})) = \mathscr{T}_f^{-1}(\operatorname{Supp}(\widetilde{gr}(\overline{\mathcal{F}})))$. Since $\mathscr{T}_f^{-1}(\operatorname{Supp}(\widetilde{gr}(\overline{\mathcal{F}}))) = \rho_f(\varpi_f^{-1}(\operatorname{Supp}(\widetilde{gr}(\overline{\mathcal{F}}))))$, we obtain the first equality of the proposition.

Suppose now that f is finite and etale. To check the second assertion we can suppose that $\overline{\mathcal{E}}$ is a coherent $\mathcal{D}^{(0)}_X$-module. Let $(\overline{\mathcal{E}}_n)$ be a good filtration of $\overline{\mathcal{E}}$. Then $(f_*\overline{\mathcal{E}}_n)$ be a good filtration of $f_{+^{(0)}}(\overline{\mathcal{E}})$ (which is isomorphic to $f_*(\overline{\mathcal{E}})$ as $\mathcal{O}_Y$-module. With this filtration, we check $\widetilde{gr}(f_{+^{(0)}}(\overline{\mathcal{E}})) \xrightarrow{\sim} \mathscr{T}_{f*}(\widetilde{gr}(\overline{\mathcal{E}}))$. From Lemma 1.2.2(2), since $\mathscr{T}_f$ is finite, we get $\operatorname{Supp}(\mathscr{T}_{f*}(\widetilde{gr}(\overline{\mathcal{E}}))) = \mathscr{T}_f(\operatorname{Supp}(\widetilde{gr}(\overline{\mathcal{E}})))$. Hence, we get the second equality of the proposition.

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Suppose now that f is finite and etale and surjective of degree d. From the first and the second equality of the proposition, we get $|\mathrm{Car}^{(0)}(f_{+^{(0)}}f^{!^{(0)}}(\overline{\mathcal{F}}))| = \mathscr{T}_f(\mathscr{T}_f^{-1}(|\mathrm{Car}^{(0)}(\overline{\mathcal{F}})|))$. Since f is surjective, then so is $\mathscr{T}_f$. Hence $|\mathrm{Car}^{(0)}(f_{+^{(0)}}f^{!^{(0)}}(\overline{\mathcal{F}}))| = |\mathrm{Car}^{(0)}(\overline{\mathcal{F}})|$. By unicity of the structure of reduced subscheme of T^*Y attached to a closed subspace (of T^*Y), we have in fact the equality $\mathrm{Car}^{(0)}(f_{+^{(0)}}f^{!^{(0)}}(\overline{\mathcal{F}})) = \mathrm{Car}^{(0)}(\overline{\mathcal{F}})$ as subvariety. It remains to compute the multiplicity (see Berthelot's definition of characteristic cycles of [10, 5.4.1]). Let $(\overline{\mathcal{F}}_n)$ be a good filtration of $\overline{\mathcal{F}}$. From what we have already checked above in the proof, we have the good filtration $(f_*f^*\overline{\mathcal{F}}_n)$ of $f_{+^{(0)}}f^{!^{(0)}}(\overline{\mathcal{F}})$ and with this filtration, $\widetilde{gr}(f_{+^{(0)}}f^{!^{(0)}}(\overline{\mathcal{F}})) \xrightarrow{\sim} \mathscr{T}_{f*}\circ\mathscr{T}_f^*(\widetilde{gr}(\overline{\mathcal{F}}))$. Then, we obtain the desired computation using Lemma [22, A.1.3] and the following fact : if $\phi\colon A\to B$ is an étale morphism, $\mathfrak{Q}$ a prime ideal of B, $\mathfrak{P} := \phi^{-1}(\mathfrak{Q})$, if M is an $A_{\mathfrak{P}}$-module then $B_{\mathfrak{Q}}\otimes_{A_{\mathfrak{P}}} M$ has the same length as $B_{\mathfrak{Q}}$-module than M as $A_{\mathfrak{P}}$-module. □

Every results of [35, 2] concerning the extraordinary inverse and direct images of complexes of filtered arithmetic $\mathcal{D}$-modules are still valid for arithmetic $\mathcal{D}$-modules at level 0 without new arguments for the check. For the reader convenience, via the following two propositions we translate in the context of arithmetic $\mathcal{D}$-modules of level 0 the corollaries [35, 2.5.1 and 2.5.2] that we need below to check Proposition 1.2.7 which will be an ingredient of the proof of Theorem 1.4.2.

Proposition 1.2.4 (Laumon). *Let $f\colon X\to Y$ be a morphism of smooth k-varieties. For any $\overline{\mathcal{F}}\in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_Y^{(0)})$ such that the restriction $\rho_f|\varpi_f^{-1}(|\mathrm{Car}^{(0)}(\overline{\mathcal{F}})|)$ is proper, we have $f^{!^{(0)}}(\overline{\mathcal{F}})\in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X^{(0)})$ and $|\mathrm{Car}^{(0)}(f^{!^{(0)}}\overline{\mathcal{F}})|\subset\rho_f(\varpi_f^{-1}(|\mathrm{Car}^{(0)}(\overline{\mathcal{F}})|))$.*

Proposition 1.2.5 (Laumon). *Let $f\colon X\to Y$ be a morphism of smooth k-varieties. For any $\overline{\mathcal{E}}\in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X^{(0)})$ such that the restriction $\varpi_f|\rho_f^{-1}(|\mathrm{Car}^{(0)}(\overline{\mathcal{E}})|)$ is proper, we have $f_{+^{(0)}}(\overline{\mathcal{E}})\in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X^{(0)})$ and $|\mathrm{Car}^{(0)}(f_{+^{(0)}}\overline{\mathcal{E}})|\subset\varpi_f(\rho_f^{-1}(|\mathrm{Car}^{(0)}(\overline{\mathcal{E}})|)) =: \mathscr{T}_f(|\mathrm{Car}^{(0)}(\overline{\mathcal{E}})|)$.*

1.2.6. Let $f\colon\mathcal{P}'\to\mathcal{P}$ be a proper morphism of smooth formal $\mathcal{V}$-schemes. Let $(\mathcal{E}',\phi)$ be a coherent F-$\mathcal{D}^\dagger_{\mathcal{P}',\mathbb{Q}}$-module. From the equivalence of categories of [9, 4.5.4], there exist (unique up to isomorphism) a coherent $\widehat{\mathcal{D}}^{(0)}_{\mathcal{P}',\mathbb{Q}}$-module $\mathcal{F}'^{(0)}$ and an isomorphism $\phi^{(0)}\colon\widehat{\mathcal{D}}^{(s)}_{\mathcal{P}',\mathbb{Q}}\otimes_{\widehat{\mathcal{D}}^{(0)}_{\mathcal{P}',\mathbb{Q}}}\mathcal{F}'^{(0)}\xrightarrow{\sim}F^*\mathcal{F}'^{(0)}$ which induced $(\mathcal{E}',\phi)$ by extension. Fix an integer $i\in\mathbb{Z}$. From the isomorphisms

$$\widehat{\mathcal{D}}^{(s)}_{\mathcal{P},\mathbb{Q}}\otimes_{\widehat{\mathcal{D}}^{(0)}_{\mathcal{P},\mathbb{Q}}}\mathcal{H}^i f_{+^{(0)}}(\mathcal{F}'^{(0)}) \underset{[10,3.5.3.1]}{\xrightarrow{\sim}} \mathcal{H}^i f_{+^{(s)}}\big(\widehat{\mathcal{D}}^{(s)}_{\mathcal{P}',\mathbb{Q}}\otimes_{\widehat{\mathcal{D}}^{(0)}_{\mathcal{P}',\mathbb{Q}}}\mathcal{F}'^{(0)}\big) \tag{1.2.6.1}$$

$$\underset{\phi^{(0)}}{\xrightarrow{\sim}} \mathcal{H}^i f_{+^{(s)}}(F^*\mathcal{F}'^{(0)}) \underset{[10,3.5.4.1]}{\xrightarrow{\sim}} F^*\mathcal{H}^i f_{+^{(0)}}(\mathcal{F}'^{(0)}), \tag{1.2.6.2}$$

we get $\mathrm{Car}(\mathcal{H}^i f_+(\mathcal{E}')) = \mathrm{Car}^{(0)}(\mathcal{H}^i f_{+^{(0)}}(\mathcal{F}'^{(0)}))$. Choose a coherent $\widehat{\mathcal{D}}^{(0)}_{\mathcal{P}'}$-module without p-torsion $\mathcal{E}'^{(0)}$ such that $\mathcal{E}'^{(0)}_{\mathbb{Q}}\xrightarrow{\sim}\mathcal{F}'^{(0)}$. Since $\mathcal{H}^i f_{+^{(0)}}(\mathcal{F}'^{(0)})\xrightarrow{\sim}(\mathcal{H}^i f_{+^{(0)}}(\mathcal{E}'^{(0)}))_{\mathbb{Q}}$, then by putting $\mathcal{G}^{(0)}$ as equal to the quotient of $\mathcal{H}^i f_{+^{(0)}}(\mathcal{E}'^{(0)})$ by its p-torsion part, $\mathrm{Car}^{(0)}(\mathcal{H}^i f_{+^{(0)}}(\mathcal{F}'^{(0)})) := \mathrm{Car}^{(0)}(\mathcal{G}^{(0)})\subset\mathrm{Car}^{(0)}(\mathcal{H}^i f_{+^{(0)}}(\mathcal{E}'^{(0)}))$ (for the equality, see the

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definition [10, 5.2.5]). Hence,

$$|\mathrm{Car}(\mathcal{H}^i f_+(\mathcal{E}'))| \subset |\mathrm{Car}^{(0)}(\mathcal{H}^i f_{+^{(0)}}(\mathcal{E}'^{(0)}))| := \left|\mathrm{Car}^{(0)}\left(k \otimes_{\mathcal{V}} \mathcal{H}^i f_{+^{(0)}}(\mathcal{E}'^{(0)})\right)\right|.$$

By using a spectral sequence (the result is given in the beginning of the proof of [40, I.5.8]), we obtain the monomorphism $k \otimes_{\mathcal{V}} \mathcal{H}^i f_{+^{(0)}}(\mathcal{E}'^{(0)}) \hookrightarrow \mathcal{H}^i(k \otimes^{\mathbb{L}}_{\mathcal{V}} f_{+^{(0)}}(\mathcal{E}'^{(0)}))$. Hence, $|\mathrm{Car}^{(0)}(k \otimes_{\mathcal{V}} \mathcal{H}^i f_{+^{(0)}}(\mathcal{E}'^{(0)}))| \subset |\mathrm{Car}^{(0)}(\mathcal{H}^i(k \otimes^{\mathbb{L}}_{\mathcal{V}} f_{+^{(0)}}(\mathcal{E}'^{(0)})))|$. We have $k \otimes^{\mathbb{L}}_{\mathcal{V}} f_{+^{(0)}} (\mathcal{E}'^{(0)}) \xrightarrow{\sim} f_{+^{(0)}}(\overline{\mathcal{E}}'^{(0)})$, where $\overline{\mathcal{E}}'^{(0)} := k \otimes_{\mathcal{V}} \mathcal{E}'^{(0)} \xrightarrow{\sim} k \otimes^{\mathbb{L}}_{\mathcal{V}} \mathcal{E}'^{(0)}$. Finally we get

$$|\mathrm{Car}(\mathcal{H}^i f_+(\mathcal{E}'))| \subset |\mathrm{Car}^{(0)}(\mathcal{H}^i f_{+^{(0)}}(\overline{\mathcal{E}}'^{(0)}))|. \tag{1.2.6.3}$$

From Proposition 1.2.5, since f is proper then $|\mathrm{Car}^{(0)}(\mathcal{H}^i f_{+^{(0)}}(\overline{\mathcal{E}}'^{(0)}))| \subset \mathscr{T}_f(|\mathrm{Car}^{(0)}(\overline{\mathcal{E}}'^{(0)})|)$. By Berthelot's definition of the characteristic variety of $\mathcal{E}'$, we have $\mathrm{Car}(\mathcal{E}') = \mathrm{Car}^{(0)}(\overline{\mathcal{E}}'^{(0)})$. Hence, by using the spectral sequence $E_2^{r,s} = \mathcal{H}^r f_+(\mathcal{H}^s(\mathcal{E})) \Rightarrow \mathcal{H}^n f_+(\mathcal{E})$ and the beginning of the remark [11, 3.7] we check the following proposition.

Proposition 1.2.7. *Let $f : \mathcal{P}' \to \mathcal{P}$ be a proper morphism of smooth formal $\mathcal{V}$-schemes. Let $(\mathcal{E}', \phi) \in F\text{-}D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathcal{P}',\mathbb{Q}})$. We have the inclusion*

$$|\mathrm{Car}(f_+(\mathcal{E}'))| \subset \mathscr{T}_f(|\mathrm{Car}(\mathcal{E}')|).$$

1.2.8. Let $f : \mathcal{P}' \to \mathcal{P}$ be a finite étale surjective morphism of smooth formal $\mathcal{V}$-schemes.

(1) Let $(\mathcal{E}', \phi)$ be a coherent $F\text{-}\mathcal{D}^{\dagger}_{\mathcal{P}',\mathbb{Q}}$-module. With the notation 1.2.6, since $f_{+^{(0)}} = f_*$ (and then preserves the property of p-torsion freeness) and $f_+ = f_*$, the inclusion (1.2.6.3) is an equality. In fact, with the usual notation of characteristic cycles (see [10, 5.4]), we get the equality

$$\mathrm{ZCar}(f_+(\mathcal{E}')) = \mathrm{ZCar}^{(0)}(f_{+^{(0)}}(\overline{\mathcal{E}}'^{(0)})). \tag{1.2.8.1}$$

(2) Moreover, let $(\mathcal{E}, \phi)$ be a coherent $F\text{-}\mathcal{D}^{\dagger}_{\mathcal{P},\mathbb{Q}}$-module. From the equivalence of categories of [9, 4.5.4], there exist (unique up to isomorphism) a coherent $\widehat{\mathcal{D}}^{(0)}_{\mathcal{P},\mathbb{Q}}$-module $\mathcal{F}^{(0)}$ and an isomorphism $\phi^{(0)} : \widehat{\mathcal{D}}^{(s)}_{\mathcal{P},\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(0)}_{\mathcal{P},\mathbb{Q}}} \mathcal{F}^{(0)} \xrightarrow{\sim} F^*\mathcal{F}^{(0)}$ which induces $(\mathcal{E}, \phi)$ by extension. Choose a coherent $\widehat{\mathcal{D}}^{(0)}_{\mathcal{P}}$-module without p-torsion $\mathcal{E}^{(0)}$ such that $\mathcal{F}^{(0)} \xrightarrow{\sim} \mathcal{E}^{(0)}_{\mathbb{Q}}$. Since $\mathcal{D}^{\dagger}_{\mathcal{P}',\mathbb{Q}} = f^*\mathcal{D}^{\dagger}_{\mathcal{P},\mathbb{Q}} = f^!\mathcal{D}^{\dagger}_{\mathcal{P},\mathbb{Q}}$ and $\widehat{\mathcal{D}}^{(0)}_{\mathcal{P}'} = f^*\widehat{\mathcal{D}}^{(0)}_{\mathcal{P}} = f^{!(0)}\widehat{\mathcal{D}}^{(0)}_{\mathcal{P}}$, we check that $f^{!(0)}\mathcal{E}^{(0)} = f^*\mathcal{E}^{(0)}$ has no p-torsion and that $f^!(\mathcal{E}) \xrightarrow{\sim} \mathcal{D}^{\dagger}_{\mathcal{P}',\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(0)}_{\mathcal{P}'}} f^{!(0)}\mathcal{E}^{(0)}$. Moreover, putting $\overline{\mathcal{E}}^{(0)} := k \otimes_{\mathcal{V}} \mathcal{E}^{(0)}$, we get

$$\mathrm{ZCar}(f^!(\mathcal{E})) = \mathrm{ZCar}(f^{!(0)}\overline{\mathcal{E}}^{(0)}). \tag{1.2.8.2}$$

Proposition 1.2.9. *Let $f : \mathcal{P}' \to \mathcal{P}$ be a finite étale surjective morphism of degree d of integral smooth formal $\mathcal{V}$-schemes. Let $(\mathcal{E}, \phi)$ be a coherent $F\text{-}\mathcal{D}^{\dagger}_{\mathcal{P},\mathbb{Q}}$-module. Then we get*

$$\mathrm{ZCar}(f_+f^!\mathcal{E}) = d\,\mathrm{ZCar}(\mathcal{E}); \tag{1.2.9.1}$$

$$\chi(\mathcal{P}, f_+f^!(\mathcal{E})) = d \cdot \chi(\mathcal{P}, \mathcal{E}). \tag{1.2.9.2}$$

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Proof. From the equality of characteristic cycles (1.2.3.1) and both equalities of the paragraph 1.2.8, we get $Z\mathrm{Car}(f_+ f^!\mathcal{E}) = dZ\mathrm{Car}(\mathcal{E})$. Moreover, the equality (1.2.9.2) is a consequence of (1.2.9.1) and of Berthelot's index theorem [10, 5.4.4]. $\square$

We need at the end of the proof of Theorem 1.4.2 the following lemma.

Lemma 1.2.10. *Let $\mathfrak{X}$ be a smooth $\mathcal{V}$-formal scheme, X be the reduction of $\mathfrak{X}$ modulo π.*

(1) *Let $\mathcal{G}$ be a coherent $\mathcal{D}_X^{(0)}$-module. Choose a good filtration $(\mathcal{G}_n)_{n\in\mathbb{N}}$ of $\mathcal{G}$. Then the following assertions are equivalent*
   (a) $\mathrm{Car}^{(0)}(\mathcal{G}) \subset T^*_X X$.
   (b) $\mathrm{gr}\mathcal{G}$ *is $\mathcal{O}_X$-coherent (for the $\mathcal{O}_X$-module structure induced by $\mathcal{O}_X \hookrightarrow \mathrm{gr}\mathcal{D}_X^{(0)}$).*
   (c) $\mathcal{G}$ *is $\mathcal{O}_X$-coherent (for the $\mathcal{O}_X$-module structure induced by $\mathcal{O}_X \hookrightarrow \mathcal{D}_X^{(0)}$).*

(2) *Let $(\mathcal{E}, \phi)$ be a coherent F-$\mathcal{D}^\dagger_{\mathfrak{X},\mathbb{Q}}$-module. The following assertions are equivalent.*
   (a) $\mathrm{Car}(\mathcal{E}) \subset T^*_X X$.
   (b) $\mathcal{E}$ *is $\mathcal{O}_{\mathfrak{X},\mathbb{Q}}$-coherent (for the $\mathcal{O}_{\mathfrak{X},\mathbb{Q}}$-module structure induced by $\mathcal{O}_{\mathfrak{X},\mathbb{Q}} \hookrightarrow \mathcal{D}^\dagger_{\mathfrak{X},\mathbb{Q}}$).*

Proof. Let us check the first part. Let us check that (a) implies (b). Since this is local, we can suppose that X affine with local coordinates $t_1, \ldots, t_d$. Let ξ_i be the global section of $\mathrm{gr}\mathcal{D}_X^{(0)}$ which is the element associated with ∂_i, the derivation with respect to t_i. Since the ideal defining the closed immersion $\mathrm{Car}^{(0)}(\mathcal{G}) \hookrightarrow T^*X$ is the radical of the annihilator of $\mathrm{gr}\mathcal{G}$, the inclusion $\mathrm{Car}^{(0)}(\mathcal{G}) \subset T^*_X X$ implies that $\xi_1^N, \ldots, \xi_d^N$ annihilate $\mathrm{gr}\mathcal{G}$ for some integer N large enough. Hence, $\mathrm{gr}\mathcal{G}$ is a coherent $\mathrm{gr}\mathcal{D}_X^{(0)}/(\xi_1, \ldots, \xi_d)^{Nd}$-module. Since $\mathrm{gr}\mathcal{D}_X^{(0)}/(\xi_1, \ldots, \xi_d)^{Nd}$ is a finite $\mathcal{O}_X$-algebra (via the composition $\mathcal{O}_X \to \mathrm{gr}\mathcal{D}_X^{(0)} \to \mathrm{gr}\mathcal{D}_X^{(0)}/(\xi_1, \ldots, \xi_d)^{Nd}$), we conclude that $\mathrm{gr}\mathcal{G}$ is $\mathcal{O}_X$-coherent. Now, suppose (b) satisfied. Then, by definition of a good filtration, this implies that $\mathcal{G}_n = \mathcal{G}$ for n large enough. Hence, $\mathcal{G}$ is $\mathcal{O}_X$-module. Finally, suppose (c). Then, the constant filtration $(\mathcal{G}_n = \mathcal{G})_{n\in\mathbb{N}}$ is a good filtration (it might be more convenient to complete the filtration by $\mathcal{G}_n = 0$ if $n < 0$). Then the action of ξ_i on $\mathrm{gr}\mathcal{G} = \mathcal{G}_0/\mathcal{G}_{-1} = \mathcal{G}$ is zero (because the action of ξ_i is induced by maps of the form $\mathcal{G}_i/\mathcal{G}_{i-1} \to \mathcal{G}_{i+1}/\mathcal{G}_i$, which are zero). Hence, $\mathrm{Car}^{(0)}(\mathcal{G}) \subset T^*_X X$ (recall that the construction of $\mathrm{Car}^{(0)}(\mathcal{G})$ does not depend on the choice of the good filtration).

