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ALGEBRAIC LIMIT CYCLES ON QUADRATIC POLYNOMIAL DIFFERENTIAL SYSTEMS

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Abstract Algebraic limit cycles in quadratic polynomial differential systems started to be studied in 1958, and a few years later the following conjecture appeared: quadratic polynomial differential systems have at most one algebraic limit cycle. We prove that a quadratic polynomial differential system having an invariant algebraic curve with at most one pair of diametrically opposite singular points at infinity has at most one algebraic limit cycle. Our result provides a partial positive answer to this conjecture.

Keywords: algebraic limit cycle; quadratic polynomial differential system; quadratic polynomial vector field

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1. Introduction

Let $\mathbb{R}[x,y]$ be the ring of all real polynomials in the variables x and y. Differential systems of the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \dot{x} = P(x,y), \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = \dot{y} = Q(x,y), \tag{1}$$

where $P, Q \in \mathbb{R}[x,y]$ with t real are called *real polynomial differential systems*. We say that system (1) has degree m if the maximum degree of the polynomials P and Q is m. When $m = 2$, system (1) is called *a quadratic system*.

The *polynomial vector field* associated with system (1) is

$$\mathcal{X} = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}. \tag{2}$$

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Let $g = g(x, y) \in \mathbb{R}[x, y]$. The algebraic curve $g(x, y) = 0$ of $\mathbb{R}^2$ is an *invariant algebraic curve* of the polynomial vector field $\mathcal{X}$ if for some polynomial $K \in \mathbb{R}[x, y]$, called the *cofactor*, we have

$$\mathcal{X}g = P\frac{\partial g}{\partial x} + Q\frac{\partial g}{\partial y} = Kg. \tag{3}$$

Note that $g = 0$ is *invariant* under the flow defined by $\mathcal{X}$.

An invariant algebraic curve $g = 0$ where g is irreducible in $\mathbb{R}[x, y]$ is called *irreducible*. A *limit cycle* of a real polynomial vector field $\mathcal{X}$ is an isolated periodic orbit in the set of all periodic orbits of $\mathcal{X}$. An *algebraic limit cycle* of degree n of $\mathcal{X}$ is an oval of a real irreducible invariant algebraic curve $g = 0$ of degree n which is a limit cycle of $\mathcal{X}$.

A point (x, y) of an algebraic curve $g = 0$ is called a *singular point* of the curve if $g(x, y) = g_x(x, y) = g_y(x, y) = 0$. In order to avoid confusion, in this paper we will distinguish between a *singular point* of a curve and an *equilibrium point* of the system, i.e. a point such that $P(x, y) = Q(x, y) = 0$. It is well known that a singular point of an invariant algebraic curve must be an equilibrium point of the system. The converse in general does not hold.

The following problem is a simpler version of the second part of Hilbert's 16th problem (see [**14**]). Let Σ_m be the set of all real polynomial vector fields (2) of degree m having invariant algebraic curves. Is there a uniform upper bound on the number of algebraic limit cycles of any polynomial vector field of Σ_m?

In [**19**], the authors give a positive answer to this question when all the invariant algebraic curves $g_j = 0$ of a vector field in Σ_m satisfy the following assumptions: $g_j = 0$ is a non-singular algebraic curve, the highest-order homogeneous terms of g_j have no repeated factors, if two curves intersect at a point in the affine plane they are transversal at this point, there are no more than two curves $g_j = 0$ meeting at any point in the affine plane, and there are no two curves having a common factor in the highest-order homogeneous terms. For related papers concerning this problem, see [**20**, **24**]. *It remains to know whether the invariant algebraic curves of a quadratic polynomial differential systems also have at most one algebraic limit cycle when they do not satisfy these generic conditions.*

In [**22**], it was proved that quadratic polynomial vector fields can have algebraic limit cycles of degree 2, and that they are unique whenever they exist. In [**10**–**12**], the author proved that quadratic vector fields do not have algebraic limit cycles of degree 3 (see also [**3**, **15**] for different and shorter proofs). In [**21**], the first class of algebraic limit cycles of degree 4 inside the quadratic vector fields was found. The second class was found in [**13**]. More recently, two new classes have been found, and in [**6**] the authors proved that there are no other algebraic limit cycles of degree 4 for quadratic vector fields. The uniqueness of these limit cycles was proved in [**2**]. It is known that there are quadratic polynomial differential systems having algebraic limit cycles of degrees 5 and 6, see [**6**], and that this limit cycle is the unique one for these differential systems.

