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# Transitive Anosov flows and Axiom-A diffeomorphisms

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*Abstract*. Let $M$ be a smooth compact Riemannian manifold without boundary, and $\phi : M \times \mathbb{R} \to M$ a transitive Anosov flow. We prove that if the time-one map of $\phi$ is $C^1$-approximated by Axiom-A diffeomorphisms with more than one attractor, then $\phi$ is topologically equivalent to the suspension of an Anosov diffeomorphism.

## 0. *Introduction*

A flow $\phi$ on a closed manifold $M$ is called an *Anosov flow* if it is *hyperbolic*: the transverse bundle of the flow splits into two invariant bundles $E^s$ and $E^u$, where the vectors in $E^s$ and $E^u$ are exponentially contracted and expanded, respectively, by the action of the flow. In the same way, a diffeomorphism on compact manifold is an *Anosov diffeomorphism* if the whole manifold is a hyperbolic set.

The global hyperbolic structure of Anosov flows or diffeomorphisms is a very strong geometric property. In fact, manifolds carrying such dynamics satisfy strong topological conditions and the list of known examples is not so long. For example, all the known Anosov diffeomorphisms are conjugated to algebraic automorphisms of infranil manifolds, and Franks [**8**] and Newhouse [**15**] proved that any codimension-one Anosov diffeomorphism (i.e. the stable bundle $E^s$ of $f$ has codimension one) is conjugated to a linear automorphism of the torus $T^n$. In the same way Verjovsky conjectured in [**24**], that every codimension-one Anosov flow on a compact manifold $M$ with $\dim M \geq 4$, is topologically equivalent to the suspension of an Anosov diffeomorphism. There are many partial results in the direction of this conjecture, and the proof of the conjecture for conservative flows has been recently announced by Simić [**21**].

However, in dimension three there are many examples of Anosov flows with unexpected behavior: for example there are non-transitive Anosov flow, see [**9**]; see also [**5**]. We refer to [**2**] and [**7**] for works in the direction of a classification of transitive Anosov flows in dimension three.

So, a natural problem is the following.

*Problem 1.* Give a dynamical characterization of Anosov flows which are obtained by the suspension of Anosov diffeomorphisms.

The time-one map $f_1 = \phi(\cdot, 1)$ of an Anosov flow $\phi$ is not hyperbolic, since the tangent direction of the orbits of the flow is neither contracted nor expanded by the differential of the iterates of $f_1$. Hence $f_1$ is not structurally stable: small perturbations of $f_1$ may be not conjugated to $f$. However [**11**] shows that perturbations of $f_1$ only change the dynamic along the orbit of the flow $\phi$: every diffeomorphism $g$, sufficiently $C^1$-close to $f_1$, is conjugated to a homeomorphism of the form $x \mapsto \phi(x, \tau_g(x))$, where $\tau_g(x) > 0$ is a continuous function. In this paper we investigate the dynamics of diffeomorphisms $C^1$-close to $f_1$.

*Question 1.* What kind of dynamical system can appear under perturbations of the time-one map of an Anosov flows?

We are specially interested in the case of a *transitive* Anosov flow (i.e. the case when the non-wandering set of $\phi$ is the whole manifold). More specifically we would like to understand when the time-one map of a transitive Anosov flow can be approximated by hyperbolic (i.e. Axiom-A) diffeomorphisms.

At the end of the sixties, Abraham and Smale [**1**] constructed a diffeomorphism $f : T^2 \times S^2 \to T^2 \times S^2$ and a $C^1$-neighborhood $N(f)$ of $f$, such that if $g \in N(f)$ then the non-wandering set of $g$ is non-hyperbolic: there is no Axiom-A diffeomorphism in $N(f)$. This example can be extended in a simple way to higher dimensions, and has been extended to dimension three in [**22**]. Later, $C^1$-open sets of non-Axiom-A and robustly transitive diffeomorphisms were described on $T^4$ by Shub (see [**19**]), on $T^3$ by Mañé (see [**14**]). All these examples show that Axiom-A diffeomorphisms are not dense, if the dimension of the manifold is greater or equal to three (it is not known if Axiom-A diffeomorphisms are $C^1$-dense in surfaces). In other words, many diffeomorphisms cannot be approximated by Axiom A, and an ambitious general question is to characterize diffeomorphisms in the $C^1$-closure of the set of Axiom-A diffeomorphisms.