Now, we deduce the second part from the first one. Let $\mathcal{E}^{(0)}$ be the coherent $\widehat{\mathcal{D}}^{(0)}_{\mathfrak{X},\mathbb{Q}}$-module endowed with an isomorphism $\phi^{(0)} \colon \widehat{\mathcal{D}}^{(s)}_{\mathfrak{X},\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(0)}_{\mathfrak{X},\mathbb{Q}}} \mathcal{E}^{(0)} \xrightarrow{\sim} F^*\mathcal{E}^{(0)}$ which induces canonically ϕ. Choose a coherent $\widehat{\mathcal{D}}^{(0)}_{\mathfrak{X}}$-module $\mathring{\mathcal{E}}^{(0)}$without p-torsion endowed with the isomorphism of the form $\mathring{\mathcal{E}}^{(0)}_{\mathbb{Q}} \xrightarrow{\sim} \mathcal{E}^{(0)}$. If $\mathcal{E}$ is $\mathcal{O}_{\mathfrak{X},\mathbb{Q}}$-coherent, then $\mathcal{E}^{(0)}$ is $\mathcal{O}_{\mathfrak{X},\mathbb{Q}}$-coherent (see [13, 2.2.14]) Then, from [7, 3.1.3], we can choose $\mathring{\mathcal{E}}^{(0)}$ so that it is $\mathcal{O}_{\mathfrak{X}}$-coherent. Since $\mathrm{Car}(\mathring{\mathcal{E}}^{(0)}/\pi\mathring{\mathcal{E}}^{(0)}) = \mathrm{Car}(\mathcal{E})$ and $\mathring{\mathcal{E}}^{(0)}/\pi\mathring{\mathcal{E}}^{(0)}$ is $\mathcal{O}_X$-coherent, this yields from the first part the inclusion $\mathrm{Car}(\mathcal{E}) \subset T^*_X X$. Conversely, suppose $\mathrm{Car}(\mathcal{E}) \subset T^*_X X$. Then, from the first part $\mathring{\mathcal{E}}^{(0)}/\pi\mathring{\mathcal{E}}^{(0)}$ is $\mathcal{O}_X$-coherent. Hence, $\mathring{\mathcal{E}}^{(0)}$ is $\mathcal{O}_{\mathfrak{X}}$-coherent, which yields that $\mathcal{E}^{(0)}$ is $\mathcal{O}_{\mathfrak{X},\mathbb{Q}}$-coherent. With [13, 2.2.14], this implies that $\mathcal{E}$ is $\mathcal{O}_{\mathfrak{X},\mathbb{Q}}$-coherent. $\square$

1.3. Generic smoothness up to Frobenius descent

We prove below in this section Proposition 1.3.9 which states that, up to some Frobenius descent (see Lemma 1.3.10), a morphism is generically smooth. This will be useful later in the proof of Theorem 1.4.2.

1.3.1 (Universal homeomorphism). Let $f: X \to Y$ be a morphism of schemes.

(1) Following Definitions [23, 3.5.4] (and Remark [23, 3.5.11]) or [26, 2.4.2], f is by definition a universal homeomorphism (respectively is universally injective) if for any morphism of schemes $g: Y' \to Y$, the morphism $f_{Y'}: X \times_Y Y' \to Y'$ is a homeomorphism (respectively is injective).

(2) Some authors use the name of 'purely inseparable' (e.g., [36, 5.3.13]) or 'radicial' (e.g., [23, 3.5.4]) instead of 'universally injective'. From definition [23, 3.5.4], proposition [23, 3.5.8] and remark [23, 3.5.11], the following conditions are equivalent:

  (a) f is universally injective;

  (b) for any field K, the map $X(K) \to Y(K)$ is injective;

  (c) f is injective and for any point x of X the monomorphism of the residue fields $k(f(x)) \to k(x)$ induced by f is purely inseparable (some authors say 'radicial' instead of 'purely inseparable').

(3) Suppose now that $f: X \to Y$ is a morphism of k-varieties. Using proposition [26, 2.4.5], we check that f is a universal homeomorphism if and only if f is finite, surjective and radicial.

Lemma 1.3.2. *Let X be a k-variety. Then the relative Frobenius $F^s_{X/k}: X \to X^\sigma$, the morphism $F^s_k: X^\sigma \to X$ (induced from F^s_k by base change) and the absolute Frobenius morphism $F^s_{X/k}: X \to X$ (equal to $F^s_X = F^s_k \circ F^s_{X/k}$) are universal homeomorphisms.*

Proof. From the characterization 1.3.1(3), $F^s_k: \operatorname{Spec} k \to \operatorname{Spec} k$ is a universal homeomorphism. Hence, by stability of this property by base change we get that $F^s_k: X^\sigma \to X$ is a universal homeomorphism. From Lemma [36, 3.2.25], we check that $F^s_{X/k}$ is finite. Hence, so is by composition F^s_X. Since F^s_X induces the identity on the underlying topological space, F^s_X is bijective. Moreover, the monomorphism of the residue fields $k(x) \to k(x)$ induced by F^s_X is the sth power of the Frobenius, hence it is radicial. From 1.3.1(2(c)), this yields that F^s_X is radicial. From 1.3.1(2(b)), this implies that $F^s_{X/k}$ is also radicial. With the characterization 1.3.1(3), we get that $F^s_{X/k}$ and F^s_X are universal homeomorphisms. □

Definition 1.3.3. Let us clarify some terminology. Let $f: X \to Y$ be a smooth morphism of schemes. Let Z be a closed subscheme of X. We say that Z is 'a strict normal crossing divisor relatively to Y' (via f) if for any point $x \in Z$, there exists an open affine set V of Y containing $y := f(x)$, there exists an open affine set U of X containing x and included in $f^{-1}(V)$, there exists an etale V-morphism of the form $U \to \mathbb{A}^n_V$ given by global sections $t_1, \ldots, t_n$ such that $Z \cap U = V(t_1 \cdots t_r)$ for some integer r.

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Remark. When Y is of the form $\operatorname{Spec} K$, with K a perfect field, a 'strict normal crossing divisor of X relatively to Y' is the same than 'a strict normal crossing divisor of X' (the latter is an absolute notion only depending on X), which might justify the terminology.

The following lemma is straightforward.

Lemma 1.3.4. *Let $f\colon X \to Y$ be a smooth morphism of schemes. Let Z be a closed subscheme of X. Let $g\colon Y' \to Y$ be morphism of schemes, $X' := X \times_Y Y'$, $Z' := Z \times_Y Y'$. If Z is a strict normal crossing divisor of X relatively to Y then Z' is a strict normal crossing divisor of X' relatively to Y'.*

Lemma 1.3.5. *Let $f\colon X \to Y$ be a smooth morphism of smooth k-varieties with Y integral. Let Z be a closed subvariety of X. Let η be the generic point of Y, $k(\eta)$ be the function field of Y, $X_\eta := X \times_Y \operatorname{Spec} k(\eta)$, $Z_\eta := Z \times_Y \operatorname{Spec} k(\eta)$. If Z_η is a strict normal crossing divisor of X_η relatively to $\operatorname{Spec} k(\eta)$ then there exists a dense open set V of Y such that, setting $X_V := f^{-1}(V)$ and $Z_V := Z \cap X_V$, the closed subvariety Z_V of X_V is a strict normal crossing of X_V relatively to V.*

Proof. We can suppose X integral. By definition, there exists a covering $U_{1,\eta}, \dots, U_{m,\eta}$ by open affine $k(\eta)$-subvarieties of X_η such that there exists a $k(\eta)$-morphism $U_{i,\eta} \to \mathbb{A}^n_\eta$ given by global section $t_{i,1}, \dots, t_{i,n}$. Consider the projective system of open affine dense k-subvarieties of Y and remark that $\operatorname{Spec} k(\eta)$ is the projective limit of this system. For any open affine k-subvariety V of Y, put $X_V := f^{-1}(V)$ and $Z_V := Z \cap X_V$. By using [27, 8.8.2(ii)], there exists an open affine k-subvariety V of Y such that there exist a scheme U_i of finite type over V (recall that from [25, 1.6] to be of finite type or of finite presentation over a locally noetherian scheme is the same) and some $k(\eta)$-isomorphism $U_i \times_V \operatorname{Spec} k(\eta) \xrightarrow{\sim} U_{i,\eta}$ for any i. By using Theorem [27, 8.8.2(i)] and Theorem [27, 8.10.5(iii)], shrinking V if necessary, we can suppose that there exists an open immersion $U_i \hookrightarrow X_V$ which induces (via the isomorphism $U_i \times_V \operatorname{Spec} k(\eta) \xrightarrow{\sim} U_{i,\eta}$) the open immersion $U_{i,\eta} \hookrightarrow X_\eta$. By using Theorem [27, 8.10.5(vi)], we can suppose that $(U_i)_{i=1,\dots,m}$ is an open covering of X_V. By using theorem [27, 8.8.2(i)] and theorem [28, 17.7.8], shrinking V is necessary, there exists an etale V-morphism of the form $U_i \to \mathbb{A}^n_V$ which induces the etale $k(\eta)$-morphism $U_{i,\eta} \to \mathbb{A}^n_\eta$. □

Lemma 1.3.6. *Let L/l be an algebraic extension of fields of characteristic p such that L is perfect. Let X be a smooth variety over l, Z be closed subvariety of X. If $(Z \times_{\operatorname{Spec} l} \operatorname{Spec} L)_{\mathrm{red}}$ is a strict normal crossing divisor of $X \times_{\operatorname{Spec} l} \operatorname{Spec} L$ then there exists a finite extension l' of l included in L such that $(Z \times_{\operatorname{Spec} l} \operatorname{Spec} l')_{\mathrm{red}}$ is a strict normal crossing divisor of $X \times_{\operatorname{Spec} l} \operatorname{Spec} l'$ relatively to $\operatorname{Spec} l'$.*

Proof. For any finite extension l' of l included in L, set $Z_{(l')} := Z \times_{\operatorname{Spec} l} \operatorname{Spec} l'$, $X_{(l')} := X \times_{\operatorname{Spec} l} \operatorname{Spec} l'$ and $Z'_{(L)} := (Z \times_{\operatorname{Spec} l} \operatorname{Spec} L)_{\mathrm{red}}$. By using [27, 8.8.2(ii)], there exists a finite extension l' of l included in L such that there exist a l'-scheme of finite type $Z'_{(l')}$ satisfying $Z'_{(L)} \xrightarrow{\sim} Z'_{(l')} \times_{\operatorname{Spec}(l')} \operatorname{Spec}(L)$. From [27, 8.7.2] we get that $Z'_{(l')}$ is reduced.

Using [27, 8.8.2(i)] and [27, 8.10.5(iv) and (vi)], increasing l' if necessary, there exist a surjective closed immersion $Z'_{(l')} \hookrightarrow Z_{(l')}$ inducing by extension $Z'_{(L)} \hookrightarrow Z_{(L)}$. Since $Z'_{(l')}$ is reduced, we get $Z'_{(l')} := Z_{(l')\,\mathrm{red}}$. Increasing l' if necessary, proceeding as in the proof of Lemma 1.3.5, we check that $Z'_{(l')}$ is a strict normal crossing divisor of $X_{(l')}$ relatively to $\operatorname{Spec} l'$. □

Lemma 1.3.7. *Let $a: X \to P$ be a dominant morphism of smooth integral k-varieties. Let $Z \hookrightarrow X$ be a proper closed subset. Then there exist a dense open subvariety U of P, a universal homeomorphism $g: U' \to U$ of k-varieties with U' normal, a projective, generically finite and etale U'-morphism of the form $f: \widetilde{V}' \to (X \times_P U')_{\mathrm{red}}$ such that $\widetilde{V}'$ is integral and $\widetilde{V}'$ is smooth over U', $f^{-1}(Z \times_P U')_{\mathrm{red}}$ is the support of a strict normal crossing divisor in $\widetilde{V}'$ relatively to U'.*

Proof. (1) Let l be the field of fractions of P and $\bar{l}$ be an algebraic closure of l, $L := \bar{l}^{Gal(\bar{l}/l)}$ the fixed field by $Gal(\bar{l}/l)$. Following [34, V.6.11], L is perfect (in other words, since $\bar{l}$ is an algebraic closure of L, $\bar{l}/L$ is separable) and L/l is purely inseparable. We put $X_{(L)} := X \times_P \operatorname{Spec}(L)$, $Y_{(L)} := (X_{(L)})_{\mathrm{red}}$ and $Z_{(L)} := Z \times_P \operatorname{Spec}(L)$.

Using theorems [27, 8.4.1] and [27, 8.10.5(v)], we get the $X \times_P \operatorname{Spec}(l)$ is irreducible and separated (we use these Theorems in the following context: consider the projective system $(U_S)_S$ of open affine dense subvarieties of P and next consider the projective system $(a^{-1}(U_S))_S$ of open integral subvarieties of X whose projective limit is $X \times_P \operatorname{Spec}(l)$). Since $X_{(L)} \to X \times_P \operatorname{Spec}(l)$ is a universal homeomorphism, we get that $X_{(L)}$ is also irreducible and separated. Hence, $Y_{(L)}$ is an integral L-variety, with L a perfect field. From the desingularization de Jong's theorem (see [20] or [8, 4.1]), this implies that there exists a projective, generically finite and etale morphism $\phi_L: Y'_{(L)} \to Y_{(L)}$ such that $Y'_{(L)}$ is integral, smooth over $\operatorname{Spec} L$ and $Z'_{(L)} := \phi_L^{-1}(Z_{(L)})_{\mathrm{red}}$ is the support of a strict normal crossing divisor in $Y'_{(L)}$.

(a) By using [27, 8.4.2], [27, 8.7.2], [27, 8.8.2(ii)] and [27, 8.10.5(v)], there exists a finite (radicial) extension l' of l included in L such that there exist two integral l'-varieties $Y_{(l')}$ and $Y'_{(l')}$ satisfying $Y_{(L)} \xrightarrow{\sim} Y_{(l')} \times_{\operatorname{Spec}(l')} \operatorname{Spec}(L)$ and $Y'_{(L)} \xrightarrow{\sim} Y'_{(l')} \times_{\operatorname{Spec}(l')} \operatorname{Spec}(L)$.

(b) We put $X_{(l')} := X \times_P \operatorname{Spec}(l')$ and $Z_{(l')} := Z \times_P \operatorname{Spec}(l')$. By increasing l' is necessary, it follows from [27, 8.8.2(i)] that there exists a morphism $\phi_{l'} := Y'_{(l')} \to Y_{(l')}$ (respectively $Y_{(l')} \to X_{(l')}$) inducing ϕ_L (respectively the surjective closed immersion $Y_{(L)} \hookrightarrow X_{(L)}$). By using [28, 17.7.8] and [27, 8.10.5] and Lemma 1.3.6, by increasing l' is necessary, we can suppose that $Y_{(l')} \to X_{(l')}$ is a surjective closed immersion (i.e., $Y_{(l')} = (X_{(l')})_{\mathrm{red}}$ since $Y_{(l')}$ is reduced), that $\phi_{l'}$ is projective, generically finite and etale morphism, that $Y'_{(l')}$ is smooth over $\operatorname{Spec} l'$ and $Z'_{(l')} := \phi_{l'}^{-1}(Z_{(l')})_{\mathrm{red}}$ is the support of a strict normal crossing divisor of $Y'_{(l')}$ relatively to $\operatorname{Spec} l'$.

(2) Let P' be the normalization of P in l' (see the definition [36, 4.1.24]). Then the canonical morphism $g: P' \to P$ is a universal homeomorphism, i.e., is finite (e.g., use [36, 4.1.27]), surjective and radicial (e.g., use the exercise [36, 5.3.9(a)]). By using [27, 8.8.2(ii)] (this time, we consider the projective system of open affine dense subvarieties of P'), there

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exists a dense open affine subvariety U' of P', two morphisms $V' \to U'$ and $\widetilde{V}' \to U'$ such that $Y_{(l')} \xrightarrow{\sim} V' \times_{U'} \mathrm{Spec}\,(l')$ and $Y'_{(l')} \xrightarrow{\sim} \widetilde{V}' \times_{U'} \mathrm{Spec}\,(l')$. Since P' is noetherian, by using propositions [27, 8.4.2] and [27, 8.7.2], we get that V' and $\widetilde{V}'$ are integral. Hence, shrinking U' is necessary, we can suppose $V' = (X \times_P U')_{\mathrm{red}}$. By shrinking U' is necessary, using [27, 8.8.2(i)], there exists a U'-morphism $f : \widetilde{V}' \to V'$ which induces $\phi_{l'}$. By shrinking U' is necessary, by using [28, 17.7.8] and [27, 8.10.5] and Lemma 1.3.5, we get the desired properties. □

Lemma 1.3.8. *Let $g : U' \to U$ be a universal homeomorphism of integral k-varieties. We suppose U normal. Then, for s large enough, there exists a unique morphism $h : U \to U'^{\sigma}$ making commutative the following diagram*

$$\begin{tikzcd} U' \arrow[r, "g"] \arrow[d, "F^s_{U'/k}"'] & U \arrow[d, "F^s_{U/k}"] \arrow[dl, "h"'] \\ U'^{\sigma} \arrow[r, "g^{\sigma}"] & U^{\sigma}. \end{tikzcd} \tag{1.3.8.1}$$

Moreover, this morphism h is a universal homeomorphism.