It turns out that the problem mentioned above is too hard to deal with, which is why a simpler version of this problem has kept the attention of the researchers for many years. It is the following conjecture which appears explicitly in [**16**, **20**], but was known many years before among the mathematicians working in this subject.

Conjecture 1. *Quadratic polynomial differential systems have at most one algebraic limit cycle.*

In this paper, we will prove this conjecture for the case in which the invariant algebraic curve has at most one pair of diametrically opposite singular points at infinity. We note that if an invariant algebraic curve has one singular point at infinity, then its diametrically opposite point at infinity is also singular. In [**18**] the authors proved the conjecture in the case in which the quadratic polynomial differential systems have two pairs of equilibrium points at infinity (which may not be singular). In the following result, we restrict ourselves to the case in which the system has three pairs of singular points at infinity.

Theorem 2. *A quadratic polynomial differential system with three pairs of equilibrium points at infinity having an invariant algebraic curve with no singular points at infinity has at most one algebraic limit cycle.*

Theorem 3. *A quadratic polynomial differential system with three pairs of equilibrium points at infinity having an invariant algebraic curve with at most one pair of diametrically opposite singular points at infinity, these being the endpoints of the y-axis, has at most one algebraic limit cycle.*

We recall that an invariant algebraic curve $g = 0$ of a quadratic system may have at most four finite singular points, and at most three pairs of infinite singular points (owing to the fact that the singular points must be equilibrium points of the system). The known cases with algebraic limit cycles that are realized with invariant algebraic curves of degrees 5 or 6 are curves with singular points: they have one finite singular point (which is the origin) and one pair of infinite singular points. In the example of degree 6, the pair of infinite singular points are the endpoints of the y-axis. See [**6**] for the explicit expressions of the known algebraic limit cycles of degrees 5 and 6.

In view of Theorem 3, it remains to prove the conjecture in the case where the system has three pairs of equilibrium points at infinity, and some of its invariant algebraic curves have either a pair of diametrically opposite singular points which are not the endpoints of the x-axis or more than one pair of diametrically opposite singular points at infinity.

The proof of Theorem 2 is given in § 2, and the proof of Theorem 3 is given in § 3. In § 2 we also state some known facts about quadratic polynomial differential systems that we shall need.

2. Quadratic systems: preliminary results

The following theorems are well known. For a proof of the first one, see [**3**, **8**].

Theorem 4. *Quadratic polynomial differential systems having an algebraic limit cycle of degree two or four have at most one limit cycle.*

In view of Theorem 4, from now on *we will consider algebraic limit cycles of degree* $n \geqslant 5$.

The next result is proved in [**1**, **9**].

Theorem 5. *Quadratic polynomial differential systems having an invariant algebraic straight line have at most one limit cycle.*

Two results were proved in [**17**]. Proposition 8 of [**17**] states the following.

Proposition 6. *All the equilibrium points of the quadratic polynomial differential system (4) and all points satisfying* $\partial g/\partial x = \partial g/\partial y = 0$ *of an invariant algebraic curve* $g = 0$ *are contained in the union of* $\{K = 0\}$ *and* $\{g = 0\}$*, where* K *is the cofactor of* $g = 0$.

From Theorem 4 of [**17**] and its proof, and Theorem 2 of [**18**], we have the following theorem.

Theorem 7. *If a quadratic polynomial differential system has an algebraic limit cycle of degree* n*, then it can be transformed, through an affine change of variables and a scaling of the time, into one of the following two systems. First,*

$$\begin{aligned} \dot{x} &= \xi x - y + ax^2 + bxy, \\ \dot{y} &= x - \xi y + dx^2 + exy + fy^2, \end{aligned} \tag{4}$$

with $d \neq 0$ *and, second,*

$$\begin{aligned} \dot{x} &= -y + ax^2 + bxy + cy^2, \\ \dot{y} &= x + exy + fy^2. \end{aligned} \tag{5}$$

The cofactor of $g = 0$ *in both cases is* ny.

It is pointed out in [**17**] that in system (4) the limit cycle, if it exists, surrounds the origin.

Proposition 8. *Let* P_2 *and* Q_2 *be the homogeneous components of* P *and* Q*, respectively. If* $yP_2 - xQ_2 \equiv 0$*, then the quadratic system has no limit cycles.*

The following result is Lemma 11 in [**17**].