Another example, built in [**3**], gives a partial answer of Question 1: for any transitive Anosov flow $\phi$, any $C^1$-neighborhood of the time-one map $f_1$ of $\phi$ contains a (non-empty) open set of non-hyperbolic and transitive diffeomorphisms.

The time-one map $f_1$ of an Anosov flow $\phi$ belongs to the closure of Axiom-A diffeomorphisms in the case that $\phi$ arises from the suspension of an Anosov diffeomorphism $g : N \to N$. Let us explain it.

The suspension manifold $N_g$ is obtained from the direct product $N \times [0, 1]$ by identifying pairs of points of the form $(x, 0)$ and $(g(x), 1)$ for $x \in N$. The suspension flow $\varphi(x, t)$ is determined by the vector field $\partial/\partial t$. The suspension of an Anosov diffeomorphism is an Anosov flow in the corresponding manifold. Besides, if the diffeomorphism is transitive so is its suspension.

The manifold $N_g$ is fibered over $S^1$ and the projection of the time-one map onto $S^1$ is the identity map. Let $f$ be a diffeomorphism preserving fibers, $C^1$-close to $\varphi(x, 1)$ such that the projection of $f$ over $S^1$ is a Morse–Smale map. We have that $f$ is an Axiom-A diffeomorphism.

Palis and Pugh asked the following question in [**16**, Problem 20].

*Question 2.* Can the time-one map of an Anosov flow be approximated by an Axiom-A diffeomorphism? If the flow is a suspension of an Anosov diffeomorphism the answer is yes.

Our main result is as follows.

THEOREM 1. *Let $M$ be a smooth compact Riemannian manifold without boundary. If the time-one map of a transitive Anosov flow $\phi$ is $C^1$-approximated by Axiom-A diffeomorphisms having more than one attractor, then $\phi$ is topologically equivalent† to the suspension of a hyperbolic diffeomorphism.*

Indeed we prove a slightly stronger statement: if $\phi$ is not topologically equivalent to a suspension, and if $f$ is an Axiom-A diffeomorphism $C^1$-close to the time-one map of $\phi$, then the unique transitive attractor and the unique transitive repeller of $f$ are connected (hence topologically mixing).

A partial result was given previously in [**10**] for codimension-one Anosov flows under some technical and restrictive assumptions related to periodic points: we asked that the number of periodic points of the Axiom-A diffeomorphism in a 'fundamental domain' of any closed central leaf is constant (see [**10**] for more details). This technical hypothesis allowed us to prove that a repeller basic set was a global section, so that the initial Anosov flow was topologically equivalent to a suspension.

An important improvement is that we remove not only this 'technical' assumption but the codimension-one hypothesis as well. Although in the general case the proof is quite different, we include some results that had appeared in [**10**]. This is done because they contain basic ideas of the proof of the main theorem.

0.1. *A stronger version of Theorem 1.* Let us restate our main result in a slightly stronger version.

Denote by $\mathcal{F}^c$ the one-dimensional foliation of $M$ whose leaves are the orbits of $\phi$. Consider the set $\mathcal{E}_\phi$ of diffeomorphisms $f: M \to M$ satisfying the following hypotheses.

- $f$ is a partially hyperbolic diffeomorphism with one-dimensional central direction: there exists a $Df$-invariant splitting $TM = E^s \oplus E^c \oplus E^u$, such that $Df|E^s$ is uniformly contracting, $Df|E^u$ is uniformly expanding, and $E^c$ is a non-hyperbolic central direction with $\dim(E^c) = 1$.
- There is an $f$-invariant foliation $\mathcal{F}^c_f$ tangent to $E^c$.

† Some authors use *orbit equivalent* instead of topologically equivalent.

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- The central foliation $\mathcal{F}_f^c$ is topologically conjugated to $\mathcal{F}^c$: there is a homeomorphism $h\colon M \to M$ such that $h(\mathcal{F}^c) = \mathcal{F}_f^c$.
- Each leaf $F_f^c(x)$ is invariant by $f$. Furthermore, the distance $d^c(x, f(x))$ in the leaf $F_f^c(x)$ is uniformly bounded on $M$: there is $K_f > 0$ such that for any $x \in M$ there is a path $\gamma \subset F_f^c(x)$ with length $\ell(\gamma) < K_f$ joining $x$ to $f(x)$ in the central leaf $F_f^c(x)$.