Proof. From 1.3.2, we know that $F^s_{U'/k}$ and $F^s_{U/k}$ are universal homeomorphisms. Hence, this is sufficient to check that there exist a unique morphism $h : U \to U'^{\sigma}$ making commutative the diagram (1.3.8.1). This is equivalent to check the existence and uniqueness of a morphism $i : U \to U'$ making commutative the diagram

$$\begin{tikzcd} U' \arrow[r, "g"] \arrow[d, "F^s_{U'}"'] & U \arrow[d, "F^s_{U}"] \arrow[dl, "i"'] \\ U' \arrow[r, "g"] & U. \end{tikzcd} \tag{1.3.8.2}$$

We can suppose U affine. We set $U = \mathrm{Spec}\, A$, $U' = \mathrm{Spec}\, A'$, $L := \mathrm{Frac} A$, $L' := \mathrm{Frac} A'$. Since g is surjective, $g^* : A \to A'$ is injective. Since A is normal and since $A \to A'$ is finite, then $A = A' \cap L$. For s large enough, we can suppose $(L')^{p^s} \subset L$. Hence the image of $F^{s*}_{U'} : A' \to A'$ is included in A. This yields the desired morphism $A' \to A$. □

Proposition 1.3.9. *Let $a : X \to P$ be a dominant morphism of smooth integral k-varieties. Let $Z \hookrightarrow X$ be a proper closed subset. Then, for s large enough, there exists a dense open subvariety U of P, such that, putting $W := (X^{\sigma} \times_{P^{\sigma}} U)_{\mathrm{red}}$ (where $X^{\sigma} \times_{P^{\sigma}} U$ means the base change of X^{σ} by the composition of $F^s_{U/k} : U \to U^{\sigma}$ with the open immersion $U^{\sigma} \subset P^{\sigma}$), there exists a projective, generically finite and etale U-morphism of the form $\phi : W' \to W$ such that W' is integral and smooth over U, $Z' := \phi^{-1}(Z^{\sigma} \times_{P^{\sigma}} U)_{\mathrm{red}}$ is the support of a strict normal crossing divisor in W' relatively to U.*

Proof. Using the Lemmas 1.3.7 (for the construction of g and f) and 1.3.8 (for the construction of h), with their notation we get the diagram of morphisms of

k-schemes

$$\begin{tikzcd}
W' \arrow[r, equal] \arrow[d, "\phi"] \arrow[dr, phantom, "\square"] & W' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \widetilde{V}'^{\sigma} \arrow[d, "f^{\sigma}"] & & \\
T_{\mathrm{red}} \arrow[r, hook] \arrow[d, equal] \arrow[dr, phantom, "\square"] & T \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\square"] & (X^{\sigma} \times_{P^{\sigma}} U'^{\sigma})_{\mathrm{red}} \arrow[d, hook] & & \\
W := (X^{\sigma} \times_{P^{\sigma}} U)_{\mathrm{red}} \arrow[r, hook] & X^{\sigma} \times_{P^{\sigma}} U \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X^{\sigma} \times_{P^{\sigma}} U'^{\sigma} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X^{\sigma} \times_{P^{\sigma}} U^{\sigma} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X^{\sigma} \arrow[d] \\
& U \arrow[r, "h"] \arrow[rr, bend right, "F^{s}_{U/k}"'] & U'^{\sigma} \arrow[r, "g^{\sigma}"] & U^{\sigma} \arrow[r] & P^{\sigma}
\end{tikzcd}$$

where $T := (X^{\sigma} \times_{P^{\sigma}} U) \times_{(X^{\sigma} \times_{P^{\sigma}} U'^{\sigma})} (X^{\sigma} \times_{P^{\sigma}} U'^{\sigma})_{\mathrm{red}}$ and $W' := \widetilde{V}'^{\sigma} \times_{U'^{\sigma}} U$. Since W' is smooth over U and U is k-smooth, then W' is k-smooth (and in particular integral since it is irreducible). Hence $W'_{\mathrm{red}} = W'$ and then $W' = W' \times_T T_{\mathrm{red}}$ (use [23, 5.1.7]), which justify the cartesianity of the left square of the top. Using [23, 5.1.7], we get the equality $T_{\mathrm{red}} = (X^{\sigma} \times_{P^{\sigma}} U)_{\mathrm{red}} =: W$ and the cartesianity of the left square of the second row. Since f^{σ} is projective, generically finite and etale then so is $\phi \colon W' \to W$. From 1.3.4, we get that $Z' := \phi^{-1}(Z^{\sigma} \times_{P^{\sigma}} U)_{\mathrm{red}}$ is the support of a strict normal crossing divisor relatively to U. $\square$

Lemma 1.3.10. *Let $f \colon \mathcal{P}' \to \mathcal{P}$ be a finite, surjective morphism of smooth formal $\mathcal{V}$-schemes. Let $\mathcal{E}$ be a coherent $\mathcal{D}^{\dagger}_{\mathcal{P},\mathbb{Q}}$-module. Then $\mathcal{E}$ is $\mathcal{O}_{\mathcal{P},\mathbb{Q}}$-coherent if and only if $f^{!}(\mathcal{E})$ is $\mathcal{O}_{\mathcal{P}',\mathbb{Q}}$-coherent.*

Proof. Since P and P' are regular, then from [36, 4.3.11], the morphism $P' \to P$ is flat. Since $\mathcal{O}_{\mathcal{P}'}$ is p-adically complete and without p-torsion, then using lemma [37, 2.1] (in the case where $I = (\pi)$), the morphism $f \colon \mathcal{P}' \to \mathcal{P}$ is also flat. Since f is also finite, then $f_{*}\mathcal{O}_{\mathcal{P}'}$ is a locally free $\mathcal{O}_{\mathcal{P}}$-module of finite type. Hence, we get that the canonical morphism $f^{*}\widehat{\mathcal{D}}^{(m)}_{\mathcal{P}} := \mathcal{O}_{\mathcal{P}'} \otimes_{f^{-1}\mathcal{O}_{\mathcal{P}}} f^{-1}\widehat{\mathcal{D}}^{(m)}_{\mathcal{P}} \to \varprojlim_i \mathcal{O}_{P'} \otimes_{f^{-1}\mathcal{O}_{P_i}} f^{-1}\mathcal{D}^{(m)}_{P_i} = \widehat{\mathcal{D}}^{(m)}_{\mathcal{P}' \to \mathcal{P}}$ is an isomorphism for any integer $m \geqslant 0$. Hence tensoring by $\mathbb{Q}$ over $\mathbb{Z}$ and passing the limits through the level, this yields that the canonical morphism $f^{*}\mathcal{D}^{\dagger}_{\mathcal{P},\mathbb{Q}} := \mathcal{O}_{\mathcal{P}'} \otimes_{f^{-1}\mathcal{O}_{\mathcal{P}}} f^{-1}\mathcal{D}^{\dagger}_{\mathcal{P},\mathbb{Q}} \to \mathcal{D}^{\dagger}_{\mathcal{P}' \to \mathcal{P},\mathbb{Q}}$ is an isomorphism. Hence, this implies that the canonical morphism $f^{*}(\mathcal{E}) \to f^{!}(\mathcal{E})$ is an isomorphism, where $f^{*}\mathcal{E} := \mathcal{O}_{\mathcal{P}'} \otimes_{f^{-1}\mathcal{O}_{\mathcal{P}}} f^{-1}\mathcal{E}$, and $f^{!}(\mathcal{E}) := \mathcal{D}^{\dagger}_{\mathcal{P}' \to \mathcal{P},\mathbb{Q}} \otimes^{\mathbb{L}}_{f^{-1}\mathcal{D}^{\dagger}_{\mathcal{P},\mathbb{Q}}} f^{-1}\mathcal{E}$. If $\mathcal{E}$ is $\mathcal{O}_{\mathcal{P},\mathbb{Q}}$-coherent this yields that $f^{!}(\mathcal{E})$ is $\mathcal{O}_{\mathcal{P}',\mathbb{Q}}$-coherent. Conversely, suppose $f^{!}(\mathcal{E})$ is $\mathcal{O}_{\mathcal{P}',\mathbb{Q}}$-coherent. Since the $\mathcal{O}_{\mathcal{P},\mathbb{Q}}$-coherence is local in $\mathcal{P}$, we can suppose $\mathcal{P}$ affine. Since $\mathcal{E}$ is a coherent $\mathcal{D}^{\dagger}_{\mathcal{P},\mathbb{Q}}$-module, this is sufficient to check that $\Gamma(\mathcal{P}, \mathcal{E})$ is of finite type over $\Gamma(\mathcal{P}, \mathcal{O}_{\mathcal{P},\mathbb{Q}})$ (see [13, 2.2.13]). Since the extension $\Gamma(\mathcal{P}, \mathcal{O}_{\mathcal{P},\mathbb{Q}}) \to \Gamma(\mathcal{P}', \mathcal{O}_{\mathcal{P}',\mathbb{Q}})$ is faithfully flat (because $f \colon \mathcal{P}' \to \mathcal{P}$ is flat and surjective), since $\Gamma(\mathcal{P}', f^{!}\mathcal{E}) \xrightarrow{\sim} \Gamma(\mathcal{P}', \mathcal{O}_{\mathcal{P}',\mathbb{Q}}) \otimes_{\Gamma(\mathcal{P}, \mathcal{O}_{\mathcal{P},\mathbb{Q}})} \Gamma(\mathcal{P}, \mathcal{E})$ is of finite type over $\Gamma(\mathcal{P}', \mathcal{O}_{\mathcal{P}',\mathbb{Q}})$, we conclude. $\square$

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1.4. The result

Lemma 1.4.1. *Let $\alpha\colon \widetilde{X} \to X$ be a finite étale surjective morphism of smooth k-varieties. Let $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{ovhol}}(X/K)$ (respectively $\widetilde{\mathcal{E}} \in D^{\mathrm{b}}_{\mathrm{ovhol}}(\widetilde{X}/K)$). The property $\widetilde{\mathcal{E}} \in D^{\mathrm{b}}_{\mathrm{isoc}}(\widetilde{X}/K)$ is equivalent to the property $\alpha_+(\widetilde{\mathcal{E}}) \in D^{\mathrm{b}}_{\mathrm{isoc}}(X/K)$. The property $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{isoc}}(X/K)$ is equivalent to the property $\alpha^+(\mathcal{E}) \in D^{\mathrm{b}}_{\mathrm{isoc}}(\widetilde{X}/K)$.*

Proof. Left to the reader. □

Theorem 1.4.2. *Let $\mathcal{P}_1$ be a smooth separated formal $\mathcal{V}$-scheme, $\mathcal{P}_2$ be a proper smooth formal $\mathcal{V}$-scheme, $\mathcal{P} := \mathcal{P}_1 \times \mathcal{P}_2$ and $\mathrm{pr}\colon \mathcal{P} \to \mathcal{P}_1$ be the projection. Let $\mathcal{E}$ be a complex of F-$D^{\mathrm{b}}_{\mathrm{ovhol}}(\mathcal{D}^{\dagger}_{\mathcal{P},\mathbb{Q}})$.*

Then there exists an open dense formal subscheme $\mathfrak{U}_1$ of $\mathcal{P}_1$ such that, for any finite étale surjective morphisms of the form $\alpha_1\colon \widetilde{\mathcal{P}}_1 \to \mathcal{P}_1$ and $\alpha_2\colon \widetilde{\mathcal{P}}_2 \to \mathcal{P}_2$, putting $\widetilde{\mathcal{P}} = \widetilde{\mathcal{P}}_1 \times \widetilde{\mathcal{P}}_2$, $\alpha\colon \widetilde{\mathcal{P}} \to \mathcal{P}$ and $\widetilde{\mathcal{E}} := \alpha^+(\mathcal{E})$, we have $(\mathrm{pr}\circ\alpha)_+(\widetilde{\mathcal{E}})|\mathfrak{U}_1 \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{O}_{\mathfrak{U}_1,\mathbb{Q}})$.

Proof. (I) We can suppose that $\alpha_1 = \mathrm{Id}$. Indeed, consider the following diagram

$$\begin{tikzcd}
\widetilde{\mathcal{P}}_1 \times \widetilde{\mathcal{P}}_2 \arrow[r, "\alpha_2"] \arrow[d, "\alpha_1"] \arrow[dr, phantom, "\square"] & \widetilde{\mathcal{P}}_1 \times \mathcal{P}_2 \arrow[r, "\mathrm{pr}"] \arrow[d, "\alpha_1"] \arrow[dr, phantom, "\square"] & \widetilde{\mathcal{P}}_1 \arrow[d, "\alpha_1"] \\
\mathcal{P}_1 \times \widetilde{\mathcal{P}}_2 \arrow[r, "\alpha_2"] & \mathcal{P}_1 \times \mathcal{P}_2 \arrow[r, "\mathrm{pr}"] & \mathcal{P}_1 .
\end{tikzcd}$$

Suppose there exists an open dense formal subscheme $\mathfrak{U}_1$ of $\mathcal{P}_1$ such that $\mathrm{pr}_+ \circ \alpha_{2+}(\alpha_2^+(\mathcal{E}))|\mathfrak{U}_1 \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{O}_{\mathfrak{U}_1,\mathbb{Q}})$. Using base change isomorphism (see for instance [3, 1.3.10]), since $\alpha_1^+ = \alpha_1^!$ (because α is finite etale), we get $\alpha_1^+ \circ (\mathrm{pr}_+ \circ \alpha_{2+}) \xrightarrow{\sim} (\mathrm{pr}_+\alpha_{2+}) \circ \alpha_1^+$. Hence, we get the first isomorphism:

$$\alpha_{1+}\alpha_1^+\mathrm{pr}_+ \circ \alpha_{2+}(\alpha_2^+(\mathcal{E})) \xrightarrow{\sim} \alpha_{1+}\mathrm{pr}_+\alpha_{2+}\alpha_1^+(\alpha_2^+(\mathcal{E})) \xrightarrow{\sim} \alpha_{1+}\mathrm{pr}_+\alpha_{2+}(\widetilde{\mathcal{E}}) \xrightarrow{\sim} \mathrm{pr}_+\alpha_+(\widetilde{\mathcal{E}}).$$

From Lemma 1.4.1, this implies that $(\mathrm{pr}\circ\alpha)_+(\widetilde{\mathcal{E}})|\mathfrak{U}_1 \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{O}_{\mathfrak{U}_1,\mathbb{Q}})$.

(II) We proceed by induction on the dimension of the support X of $\mathcal{E}$. The case where $X \to P_1$ is not surjective is obvious (indeed, since $\alpha_+\alpha^+(\mathcal{E})$ has is support in X, we can choose $\mathfrak{U}_1$ to be the open dense subset of $\mathcal{P}_1$ complementary to $\mathrm{pr}(X)$). Hence, we can suppose that $X \to P_1$ is surjective. There exists a smooth dense open subvariety Y of X such that $\mathcal{E}|Y \in F$-$D^{\mathrm{b}}_{\mathrm{isoc}}(Y, \mathcal{P}/K)$ (see the notation [3, 1.2.14] and use [14, 3.1.1]). We put $Z := X \setminus Y$, endowed with its canonical structure of subvariety of X (recall that varieties are reduced following the convention of the paper). Let $j\colon Y \subset X$ be the inclusion and $i\colon Z \hookrightarrow X$ the corresponding closed immersion. By using the exact triangle of localization of the form $i_+i^!(\mathcal{E}) \to \mathcal{E} \to j_+j^+(\mathcal{E}) \to +1$ (see [3, 1.1.8(ii)]), by devissage and by induction hypothesis, we can suppose Y integral, that $\mathcal{E}$ is a module and that $\mathcal{E} \xrightarrow{\sim} j_+j^+(\mathcal{E})$ with $j^+(\mathcal{E}) \in F\text{-Isoc}^{\dagger\dagger}(Y, \mathcal{P}/K)$. By abuse of notation (to simplify them), $\mathcal{P}_1$ will mean a dense open set $\mathfrak{U}_1$ of $\mathcal{P}_1$ (be careful that the open set has to be independent of the choice of α_2), X and $\mathcal{P}$ will mean the base change of X and $\mathcal{P}$ by the inclusion $\mathfrak{U}_1 \subset \mathcal{P}_1$.

(1) From de Jong desingularization theorem, there exists a surjective, projective, generically finite étale morphism $a\colon X' \to X$, with X' integral and smooth such that

$Z' := a^{-1}(Z)$ is the support of a strict normal crossing divisor of X'. Since a is projective, there exists a closed immersion of the form $u' \colon X' \hookrightarrow \widehat{\mathbb{P}}^N \times \mathcal{P}$ (this is the product in the category of formal schemes over $\mathcal{V}$) such that the composition of u' with the projection $f \colon \widehat{\mathbb{P}}^N \times \mathcal{P} \to \mathcal{P}$ is equal to the composition of a with the closed immersion $X \hookrightarrow \mathcal{P}$. Since $\mathcal{E} \xrightarrow{\sim} j_+ j^+(\mathcal{E})$, by setting $\mathcal{E}' := a^!(\mathcal{E})$, $Y' := a^{-1}(Y)$, $\mathcal{P}' := \widehat{\mathbb{P}}^N \times \mathcal{P}$, we get $\mathcal{E}' \xrightarrow{\sim} j_+ j^+(\mathcal{E}')$ (use the base change isomorphism [3, 1.3.10]) and $j^+(\mathcal{E}') \in F\text{-Isoc}^{\dagger\dagger}(Y', \mathcal{P}'/K)$ (with the convention given at the beginning of the paper, recall that j means also the morphisms induced by base change form j). We denote by $(X', M_{Z'})$ the smooth log whose the underlying scheme is X' and the log structure $M_{Z'}$ comes canonically from the strict normal crossing divisor Z'. Sometimes we simply denote it by (X', Z') if the notation is not confusing. From Kedlaya's semistable reduction theorem (more precisely the global one, i.e., [33, 2.4.4]), we can suppose that the overconvergent isocrystal $\mathcal{E}'|Y'$ on Y' extends to a convergent log-F-isocrystal on $(X', M_{Z'})$. Using the properties satisfied by a, using for instance [12, 6.3.1], we get that $a_+ \circ a^!(\widetilde{\mathcal{E}})$ is a direct factor of $\widetilde{\mathcal{E}}$. Since by transitivity $\alpha^+ a^! \xrightarrow{\sim} a^! \alpha^+$ (recall $\alpha^+ = \alpha^!$) and $a_+ \alpha_+ \xrightarrow{\sim} \alpha_+ a_+$ then $\mathrm{pr}_+ a_+ \alpha_+ \alpha^+(a^+(\mathcal{E}))$ is a direct factor of $\mathrm{pr}_+ \alpha_+ \widetilde{\mathcal{E}}$. By definition of cohomological operations (see the beginning of the paper), $\mathrm{pr}_+ a_+ \alpha_+ \alpha^+(a^+(\mathcal{E})) = (\mathrm{pr} \circ f)_+ \alpha_+ \alpha^+(\mathcal{E}')$, where in the latter term $\mathrm{pr} \circ f \colon \mathcal{P}' = \widehat{\mathbb{P}}^N \times \mathcal{P}_2 \times \mathcal{P}_1 \to \mathcal{P}_1$ is the projection and $\alpha \colon \widehat{\mathbb{P}}^N \times \widetilde{\mathcal{P}} \to \widehat{\mathbb{P}}^N \times \mathcal{P} = \mathcal{P}'$. This implies that we can reduce to the case where X is smooth, Z is the support of a strict normal crossing divisor of X and $\mathcal{E}|Y$ extends to a convergent log-F-isocrystal on (X, M_Z) and we can forget the notation of part (1) of the proof.