Proposition 9. *The invariant algebraic curve* $g = 0$ *must intersect the infinity at least in one point, eventually complex. All the intersection points must be equilibrium points of the extended vector field in the projective space.*

The following proposition, from Christopher [**5**], provides a result about the higher-degree terms of an invariant algebraic curve $g = 0$ of a polynomial differential system (1).

Proposition 10. *Suppose that a polynomial differential system (1) of degree* 2 *has the invariant algebraic curve* $g = 0$ *of degree* n*. Let* P_2*,* Q_2 *and* g_n *be the homogeneous components of* P, Q *and* g *of degree* 2 *and* n*, respectively. Then the irreducible factors of* g_n *must be factors of* $yP_2 - xQ_2$.

The proof of the next theorem can be found in [**5**] (see also Theorem A2 in [**2**]).

Theorem 11 (Christopher [5]). *The system*

$$\dot{x} = -y + a_1x + a_2x^2, \qquad \dot{y} = x(1 + a_3x + a_4y)$$

has at most one limit cycle surrounding the origin.

The proof of the next theorem can be found in [18] (see Theorem 2).

Theorem 12. *A quadratic polynomial differential system with at most one pair of equilibrium points at infinity has at most one limit cycle.*

In view of Theorem 12, in order to prove Theorem 3 we can restrict ourselves to the case in which the quadratic polynomial differential system has at least two pairs of equilibrium points at infinity.

The proof of the next theorem can be found in [18] (see Theorem 2).

Theorem 13. *A quadratic polynomial differential system with two pairs of equilibrium points at infinity has at most one limit cycle.*

In view of Theorem 13, in order to prove Theorem 3 we can restrict ourselves to the case in which the quadratic polynomial differential system has three pairs of equilibrium points at infinity.

2.1. Proof of Theorem 2

Let $g = 0$ be an invariant algebraic curve of degree $n \geqslant 5$ of the quadratic polynomial differential system (4). Then we can write $g = (\sum_{i=0}^{m} g_{n-i,i}x^{n-i}y^i) + \cdots$, $0 \leqslant m \leqslant n$ with $g_{n-m,m} \neq 0$, where the dot denotes terms of order $n-1$ and lower. From Proposition 10, we have

$$\sum_{i=0}^{m} g_{n-i,i}x^{n-i}y^i = x^{n-m}(x - x_1y)^k(x - x_2y)^{m-k}, \tag{6}$$

where

$$x_{1,2} = \frac{a - e \pm \sqrt{\Delta}}{2d} \quad \text{with} \quad \Delta = (a-e)^2 + 4d(b-f) \geqslant 0 \tag{7}$$

are the roots of the polynomial $dx^2 - (a-e)x - (b-f) = 0$. The case in which $\Delta \leqslant 0$ was proved in [18], because in that case there are at most two pairs of equilibrium points at infinity. Therefore, $x_1 \neq x_2$ and $x_i \neq 0$ for $i = 1, 2$. Note that in order that system (4) has no singular points at infinity, we must have

$$n - m \leqslant 1, k \leqslant 1, m - k \leqslant 1 \quad \text{that is} \quad m \leqslant k + 1 \leqslant 2,\ n \leqslant m + 1 \leqslant 3$$

which is not possible because $n \geqslant 5$. Hence this case is not possible.

Let $g = 0$ be an invariant algebraic curve of degree $n \geqslant 5$ of the quadratic polynomial differential system (5). Then we can write $g = (\sum_{i=0}^{m} g_{n-i,i}y^{n-i}x^i) + \cdots$ with $g_{n-m,m} \neq 0$, where the dots indicate terms of degree $n-1$ and lower. By Propositions 8

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and 10, we have that $\sum_{i=0}^{m} g_{n-i,i} y^{n-i} x^i = y^{n-m}(y - y_1 x)^k (y - x_2 x)^{m-k}$ with $0 \leqslant k \leqslant m$, where

$$y_{1,2} = \frac{b - f \pm \sqrt{\Delta_1}}{2d} \quad \text{with} \quad \Delta_1 = (b-f)^2 + 4c(a-e) \geqslant 0$$

are the roots of the polynomial $cy^2 + (b-f)y + (a-e) = 0$. The case in which $\Delta \leqslant 0$ was proved in [18], because in that case there are at most two pairs of equilibrium points at infinity. Therefore $y_1 \neq y_2$ and $y_i \neq 0$ for $i = 1, 2$. Note that in order that system (5) has no singular points at infinity, we must have

$$n - m \leqslant 1, k \leqslant 1, m - k \leqslant 1 \quad \text{that is,} \quad m \leqslant k + 1 \leqslant 2, \; n \leqslant m + 1 \leqslant 3$$

which is not possible because $n \geqslant 5$. Hence this case is not possible. This completes the proof.