We will show that for $f \in \mathcal{E}_\phi$ there is a continuous function $\tau(x) \neq 0$ such that $f$ is conjugated to the homeomorphism $x \mapsto \phi(x, \tau(x))$.

Hirsch, Pugh and Shub (see §1) proved that $\mathcal{E}_\phi$ contains a $C^1$-neighborhood of the time-one map $f_1$ of the flow $\phi$. Hence Theorem 1 is a direct consequence of the following.

THEOREM 2. *Let $\phi$ be a transitive Anosov flow on a smooth compact Riemannian manifold $M$ without boundary. If the set $\mathcal{E}_\phi$ contains Axiom-A diffeomorphisms which have more than one attractor, then $\phi$ is topologically equivalent to the suspension of a hyperbolic diffeomorphism.*

In a future work, we will show that for every transitive Anosov flow on a 3-manifold $M$, the set $\mathcal{E}_\phi$ contains an Axiom-A diffeomorphism whose non-wandering set is the union of only two basic sets. One of them is an attractor and the other is a repeller set.

0.2. *Sketch of the proof of Theorem 2 and organization of the paper.* Assume that $f \in \mathcal{E}_\phi$ is an Axiom-A diffeomorphism.

Using arguments in [**10**] we prove in §3 the following.

- Every attracting or repelling basic set of $f$ meets every central segment (i.e. segment of $\mathcal{F}_f^c$) of length $K_f$; in particular they meet every compact central leaf.
- The local central direction is contracting for every point in any attracting basic set and it is expanding for the points in any repelling basic set.

Then, in §4 we prove the following.

- Let $A$ be an attracting basic set and $W^s(A)$ its basin. Then there is an open and dense subset $U$ of $W^s(A)$ such that for $x \in U$ the connected component of $F_f^c(x) \cap W^s(A)$ containing $x$ meets $A$. Furthermore, there is a residual subset of $U$ (hence of $W^s(A)$) for which this connected component meets $A$ in precisely one point.

Using these properties, in §5, we prove that for any attractor $A$ there are two repellers $\Lambda_-$ and $\Lambda_+$, called the predecessor and the successor of $A$, respectively, such that for generic point $x \in W^s(A)$ the closure of the connected component of $F_f^c(x) \cap W^s(A)$ containing $x$ is a segment $[a(x), b(x)]$ of central leaf with $a(x) \in \Lambda_-$ and $b(x) \in \Lambda_+$. An analogous fact holds for generic points in the basin of any repeller.

As $f$ is an Axiom-A diffeomorphism, the union of the basin of attracting basic sets is a dense open subset of $M$. As a consequence one proves that for a generic point $x$ of $M$, the intersection of the (dynamically oriented) central leaf with the union of attracting and repelling basic sets of $f$ form a sequence $x_i$ which belongs alternately to attracting and repelling basic sets; furthermore if $x_i$ belongs to an attractor $A$ (respectively a repeller $\Lambda$), then $x_{i+1}$ belongs to the repeller $\Lambda_+$ (respectively the attractor $A_+$) which is the successor of $A$ (respectively of $\Lambda$) (see Lemma 5.3).

This relation of successor induces a family of cycles in the set of attracting and repelling basic sets of $f$. We prove that there is a unique cycle (Lemma 5.4); in other words there is an indexation $A_i$, $\Lambda_i$ of the set of attractors and repellers such that $\Lambda_i$ is the successor of $A_i$ and the predecessor of $A_{i+1}$. As a consequence, generic central leaves intersect the attractors and repellers following this cycle order. One deduces (Proposition 5.1) that the same holds for all the central leaves, up to allowing repetitions: a leaf may cut an attractor in more than one point before crossing the successor.

Assume now that $f$ has more than one attracting basic set (or equivalently more than one hyperbolic repelling basic set). We show (Corollary 7.1) that the boundary of the basin of a repelling basic set, $\Lambda_0$, is the union of two disjoint compact sets $K_0$ and $K_1$, which verifies that $A_0 \subset K_0$ and $A_1 \subset K_1$.

Then we build a continuous and surjective function $\rho\colon M \to S^1 = \mathbb{R}/\mathbb{Z}$, non-homotopic to a constant map, mapping $K_0$ on $0 = 1$ and $K_1$ on $1/2$. Since every segment of central leaf of length greater than $K_f$ meets every attractor $A_i$, then there exists $L$ such that every segment of central leaf $\gamma$ of length greater than $L$ verifies that $\rho(\gamma) = S^1$.