(2) From Proposition 1.3.9, replacing $\mathcal{P}_1$ by an open affine dense formal subscheme if necessary and for s large enough, putting $W := (X^\sigma \times_{P_1^\sigma} P_1)_{\mathrm{red}}$, there exists a projective, surjective, generically finite and etale P_1-morphism of the form $\phi \colon W' \to W$ such that W' is integral and smooth over P_1, $Z' := \phi^{-1}(Z^\sigma \times_{P_1^\sigma} P_1)_{\mathrm{red}}$ is the support of a strict normal crossing divisor in W' relatively to P_1. Let $a \colon W = (X^\sigma \times_{P_1^\sigma} P_1)_{\mathrm{red}} \to X^\sigma$ be canonical morphism. We set $Y' := \phi^{-1}(Y^\sigma \times_{P_1^\sigma} P_1)_{\mathrm{red}}$. Put $\psi := Y' \to (Y^\sigma \times_{P_1^\sigma} P_1)_{\mathrm{red}}$ and $b \colon (Y^\sigma \times_{P_1^\sigma} P_1)_{\mathrm{red}} \to Y^\sigma$ the morphisms induced respectively by ϕ and a. Since ϕ is projective, for some integer N putting $\mathcal{P}_3 := \widehat{\mathbb{P}}^N \times \mathcal{P}_2^\sigma$ and taking $f \colon \mathcal{P}_3 \to \mathcal{P}_2^\sigma$ to be the projection, we get the commutative diagram of the left:

$$
\begin{tikzcd}
Z' \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square_{\mathrm{red}}"] & W' \arrow[r, hook] \arrow[d, "\phi"] & \mathcal{P}' = \mathcal{P}_3 \times \mathcal{P}_1 \arrow[r, "\mathrm{pr}'"] \arrow[d, "f"] & \mathcal{P}_1 \arrow[d, equal] \\
(Z^\sigma \times_{P_1^\sigma} P_1)_{\mathrm{red}} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square_{\mathrm{red}}"] & W = (X^\sigma \times_{P_1^\sigma} P_1)_{\mathrm{red}} \arrow[r, hook] \arrow[d, "a"] \arrow[dr, phantom, "\square_{\mathrm{red}}"] & \mathcal{P}_2^\sigma \times \mathcal{P}_1 \arrow[r, "\mathrm{pr}^\sigma"] \arrow[d, "F^s_{\mathcal{P}_1/\mathcal{V}}"] \arrow[dr, phantom, "\square"] & \mathcal{P}_1 \arrow[d, "F^s_{\mathcal{P}_1/\mathcal{V}}"] \\
Z^\sigma \arrow[r, hook] & X^\sigma \arrow[r, hook] & \mathcal{P}_2^\sigma \times \mathcal{P}_1^\sigma \arrow[r, "\mathrm{pr}^\sigma"] & \mathcal{P}_1^\sigma,
\end{tikzcd}
\qquad
\begin{tikzcd}
Y' \arrow[r, hook, "j^\sigma"] \arrow[d, "\psi"] \arrow[dr, phantom, "\square"] & W' \arrow[d, "\phi"] \\
(Y^\sigma \times_{P_1^\sigma} P_1)_{\mathrm{red}} \arrow[r, hook, "j^\sigma"] \arrow[d, "b"] \arrow[dr, phantom, "\square"] & W \arrow[d, "a"] \\
Y^\sigma \arrow[r, hook, "j^\sigma"] & X^\sigma
\end{tikzcd}
\tag{1.4.2.1}
$$

where the symbol '$\square_{\mathrm{red}}$' means the cartesianity in the category of reduced schemes, where $\mathrm{pr}' \colon \mathcal{P}_3 \to \mathrm{Spf}\, \mathcal{V}$ is the structural morphism. Since $(Y^\sigma \times_{P_1^\sigma} P_1)_{\mathrm{red}} = a^{-1}(Y^\sigma)$, we get the morphism $j^\sigma \colon (Y^\sigma \times_{P_1^\sigma} P_1)_{\mathrm{red}} \to (X^\sigma \times_{P_1^\sigma} P_1)_{\mathrm{red}} = W$. We have also $j^\sigma \colon Y' \to W'$. Hence, this justifies the cartesianity of the diagram of the right of (1.4.2.1).

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We put $\mathcal{G} := a^!(\mathcal{E}^\sigma) = (F^s_{\mathcal{P}_1/\mathcal{V}})^!(\mathcal{E}^\sigma)$ and $\mathcal{G}' := \phi^!(\mathcal{G}) = \mathbb{R}\underline{\Gamma}^\dagger_{W'} f^!(\mathcal{G})$. Since $a \circ \phi$ induces the morphism of smooth log-schemes $(W', M_{Z'}) \to (X^\sigma, M_{Z^\sigma})$, since $\mathcal{E}^\sigma|Y^\sigma$ extends to a convergent log-F-isocrystal on (X^σ, M_{Z^σ}) then $\mathcal{G}'|Y'$ extends to a convergent log-F-isocrystal on $(W', M_{Z'})$.

(3) Let $Z'_1, \ldots, Z'_{r'}$ be the irreducible components of Z'. For any subset I' of $\{1, \ldots, r'\}$, we set $Z'_{I'} := \bigcap_{i' \in I'} Z_{i'}$. Then $|\mathrm{Car}(\mathcal{G}')| \subset \bigcup_{I' \subset \{1,\ldots,r'\}} T^*_{Z'_{I'}} P'$.

Proof. Since the check is local on P', we can suppose P' affine with local coordinates $t'_1, \ldots, t'_{d'}$ inducing local coordinates $\bar{t}'_1, \ldots, \bar{t}'_{n'}$ of W' and such that $Z'_{i'} = V(\bar{t}'_{i'})$ for $i' = 1, \ldots, r'$. From [39], there exists a smooth affine formal $\mathcal{V}$-scheme $\mathfrak{W}'$ whose special fibre is W'. Let $u\colon \mathfrak{W}' \hookrightarrow \mathcal{P}'$ be a lifting of $W' \hookrightarrow P'$ and let $\mathcal{F}' := u^!(\mathcal{G}')$. From [11, 1.4.3.1], we have $|\mathrm{Car}(\mathcal{F}')| \subset \bigcup_{I' \subset \{1,\ldots,r'\}} T^*_{Z'_{I'}} W'$. Since $\mathcal{G}' \xrightarrow{\sim} u_+(\mathcal{F}')$ (this comes from Berthelot–Kashiwara's theorem), from [10, 5.3.3], we get $|\mathrm{Car}(\mathcal{G}')| = \mathscr{T}_u(|\mathrm{Car}(\mathcal{F}')|)$. Using Lemma 1.1.2, we get $\mathscr{T}_u(T^*_{Z'_{I'}} W') = T^*_{Z'_{I'}} P'$, which gives the desired result.

(4) We put $\widetilde{\mathcal{G}}' := \alpha_2^{\sigma+}(\mathcal{G}')$. We have $|\mathrm{Car}(\mathrm{pr}'_+ \alpha^\sigma_{2+}(\widetilde{\mathcal{G}}'))| \subset T^*_{P_1} P_1$.

Proof. Since pr' is proper, from Proposition 1.2.7, we get the inclusion

$$|\mathrm{Car}(\mathrm{pr}'_+ \alpha^\sigma_{2+}(\widetilde{\mathcal{G}}'))| \subset \mathscr{T}_{\mathrm{pr}'}(|\mathrm{Car}(\alpha^\sigma_{2+}(\widetilde{\mathcal{G}}'))|).$$

Using (1.2.9.1), we get $|\mathrm{Car}(\alpha^\sigma_{2+}\alpha_2^{\sigma+}(\mathcal{G}')| = |\mathrm{Car}(\mathcal{G}')|$. Hence,

$$\mathscr{T}_{\mathrm{pr}'}(|\mathrm{Car}(\alpha^\sigma_{2+}(\widetilde{\mathcal{G}}'))|) = \mathscr{T}_{\mathrm{pr}'}(|\mathrm{Car}(\mathcal{G}')|) \subset \mathscr{T}_{\mathrm{pr}'}\left(\bigcup_{I' \subset \{1,\ldots,r'\}} T^*_{Z'_{I'}} P'\right) = \bigcup_{I' \subset \{1,\ldots,r'\}} (\mathscr{T}_{\mathrm{pr}'}(T^*_{Z'_{I'}} P')),$$

where the inclusion comes from the step (3). Moreover, from Lemma 1.1.2, $\mathscr{T}_{u_{I'}}(T^*_{Z'_{I'}} Z'_{I'}) = T^*_{Z'_{I'}} P'$, where $u_{I'}\colon Z'_{I'} \hookrightarrow P'$ is the closed immersion. By transitivity of the application $\mathscr{T}$ (see 1.1.1), $\mathscr{T}_{\mathrm{pr}'}(T^*_{Z'_{I'}} P') = \mathscr{T}_{\mathrm{pr}'}(\mathscr{T}_{u_{I'}}(T^*_{Z'_{I'}} Z'_{I'})) = \mathscr{T}_{pr' \circ u_{I'}}(T^*_{Z'_{I'}} Z'_{I'})$. Since $pr' \circ u_{I'}\colon Z'_{I'} \to P_1$ is smooth, using 1.1.4(2), we get the inclusion $\mathscr{T}_{pr' \circ u_{I'}}(T^*_{Z'_{I'}} Z'_{I'}) \subset T^*_{P_1} P_1$, which yields the desired result.

(5) $\mathcal{G}$ is a direct factor of $\phi_+(\mathcal{G}')$ (which is by definition, if we look at the left diagram of (1.4.2.1), equal to $f_+(\mathcal{G}')$).

(a) We have the isomorphism $b^! \xrightarrow{\sim} b^+$. Indeed, from [3, 1.3.12], since b is a universal homeomorphism, the functors $b^!$ and b_+ induce quasi-inverse equivalences of categories (for categories of overholonomic complexes). Since b is proper, then $b_+ = b_!$ (i.e., via the biduality isomorphism, b_+ commutes with dual functors). Hence, we get that $b^!$ commutes also with dual functors.

(b) Since $\theta := b \circ \psi$ is a morphism of smooth varieties and $\mathcal{E}^\sigma|Y^\sigma$ is an isocrystal, then $\theta^!(\mathcal{E}^\sigma|Y^\sigma) \xrightarrow{\sim} \theta^+(\mathcal{E}^\sigma|Y^\sigma)$. Hence, since θ is proper, we get the morphisms by adjunction (see [3, 1.3.14(viii)]) $\mathcal{E}^\sigma|Y^\sigma \to \theta_+\theta^+(\mathcal{E}^\sigma|Y^\sigma) \xrightarrow{\sim} \theta_+\theta^!(\mathcal{E}^\sigma|Y^\sigma) \to \mathcal{E}^\sigma|Y^\sigma$. The composition is an isomorphism. Indeed, since $\mathcal{E}^\sigma|Y^\sigma$ is an isocrystal, we reduce to check it on a dense open subset of Y^σ. Hence, we can suppose that ψ and b are morphisms of smooth varieties. Using [3, 1.3.12] and the transitivity of the adjunction morphisms, we reduce to check such property for ψ. Since ψ is generically finite and etale, this is already known (e.g., see [12, 6.3.1]).

(c) We have just checked that $\mathcal{E}^\sigma|Y^\sigma$ is a direct factor of $\theta_+\theta^!(\mathcal{E}^\sigma|Y^\sigma)$. This implies that $b^!(\mathcal{E}^\sigma|Y^\sigma)$ is a direct factor of $b^!\theta_+\theta^!(\mathcal{E}^\sigma|Y^\sigma) \xrightarrow{\sim} b^!b_+\psi_+\theta^!(\mathcal{E}^\sigma|Y^\sigma) \underset{[3,1.3.12]}{\xrightarrow{\sim}} \psi_+\theta^!(\mathcal{E}^\sigma|Y^\sigma) \xrightarrow{\sim} \psi_+(\mathcal{G}'|Y')$. We get $j_+^\sigma b^!(\mathcal{E}^\sigma|Y^\sigma)$ is a direct factor of $j_+^\sigma\psi_+(\mathcal{G}'|Y')$. By base change isomorphism (e.g., see [3, 1.3.10]), by using the cartesianity of the right diagram of (1.4.2.1), we get the isomorphism $(a\circ\phi)^!j_+^\sigma \xrightarrow{\sim} j_+^\sigma\theta^!$. By applying the functor $(a\circ\phi)^!$ to the isomorphism $\mathcal{E}^\sigma \xrightarrow{\sim} j_+^\sigma j^{\sigma!}\mathcal{E}^\sigma$ we obtain $\mathcal{G}' \xrightarrow{\sim} j_+^\sigma j^{\sigma!}\mathcal{G}'$. This yields $\phi_+(\mathcal{G}') \xrightarrow{\sim} \phi_+ j_+^\sigma(\mathcal{G}'|Y') \xrightarrow{\sim} j_+^\sigma\psi_+(\mathcal{G}'|Y')$ and $\mathcal{G} \xrightarrow{\sim} a^! j_+^\sigma(\mathcal{E}^\sigma|Y^\sigma) \xrightarrow{\sim} j_+^\sigma b^!(\mathcal{E}^\sigma|Y^\sigma)$, which gives the desired result.

(6) Putting $\widetilde{\mathcal{G}} := \alpha_{2+}^\sigma(\mathcal{G})$, (using some base change isomorphism) the step (5) implies that $\widetilde{\mathcal{G}}$ is a direct factor of $\phi_+(\widetilde{\mathcal{G}}') = f_+(\widetilde{\mathcal{G}}')$. Since $\mathrm{pr}'_+\alpha_{2+}^\sigma(\widetilde{\mathcal{G}}') \xrightarrow{\sim} \mathrm{pr}_+^\sigma\alpha_{2+}^\sigma\phi_+(\widetilde{\mathcal{G}}')$, we obtain $|\mathrm{Car}(\mathrm{pr}_+^\sigma\alpha_{2+}^\sigma(\widetilde{\mathcal{G}}))| \subset |\mathrm{Car}(\mathrm{pr}'_+\alpha_{2+}^\sigma(\widetilde{\mathcal{G}}'))|$. From part (4), this yields $|\mathrm{Car}(\mathrm{pr}_+^\sigma\alpha_{2+}^\sigma(\widetilde{\mathcal{G}}))| \subset T^*_{P_1}P_1$. Let $\widetilde{\mathrm{pr}}\colon \widetilde{\mathcal{P}}_2 \to \mathrm{Spf}\,\mathcal{V}$ be the structural morphism. Since $\mathrm{pr}_+^\sigma\alpha_{2+}^\sigma(\widetilde{\mathcal{G}}) = \widetilde{\mathrm{pr}}_+^\sigma(\widetilde{\mathcal{G}})$, using the second part of the Lemma 1.2.10, this inclusion is equivalent to say that $\widetilde{\mathrm{pr}}_+^\sigma(\widetilde{\mathcal{G}}) \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{O}_{\mathcal{P}_1,\mathbb{Q}})$. Since $\widetilde{\mathcal{G}} \xrightarrow{\sim} a^!(\widetilde{\mathcal{E}}^\sigma)$, we obtain $\widetilde{\mathrm{pr}}_+^\sigma(\widetilde{\mathcal{G}}) \xrightarrow{\sim} \widetilde{\mathrm{pr}}_+^\sigma(a^!(\widetilde{\mathcal{E}}^\sigma)) \xrightarrow{\sim} (F^s_{\mathcal{P}_1/\mathcal{V}})^!\widetilde{\mathrm{pr}}_+^\sigma(\widetilde{\mathcal{E}}^\sigma)$. From Lemma 1.3.10, this implies that $\widetilde{\mathrm{pr}}_+^\sigma(\widetilde{\mathcal{E}}^\sigma) \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{O}_{\mathcal{P}_1^\sigma,\mathbb{Q}})$, which yields $\widetilde{\mathrm{pr}}_+(\widetilde{\mathcal{E}}) \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{O}_{\mathcal{P}_1,\mathbb{Q}})$. ☐

2. Betti numbers estimates

2.1. The curve case

Lemma 2.1.1. *Let $f\colon Y \to X$ be a smooth morphism of integral smooth k-varieties of relative dimension d (i.e., $d = \dim Y - \dim X$). If $\mathcal{F} \in D^{\geqslant 0}_{\mathrm{ovhol}}(Y/K)$ then $f_+(\mathcal{F}) \in D^{\geqslant -d}_{\mathrm{ovhol}}(X/K)$. If $\mathcal{G} \in D^{\leqslant 0}_{\mathrm{ovhol}}(Y/K)$ then $f_!(\mathcal{G}) \in D^{\leqslant d}_{\mathrm{ovhol}}(X/K)$.*

Proof. Since the second statement follows by duality (recall $f_! = \mathbb{D}_X \circ f_+ \circ \mathbb{D}_Y$ and dual functors exchange $D^{\geqslant n}$ with $D^{\leqslant -n}$), let us check the first one. As explained in the convention of the paper, since our varieties are realizable, there exist a smooth morphism $\phi\colon \mathcal{Q} \to \mathcal{P}$ of proper smooth formal $\mathcal{V}$-schemes, immersions $\iota\colon X \hookrightarrow \mathcal{P}$, $\iota'\colon Y \hookrightarrow \mathcal{Q}$ so that $\phi\circ\iota' = \iota\circ f$. The morphism f is the composition of an immersion of the form $u'\colon Y \hookrightarrow \phi^{-1}(X)$ followed by the morphism $\phi^{-1}(X) \to X$ induced by ϕ. Since $u'_+\colon D^{\geqslant 0}_{\mathrm{ovhol}}(Y/K) \to D^{\geqslant 0}_{\mathrm{ovhol}}(\phi^{-1}(X)/K)$, we reduce to the case where $Y = \phi^{-1}(X)$. Let $\mathfrak{U}$ be an open set of $\mathcal{P}$ such that ι factors through a closed immersion $X \hookrightarrow \mathfrak{U}$. Put $\mathfrak{V} := \phi^{-1}(\mathfrak{U})$ and $\psi\colon \mathfrak{V} \to \mathfrak{U}$ be the morphism induced by ϕ. By definition, $\mathcal{F} \in D^{\geqslant 0}_{\mathrm{ovhol}}(Y/K) = D^{\geqslant 0}_{\mathrm{ovhol}}(Y, \mathcal{Q}/K)$, $f_+(\mathcal{F}) = \phi_+(\mathcal{F})$ and $\phi_+(\mathcal{F}) \in D^{\geqslant -d}_{\mathrm{ovhol}}(X, \mathcal{P}/K)$ means that $\phi_+(\mathcal{F})|\mathfrak{U} \in D^{\geqslant -d}_{\mathrm{ovhol}}(\mathcal{D}^\dagger_{\mathfrak{U},\mathbb{Q}})$. Since this last property is local in $\mathfrak{U}$, we can suppose $\mathfrak{U}$ affine. Hence, there exists a closed immersion of smooth formal $\mathcal{V}$-schemes $u\colon \mathfrak{X} \hookrightarrow \mathfrak{U}$ which lifts $X \hookrightarrow \mathfrak{U}$. Set $\mathcal{Y} := \mathfrak{V} \times_{\mathfrak{U}} \mathfrak{X}$, which is a smooth lifting of Y. Put $v\colon \mathcal{Y} \hookrightarrow \mathfrak{V}$ the projection, which is a closed immersion and $\theta\colon \mathcal{Y} \to \mathfrak{X}$ the second projection. We have $\psi_+(\mathcal{F}|\mathfrak{V}) \xrightarrow{\sim} \phi_+(\mathcal{F})|\mathfrak{U}$ and $\mathcal{F}|\mathfrak{V} \in D^{\geqslant 0}_{\mathrm{ovhol}}(Y, \mathfrak{V}/K) \cong D^{\geqslant 0}_{\mathrm{ovhol}}(\mathcal{D}^\dagger_{\mathcal{Y},\mathbb{Q}})$, where the latter equivalence of categories are given by the quasi-inverse functors v_+ and $v^!$ (this is a form of

Berthelot–Kashiwara's theorem). Hence, we reduce to check that the functor θ_+ induces the factorization $\theta_+ \colon D^{\geqslant 0}_{\mathrm{ovhol}}(\mathcal{D}^\dagger_{\mathcal{Y},\mathbb{Q}}) \to D^{\geqslant -d}_{\mathrm{ovhol}}(\mathcal{D}^\dagger_{\mathfrak{X},\mathbb{Q}})$, which is obvious because, since θ is a smooth morphism of smooth formals $\mathcal{V}$-schemes of relative dimension d, then we have $\theta_+(\mathcal{E}) = \mathbb{R}\theta_*(\mathcal{E} \otimes_{\mathcal{O}_{\mathcal{Y}}} \Omega^\bullet_{\mathcal{Y}/\mathfrak{X}})[d]$ for any $\mathcal{E} \in D^{\geqslant 0}_{\mathrm{ovhol}}(\mathcal{D}^\dagger_{\mathcal{Y},\mathbb{Q}})$. □

Lemma 2.1.2. *Let X be an integral k-variety of dimension d. If $\mathcal{F} \in D^{\geqslant 0}_{\mathrm{ovhol}}(X/K)$ then $f_+(\mathcal{F}) \in D^{\geqslant -d}_{\mathrm{ovhol}}(\operatorname{Spec} k/K)$. If $\mathcal{G} \in D^{\leqslant 0}_{\mathrm{ovhol}}(X/K)$ then $f_!(\mathcal{G}) \in D^{\leqslant d}_{\mathrm{ovhol}}(\operatorname{Spec} k/K)$.*