3. Proof of Theorem 3

Let $g = 0$ be an invariant algebraic curve of degree $n \geqslant 5$ of the quadratic polynomial differential system (4). We write it as $g = (\sum_{i=0}^{m} g_{n-i,i} x^{n-i} y^i) + \cdots$, $0 \leqslant m \leqslant n$ with $g_{n-m,m} \neq 0$, where the dot denotes terms of order $n-1$ and lower. The coefficient of the term $x^{n-m} y^{m+1}$ in the expression of $\dot{g} = nyg$ is equal to

$$g_{n-m,m}((n-m)b + mf - n) = 0.$$

Therefore

$$b = (n - mf)/(n-m) \text{ if } m \neq n \quad \text{and} \quad f = 1 \text{ if } m = n.$$

From Proposition 10, it can be written as in (6) and (7), i.e.

$$\sum_{i=0}^{m} g_{n-i,i} x^{n-i} y^i = x^{n-m}(x - x_1 y)^k (x - x_2 y)^{m-k}.$$

Note that the endpoint of $y = 0$ is never a singular point of system (4) at infinity, and so system (4) cannot have any singular point at infinity; thus, this case is not possible.

So, we must assume that we have system (5). We will prove the following theorem.

Theorem 14. *Let $g = 0$ be an invariant algebraic curve with at most one singular point at infinity of the quadratic polynomial differential system (5). Then system (5) has at most one limit cycle.*

Proof. Let $g = (\sum_{i=0}^{m} g_{n-i,i} y^{n-i} x^i) + \cdots$ with $g_{n-m,m} \neq 0$, where the dots indicate terms of degree $n-1$ and lower. The coefficient of the term $y^{n-m} x^{m+1}$ in the expression $\dot{g} - nyg$ is equal to $am + e(n-m) = 0$. Therefore,

$$e = \frac{am}{m-n} \text{ if } m \neq n, \quad \text{and} \quad a = 0 \text{ if } m = n.$$

By Propositions 8 and 10 (we are assuming that the line at infinity is not formed by equilibrium points, otherwise the system cannot have a limit cycle), we have that

$\sum_{i=0}^{m} g_{n-i,i} y^{n-i} x^i = y^{n-m}(y - y_1 x)^k (y - x_2 x)^{m-k}$ with $0 \leqslant k \leqslant m$, where y_1, y_2 are the roots of $cy^2 + (b - f)y + (a - e) = 0$.

We now consider two different cases: $m = n$ and $m < n$.

Case 1*:* $m = n$. In this case $a = 0$. Proceeding as in [**17**], there are three reasons for having the condition $m = n$. First, we simply have chosen the wrong system of coordinates, and there is some other real singular point of the system at infinity through which $g = 0$ passes. In that case, the system can be transformed into system (4) with $m < n$.

The second reason for having $m = n$ is that all the branches of $g = 0$ go through non-real equilibrium points of the system at infinity. This means that $y_1 = \overline{y_2} \notin \mathbb{R}$. In that case, system (5) would have only a pair of equilibrium points at infinity, and in view of Theorem 12 it is proved that in this case the system has at most one limit cycle.

The third reason for having $m = n$ is that $c = 0$. Then system (5) becomes

$$\dot{x} = -y + bxy, \qquad \dot{y} = x + exy + fy^2.$$

If $b \neq 0$, system (5) has the invariant straight line $x = 1/b$ and, in view of Proposition 5, has at most one limit cycle. So, $b = 0$. Moreover, if $f = 0$, then either the system has no equilibrium points at infinity (if $e \neq 0$) or the line at infinity is formed by equilibrium points (if $e = 0$). In both cases it follows, respectively, from Propositions 9 and 8 that system (5) has no limit cycles.

In short, $b = 0$ and $f \neq 0$. In this case, $g_n = (y - (e/f)x)^n$. Imposing that

$$(exy + fy^2)\frac{\partial g_n}{\partial y} - nyg_n = 0,$$

we get

$$y\left(y - \frac{e}{f}x\right)^{n-1}\left(\frac{e(f+1)}{f}x + (f-1)y\right) = 0,$$

and so

$$f = 1 \quad \text{and} \quad e = 0.$$

It follows from Proposition 11 (making the change $x \to y$ and $y \to x$) that system (5) has at most one limit cycle.