Let $\Pi\colon \tilde{M} \to M$ be the infinite cyclic cover of $M$, obtained by pullback by $\rho$ of the universal cover $\mathbb{R} \to S^1$. Consider the lift $\tilde{\mathcal{F}}^c_f$ of the foliation $\mathcal{F}^c_f$ on this cyclic cover. We prove that the leaves of the foliation $\tilde{\mathcal{F}}^c_f$ are going uniformly from $-\infty$ to $+\infty$ (Lemma 7.6). The same holds for the lift of the initial foliation $\mathcal{F}^c$ generated by the Anosov flow $\phi$. An argument of Schwartzman (see [**18**] or [**25**]) allows us to conclude that the flow $\phi$ admits a global section, hence it is topologically equivalent to the suspension of an Anosov diffeomorphism.

## 1. *Definitions and classical results*

We begin by recalling some basic definitions about flows and diffeomorphisms.

### 1.1. *Anosov flows.*

A good reference for basic properties of Anosov flows is [**25**].

*Definition 1.1.* Let $\phi\colon M \times \mathbb{R} \to \mathbb{R}$ be a $C^1$-flow on a compact Riemannian manifold $M$. A compact $\phi_t$-invariant set, $\Lambda \subset M$, is called a *hyperbolic set for the flow* $\phi$ if there exist $C > 0$ and $0 < \lambda < 1 < \mu$ such that for all $x \in \Lambda$ there is a decomposition

$$T_x M = E^{ss}_x \oplus E^{uu}_x \oplus E^c_x$$

such that $\partial_t \phi(x, t)|_{t=0} \in E^c_x - \{0\}$, $\dim(E^c(x)) = 1$, $D_x\phi_t(x)(E^i_x) = E^i_{\phi(x,t)}$, with $i = ss$, $uu$, and

$$\|D_x\phi(x, t)|_{E^{ss}(x)}\| \leq C\lambda^t \quad \text{for } t \geq 0,$$
$$\|D_x\phi(x, t)|_{E^{uu}(x)}\| \leq C\mu^t \quad \text{for } t \leq 0.$$

A $C^1$ flow $\phi : M \times \mathbb{R} \to M$, is called an *Anosov flow* if $M$ is a hyperbolic set for $\phi$.

Let $\phi$ be an Anosov flow on a compact connected manifold $M$. The bundles $E^{ss}$ and $E^{uu}$ are called strong stable and strong unstable bundles of $\phi$. We fix $k$ such that $\dim E^{ss}_x = n - k - 1$ and $\dim E^{uu}_x = k$ for all $x \in M$.

We call $E^{cs} = E^{ss} \oplus E^c$ and $E^{cu} = E^{uu} \oplus E^c$ the *central stable* and *central unstable bundles*, respectively.

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There are $\phi$-invariant foliations $\mathcal{F}^{cs}$, $\mathcal{F}^{cu}$, $\mathcal{F}^{ss}$, $\mathcal{F}^{uu}$ and $\mathcal{F}^{c}$ tangent to the bundles $E^{cs}$, $E^{cu}$, $E^{ss}$, $E^{uu}$ and $E^{c}$, respectively, and they are called center stable, center unstable, strong stable, strong unstable and central foliations, respectively. The leaves of these foliations are $C^1$-manifolds varying continuously in the $C^1$ topology, but in general they fail to be $C^1$-foliations: the holonomy maps are in general not $C^1$.

The leaves of the central foliation (called the central leaves) are the orbits of $\phi$, so that the central foliation $\mathcal{F}^c$ is a $C^1$ foliation. For any point $x \in M$ the strong stable leaf $F^{ss}(x)$ is the stable manifold $W^{ss}(x)$ of $x$, that is the set of points $y$ such that the distance $d(\phi(x,t), \phi(y,t))$ tends to 0 when $t$ tends to $+\infty$. The central stable leaf $F^{cs}(x)$ is the stable manifold of the orbit of $x$, that is union of the strong stable leaves through the orbit of $x$:

$$F^{cs}(x) = \bigcup_{y \in F^c(x)} F^{ss}(y).$$

In the same way, the central unstable leaf of $x$ is the union of the strong unstable leaves through the orbit of $x$:

$$F^{cu}(x) = \bigcup_{y \in F^c(x)} F^{uu}(y).$$

In particular, if $\mathcal{O}$ is a closed orbit of $\phi$ then, for $x \in \mathcal{O}$ the central stable and central unstable leaves $F^{cs}(x)$ and $F^{cu}(x)$ are the stable and unstable manifolds $W^s(\mathcal{O})$ and $W^u(\mathcal{O})$ of the closed orbit $\mathcal{O}$, respectively.