Proof. As for the proof of Lemma 2.1.1, we reduce to check the first statement. We proceed by induction on the dimension of X. Choose an open smooth dense subvariety U of X. Put $Z := X \setminus U$. Let $j \colon U \hookrightarrow X$, $i \colon Z \hookrightarrow X$ be the corresponding morphisms of k-varieties. Since $i^!(\mathcal{F}) \in D^{\geqslant 0}_{\mathrm{ovhol}}(Z/K)$, then, by induction hypothesis, we get $p_{Z+} i^!(\mathcal{F}) \in D^{\geqslant -d}_{\mathrm{ovhol}}(\operatorname{Spec} k/K)$. From Lemma 2.1.1, we get $p_{U+} j^!(\mathcal{F}) \in D^{\geqslant -d}_{\mathrm{ovhol}}(\operatorname{Spec} k/K)$. Applying p_{X+} to the exact triangle of localization $i_+ i^!(\mathcal{F}) \to \mathcal{F} \to j_+ j^!(\mathcal{F}) \to +1$ (see [3, 1.1.8(ii)]), we get the exact triangle $p_{Z+} i^!(\mathcal{F}) \to p_{X+}(\mathcal{F}) \to p_{U+} j^!(\mathcal{F}) \to +1$. □

Lemma 2.1.3. *Let $u \colon Z \hookrightarrow X$ be a closed immersion of k-varieties. Suppose X smooth and integral. Let $\mathcal{E} \in F\text{-}\mathrm{Isoc}^{\dagger\dagger}(X/K)$. Then $u^!(\mathcal{E}) \in F\text{-}D^{\geqslant r}_{\mathrm{ovhol}}(Z/K)$ (see the notation of [3, 1.2]), with $r = \dim X - \dim Z$.*

Proof. First, suppose that Z is smooth. Then, we get the exact functor $u^![r] \colon F\text{-}\mathrm{Isoc}^{\dagger\dagger}(X/K) \to F\text{-}\mathrm{Isoc}^{\dagger\dagger}(Z/K)$ and the Lemma follows. More generally, we prove the Lemma by induction on the dimension of Z. When the dimension of Z is 0 then Z is a finite etale over $\operatorname{Spec} k$ (recall a reduced k-scheme of finite type of dimension 0 is finite etale over $\operatorname{Spec} k$, and by our convention k-varieties are assumed to be reduced) and then this case has already been checked. Suppose $\dim Z > 1$. Then there exists a dense open smooth subset Z_0 of Z. Put $T := Z \setminus Z_0$. Let $j \colon Z_0 \to Z$ and $i \colon T \to Z$ be the corresponding immersions. Put $\mathcal{F} := u^!(\mathcal{E})$. Then we conclude by using the induction hypothesis and consider the exact triangle of localization $i_+ i^!(\mathcal{F}) \to \mathcal{F} \to j_+ j^!(\mathcal{F}) \to +1$ (see [3, 1.1.8(ii)]). From the smooth case, $j^!(\mathcal{F})[r] \in F\text{-}\mathrm{Isoc}^{\dagger\dagger}(Z_0/K)$ and then $j_+ j^!(\mathcal{F}) \in F\text{-}D^{\geqslant r}_{\mathrm{ovhol}}(Z/K)$. By induction hypothesis, we get $i^!(\mathcal{F}) \in F\text{-}D^{\geqslant r'}_{\mathrm{ovhol}}(T/K)$, with $r' := \dim X - \dim T \geqslant r$. Since i_+ is exact, $i_+ i^!(\mathcal{F}) \in F\text{-}D^{\geqslant r'}_{\mathrm{ovhol}}(Z/K) \subset F\text{-}D^{\geqslant r}_{\mathrm{ovhol}}(Z/K)$. □

Notation 2.1.4. Let X be an integral variety and $\mathcal{E} \in F\text{-}\mathrm{Ovhol}(X/K)$. Then, there exists a smooth dense open subvariety Y of X such that $\mathcal{E}|Y \in F\text{-}\mathrm{Isoc}^{\dagger\dagger}(Y/K)$ (see the notation [3, 1.2.14] and use [14, 3.1.1]). Then, by definition, $\mathrm{rk}(\mathcal{E})$ means the rank of the corresponding overconvergent isocrystal associated to $\mathcal{E}|Y$ (which does not depend on the choice of such open dense subvariety Y).

Lemma 2.1.5. *We suppose that k is infinite. Let $f \colon Y \to X$ be a smooth morphism of integral smooth k-varieties. We suppose there exists a k-valued point x of X such that $Y_x := f^{-1}(x)$ is an integral k-variety of dimension d. Let $\mathcal{F} \in F\text{-}\mathrm{Isoc}^{\dagger\dagger}(Y/K)$ such that $f_+(\mathcal{F}) \in F\text{-}D^{\mathrm{b}}_{\mathrm{isoc}}(X/K)$. Then we have the inequalities:*

$$\mathrm{rk} H^{-d} f_+(\mathcal{F}) \leqslant \mathrm{rk}(\mathcal{F}); \tag{2.1.5.1}$$

$$\mathrm{rk} H^{d} f_!(\mathcal{F}) \leqslant \mathrm{rk}(\mathcal{F}). \tag{2.1.5.2}$$

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Proof. (0) We remark that $\mathrm{rk}(\mathcal{F}) = \mathrm{rk}(\mathbb{D}_Y(\mathcal{F}))$ (recall [1, 3.12]). Since $\mathbb{D}_X H^{-d} f_+(\mathcal{F}) = H^d \mathbb{D}_X f_+(\mathcal{F}) = H^d f_! \mathbb{D}_Y(\mathcal{F})$ (recall $f_!(\mathcal{F}) := \mathbb{D}_X(f_+(\mathbb{D}_Y(\mathcal{F})))$ and $\mathbb{D}_Y \circ \mathbb{D}_Y \xrightarrow{\sim} \mathrm{Id}$), we get similarly $\mathrm{rk} H^{-d} f_+(\mathcal{F}) = \mathrm{rk} H^d f_!(\mathbb{D}_Y(\mathcal{F}))$. Hence, we reduce to check the first inequality. Let $i_x \colon x \hookrightarrow X$ be the canonical closed immersion. Recall that from the convention of the paper $i_x \colon Y_x \hookrightarrow Y$ (respectively $f \colon Y_x \to x$) means the morphism induced from i_x (respectively f) by base change. The functors $i_x^![\dim X] \colon F\text{-Isoc}^{\dagger\dagger}(X/K) \to F\text{-Isoc}^{\dagger\dagger}(x/K)$ and $i_x^![\dim X] \colon F\text{-Isoc}^{\dagger\dagger}(Y/K) \to F\text{-Isoc}^{\dagger\dagger}(Y_x/K)$ are exact and preserve the rank (and in particular an isocrystal $\mathcal{G}$ is null if and only if $i_x^![\dim X](\mathcal{G}) = 0$). We have the base change isomorphism $i_x^![\dim Y] \circ f_+(\mathcal{F}) \xrightarrow{\sim} f_+ \circ i_x^![\dim Y](\mathcal{F})$ (e.g., see [3, 1.3.10]). Hence, we reduce to the case where $X = \mathrm{Spec}\, k$, i.e., $f = p_Y$ (see the notation of the paper).

(1) We check that for any open dense subset V of Y, we have $H^{-d} p_{Y+}(\mathcal{F}) \xrightarrow{\sim} H^{-d} p_{V+}(\mathcal{F}|V)$. Indeed, let V be a open dense subset of Y and put $Z := Y \setminus V$. We denote by $j \colon V \hookrightarrow Y$ the canonical open immersion and $i \colon Z \hookrightarrow Y$ the canonical closed immersion. Since V is dense in Y, then $d_Z := \dim Z < d$. Since $\mathcal{F}$ is an isocrystal, then from Lemma 2.1.3 we have $i^!(\mathcal{F}) \in F\text{-}D^{\geqslant d-d_Z}_{\mathrm{ovhol}}(Z/K)$. Hence, via Lemma 2.1.2, this implies $p_{Z+} i^!(\mathcal{F}) \in F\text{-}D^{\geqslant d-2d_Z}_{\mathrm{ovhol}}(\mathrm{Spec}\, k/K)$. Since $d_Z \leqslant d-1$, we get $d - 2d_Z \geqslant -d+2$. This yields $H^{-d} p_{Z+} i^!(\mathcal{F}) = 0$ and $H^{-d+1} p_{Z+} i^!(\mathcal{F}) = 0$. Applying p_{Y+} to the exact triangle of localization $i_+ i^!(\mathcal{F}) \to \mathcal{F} \to j_+ j^!(\mathcal{F}) \to +1$ (see [3, 1.1.8(ii)]), we get the exact triangle $p_{Z+} i^!(\mathcal{F}) \to p_{Y+}(\mathcal{F}) \to p_{V+}(\mathcal{F}|V) \to +1$. By considering the long exact sequence associated with the latter exact triangle, we conclude.

(2) We check the lemma in the case where $d = 1$. From (1), we can suppose that Y is affine. Choose a smooth compactification $\overline{Y}$ of Y and put $D := \overline{Y} \setminus Y$. Choose a closed point y of D. With the notation [19, 7.1.1], we associate to $\mathcal{F}$ an $A_Y^\dagger$-module M endowed with a connexion. Then $H^{-1} p_{Y+}(\mathcal{F})$ is equal to the horizontal sections of M. Let $A(y)$ be the Robba ring (or local algebra following the terminology of [19]) corresponding to y (see the notation of [19, 7.3]). Using [19, 6.2] we get that the dimension over K of the K-vector space of the horizontal sections of $M \otimes_{A_Y^\dagger} A(y)$ (which is bigger than that of the horizontal sections of M) is less or equal to the rank of M (which is also the rank of $\mathcal{F}$).

(3) Now we prove the lemma by induction on d. Suppose $d \geqslant 2$. From part (1) of the proof, using [31], we can suppose that there exists a finite etale morphism of the form $\phi \colon Y \to \mathbb{A}_k^d$. Let g be the composite of ϕ with the projection $\mathbb{A}_k^1 \times \mathbb{A}_k^{d-1} \to \mathbb{A}_k^{d-1}$. There exists a dense open subvariety U of $\mathbb{A}_k^{d-1}$ such that $g_+(\mathcal{F})|U \in F\text{-}D^{\mathrm{b}}_{\mathrm{isoc}}(U/K)$ (see the notation [3, 1.2.14] and use [14, 3.1.1]). Let $V := g^{-1}(U)$ and $h \colon V \to U$ the induced smooth morphism of relative dimension 1. Since k is infinite and U is dense in $\mathbb{A}_k^{d-1}$, there exists a k-valued point x of U. Since g is surjective, $g^{-1}(x)$ is a smooth variety of dimension 1. Shrinking V (from now V is only a open dense subset of $g^{-1}(U)$) if necessary, we can assume that $h^{-1}(x)$ is an integral smooth variety of dimension 1 (use again part (1) of the proof). Proceeding as in part (0) and using part (2) of the proof, we get $\mathrm{rk}\, \mathcal{H}^{-1} h_+(\mathcal{F}|V) \leqslant \mathrm{rk}\,(\mathcal{F}|V) = \mathrm{rk}\,(\mathcal{F})$. By using the induction hypothesis, we obtain $\mathrm{rk}\, H^{-d+1} p_{U+}(\mathcal{H}^{-1} h_+(\mathcal{F}|V)) \leqslant \mathrm{rk}\, \mathcal{H}^{-1} h_+(\mathcal{F}|V)$.

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Using Lemma 2.1.1, the functors $p_{U+}[-d+1]$ and $h_+[-1]$ are left exact. Hence, we get $H^{-d}p_{V+}(\mathcal{F}|V) \xrightarrow{\sim} H^{-d+1}p_{U+}(\mathcal{H}^{-1}h_+(\mathcal{F}|V))$ and we are done. □

Lemma 2.1.6. *We suppose that k is algebraically closed. Let X be a smooth irreducible curve, $j\colon U \hookrightarrow X$ an open immersion such that $Z := X \setminus U$ is a closed point. Let $\mathcal{F} \in F\text{-Isoc}^{\dagger\dagger}(U/K)$. Let $i\colon Z \hookrightarrow X$ be the closed immersion. Then, for $n = 0, 1$, we have the inequality*

$$\dim_K H^n i^! j_!(\mathcal{F}) \leqslant \mathrm{rk}(\mathcal{F}). \tag{2.1.6.1}$$

Proof. Let $\theta_j\colon j_!(\mathcal{F}) \to j_+(\mathcal{F})$ be the canonical morphism. Let $\mathcal{C}$ be the mapping cone of θ_j. By definition, $\mathcal{H}^{-1}(\mathcal{C}) \xrightarrow{\sim} \ker(\theta_j)$ and $\mathcal{H}^0(\mathcal{C}) \xrightarrow{\sim} \mathrm{coker}(\theta_j)$. Since $j^!(\theta_j)$ is an isomorphism (use $j^! = j^+$), then $j^!(\mathcal{C}) = 0$. By using the triangle of localization $i_+i^!(\mathcal{C}) \to \mathcal{C} \to j_+j^!(\mathcal{C}) \to +1$, this implies that the canonical morphism $i_+i^!(\mathcal{C}) \to \mathcal{C}$ is an isomorphism, i.e . the cone of θ_j has its support in Z (see the terminology of [3, 1.3.2(iii)]). Since $i^!j_+ = 0$, since the cone of θ_j has its support in Z, then by using Berthelot–Kashiwara theorem (in the form of [3, 1.3.2(iii)]), we get $\ker(\theta_j) \xrightarrow{\sim} i_+H^0i^!j_!(\mathcal{F})$ and $\mathrm{coker}(\theta_j) \xrightarrow{\sim} i_+H^1i^!j_!(\mathcal{F})$. From the last isomorphism of corollary [3, 1.4.3], we obtain $\mathbb{D}_X i_+H^0i^!j_!(\mathcal{F}) \xrightarrow{\sim} i_+H^1i^!j_!(\mathbb{D}_U(\mathcal{F}))$. Hence, by using again Berthelot–Kashiwara theorem (in the form of [3, 1.3.2(iii)]), and the relative duality isomorphism (i.e., the isomorphism [3, 1.3.14(vi)]), we get $H^0i^!j_!(\mathcal{F}) \xrightarrow{\sim} \mathbb{D}_Z H^1i^!j_!(\mathbb{D}_U(\mathcal{F}))$ and we reduce to check the case $n = 1$.

We have the exact sequence $0 \to j_{!+}(\mathcal{F}) \to j_+(\mathcal{F}) \to i_+H^1i^!j_!(\mathcal{F}) \to 0$, where $j_{!+}(\mathcal{F})$ is the intermediate extension as defined in [3, 1.4.1], i.e., $j_{!+}(\mathcal{F})$ is the image of θ_j. From Kedlaya's semistable theorem [30], there exists a finite surjective morphism $f\colon P' \to X$, with P' smooth integral, such that $f^!(\mathcal{F})$ comes from a convergent isocrystal on P' with logarithmic poles along $f^{-1}(Z)$. Since P' and X are smooth, since f is finite and surjective then f is flat (e.g., see [28, IV.15.4.2]). Let X' be an open dense subset of P' such that $Z' := f^{-1}(Z) \cap X'$ is a closed point. We get the (quasi-finite) flat morphism $a\colon X' \to X$ and the open immersion $j\colon U' := a^{-1}(U) \to X'$ and $a\colon U' \to U$. Since a is flat and quasi-finite, then $a^!$ is exact. Hence, we get the exact sequence $0 \to a^!j_{!+}(\mathcal{F}) \to a^!j_+(\mathcal{F}) \to a^!i_+H^1i^!j_!(\mathcal{F}) \to 0$. From the base change isomorphism (e.g., see [3, 1.3.10]), we get $a^!j_+(\mathcal{F}) \xrightarrow{\sim} j_+a^!(\mathcal{F})$. Hence, $a^!j_{!+}(\mathcal{F})$ is a subobject of $j_+a^!(\mathcal{F})$. Moreover, $j^!a^!j_{!+}(\mathcal{F}) \xrightarrow{\sim} a^!j^!j_{!+}(\mathcal{F}) \xrightarrow{\sim} a^!(\mathcal{F})$. This yields, from [3, 1.4.8], that the inclusion $j_{!+}(a^!(\mathcal{F})) \hookrightarrow j_+(a^!(\mathcal{F}))$ factors through the composition $a^!j_{!+}(\mathcal{F}) \hookrightarrow a^!j_+(\mathcal{F}) \xrightarrow{\sim} j_+a^!(\mathcal{F})$. Then, we get the epimorphism

$$\begin{aligned} i_+H^1i^!j_!(a^!\mathcal{F}) &\xrightarrow{\sim} j_+(a^!(\mathcal{F}))/j_{!+}(a^!(\mathcal{F})) \twoheadrightarrow a^!j_+(\mathcal{F})/a^!j_{!+}(\mathcal{F}) \\ &\xrightarrow{\sim} a^!i_+H^1i^!j_!(\mathcal{F}) \underset{[3, 1.3.10]}{\xrightarrow{\sim}} i_+H^1i^!j_!(\mathcal{F}), \end{aligned}$$

where for the last isomorphism we use also that a induces the isomorphism $a\colon a^{-1}(Z) \xrightarrow{\sim} Z$ (because k is algebraically closed). By applying $i^!$ (and by using Berthelot–Kashiwara theorem), we get $\dim_K H^1i^!j_!(\mathcal{F}) \leqslant \dim_K H^1i^!j_!(a^!\mathcal{F})$. Since $a^!\mathcal{F}$ is log-extendable, then from [3, 3.4.19.1] we obtain the inequality $\dim_K H^1i^!j_!(a^!\mathcal{F}) \leqslant \mathrm{rk}(a^!\mathcal{F}) = \mathrm{rk}(\mathcal{F})$. □

Since the proof of the main result on Betti estimate (see Theorem 2.2.6) in the case of curves is easier (e.g., remark that we do not need in this case the Lemma 1.4.2) and since its proof is made by induction, we first check separately this curve case via the following proposition.

Proposition 2.1.7 (Curve case). *Suppose k is algebraically closed. Let X_1 be a projective, smooth and connected curve, $\mathcal{E} \in F\text{-}D^{\leqslant 0}(X_1/K)$ (see the notation of [3, 1.2]). There exists a constant $c(\mathcal{E})$ such that, for any finite étale morphism of degree d_1 of the form $\alpha_1 \colon \widetilde{X}_1 \to X_1$ with $\widetilde{X}_1$ connected, by putting $\widetilde{\mathcal{E}} := \alpha_1^{+}(\mathcal{E})$, we have*

(1) $\dim_K H^1 p_{\widetilde{X}_1 +}(\widetilde{\mathcal{E}}) \leqslant c(\mathcal{E})$;

(2) *For any integer $r \leqslant 0$, $\dim_K H^r p_{\widetilde{X}_1 +}(\widetilde{\mathcal{E}}) \leqslant c(\mathcal{E}) d_1$.*

Proof. There exists an open dense affine subvariety U_1 of X_1 such that $\mathcal{E}|U_1 \in F\text{-}D^{\mathrm{b}}_{\mathrm{isoc}}(U_1/K)$ (see the notation [3, 1.2.14] and use [14, 3.1.1]). Let Z_1 be the closed subvariety $X_1 \setminus U_1$, $j \colon U_1 \hookrightarrow X_1$ and $i \colon Z_1 \hookrightarrow X_1$ be the immersions. We put $\widetilde{U}_1 := \alpha_1^{-1}(U_1)$, $\widetilde{Z}_1 := \alpha_1^{-1}(Z_1)$ i.e we get the Cartesian squares:

$$\begin{tikzcd}
\widetilde{Z}_1 \arrow[r, "i"] \arrow[d, "\alpha_1"] \arrow[dr, phantom, "\square"] & \widetilde{X}_1 \arrow[d, "\alpha_1"] \arrow[dr, phantom, "\square"] & \widetilde{U}_1 \arrow[l, "j"'] \arrow[d, "\alpha_1"] \\
Z_1 \arrow[r, "i"] & X_1 & U_1. \arrow[l, "j"']
\end{tikzcd}$$

By considering the exact triangle $j_! j^!(\mathcal{E}) \to \mathcal{E} \to i_+ i^+(\mathcal{E}) \to +1$ (see [3, 1.1.8(ii)]) we reduce to check the proposition for $\mathcal{E} = j_! j^!(\mathcal{E})$ or $\mathcal{E} = i_+ i^+(\mathcal{E})$ (and because the functors $j_! j^!$ and $i_+ i^+$ preserve $D^{\leqslant 0}$).