Case 2*:* $m < n$. If $a = 0$ then $e = 0$, and it follows from Proposition 11 (making the change $x \to y$ and $y \to x$) that system (5) has at most one limit cycle.

We can thus assume that $a \neq 0$ and so $e = am/(m - n)$. We distinguish between the cases $m = 0$ (and then $k = 0$), $m = 1$ (with $k = 0$ or $k = 1$) and $m = 2$ (with $k = 1$). Note that if $c = 0$ then we only have the cases $m = k = 0$ and $m = 1$ with $k = 0$.

Subcase 2.1*:* $m = 0$. In this case, $g_n = y^n$ and imposing that g_n satisfies

$$(ax^2 + bxy + cy^2)\frac{\partial g_n}{\partial x} + \left(\frac{am}{m-n}xy + fy^2\right)\frac{\partial g_n}{\partial y} = nyg_n \tag{8}$$

with $m = 0$ we get $f = 1$. We write g in powers of x as $g = \sum_{j=0}^{k} g_j(y)x^j$ where $g_k \neq 0$ (otherwise $g = 0$, which is not possible). We obtain that the coefficient of x^{k+1} satisfies

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$kag_k(y) = 0$. Since $a \neq 0$ then $k = 0$, and so $g = g_0(y)$ and satisfies

$$(x + y^2)\frac{\mathrm{d}g_0}{\mathrm{d}y} = nyg_0.$$

The coefficient of x satisfies $\mathrm{d}g_0/\mathrm{d}y = 0$ and so g_0 is constant, which is not possible because it is an invariant algebraic curve. Therefore, this case is not possible.

Subcase 2.2*:* $m = 1$. We consider two different subcases, $c = 0$ and $c \neq 0$.

Subcase 2.2.1*:* $c = 0$. We have that $g_n = y^{n-1}(y + (an/(n-1)(b-f))x)$. Imposing that g_n satisfies (8) we get

$$\frac{(f-1)n}{(b-f)(n-1)}y^n(anx + (f - b + bn - fn)y) = 0$$

which yields, in particular, that $a = 0$; this is not possible.

Subcase 2.2.2*:* $c \neq 0$. We consider two different subcases: $k = 0$ and $k = 1$. In both cases $e = a/(1-n)$ and system (5) becomes

$$\begin{aligned} \dot{x} &= -y + ax^2 + bxy + cy^2, \\ \dot{y} &= x + \frac{a}{1-n}xy + fy^2. \end{aligned} \tag{9}$$

We have $g_n = y^{n-1}(y + ((b-f) \pm \sqrt{b^2 - 4ac - 2bf + f^2 + 4ac/(1-n)}/2c)x)$ (+ for $k =$ 0 and $-$ for $k = 1$). Imposing that g_n satisfies (8) we get

$$\begin{aligned} &\frac{1}{2c(n-1)}y^n\big((\pm(n-1)(b - f - n + fn)\sqrt{b^2 - 4ac - 2bf + f^2 + 4ac/(1-n)} \\ &\quad - b^2 + 2bf - f^2 + bn + b^2n - 2acn - fn - 3bfn + 2f^2n - bn^2 + fn^2 \\ &\quad + bfn^2 - f^2n^2)x \pm c(n-1)(\sqrt{b^2 - 4ac - 2bf + f^2 + 4ac/(1-n)} + b \\ &\quad - f - 2n + 2fn)y\big) = 0 \end{aligned}$$

and so either $c = 0$, or $a = 0, b = f + n - fn$, or $c = (f-1)(1-n)(b - f - n + fn)/a$. Since $ac \neq 0$, we must have $c = (f-1)(1-n)(b - f - n + fn)/a$.

Assume system (9) has a Darboux polynomial g with cofactor $K = ny$. Then writing g in powers of x as $g = \sum_{j=0}^{k} g_j(y)x^j$ we obtain that the coefficient of x^{k+1} satisfies

$$kag_k(y) + \left(1 + \frac{a}{1-n}y\right)\frac{\mathrm{d}g_k}{\mathrm{d}y} = 0$$

and so

$$g_k(y) = c_k(1 - n + ay)^{(k(n-1)}.$$

Therefore

$$g = c_kx^k(1 - n + ay)^{k(n-1)} + l.o.t$$

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where $l.o.t$ means terms of lower order in the variable x. Since the degree of g is n, we must have that

$$k + k(n-1) \leqslant n, \quad \text{which implies} \quad k \leqslant 1.$$

Then

$$g = xg_1(y) + g_0(y), \qquad g_1(y) = c_1(1-n+ay)^{n-1}, \qquad c_1 \in \mathbb{R}.$$