*Remark 1.1.* Let $\phi$ be an Anosov flow on a closed manifold $M$. The set of closed orbits of $\phi$ is countable (in fact, for every $K > 0$, the set of closed orbits of period bounded by $K$ is finite). In particular, the complement of the union of the periodic orbits is connected.

Assume now that the Anosov flow $\phi$ is transitive. Then we have the following.

- The periodic orbits of $\phi$ are dense in $M$. In other words,

$$\{x \mid F^c(x) \text{ is a closed set}\} \text{ is dense in } M.$$

- Generic points of $M$ have a dense forward and backward orbit. That is,

$$\{x \mid F^c(x) \text{ is dense in } M\} \text{ is a residual set.}$$

- For every point $x \in M$ the central stable and central unstable leaves $F^{cs}(x)$ and $F^{cu}(x)$ are both dense in $M$.

1.2. *Axiom-A diffeomorphisms.* The proof of Theorems 1 and 2 uses many properties of Axiom-A diffeomorphisms. Here we just recall briefly some basic definitions and properties of Smale's hyperbolic theory. The reader will find more complete information on hyperbolic dynamics in [**12**, Part 4, **17**, Ch. 0, **20**, **23**].

*Definition 1.2.* Let $f : M \to M$ be a $C^r$ diffeomorphism. An $f$-invariant set $\Lambda$ is called *hyperbolic* if there exists a $Df$-invariant decomposition of $T_\Lambda M$ such that

$$T_\Lambda M = E^s \oplus E^u$$

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and $Df|E^s$ is uniformly contracting and $Df|E^u$ is uniformly expanding. More precisely, there are $c > 0$ and $\lambda$, with $0 < \lambda < 1$ such that for all $x \in \Lambda$

$$\|D_x f^n | E^s(x)\| < c\lambda^n$$

and

$$\|D_x f^{-n} | E^u(x)\| < c\lambda^n.$$

A diffeomorphism $f : M \to M$ is called an *Anosov diffeomorphism* if $M$ is a hyperbolic set for $f$.

A diffeomorphism $f : M \to M$ satisfies the *Axiom A* if the non-wandering set $\Omega(f)$ is hyperbolic and the set of periodic points is dense in $\Omega(f)$.

A compact hyperbolic set $K$ of $f$ is called a *basic set* if it is *transitive* (i.e. there is a point $x \in K$ whose positive orbit is dense in $K$) and it is the maximal invariant set of $f$ in an open neighborhood $U$ of $K$, i.e. $K = \bigcap_{n\in\mathbb{Z}} f^n(U)$. The stable manifold $W^s(K)$ of a basic set $K$ is the set of points whose $\omega$-limit (limit of the forward orbit) is contained in $K$; according to the shadowing lemma, $W^s(K)$ is the union of the stable manifolds $W^s(x)$ of the points $x \in K$, where the stable manifold of $x$ is the set of points $y$ such that the distance $d(f^n(x), f^n(y))$ tends to 0 when $n$ tends to $+\infty$. The stable manifold $W^s(x)$ is $C^1$-immersion of $E^s(x)$, and it is tangent at $x$ to $E^s(x)$. Furthermore it depends continuously on $x$ for the $C^1$ topology.

Smale proved the following in [**23**].

- The non-wandering set of an Axiom-A diffeomorphism is the union of finitely many disjoint basic sets $K_i$.
- For any point $x \in M$, there exist $i, j$ such that $x$ belongs to the stable manifold of $K_i$ and to the unstable manifold of $K_j$.
- For any $i$, the intersection between the stable and the unstable manifold of $K_i$ is equal to $K_i$:
$$W^s(K_i) \cap W^u(K_i) = K_i.$$
- Some of the $K_i$ are *attractors*: this means that $K_i$ admits an open neighborhood $U$ such that $f(\bar{U}) \subset U$ and such that $K_i = \bigcap_{n>0} f^n(U)$. The basin (i.e. stable manifold) of the attractors are open sets whose union is dense in $M$.
- A *repeller* of $f$ is an attractor of the Axiom-A diffeomorphism $f^{-1}$.