(1) In the case where $\mathcal{E} = i_+ i^+(\mathcal{E})$, we can suppose that Z_1 is a point. We put $\mathcal{G} := i^+(\mathcal{E})$. Since $\widetilde{Z}_1$ is d_1 copies of Z_1, then we get $\alpha_{1+}\alpha_1^{+}\mathcal{G} \xrightarrow{\sim} \mathcal{G}^{d_1}$. Since $i_+ \xrightarrow{\sim} i_!$ and $\alpha_1^{+} \xrightarrow{\sim} \alpha_1^{!}$, then we get from the base change isomorphism (e.g., see [3, 1.3.10]): $\alpha_1^{+} i_+ \xrightarrow{\sim} i_+ \alpha_1^{+}$. Since $\mathcal{E} = i_+ i^+(\mathcal{E})$, this implies $\alpha_{1+}(\widetilde{\mathcal{E}}) = \alpha_{1+}\alpha_1^{+}(\mathcal{E}) \xrightarrow{\sim} i_+ \alpha_{1+} \alpha_1^{+} i^{+}(\mathcal{E}) = i_+ \alpha_{1+} \alpha_1^{+}(\mathcal{G}) \xrightarrow{\sim} i_+ \mathcal{G}^{d_1}$. Hence, $p_{\widetilde{X}_1 +}(\widetilde{\mathcal{E}}) \xrightarrow{\sim} p_{X_1 +}(i_+ \mathcal{G}^{d_1}) \xrightarrow{\sim} \mathcal{G}^{d_1}$, which gives the desired result.

(2) Suppose now $\mathcal{E} = j_! j^!(\mathcal{E})$. We put $\mathcal{F} = j^!(\mathcal{E})$, $\widetilde{\mathcal{F}} = j^!(\widetilde{\mathcal{E}})$. Using the spectral sequence $E_2^{r,s} = H^r p_{\widetilde{X}_1 +}(\mathcal{H}^s(\widetilde{\mathcal{E}})) \Rightarrow H^{r+s} p_{\widetilde{X}_1 +}(\widetilde{\mathcal{E}})$, we reduce to the case where $\mathcal{F} \in F\text{-Isoc}^{\dagger\dagger}(U_1/K)$ (and then $\widetilde{\mathcal{F}} \in F\text{-Isoc}^{\dagger\dagger}(\widetilde{U}_1/K)$). Since X_1 is proper and smooth integral of dimension 1, then $p_{\widetilde{X}_1 +}(\widetilde{\mathcal{E}}) \xrightarrow{\sim} p_{\widetilde{X}_1 !}(\widetilde{\mathcal{E}})$ is a complex concentrated in degree $-1, 0, 1$ (use Lemma 2.1.1). Since U_1 is affine, then $p_{\widetilde{U}_1 !}$ is left t-exact (see [3, 1.3.13(i)]). Since $\widetilde{\mathcal{E}} \xrightarrow{\sim} j_!(\widetilde{\mathcal{F}})$, then $p_{\widetilde{X}_1 +}(\widetilde{\mathcal{E}}) \xrightarrow{\sim} p_{\widetilde{X}_1 !}(\widetilde{\mathcal{E}}) \xrightarrow{\sim} p_{\widetilde{U}_1 !}(\widetilde{\mathcal{F}})$. Hence, we get $H^{-1} p_{\widetilde{X}_1 +}(\widetilde{\mathcal{E}}) = 0$. From (2.1.5.2), we get $\dim_K H^1 p_{\widetilde{X}_1 +}(\widetilde{\mathcal{E}}) = \dim_K H^1 p_{\widetilde{U}_1 !}(\widetilde{\mathcal{F}}) \leqslant \mathrm{rk}(\widetilde{\mathcal{F}}) = \mathrm{rk}(\mathcal{F})$. It remains to estimate $|\chi(\widetilde{X}_1, \widetilde{\mathcal{E}})|$. Since $p_{\widetilde{X}_1 +}(\widetilde{\mathcal{E}}) \xrightarrow{\sim} p_{X_1 +}(\alpha_{1+}\alpha_1^{+}(\mathcal{E}))$, we get the equality $\chi(\widetilde{X}_1, \widetilde{\mathcal{E}}) = \chi(X_1, \alpha_{1+}\alpha_1^{+}(\mathcal{E}))$. From Lemma 1.2.9 (recall also that from [39, III.6.10], there exist some smooth proper formal $\mathcal{V}$-schemes $\mathfrak{X}_1$ and $\widetilde{\mathfrak{X}}_1$ which are respectively a lifting of X_1 and $\widetilde{X}_1$, so we are in the geometrical context of Lemma 1.2.9), we have the formula $\chi(X_1, \alpha_{1+}\alpha_1^{+}(\mathcal{E})) = d_1 \cdot \chi(X_1, \mathcal{E})$. Hence, we can choose in that case $c(\mathcal{E}) = \max\{|\chi(X_1, \mathcal{E})|; \mathrm{rk}(\mathcal{F})\}$. □

2.2. The result and some applications

In this subsection, we need the Fourier transform (see 2.2.2). Hence, we assume here that there exists $\pi_0 \in K$ such that $\pi_0^{p-1} = -p$ and such that $\sigma(\pi_0) = \pi_0$. Let $q := p^s$, $W(\mathbb{F}_q)$ be the ring of Witt vectors of $\mathbb{F}_q$. We get a complete discrete valuation ring of residue field $\mathbb{F}_q$ by setting $\mathcal{V}_0 := W(\mathbb{F}_q)[X]/(X^{p-1}+p)$ (indeed, $X^{p-1}+p$ is an Eisenstein polynomial). The class of X is a uniformizer of $\mathcal{V}_0$. Remark that the canonical lifting of the sth Frobenius power of $\mathbb{F}_q$ is the identity of $\mathcal{V}_0$. We define the extension $\rho\colon \mathcal{V}_0 \to \mathcal{V}$ by sending the class of X to π_0. Since $\sigma(\pi_0) = \pi_0$, this homomorphism ρ is compatible with Frobenius liftings i.e., $\sigma \circ \rho = \rho$. Let K_0 be the field of fraction of $\mathcal{V}_0$. We know the $K_0 = \mathrm{Frac}(W(\mathbb{F}_q))(\mu_p)$, where $\mu_p \subset \overline{\mathbb{Q}}_p$ are the group of p-rooth of unity (see [6, 1.3]).

We fix a nontrivial additive character $\psi\colon \mathbb{F}_q \to \mu_p \subset K_0$.

2.2.1. We denote by $\mathcal{L}_\psi$ the Artin–Schreier isocrystal in $F\text{-Isoc}^{\dagger\dagger}(\mathbb{A}^1_{\mathbb{F}_q}/K_0)$ (see Proposition [6, 1.5]). The extension $\mathcal{V}_0 \to \mathcal{V}$ induces the morphism $\widehat{\mathbb{P}}^1_{\mathcal{V}} \to \widehat{\mathbb{P}}^1_{\mathcal{V}_0}$. We obtain the morphism of ringed spaces $f\colon (\widehat{\mathbb{P}}^1_{\mathcal{V}}, \mathcal{O}_{\widehat{\mathbb{P}}^1_{\mathcal{V}}}({}^\dagger\infty)_{\mathbb{Q}}) \to (\widehat{\mathbb{P}}^1_{\mathcal{V}_0}, \mathcal{O}_{\widehat{\mathbb{P}}^1_{\mathcal{V}_0}}({}^\dagger\infty)_{\mathbb{Q}})$, where ∞ is the closed point $\mathbb{P}^1_{\mathbb{F}_q} \setminus \mathbb{A}^1_{\mathbb{F}_q}$ and respectively $\mathbb{P}^1_k \setminus \mathbb{A}^1_k$. Recall (see the convention of the paper) that $F\text{-Isoc}^{\dagger\dagger}(\mathbb{A}^1_{\mathbb{F}_q}/K_0)$ ($F\text{-Isoc}^{\dagger\dagger}(\mathbb{A}^1_k/K)$) is equivalent to the category of coherent $\mathcal{O}_{\widehat{\mathbb{P}}^1_{\mathcal{V}_0}}({}^\dagger\infty)_{\mathbb{Q}}$-modules $\mathcal{E}$ (respectively coherent $\mathcal{O}_{\widehat{\mathbb{P}}^1_{\mathcal{V}}}({}^\dagger\infty)_{\mathbb{Q}}$-modules $\mathcal{E}$) endowed with an integrable connexion and a Frobenius structure i.e., an isomorphism of the form $F^*(\mathcal{E}) \xrightarrow{\sim} \mathcal{E}$. Hence, we get the functor $f^*\colon F\text{-Isoc}^{\dagger\dagger}(\mathbb{A}^1_{\mathbb{F}_q}/K_0) \to F\text{-Isoc}^{\dagger\dagger}(\mathbb{A}^1_k/K)$. We still denote by $\mathcal{L}_\psi$ the object of $F\text{-Isoc}^{\dagger\dagger}(\mathbb{A}^1_k/K)$ which is the image by f^* of the Artin–Schreier isocrystal $\mathcal{L}_\psi$ in $F\text{-Isoc}^{\dagger\dagger}(\mathbb{A}^1_{\mathbb{F}_q}/K_0)$.

2.2.2 (Fourier transform). Let S be a k-variety. Let us briefly review the geometric Fourier transform defined by Noot-Huyghe in [38], but only in the specific case of $\mathbb{A}^1_S/S$. Let $\mu\colon \mathbb{A}^1_k \times (\mathbb{A}^1_k)' \to \mathbb{A}^1_k$ be the canonical duality bracket given by $t \mapsto xy$, where $(\mathbb{A}^1_S)'$ is the 'dual affine space over S', which is nothing but $\mathbb{A}^1_S$ (we have $\mathbb{A}^1_S = \mathrm{Spec}\,\mathcal{O}_S[x]$ and $(\mathbb{A}^1_S)' = \mathrm{Spec}\,\mathcal{O}_S[y]$). We denote the composition by $\mu_S\colon \mathbb{A}^1_S \times_S (\mathbb{A}^1_S)' \to \mathbb{A}^1_k \times (\mathbb{A}^1_k)' \to \mathbb{A}^1_k$.

Now, consider the following diagram: $(\mathbb{A}^1_S)' \xleftarrow{p_2} \mathbb{A}^1_S \times_S (\mathbb{A}^1_S)' \xrightarrow{p_1} \mathbb{A}^1_S$. Similarly to Katz and Laumon in [29, 7.1.4, 7.1.5] (in fact, here is the particular case where $r = 1$), for any $\mathcal{E} \in F\text{-}D^{\mathrm{b}}_{\mathrm{ovhol}}(\mathbb{A}^1_S)$, the geometric Fourier transform $\mathscr{F}_\psi(\mathcal{E})$ is defined to be

$$\mathscr{F}_\psi(\mathcal{E}) := p_{2+}(p_1^! \mathcal{E} \widetilde{\otimes}_{\mathbb{A}^2_S} \mu_S^! \mathcal{L}_\psi[-1]) \tag{2.2.2.1}$$

(cf. [38, 3.2.1][1]). Here $\widetilde{\otimes}$ is compatible with Laumon's notation (see [29, 7.0.1, p. 192]) and was defined in the context of arithmetic $\mathcal{D}$-modules in [3, 1.1.6].

2.2.3. An important property for us of Fourier transform is the following. The functor $\mathscr{F}_\psi[1]$ is acyclic, i.e., if $\mathcal{E} \in F\text{-Ovhol}(\mathbb{A}^1_S/K)$ then $\mathscr{F}_\psi(\mathcal{E})[1] \in F\text{-Ovhol}((\mathbb{A}^1_S)'/K)$ (cf. [38, Theorem 5.3.1]).

[1]Notice that our twisted tensor product and hers are the same.

Remark. This might be simpler in our case to define the Fourier transform by setting $\mathscr{F}_\psi(\mathcal{E}) := p_{2+}(p_1^! \mathcal{E} \widetilde{\otimes}_{\mathbb{A}^2_S} \mu_S^! \mathcal{L}_\psi)$ (and then $\mathscr{F}_\psi(\mathcal{E}) \in F\text{-Ovhol}((\mathbb{A}^1_S)'/K))$ but, to avoid confusion with the standard notation, we stick with the convention of [4, 3.2.2] or [38, 3.2.1] (for this latter reference, remark that there is a typo in [38, 3.1.1] : $K_\pi = \delta^* L_\pi[2N-2]$ and not $K_\pi = \delta^* L_\pi[2-2N]$).

Lemma 2.2.4. *Let $f: T \to S$ be a morphism of k-varieties. Let $\mathcal{E} \in F\text{-}D^{\mathrm{b}}_{\mathrm{ovhol}}(\mathbb{A}^1_S/K)$ and $\mathcal{F} \in F\text{-}D^{\mathrm{b}}_{\mathrm{ovhol}}(\mathbb{A}^1_T/K)$. We have the canonical isomorphisms*

$$f^! \mathscr{F}_\psi(\mathcal{E}) \xrightarrow{\sim} \mathscr{F}_\psi(f^! \mathcal{E}); \tag{2.2.4.1}$$

$$f_+ \mathscr{F}_\psi(\mathcal{F}) \xrightarrow{\sim} \mathscr{F}_\psi(f_+ \mathcal{F}). \tag{2.2.4.2}$$

Proof. We have the canonical isomorphisms:

$$\begin{aligned} f^! \mathscr{F}_\psi(\mathcal{E}) = f^! p_{2+}\big(p_1^! \mathcal{E} \widetilde{\otimes}_{\mathbb{A}^2_S} \mu_S^! \mathcal{L}_\psi[-1]\big) &\underset{[3,1.3.10]}{\xrightarrow{\sim}} p_{2+} f^! \big(p_1^! \mathcal{E} \widetilde{\otimes}_{\mathbb{A}^2_S} \mu_S^! \mathcal{L}_\psi[-1]\big) \\ &\underset{[3,1.1.9.1]}{\xrightarrow{\sim}} p_{2+}\big(f^! p_1^! \mathcal{E} \widetilde{\otimes}_{\mathbb{A}^2_T} f^! \mu_S^! \mathcal{L}_\psi[-1]\big). \end{aligned}$$

Since $\mu_T = \mu_S \circ f$, by transitivity of the extraordinary inverse image, we obtain the isomorphism $f^! \mu_S^! \mathcal{L}_\psi[-1] \xrightarrow{\sim} \mu_T^! \mathcal{L}_\psi[-1]$. Hence, $p_{2+}\big(f^! p_1^! \mathcal{E} \widetilde{\otimes}_{\mathbb{A}^2_T} f^! \mu_S^! \mathcal{L}_\psi[-1]\big) \xrightarrow{\sim} p_{2+}\big(p_1^! (f^! \mathcal{E}) \widetilde{\otimes}_{\mathbb{A}^2_T} \mu_T^! \mathcal{L}_\psi[-1]\big) = \mathscr{F}_\psi(f^! \mathcal{E})$, which gives 2.2.4.1. Moreover, by transitivity of the push-forward, we get the first isomorphism:

$$\begin{aligned} f_+ \mathscr{F}_\psi(\mathcal{F}) \quad &= \quad f_+ p_{2+}\big(p_1^!(\mathcal{F}) \widetilde{\otimes}_{\mathbb{A}^2_T} \mu_T^! \mathcal{L}_\psi[-1]\big) \xrightarrow{\sim} p_{2+} f_+ \big(p_1^!(\mathcal{F}) \widetilde{\otimes}_{\mathbb{A}^2_T} f^! \mu_S^! \mathcal{L}_\psi[-1]\big) \\ &\underset{[3,A.6]}{\xrightarrow{\sim}} p_{2+}\big(f_+ p_1^!(\mathcal{F}) \widetilde{\otimes}_{\mathbb{A}^2_S} \mu_S^! \mathcal{L}_\psi[-1]\big) \\ &\underset{[3,1.3.10]}{\xrightarrow{\sim}} p_{2+}\big(p_1^!(f_+ \mathcal{F}) \widetilde{\otimes}_{\mathbb{A}^2_S} \mu_S^! \mathcal{L}_\psi[-1]\big) = \mathscr{F}_\psi(f_+ \mathcal{F}). \end{aligned}$$ □

We use the following remark during the proof of the main theorem.

Remark 2.2.5. Let $\mathcal{R}_K$ be the Robba ring over K (e.g., see [32, 15.1.4]). Let M be a differential module on $\mathcal{R}_K$, i.e., a free $\mathcal{R}_K$-module of finite type endowed with an integrable connexion (e.g., see the beginning of [18, 3]). We suppose M solvable (e.g., see Definition [18, 8.7]). We get the differential slope decomposition $M = \bigoplus M_\beta$, where M_β is purely of differential slope β (see Theorem [16, 2.4-1]). By definition $\mathrm{Irr}(M) := \sum_{\beta \geqslant 0} \beta \cdot \mathrm{rk}(M_\beta)$ (see both definitions in [4, 2.3.1] and the formula [4, 2.3.2.2] which compare both definitions). Hence, we get that

$$\mathrm{Irr}(M) \leqslant \mathrm{rk}(M) + \mathrm{Irr}(M_{]1,\infty[}), \tag{2.2.5.1}$$

where $M_{]1,\infty[} := \bigoplus_{\beta \in]1,\infty[} M_\beta$.

Theorem 2.2.6. *Suppose k is algebraically closed. Let $(X_a)_{1 \leqslant a \leqslant n}$ be projective, smooth and connected curves, $X = \prod_{a=1}^n X_a$, $\mathcal{E} \in F\text{-}D^{\leqslant 0}(X/K)$ (see the notation of [3, 1.2]). There*

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exists a constant $c(\mathcal{E})$ such that, for any finite étale morphism of degree d_a of the form $\alpha_a\colon \widetilde{X}_a \to X_a$ with $\widetilde{X}_a$ connected, by putting $\widetilde{X} = \prod_{a=1}^n \widetilde{X}_a$, $\alpha\colon \widetilde{X} \to X$ and $\widetilde{\mathcal{E}} := \alpha^+(\mathcal{E})$, we have

(1) *For any integer r, $\dim_K H^r p_{\widetilde{X}+}(\widetilde{\mathcal{E}}) \leqslant c(\mathcal{E}) \prod_{a=1}^n d_a$.*

(2) *For any integer $r \geqslant 1$, $\dim_K H^r p_{\widetilde{X}+}(\widetilde{\mathcal{E}}) \leqslant c(\mathcal{E}) \max\{\prod_{a\in A} d_a \mid A \subset \{1,\dots,n\}$ and $|A| = n-r\}$.*

Proof. We proceed by induction on $n \in \mathbb{N}$. The case $n = 1$ has already been checked in Proposition 2.1.7. Suppose $n \geqslant 2$. Let $\alpha_a\colon \widetilde{X}_a \to X_a$ be some finite étale morphism of degree d_a with $\widetilde{X}_a$ connected, $\widetilde{X} = \prod_{a=1}^n \widetilde{X}_a$, $\alpha\colon \widetilde{X} \to X$ and $\widetilde{\mathcal{E}} := \alpha^+(\mathcal{E})$. We put $Y := \prod_{a\neq 1} X_a$, $\widetilde{Y} := \prod_{a\neq 1} \widetilde{X}_a$, $\beta\colon \widetilde{Y} \to Y$. Let $\mathrm{pr}\colon Y \to \operatorname{Spec} k$ and $\widetilde{\mathrm{pr}}\colon \widetilde{Y} \to \operatorname{Spec} k$ be the projections (recall that from the convention of this paper, for instance, $\widetilde{\mathrm{pr}}$ means also the projection $\widetilde{\mathrm{pr}}\colon \widetilde{X} = \widetilde{X}_1 \times \widetilde{Y} \to \widetilde{X}_1$ etc.). From Lemma 1.4.2 (recall also that from [39, III.6.10], since X_i and $\widetilde{X}_i$ are smooth curves, there exist some smooth proper formal $\mathcal{V}$-schemes $\mathfrak{X}_i$ and $\widetilde{\mathfrak{X}}_i$ which lift them, which reduce us to the geometrical situation of Lemma 1.4.2) there exists an affine open dense subvariety U_1 (independent of the choice of α_i) of X_1 such that $\mathrm{pr}_+(\alpha_+\widetilde{\mathcal{E}})|U_1 \in D^{\mathrm{b}}_{\mathrm{isoc}}(U_1/K)$ (use [13, 2.2.12] to check this latter property). Let Z_1 be the closed subvariety $X_1 \setminus U_1$, $\widetilde{U}_1 := \alpha_1^{-1}(U_1)$, $\widetilde{Z}_1 := \alpha_1^{-1}(Z_1)$. Let $j\colon U_1 \hookrightarrow X_1$ $i\colon Z_1 \hookrightarrow X_1$ be the inclusions.