Moreover, computing the terms of x and the independent term, we get that $g_0(y)$ must satisfy

$$\begin{aligned}
&byg_1 + fy^2g_1'(y) + \Big(1 + \frac{a}{1-n}y\Big)g_0'(y) = nyg_1(y),\\
&\Big(\frac{(f-1)(1-n)(b-f-n+fn)}{a}y^2 - y\Big)g_1(y) + fy^2g_0'(y) = nyg_0(y).
\end{aligned}$$

We have

$$\begin{aligned}
g_0(y) = c_0 + \frac{c_1(n-1)}{a^2n(n-2)}&(1-n+ay)^{n-2}(-b(-2+n)(-1+n-ay)\\
&\times(1+ay) + (-2+n)n(-1+n-ay)(1+ay) + f(-1+n)\\
&\times(-2(1+ay)^2 + n(2+2ay+a^2y^2))).
\end{aligned}$$

From the third relation, we get

$$\begin{aligned}
&-c_0ny + \frac{c_1}{a^2(2-n)}y(1-n+ay)^{n-3}(-2a^2+2b-2f-2n+5a^2n-7bn\\
&\quad+8fn+7n^2-4a^2n^2+9bn^2-12fn^2-9n^3+a^2n^3-5bn^3+8fn^3+5n^4\\
&\quad+bn^4-2fn^4-n^5+(-4a^3+4ab-4af+2abf-2af^2-10abn+9afn\\
&\quad-7abfn+9af^2n+10an^2-2a^3n^2+8abn^2-afn^2+9abfn^2-16af^2n^2\\
&\quad-8an^3-2abn^3-11afn^3-5abfn^3+14af^2n^3+2an^4+9afn^4+abfn^4\\
&\quad-6af^2n^4-2afn^5+af^2n^5)y+(-2a^4+2a^2b-2a^2f+2a^2bf-4a^2f^2\\
&\quad-2a^2n+a^4n-3a^2bn+a^2fn-5a^2bfn+14a^2f^2n+3a^2n^2-4a^4n^2\\
&\quad+a^2bn^2+6a^2fn^2+4a^2bfn^2-18a^2f^2n^2-a^2n^3-7a^2fn^3-a^2bfn^3\\
&\quad+10a^2f^2n^3+2a^2fn^4-2a^2f^2n^4)y^2)=0.
\end{aligned}$$

Solving it we obtain that either $a = 0, b = (n^2 + 2fn - 2n - 2f)/(n-2)$, or $a = \pm\sqrt{b - n - bn + n^2}, f = 0$. The first case is not possible. For the second and third cases, we have that $f = 0$ and so, since $e \neq 0$ (because $a \neq 0$), $y = -1/e$ is an invariant straight line and, in view of Proposition 5, system (5) has at most one limit cycle. This completes the proof in this case.

Subcase 2.3*:* $m = 2$ and $k = 1$. We recall that $c \neq 0$ and $g_n = y^{n-2}(y^2 + (anx^2/c(n-2))x\, y + y^2)$. Imposing that g_n satisfies (8), we get

$$\frac{(b-f-n+fn)y^{n-1}}{c(n-2)}\big(anx^2 + (2f-2b+bn-fn)xy + c(n-2)y^2\big) = 0$$

which yields $b = f(1-n) + n$ because $ac \neq 0$.

Then $e = 2a/(2-n)$, $b = n + (1-n)f$, $f \neq 0$ (because $m \neq 0$) and system (5) becomes

$$\begin{aligned} \dot{x} &= -y + ax^2 + (n + f(1-n))xy + cy^2, \\ \dot{y} &= x + \frac{2a}{2-n}xy + fy^2. \end{aligned} \tag{10}$$

Assume system (10) has a Darboux polynomial g with cofactor $K = ny$. Then writing g in powers of x as $g = \sum_{j=0}^{k} g_j(y)x^j$ we obtain that the coefficient of x^{k+1} satisfies

$$kag_k(y) + \Big(1 + \frac{2a}{2-n}y\Big)\frac{\mathrm{d}g_k}{\mathrm{d}y} = 0$$

and so

$$g_k(y) = c_k\Big(1 + \frac{2a}{2-n}y\Big)^{k(n-2)/2}.$$

Therefore

$$g = c_k x^k\Big(1 + \frac{2a}{2-n}y\Big)^{k(n-2)/2} + l.o.t$$

Since the degree of g is n, we must have that

$$k + \frac{k(n-2)}{2} \leqslant n, \quad \text{which implies} \quad k \leqslant 2.$$

Then

$$g = x^2 g_2(y) + x_1 g_1(y) + g_0(y)$$

with $g_2(y) = c_2(1 + (2a/2-n)y)^{n-2}$, where $c_2 \in \mathbb{R}$. Moreover, computing the terms of x^2, x and the independent term, we get that f_1 and f_0 must satisfy

$$\begin{aligned} &ag_1(y) + 2(n + (1-n)f)yg_2(y) + \Big(1 + \frac{2a}{2-n}y\Big)\frac{\mathrm{d}g_1}{\mathrm{d}y} + fy^2\frac{\mathrm{d}g_2}{\mathrm{d}y} = nyg_2, \\ &(n + (1-n)f)yg_1(y) + 2(cy^2 - y)g_2 + \Big(1 + \frac{2a}{2-n}y\Big)\frac{\mathrm{d}g_0}{\mathrm{d}y} + fy^2\frac{\mathrm{d}g_1}{\mathrm{d}y} = nyg_1, \\ &(cy^2 - y)g_1(y) + fy^2\frac{\mathrm{d}g_0}{\mathrm{d}y} = nyg_0. \end{aligned} \tag{11}$$

Solving the first equation in (11), we get

$$\begin{aligned} g_1(y) = c_1(2-n+2ay)^{n/2-1} &+ \frac{c_2(n-2)}{a^2(n-4)}(2-n+2ay)^{n-3}((4+2f(n-3) \\ &- n)(n-2) + a(4+2f(n-3)-n)(n-4)y - 2a^2(f-1)(n-4)y^2), \end{aligned}$$

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with $c_1 \in \mathbb{R}$, and solving the second equation in (11), we obtain

$$\begin{aligned} g_0 = {} & \frac{n-2}{a^2(n-4)}\left(\frac{c_1}{4}\left(\frac{f}{n}(14n-8-3n^2)+2(n-4)\right)(2-n+2ay)^{n/2}\right. \\ & \left. + f(n-2)^3(2-n+2ay)^{n/2-2} - 2(f-1)n(n-4)(2-n+2ay)^{n/2-1}\right) \\ & + \frac{c_2}{8a^2}\left(\frac{2ac+(f-1)(f(n-1)-n)(n-2)(n-4)}{n}(2-n+2ay)^n\right. \\ & + \frac{f^2(n-3)(n-2)^6}{n-4}(2-n+2ay)^{n-4} + \frac{2f(n-2)^4(8f-3(f+1)n+n^2)}{n-3} \\ & \quad (2-n+2ay)^{n-3} + (-4a^2(n-4)+2ac(8+n(n-6))-(n-2)(-(n-4)n^2 \\ & + f(8+n(n(15-2n)-34)) + 2f^2(6+n(n(n-5)+5))))(2-n+2ay)^{n-2} \\ & \left. - \frac{2(n-4)}{n-1}(2a^2-2ac(n-2)-(f-1)(f+1-n)(n-2)n)(2-n+2ay)^{n-1}\right) \\ & + c_0, \end{aligned}$$

with $c_0 \in \mathbb{R}$.

We write the last relation in (11) as

$$\mathcal{A}(y) := (cy^2-y)g_1(y) + fy^2\frac{\mathrm{d}g_0}{\mathrm{d}y} - nyg_0 = 0.$$

Note that $\mathcal{A}(y)$ is a polynomial in the variable y. Computing in $\mathcal{A}(y)=0$ the coefficient of the highest-order term in y (which is y^{n+1}), we get

$$2^{n-3}a^{n-4}c_2(f-1)^2(n-2)^2((n-1)f-n) = 0.$$

Since $a(n-2)\neq 0$, we have three possibilities: $c_2=0$, or $f=1$ or $f=n/(n-1)$. We consider the three subcases separately.