Step (0) We have $j^!\widetilde{\mathrm{pr}}_+(\widetilde{\mathcal{E}}) = \widetilde{\mathrm{pr}}_+(\widetilde{\mathcal{E}})|\widetilde{U}_1 \in F\text{-}D^{\mathrm{b}}_{\mathrm{isoc}}(\widetilde{U}_1/K)$. Indeed, from Lemma 1.4.1, this is equivalent to prove $\alpha_{1+}(\widetilde{\mathrm{pr}}_+(\widetilde{\mathcal{E}}))|U_1 \in D^{\mathrm{b}}_{\mathrm{isoc}}(U_1/K)$. Then, we get the desired property from the isomorphism $\alpha_{1+}(\widetilde{\mathrm{pr}}_+(\widetilde{\mathcal{E}})) \xrightarrow{\sim} \mathrm{pr}_+(\alpha_+\widetilde{\mathcal{E}})$ (checked by transitivity of the push-forwards).

Step (I) With the notation 2.1.4, we check that there exists a constant c (only depending on $\mathcal{E}$) such that

- For any integer s, $\operatorname{rk} \mathcal{H}^s \widetilde{pr}_+(\widetilde{\mathcal{E}}) \leqslant c \prod_{b=2}^n d_b$.
- For any integer $s \geqslant 1$, $\operatorname{rk} \mathcal{H}^s \widetilde{pr}_+(\widetilde{\mathcal{E}}) \leqslant c \max\{\prod_{b\in B} d_b \mid B \subset \{2,\dots,n\}$ and $|B| = n-1-s\}$.

Proof. Let t be a closed point of U_1, $\widetilde{t}$ be a closed point of $\widetilde{U}_1$ such that $\alpha_1(\widetilde{t}) = t$. Let $i_t\colon t \hookrightarrow X_1$, $i_{\widetilde{t}}\colon \widetilde{t} \hookrightarrow \widetilde{X}_1$, $\iota_{\widetilde{t}}\colon \widetilde{t} \hookrightarrow \widetilde{U}_1$ be the closed immersions. Since the functor $\iota_{\widetilde{t}}^![1]$ is acyclic on $F\text{-}D^{\mathrm{b}}_{\mathrm{isoc}}(\widetilde{U}_1/K)$, since $i_{\widetilde{t}}^![1] \xrightarrow{\sim} (\iota_{\widetilde{t}}^![1]) \circ j^!$, we obtain $i_{\widetilde{t}}^![1](\mathcal{H}^s \widetilde{pr}_+(\widetilde{\mathcal{E}})) \xrightarrow{\sim} \mathcal{H}^s(i_{\widetilde{t}}^! \widetilde{pr}_+(\widetilde{\mathcal{E}}[1]))$. Moreover, for such $\widetilde{t}$, we have $\operatorname{rk}(\mathcal{H}^s \widetilde{pr}_+(\widetilde{\mathcal{E}})) = \dim_K i_{\widetilde{t}}^![1]\mathcal{H}^s \widetilde{pr}_+(\widetilde{\mathcal{E}})$.

We put $\mathcal{E}_1 := i_t^!(\mathcal{E})[1]$ and $\widetilde{\mathcal{E}}_1 := \beta^+(\mathcal{E}_1)$. Since $t \times Y$ is a smooth divisor of X, then $i_t^![1]$ is right exact. Hence $\mathcal{E}_1 \in F\text{-}D^{\leqslant 0}(Y/K)$ (we identify Y with $t \times Y$). To simplify the notation, we avoid to mention the isomorphism $\widetilde{t} \xrightarrow{\sim} t$ induced by α_1 in other words, we identify t and $\widetilde{t}$ via this isomorphism). We get $i_t = \alpha_1 \circ i_{\widetilde{t}}$ and then $\alpha \circ i_{\widetilde{t}} = i_t \circ \beta\colon \widetilde{t} \times \widetilde{Y} \to X$. Since $\beta^+ = \beta^!$, we get by transitivity of the extraordinary inverse image the isomorphism

$$i_{\widetilde{t}}^! \widetilde{\mathcal{E}}[1] = i_{\widetilde{t}}^! \alpha^+(\mathcal{E})[1] \xrightarrow{\sim} \beta^+ i_t^!(\mathcal{E})[1] = \beta^+(\mathcal{E}_1) = \widetilde{\mathcal{E}}_1. \tag{2.2.6.1}$$

Hence,

$$i_{\widetilde{t}}^! \widetilde{pr}_+(\widetilde{\mathcal{E}})[1] \underset{[3, 1.3.10]}{\xrightarrow{\sim}} \widetilde{pr}_+ i_{\widetilde{t}}^!(\widetilde{\mathcal{E}})[1] \underset{(2.2.6.1)}{\xrightarrow{\sim}} \widetilde{pr}_+(\widetilde{\mathcal{E}}_1) = p_{\widetilde{Y}+}(\widetilde{\mathcal{E}}_1), \tag{2.2.6.2}$$

where we have identified $\widetilde{Y}$ with $\widetilde{t} \times \widetilde{Y}$ in the last equality. By composition, we obtain

$$i^!_{\widetilde{t}}[1](\mathcal{H}^s \widetilde{pr}_+(\widetilde{\mathcal{E}})) \xrightarrow{\sim} \mathcal{H}^s(i^!_{\widetilde{t}}\widetilde{pr}_+(\widetilde{\mathcal{E}}[1])) \underset{(2.2.6.2)}{\xrightarrow{\sim}} \mathcal{H}^s p_{\widetilde{Y}+}(\widetilde{\mathcal{E}}_1)$$

and then $\mathrm{rk}(\mathcal{H}^s \widetilde{pr}_+(\widetilde{\mathcal{E}})) = \dim_K \mathcal{H}^s p_{\widetilde{Y}+}(\widetilde{\mathcal{E}}_1)$. We conclude by applying the induction hypothesis to $\mathcal{E}_1$ (notice that we do need Theorem 1.4.2: since U_1 is independent of the choice of α_i then so is t and then $\mathcal{E}_1$).

Step (II)
(1) By considering the exact triangle $j_! j^!(\mathcal{E}) \to \mathcal{E} \to i_+ i^+(\mathcal{E}) \to +1$ (see [3, 1.1.8(ii)]) we reduce to check the proposition for $\mathcal{E} = j_! j^!(\mathcal{E})$ or $\mathcal{E} = i_+ i^+(\mathcal{E})$ (and because the functors $j_! j^!$ and $i_+ i^+$ preserve $D^{\leqslant 0}$).
(2) Suppose $\mathcal{E} = i_+ i^+(\mathcal{E})$. We can suppose that Z_1 is irreducible (i.e., since k is algebraically closed, $Z_1 \xrightarrow{\sim} \operatorname{Spec} k$). Consider the diagram with Cartesian squares:

$$\begin{tikzcd}
\widetilde{Z}_1 \times \widetilde{Y} \arrow[r, "\alpha_1"] \arrow[d, hook, "i"] \arrow[dr, phantom, "\square"] & Z_1 \times \widetilde{Y} \arrow[r, "\beta"] \arrow[d, hook, "i"] \arrow[dr, phantom, "\square"] & Z_1 \times Y \arrow[r, "\mathrm{pr}"] \arrow[d, hook, "i"] \arrow[dr, phantom, "\square"] & Z_1 \arrow[d, hook, "i"] \\
\widetilde{X} \arrow[r, "\alpha_1"] & X_1 \times \widetilde{Y} \arrow[r, "\beta"] & X \arrow[r, "\mathrm{pr}"] & X_1 .
\end{tikzcd}$$

We put $\mathcal{G} := i^+(\mathcal{E})$, $\widetilde{\mathcal{G}} := \beta^+(\mathcal{G})$. Since $\widetilde{Z}_1$ is d_1 copies of Z_1, then we get $\alpha_{1+}\alpha_1^+\widetilde{\mathcal{G}} \xrightarrow{\sim} (\widetilde{\mathcal{G}})^{d_1}$. This implies

$$p_{\widetilde{X}+}(\widetilde{\mathcal{E}}) = p_{\widetilde{X}+}(\alpha^+(i_+\mathcal{G})) \underset{[3,1.3.10]}{\xrightarrow{\sim}} p_{\widetilde{X}+}(i_+\alpha_1^+\beta^+\mathcal{G}) \xrightarrow{\sim} p_{\widetilde{Y}+}\alpha_{1+}\alpha_1^+(\widetilde{\mathcal{G}}) \xrightarrow{\sim} (p_{\widetilde{Y}+}(\widetilde{\mathcal{G}}))^{d_1}, \tag{2.2.6.3}$$

where in the second isomorphism we have identified $\widetilde{Y}$ with $Z_1 \times \widetilde{Y}$. We conclude by applying the induction hypothesis to $\mathcal{G}$.

(3) Suppose now $\mathcal{E} = j_! j^!(\mathcal{E})$.
(a) We check that there exists a constant c (only depending on $\mathcal{E}$) such that

- For any s, $\dim_K H^1 p_{\widetilde{X}_1+}(\mathcal{H}^s \widetilde{pr}_+(\widetilde{\mathcal{E}})) \leqslant c \prod_{b=2}^n d_b$.
- For any $s \geqslant 1$, $\dim_K H^1 p_{\widetilde{X}_1+}(\mathcal{H}^s \widetilde{pr}_+(\widetilde{\mathcal{E}})) \leqslant c \max\{\prod_{b\in B} d_b \mid B \subset \{2, \ldots, n\}$ and $|B| = n-1-s\}$.

Proof. We put $\mathcal{F} = j^!(\mathcal{E})$, $\widetilde{\mathcal{F}} := \alpha^+(\mathcal{F})$. By transitivity of the extraordinary push-forward, we get the first isomorphism

$$j_! \widetilde{pr}_+(\widetilde{\mathcal{F}}) \xrightarrow{\sim} \widetilde{pr}_+ j_!(\widetilde{\mathcal{F}}) \underset{[3,1.3.10]}{\xrightarrow{\sim}} \widetilde{pr}_+(\widetilde{\mathcal{E}}). \tag{2.2.6.4}$$

Moreover, since $j^! j_! \xrightarrow{\sim} \mathrm{Id}$, we get the first isomorphism $\widetilde{pr}_+(\widetilde{\mathcal{F}}) \xrightarrow{\sim} j^! j_! \widetilde{pr}_+(\widetilde{\mathcal{F}}) \underset{(2.2.6.4)}{\xrightarrow{\sim}} j^! \widetilde{pr}_+(\widetilde{\mathcal{E}})$. Hence, from the Step 0, this implies $\widetilde{pr}_+(\widetilde{\mathcal{F}}) \in D^{\mathrm{b}}_{\mathrm{isoc}}(\widetilde{U}_1/K)$.

Since $j_!$ is exact, we get from (2.2.6.4) the isomorphism $j_! \mathcal{H}^s \widetilde{pr}_+(\widetilde{\mathcal{F}}) \xrightarrow{\sim} \mathcal{H}^s \widetilde{pr}_+(\widetilde{\mathcal{E}})$. By applying the functor $p_{\widetilde{X}_1+}$ to this last isomorphism, we get by transitivity of the extraordinary push-forward

$$p_{\widetilde{U}_1 !}(\mathcal{H}^s \widetilde{pr}_+(\widetilde{\mathcal{F}})) \xrightarrow{\sim} p_{\widetilde{X}_1+}(\mathcal{H}^s \widetilde{pr}_+(\widetilde{\mathcal{E}})). \tag{2.2.6.5}$$

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Applying H^1 to (2.2.6.5), we get the first equality:

$$\dim_K H^1 p_{\widetilde{X}_1+}(\mathcal{H}^s \widetilde{pr}_+(\widetilde{\mathcal{E}})) = \dim_K H^1 p_{\widetilde{U}_1!}(\mathcal{H}^s \widetilde{pr}_+(\widetilde{\mathcal{F}})) \leqslant \mathrm{rk}\mathcal{H}^s \widetilde{pr}_+(\widetilde{\mathcal{F}}) = \mathrm{rk}\mathcal{H}^s \widetilde{pr}_+(\widetilde{\mathcal{E}}),$$

the inequality in the middle is a consequence of (2.1.5.2). From the step (I), we get the desired estimate.

(b) We have the spectral sequence

$$E_2^{r,s} = H^r p_{\widetilde{X}_1+}\mathcal{H}^s \widetilde{pr}_+(\widetilde{\mathcal{E}}) \Rightarrow H^{r+s} p_{\widetilde{X}+}(\widetilde{\mathcal{E}}).$$

Since $\widetilde{U}_1$ is affine of dimension 1, using the isomorphism (2.2.6.5), we get $E_2^{r,s} = 0$ when $r \notin \{0, 1\}$ (use also Lemma 2.1.1 and the respective case of [3, 1.3.13(i)]). Hence, by using the step (a) (still valid if we vary the order of $X_1, \ldots, X_n$), it remains to check that there exists a constant $c(\mathcal{E})$ such that

- For any s, $|\chi(\widetilde{X}_1, \mathcal{H}^s \widetilde{pr}_+(\widetilde{\mathcal{E}}))| \leqslant c(\mathcal{E}) \prod_{a=1}^n d_a$.
- For any $s \geqslant 1$, $|\chi(\widetilde{X}_1, \mathcal{H}^s \widetilde{pr}_+(\widetilde{\mathcal{E}}))| \leqslant c(\mathcal{E}) \max\{\prod_{a\in A} d_a \mid A \subset \{1, \ldots, n\}$ and $|A| = n - s\}$.

(i) In this step, we reduce to the case where α_1 is the identity. For this purpose, consider the following diagram

$$\begin{tikzcd}
\widetilde{X}_1 \times \widetilde{Y} \arrow[r, "\alpha_1"] \arrow[d, "\widetilde{pr}"] \arrow[dr, phantom, "\square"] & X_1 \times \widetilde{Y} \arrow[r, "\beta"] \arrow[d, "\widetilde{pr}"] & X \arrow[dl, "\mathrm{pr}"] \\
\widetilde{X}_1 \arrow[r, "\alpha_1"] & X_1. &
\end{tikzcd}$$

We have $\chi(\widetilde{X}_1, \mathcal{H}^s \widetilde{pr}_+(\widetilde{\mathcal{E}})) = \chi(X_1, \alpha_{1+}\mathcal{H}^s \widetilde{pr}_+(\widetilde{\mathcal{E}}))$. By using the transitivity of the pull-back, we get the first isomorphism: $\alpha_{1+}\widetilde{pr}_+(\widetilde{\mathcal{E}}) \xrightarrow{\sim} \alpha_{1+}\widetilde{pr}_+\alpha_1^+(\beta^+(\mathcal{E})) \underset{[3, 1.3.10]}{\xrightarrow{\sim}} \alpha_{1+}\alpha_1^+\widetilde{pr}_+(\beta^+(\mathcal{E}))$. Since α_{1+} and α_1^+ are exact, this implies the isomorphism $\alpha_{1+}\mathcal{H}^s \widetilde{pr}_+(\widetilde{\mathcal{E}}) \xrightarrow{\sim} \alpha_{1+}\alpha_1^+\mathcal{H}^s \widetilde{pr}_+(\beta^+(\mathcal{E}))$. Hence,

$$\chi(X_1, \alpha_{1+}\mathcal{H}^s \widetilde{pr}_+(\widetilde{\mathcal{E}})) = \chi(X_1, \alpha_{1+}\alpha_1^+\mathcal{H}^s \widetilde{pr}_+(\beta^+(\mathcal{E}))).$$

From Lemma 1.2.9 (recall also that from [39, III.6.10], there exist some smooth proper formal $\mathcal{V}$-scheme $\mathfrak{X}_1$ which is a lifting of X_1), we have $\chi(X_1, \alpha_{1+}\alpha_1^+\mathcal{H}^s \widetilde{pr}_+(\beta^+(\mathcal{E}))) = d_1\chi(X_1, \mathcal{H}^s \widetilde{pr}_+(\beta^+(\mathcal{E})))$. Hence, we have checked that $\chi(\widetilde{X}_1, \mathcal{H}^s \widetilde{pr}_+(\widetilde{\mathcal{E}})) = d_1\chi(X_1, \mathcal{H}^s \widetilde{pr}_+(\beta^+(\mathcal{E})))$, which yields the desired result.

(ii) We suppose from now that α_1 is the identity. We prove that we can reduce to the case where $X_1 = \mathbb{P}_k^1$ and $U_1 = \mathbb{A}_k^1$. Indeed, from Kedlaya's main theorem of [31], by shrinking U_1 is necessary, there exists a finite morphism $f : X_1 \to \mathbb{P}_k^1$ such that $U_1 = f^{-1}(\mathbb{A}_k^1)$ and the induced morphism $g : U_1 \to \mathbb{A}_k^1$ is etale. We get the Cartesian squares:

$$\begin{tikzcd}
\widetilde{X} \arrow[r, "\alpha"'] \arrow[rr, bend left, "\widetilde{pr}"] \arrow[d, "f"] \arrow[dr, phantom, "\square"] & X \arrow[r, "pr"'] \arrow[d, "f"] \arrow[dr, phantom, "\square"] & X_1 \arrow[d, "f"] \arrow[dr, phantom, "\square"] & U_1 \arrow[l, "j"] \arrow[d, "g"] \\
\mathbb{P}_k^1 \times \widetilde{Y} \arrow[r, "\alpha"] \arrow[rr, bend right, "\widetilde{pr}"'] & \mathbb{P}_k^1 \times Y \arrow[r, "pr"] & \mathbb{P}_k^1 & \mathbb{A}_k^1. \arrow[l, "j"']
\end{tikzcd} \tag{2.2.6.6}$$

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Set $\mathcal{E}' := f_+(\mathcal{E}) = f_+ j_! j^!(\mathcal{E}) \xrightarrow{\sim} j_! g_+ j^!(\mathcal{E}) \underset{[3, 1.3.10]}{\xrightarrow{\sim}} j_! j^! f_+(\mathcal{E}) = j_! j^!(\mathcal{E}')$. Set $\widetilde{\mathcal{E}}' := \alpha^+(\mathcal{E}')$. We have the isomorphisms $j^! \mathrm{pr}_+ \alpha_+(\widetilde{\mathcal{E}}') \underset{[3, 1.3.10]}{\xrightarrow{\sim}} j^! \mathrm{pr}_+ \alpha_+ f_+(\widetilde{\mathcal{E}}) \xrightarrow{\sim} j^! f_+ \mathrm{pr}_+ \alpha_+(\widetilde{\mathcal{E}}) \underset{[3, 1.3.10]}{\xrightarrow{\sim}} g_+ j^! \mathrm{pr}_+ \alpha_+(\widetilde{\mathcal{E}})$. Since $j^! \mathrm{pr}_+ \alpha_+(\widetilde{\mathcal{E}}) \in D^{\mathrm{b}}_{\mathrm{isoc}}(U_1/K)$, since g is finite and etale, using Lemma 1.4.1, this yields that $j^! \mathrm{pr}_+ \alpha_+(\widetilde{\mathcal{E}}') \in D^{\mathrm{b}}_{\mathrm{isoc}}(\mathbb{A}^1_k/K)$.

To finish this step (ii), it remains to compare the Euler–Poincare characteristic. Since $f_+ j_! \xrightarrow{\sim} j_! g_+$, since g_+ (because g is finite and etale) and $j_!$ are exact, then we get

$$\begin{aligned}\mathcal{H}^s f_+ \widetilde{pr}_+(\widetilde{\mathcal{E}}) &\underset{(2.2.6.4)}{\xrightarrow{\sim}} \mathcal{H}^s f_+ j_! \widetilde{pr}_+(\widetilde{\mathcal{F}}) \xrightarrow{\sim} j_! g_+ \mathcal{H}^s \widetilde{pr}_+(\widetilde{\mathcal{F}}) \xrightarrow{\sim} f_+ j_! \mathcal{H}^s \widetilde{pr}_+(\widetilde{\mathcal{F}}) \\ &\underset{(2.2.6.4)}{\xrightarrow{\sim}} f_+ \mathcal{H}^s \widetilde{pr}_+(\widetilde{\mathcal{E}}).\end{aligned} \tag{2.2.6.7}$$

This yields $\mathcal{H}^s \widetilde{pr}_+(\widetilde{\mathcal{E}}') \underset{[3, 1.3.10]}{\xrightarrow{\sim}} \mathcal{H}^s f_+ \widetilde{pr}_+(\widetilde{\mathcal{E}}) \underset{2.2.6.7}{\xrightarrow{\sim}} f_+ \mathcal{H}^s \widetilde{pr}_+(\widetilde{\mathcal{E}})$ and then we get the last equality

$$\chi(X_1, \mathcal{H}^s \widetilde{pr}_+(\widetilde{\mathcal{E}})) = \chi(\mathbb{P}^1_k, f_+ \mathcal{H}^s \widetilde{pr}_+(\widetilde{\mathcal{E}})) = \chi(\mathbb{P}^1_k, \mathcal{H}^s \widetilde{pr}_+(\widetilde{\mathcal{E}}')).$$

(iii) We suppose from now $X_1 = \mathbb{P}^1_k$ and $U_1 = \mathbb{A}^1_k$. Recall that $\alpha_1 = \mathrm{id}$ and that we have checked in Step II.3(a) that $\widetilde{pr}_+(\widetilde{\mathcal{F}}) \in D^{\mathrm{b}}_{\mathrm{isoc}}(U_1/K)$. Hence we can apply Lemma 2.1.6: for any $m \in \{0, 1\}$ we have the inequality

$$\mathcal{H}^m(i^! j_! \mathcal{H}^s \widetilde{pr}_+(\widetilde{\mathcal{F}})) \leqslant \mathrm{rk}(\mathcal{H}^s \widetilde{pr}_+(\widetilde{\mathcal{F}})) \underset{(2.2.6.4)}{=} \mathrm{rk}(\mathcal{H}^s \widetilde{pr}_+(\widetilde{\mathcal{E}})).$$

From the step (I), this latter is well estimated. This implies that $\chi(X_1, i_+ i^! j_! \mathcal{H}^s \widetilde{pr}_+(\widetilde{\mathcal{F}}))$ is well estimated.

From (2.2.6.4), we obtain $\chi(X_1, \mathcal{H}^s \widetilde{pr}_+(\widetilde{\mathcal{E}})) = \chi(X_1, j_! \mathcal{H}^s \widetilde{pr}_+(\widetilde{\mathcal{F}}))$. Moreover, by using the exact triangle $i_+ i^! \to \mathrm{id} \to j_+ j^! \to +1$ for $j_! \mathcal{H}^s \widetilde{pr}_+(\widetilde{\mathcal{F}})$ (see [3, 1.1.8(ii)]), since $j^! j_! = \mathrm{id}$, we get the equality $\chi(X_1, j_! \mathcal{H}^s \widetilde{pr}_+(\widetilde{\mathcal{F}})) = \chi(X_1, i_+ i^! j_! \mathcal{H}^s \widetilde{pr}_+(\widetilde{\mathcal{F}})) + \chi(X_1, j_+ \mathcal{H}^s \widetilde{pr}_+(\widetilde{\mathcal{F}}))$. Hence, we reduce to estimate $\chi(X_1, j_+ \mathcal{H}^s \widetilde{pr}_+(\widetilde{\mathcal{F}}))$.

From Christol–Mebkhout's theorem [17, 5.0-10] (as described in the introduction), we have the following p-adic Euler–Poincare formula:

$$\chi(X_1, j_+ \mathcal{H}^s \widetilde{pr}_+(\widetilde{\mathcal{F}})) = \chi(U_1, \mathcal{H}^s \widetilde{pr}_+(\widetilde{\mathcal{F}})) = \mathrm{rk}\,(\mathcal{H}^s \widetilde{pr}_+(\widetilde{\mathcal{F}}))\chi(U_1) - \mathrm{Irr}_\infty(\mathcal{H}^s \widetilde{pr}_+(\widetilde{\mathcal{F}})),$$

where ∞ is the complement of U_1 in X_1, i.e., of $\mathbb{A}^1_k$ in $\mathbb{P}^1_k$.

To simplify notation, we put $\mathscr{F} := \mathscr{F}_\psi[1]$ (see the notation 2.2.2) and then from 2.2.3 the image by $\mathscr{F}$ of a module is a module. Since $\mathcal{G}^s := \mathcal{H}^s \widetilde{pr}_+(\widetilde{\mathcal{F}}) \in F\text{-Isoc}(U_1/K)$, then it has no singular points (see Definition [4, 2.4.2]). Hence, Abe–Marmora's formula [4, 4.1.6(i)] can be formulated of the form (see also the notation [4, 2.4.1, 4.1.1]):

$$-\mathrm{rk}(\mathscr{F}(\mathcal{G}^s)) = \mathrm{rk}((\mathcal{G}^s|\eta_\infty)_{]1,\infty[}) - \mathrm{Irr}((\mathcal{G}^s|\eta_\infty)_{]1,\infty[}). \tag{2.2.6.8}$$

With [4, 2.3.2.2] (respectively (2.2.5.1)), we get the equality (respectively inequality):

$$\mathrm{Irr}_\infty(\mathcal{G}^s) = \mathrm{Irr}(\mathcal{G}^s|\eta_\infty) \leqslant \mathrm{rk}(\mathcal{G}^s|\eta_\infty) + \mathrm{Irr}((\mathcal{G}^s|\eta_\infty)_{]1,\infty[}). \tag{2.2.6.9}$$

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Since $\mathrm{rk}(\mathcal{G}^s) = \mathrm{rk}(\mathcal{G}^s|\eta_\infty)$ and $\mathrm{rk}((\mathcal{G}^s|\eta_\infty)_{]1,\infty[}) \leqslant \mathrm{rk}(\mathcal{G}^s)$, we get from (2.2.6.8) and (2.2.6.9) the inequality:

$$\mathrm{Irr}_\infty(\mathcal{H}^s \widetilde{pr}_+(\widetilde{\mathcal{F}})) \leqslant \mathrm{rk}(\mathscr{F}(\mathcal{H}^s \widetilde{pr}_+(\widetilde{\mathcal{F}}))) + 2\mathrm{rk}(\mathcal{H}^s \widetilde{pr}_+(\widetilde{\mathcal{F}})).$$

Hence, we reduce to check the step (iv).

(iv) In this step, we estimate $\mathrm{rk}(\mathscr{F}(\mathcal{H}^s \widetilde{pr}_+(\widetilde{\mathcal{F}})))$. Since $\beta = \alpha$, we get the diagram

$$\begin{tikzcd}
\widetilde{X} \arrow[r, "\alpha"'] \arrow[rr, bend left, "\widetilde{pr}"] & X \arrow[r, "pr"'] & X_1 \\
\widetilde{U} \arrow[u, "j"] \arrow[r, "\alpha"] \arrow[rr, bend right, "\widetilde{pr}"'] \arrow[ur, phantom, "\square"] & U \arrow[u, "j"] \arrow[r, "pr"] \arrow[ur, phantom, "\square"] & U_1, \arrow[u, "j"]
\end{tikzcd} \tag{2.2.6.10}$$

where $U = \mathbb{A}^1_Y$, $\widetilde{U} = \mathbb{A}^1_{\widetilde{Y}}$. Set $\mathcal{M} := j_+\mathscr{F}(j^!\mathcal{E})$, where $\mathscr{F} := \mathscr{F}_\psi[1]\colon F\text{-}D^{\leqslant 0}_{\mathrm{ovhol}}(\mathbb{A}^1_Y) \to F\text{-}D^{\leqslant 0}_{\mathrm{ovhol}}(\mathbb{A}^1_Y)$ (see the notation 2.2.2 and 2.2.3 for the acyclicity). From the step (I) applied to $\mathcal{M}$, there exists a constant c (only depending on $\mathcal{E}$) such that

- For any s, $\mathrm{rk}(\mathcal{H}^s \widetilde{pr}_+(\alpha^+\mathcal{M})) \leqslant c\prod_{b=2}^n d_b$.
- For any $s \geqslant 1$, $\mathrm{rk}(\mathcal{H}^s \widetilde{pr}_+(\alpha^+\mathcal{M})) \leqslant c\max\{\prod_{b\in B} d_b \mid B \subset \{2,\ldots,n\} \text{ and } |B| = n-1-s\}$.

It remains to check that $\mathrm{rk}(\mathscr{F}(\mathcal{H}^s \widetilde{pr}_+(\widetilde{\mathcal{F}}))) = \mathrm{rk}(\mathcal{H}^s \widetilde{pr}_+(\alpha^+\mathcal{M}))$. By base change (recall that $\alpha^! = \alpha^+$) and next by using 2.2.4, we have

$$\begin{aligned}
\widetilde{pr}_+(\alpha^+\mathcal{M}) = \widetilde{pr}_+(\alpha^+ j_+\mathscr{F}(\mathcal{F})) &\xrightarrow{\sim} \widetilde{pr}_+ j_+\alpha^+(\mathscr{F}(\mathcal{F})) \xrightarrow{\sim} j_+\widetilde{pr}_+\alpha^+(\mathscr{F}(\mathcal{F})) \\
&\xrightarrow{\sim} j_+\widetilde{pr}_+(\mathscr{F}(\widetilde{\mathcal{F}})) \xrightarrow{\sim} j_+(\mathscr{F}(\widetilde{pr}_+\widetilde{\mathcal{F}})),
\end{aligned}$$

where $\mathscr{F} := \mathscr{F}_\psi[1]\colon F\text{-}D^{\mathrm{b}}_{\mathrm{ovhol}}(\mathbb{A}^1_S) \to F\text{-}D^{\mathrm{b}}_{\mathrm{ovhol}}(\mathbb{A}^1_S)$ is the shifted Fourier transform for respectively $S = Y$, $S = \widetilde{Y}$ or $S = \operatorname{Spec} k$. Since $\mathscr{F}$ and j_+ are acyclic (see 2.2.3), then we get $\mathcal{H}^s \widetilde{pr}_+(\alpha^+\mathcal{M}) \xrightarrow{\sim} j_+(\mathscr{F}(\mathcal{H}^s \widetilde{pr}_+\widetilde{\mathcal{F}}))$. Since $\mathrm{rk}(\mathscr{F}(\mathcal{H}^s \widetilde{pr}_+\widetilde{\mathcal{F}})) = \mathrm{rk}(j_+(\mathscr{F}(\mathcal{H}^s \widetilde{pr}_+\widetilde{\mathcal{F}})))$ (recall the notation of 2.1.4), we can conclude. □

From Theorem 2.2.6, the reader can check the p-adic analogues of corollaries [5, 4.5.2–5] by copying the proofs. Moreover, from [3], we have a theory of weight in the framework of arithmetic $\mathcal{D}$-modules. For instance, we have checked the stability of the weight under Grothendieck six operations (i.e., the p-adic analogue of Deligne famous work in [21]), which is also explained in [5, 5.1.14]. In [5, 5.2.1], a reverse implication was proved. The reader can check that we can copy the proof without further problems (i.e., we only have to check that we have nothing new to check, e.g., we have already Lemma 2.1.5 or the purity of the middle extension of some pure unipotent F-isocrystal as given in [3, 3.6.3]). For the reader, let us write this p-adic version and its important corollary [5, 5.3.1] (this corollary is proved in [3] in another way, but Theorem 2.2.7 below is a new result).

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Theorem 2.2.7 [5, 5.2.1]. *We suppose $k = \mathbb{F}_{p^s}$ is finite and that F means the sth power of Frobenius. Choose an isomorphism of the form $\iota\colon \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$. Let X be a k-variety and $\mathcal{E} \in F\text{-}\mathrm{Ovhol}(X/K)$. We suppose that, for any etale morphism $\alpha\colon U \to X$ with U affine, the K-vector space $H^0(p_{U+}(\alpha^+(\mathcal{E}))$ is ι-mixed of weight $\geqslant w$. Then $\mathcal{E}$ is ι-mixed of weight $\geqslant w$.*

Corollary 2.2.8 [5, 5.3.1]. *With the notation 2.2.7, if $\mathcal{E}$ is ι-mixed of weight $\geqslant w$ (respectively $\leqslant w$), then any subquotient of $\mathcal{E}$ is ι-mixed of weight $\geqslant w$ (respectively $\leqslant w$).*

Finally, except [5, 5.4.7–8], the reader can check easily the other results of the Chapter 5 of [5] by translating the proofs in our p-adic context.

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References

1. T. Abe, Explicit calculation of Frobenius isomorphisms and Poincaré duality in the theory of arithmetic $\mathcal{D}$-modules, *Rend. Semin. Mat. Univ. Padova* **131** (2014), 89–149.
2. T. Abe and D. Caro, On Beilinson's equivalence for p-adic cohomology. 09 2013.
3. T. Abe and D. Caro, Theory of weights in p-adic cohomology. 03 2013.
4. T. Abe and A. Marmora, Product formula for p-adic epsilon factors, *J. Inst. Math. Jussieu* **14**(2) (2015), 275–377.
5. A. A. Beĭlinson, J. Bernstein and P. Deligne, Faisceaux pervers, in *Analysis and Topology on Singular Spaces, I (Luminy 1981)*, Astérisque, Volume 100, pp. 5–171 (Soc. Math. France, Paris, 1982).
6. P. Berthelot, Cohomologie rigide et théorie de Dwork: le cas des sommes exponentielles, *Astérisque,* (119–120) **3** (1984), 17–49. p-adic cohomology.
7. P. Berthelot, Cohomologie rigide et théorie des $\mathcal{D}$-modules, in *p-adic Analysis (Trento, 1989)*, pp. 80–124 (Springer, Berlin, 1990).
8. P. Berthelot, Altérations de variétés algébriques (d'après A. J. de Jong), *Astérisque,* (241), Exp. No. 815 **5** (1997), 273–311. Séminaire Bourbaki, Vol. 1995/96.
9. P. Berthelot, $\mathcal{D}$-modules arithmétiques. II. Descente par Frobenius, *Mém. Soc. Math. Fr. (N.S.)* **81** (2000), vi+136.
10. P. Berthelot, Introduction à la théorie arithmétique des $\mathcal{D}$-modules, *Astérisque* **279** (2002), 1–80. Cohomologies p-adiques et applications arithmétiques, II.
11. D. Caro, Lagrangianity for log extendable overconvergent F-isocrystals, *Math. Z.*, available online in December 2016, 15 pages.
12. D. Caro, Dévissages des F-complexes de $\mathcal{D}$-modules arithmétiques en F-isocristaux surconvergents, *Invent. Math.* **166**(2) (2006), 397–456.
13. D. Caro, Fonctions L associées aux $\mathcal{D}$-modules arithmétiques. Cas des courbes, *Compos. Math.* **142**(01) (2006), 169–206.
14. D. Caro, Overconvergent F-isocrystals and differential overcoherence, *Invent. Math.* **170**(3) (2007), 507–539.

15. D. Caro and N. Tsuzuki, Overholonomicity of overconvergent F-isocrystals over smooth varieties, *Ann. of Math. (2)* **176**(2) (2012), 747–813.
16. G. Christol and Z. Mebkhout, Sur le théorème de l'indice des équations différentielles p-adiques. IV, *Invent. Math.* **143**(3) (2001), 629–672.
17. G. Christol and Z. Mebkhout, Sur le théorème de l'indice des équations différentielles p-adiques. IV, *Invent. Math.* **143**(3) (2001), 629–672.
18. G. Christol and Z. Mebkhout, Équations différentielles p-adiques et coefficients p-adiques sur les courbes, *Astérisque* **279** (2002), 125–183. Cohomologies p-adiques et applications arithmétiques, II.
19. R. Crew, Finiteness theorems for the cohomology of an overconvergent isocrystal on a curve, *Ann. Sci. Éc. Norm. Supér. (4)* **31**(6) (1998), 717–763.
20. A. J. de Jong, Smoothness, semi-stability and alterations, *Inst. Hautes Études Sci. Publ. Math.* **83** (1996), 51–93.
21. P. Deligne, La conjecture de Weil. II, *Inst. Hautes Études Sci. Publ. Math.* **52** (1980), 137–252.
22. W. Fulton, *Intersection Theory*, second edition, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], Volume 2 (Springer, Berlin, 1998).
23. A. Grothendieck, Éléments de géométrie algébrique. I. Le langage des schémas, *Inst. Hautes Études Sci. Publ. Math.* **4** (1960), 228.
24. A. Grothendieck, Éléments de géométrie algébrique. II. Étude globale élémentaire de quelques classes de morphismes, *Inst. Hautes Études Sci. Publ. Math.* **8** (1961), 222.
25. A. Grothendieck, Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. I, *Inst. Hautes Études Sci. Publ. Math.* **20** (1964), 259.
26. A. Grothendieck, Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. II, *Inst. Hautes Études Sci. Publ. Math.* (24) (1965), 231.
27. A. Grothendieck, Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. III, *Inst. Hautes Études Sci. Publ. Math.* (28) (1966), 255.
28. A. Grothendieck, Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas IV, *Inst. Hautes Études Sci. Publ. Math.* (32) (1967), 361.
29. N. M. Katz and G. Laumon, Transformation de Fourier et majoration de sommes exponentielles, *Inst. Hautes Études Sci. Publ. Math.* **62** (1985), 361–418.
30. K. S. Kedlaya, Semistable reduction for overconvergent F-isocrystals on a curve, *Math. Res. Lett.* **10**(2–3) (2003), 151–159.
31. K. S. Kedlaya, More étale covers of affine spaces in positive characteristic, *J. Algebraic Geom.* **14**(1) (2005), 187–192.
32. K. S. Kedlaya, *p-adic Differential Equations*, Cambridge Studies in Advanced Mathematics, Volume 125 (Cambridge University Press, Cambridge, 2010).
33. K. S. Kedlaya, Semistable reduction for overconvergent F-isocrystals, IV: local semistable reduction at nonmonomial valuations, *Compos. Math.* **147**(2) (2011), 467–523.
34. S. Lang, *Algebra*, third edition, Graduate Texts in Mathematics, Volume 211 (Springer, New York, 2002).
35. G. Laumon, Transformations canoniques et spécialisation pour les rD-modules filtrés, *Astérisque* (130) (1985), 56–129. Differential systems and singularities (Luminy, 1983).
36. Q. Liu, *Algebraic Geometry and Arithmetic Curves*, Oxford Graduate Texts in Mathematics, Volume 6 (Oxford University Press, Oxford, 2002). Translated from the French by Reinie Erné, Oxford Science Publications.
37. P. Monsky and G. Washnitzer, Formal cohomology. I, *Ann. of Math. (2)* **88** (1968), 181–217.

38. C. Noot-Huyghe, Transformation de Fourier des $\mathcal{D}$-modules arithmétiques. I, in *Geometric Aspects of Dwork Theory. Vols I, II*, pp. 857–907 (Walter de Gruyter GmbH & Co., KG, Berlin, 2004).
39. *Revêtements étales et groupe fondamental (SGA 1).* Documents Mathématiques (Paris) [Mathematical Documents (Paris)], 3. Société Mathématique de France, Paris, 2003. Séminaire de géométrie algébrique du Bois Marie 1960–1961. [Geometric Algebra Seminar of Bois Marie 1960–1961], Directed by A. Grothendieck, With two papers by M. Raynaud, Updated and annotated reprint of the 1971 original [Lecture Notes in Mathematics, 224, Springer, Berlin].
40. A. Virrion, Dualité locale et holonomie pour les $\mathcal{D}$-modules arithmétiques, *Bull. Soc. Math. France* **128**(1) (2000), 1–68.