Subcase 2.3.1*:* $c_2=0$. In this case

$$\begin{aligned} \mathcal{A}(y)|_{c_2=0} = {} & -c_0ny - \frac{c_1}{a^2(n-4)}y(2-n+2ay)^{n/2-3}\big(-a^3(n-4)\big(-3n^2f^2 \\ & + 8nf^2 - 4f^2 + 8n^2f - 20nf + 8f - 4n^2 + 4ac + 8n\big)y^3 \\ & + 2a^2(n-4)\big(-f^2n^3 + 4fn^3 - 2n^3 + 5f^2n^2 - 21fn^2 + 10n^2 \\ & - 8f^2n + 2acn + 32fn - 12n + 2a^2 + 4f^2 - 4ac - 12f\big)y^2 \\ & - a(n-2)\big(2fn^4 - n^4 - 24fn^3 + 12n^3 + acn^2 + 92fn^2 - 44n^2 \\ & + 4a^2n - 6acn - 128fn + 48n - 16a^2 + 8ac + 48f\big)y \\ & + (n-2)^2\big(-2fn^3 + n^3 + 12fn^2 - 6n^2 + a^2n - 20fn + 8n \\ & - 4a^2 + 8f\big)\big). \end{aligned}$$

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Note that n must be even and $n \geqslant 5$, so $n \geqslant 6$. Then we compute the coefficients of $y^2, y^3, y^4, \dots$ for $n > 6$ and the coefficients of y, y^2, y^3, y^4 when $n = 6$. Equating to zero the set of these coefficients and using that $f \neq 0$ and c, a are real, we obtain that this system has no solution. So this subcase is not possible.

Subcase 2.3.2*:* $c_2 \neq 0$ *and* $f = 1$. In this case

$$\begin{aligned}
\mathcal{A}(y)|_{f=1} = {}& -c_0 n y - \frac{c_2(n-3)(n-2)^7 n}{8a^4(n-4)^2} y(2-n+2a)^{n-4} + \frac{c_2(n-2)^5 n}{4a^4(n-4)(n-3)} \\
& (-8+6n-n^2)y(2-n+2a)^{n-3} + \frac{c_2 n(n-2)}{8a^4(n-4)}(-40-16a^2-16ac \\
& + 68n + 4a^2 n + 12acn - 42n^2 - 2acn^2 + 11n^3 - n^4)y(2-n+2a)^{n-2} \\
& + \frac{c_2(n-2)n(a+2c-cn)}{2a^3(n-1)} y(2-n+2a)^{n-1} - \frac{cc_2(n-2)}{4a^3} \\
& y(2-n+2a)^n + c_1(-1+cy)y(2-n+2a)^{n/2-1} + c_1(n-2)(4(1-n) \\
& + n^2 + (2-n)ay)y^3(2-n+2a)^{n/2-3} - \frac{c_2(n-2)(-1+cy)}{a^2(n-4)} \\
& (4(n-1) - n^2 + (6n-8)ay - an^2 y)y(2-n+2a)^{n-3} + \frac{c_1(n-2)}{a^2(n-4)} \\
& (8-12n+6n^2-n^3 + a(16-20n+8n^2-n^3)y + a^2(8-6n+n^2)y^2) \\
& y(2-n+2a)^{n/2-2} + \frac{c_2(n-2)}{a^2(n-4)}(32(a^2-1) + 80n - 40a^2 n - 80n^2 \\
& + 16a^2 n^2 + 40n^3 - 2a^2 n^3 - 10n^4 + n^5 + (-80a + 64a^3 - 32a^2 c \\
& + 176an - 48a^3 n + 40a^2 cn - 152an^2 + 8a^3 n^2 - 16a^2 cn^2 + 64an^3 \\
& + 2a^2 cn^3 - 13an^4 + an^5)y + 8a^2(4a^2 - 8ac - a^2 n + 6acn - acn^2)y^2 \\
& + 8a^4 c(n-4)y^3)y^3(2-n+2a)^{n-5}.
\end{aligned}$$

Computing the coefficient of the highest power in y, which is y^n, we get

$$\frac{2^{n-2}c_2(n-2)(a+4c(1-n)+cn^2)a^{n-4}}{n-1} = 0.$$

Taking into account that $c_2(n-2)a \neq 0$, we must have

$$a = -c(n-2)^2.$$

Now, computing the coefficient of y^{n-1}, we obtain

$$-\frac{2^{n-1}c_2^{n+1}(n-2)^{n-4}(n-5/2)}{c^3(n-3)} = 0$$

and this is not possible. So this case is not possible.

Subcase 2.3.3*:* $c_2 \neq 0$ *and* $f = n/(n-1)$. In this case, $\mathcal{A}(y)|_{f=n/(n-1)}$ is very large. We only write the coefficient of its highest power in y, which is y^n, and we get

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$2^{n-2}c_2(n-3)(-1)^n(n-2)^{2n-7}c^{n-3}$, which is never zero. So this subcase is not possible. This completes the proof of Case 2 and concludes the proof of Theorem 14. □

The proof of Theorem 3 for system (5) follows directly from Theorem 14.

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