Furthermore, a basic set $K$ is an attractor if and only if it contains its unstable manifold, that is $K = W^u(K)$.

*Remark 1.2.*

(1) A transitive hyperbolic attractor (i.e. an attracting basic set) has finitely many connected components, which are exchanged by $f$.
(2) For any $n > 0$, $f^n$ is an Axiom-A diffeomorphism.
(3) Any transitive hyperbolic attractor of $f^n$ is the orbit by $f^n$ of a connected component of an attractor of $f$.
(4) Hence, there is $n > 0$ such that each transitive attractor and repeller of $f^n$ is connected.

1.3. *Partially hyperbolic diffeomorphisms.* We refer to [**11**] and to [**4**, Appendix B] for the basic properties of partially hyperbolic dynamics.

*Definition 1.3.* A $C^1$ diffeomorphism $f : M \to M$ is called *partially hyperbolic* if there exists a $Df$-invariant decomposition of

$$T_x M = E^s_x \oplus E^c_x \oplus E^u_x$$

such that the dimensions of the spaces $E^s_x$, $E^c_x$, and $E^u_x$ do not depend on $x \in M$, furthermore $Df|E^s$ is uniformly contracting, $Df|E^u$ is uniformly expanding, and the expansion in $E^c$ is stronger than the expansion in $E^s$ and less than the expansion in $E^u$. More precisely, there are $c > 0$ and $\lambda$, with $0 < \lambda < 1$ such that for all $x \in \Lambda$

$$\|D_x f^n|E^s(x)\| < c\lambda^n,$$
$$\|D_x f^{-n}|E^u(x)\| < c\lambda^n,$$
$$\|D_x f^n|E^s(x)\|\|D_x f^{-n}|E^c(x)\| < c\lambda^n, \quad \text{and}$$
$$\|D_x f^n|E^c(x)\|\|D_x f^{-n}|E^u(x)\| < c\lambda^n.$$

The bundles $E^s$, $E^c$ and $E^u$ are always continuous. The partial hyperbolicity is an $C^1$-open structure: if $f$ is partially hyperbolic then there is a $C^1$-neighborhood $\mathcal{U}$ of $f$ such that any $g \in \mathcal{U}$ admits a (unique) partially hyperbolic structure $T_x M = E^s_{x,g} \oplus E^c_{x,g} \oplus E^u_{x,g}$ such that the dimension of the spaces are the same for $f$ and for $g$. Furthermore the bundles $E^s_g$ $E^c_g$ and $E^u_g$ depend continuously on $g$.

As in the hyperbolic case, if $f$ is partially hyperbolic there are invariant foliations $\mathcal{F}^{ss}_f$ and $\mathcal{F}^{uu}_f$ tangent to $E^s_f$ and $E^u_f$, respectively, whose leaves are $C^1$ immersed manifolds. In general there is no invariant foliation tangent to $E^c$ (there are counter-examples when $\dim E^c > 1$, and the existence of a central foliation is unknown for $\dim E^c = 1$). However, the existence of a *central foliation* $\mathcal{F}^c$ tangent to $E^c$ and $f$ invariant leads to a strong rigidity property of $f$. Hirsch, Pugh and Shub showed the following.

THEOREM 3. *Let $f$ be a partially hyperbolic diffeomorphism of a compact manifold. Assume that $f$ admits a $C^1$-foliation $\mathcal{F}^c_f$ tangent to $E^c$. Then, there is a $C^1$-neighborhood $\mathcal{U}$ of $f$ such that every $g \in \mathcal{U}$ is a partially hyperbolic diffeomorphism having an invariant central foliation $\mathcal{F}^c_g$; furthermore, there is a homeomorphism $h_g : M \to M$ and a constant $c_g > 0$ such that*

$$h_g(\mathcal{F}^c_g) = \mathcal{F}^c_f,$$

*and the homeomorphism $h_g \circ g \circ h_g^{-1}$ satisfies the following property: for every $x \in M$ the point $h_g \circ g \circ h_g^{-1}(x)$ belongs to the same leaf of $\mathcal{F}^c_f$ as $f(x)$ and the distance between $h_g \circ g \circ h_g^{-1}(x)$ and $f(x)$ in the central leaf is bounded by $c_g$,*

$$d^c(h_g \circ g \circ h_g^{-1}(x), f(x)) < c_g.$$

*Finally, if the neighborhood $\mathcal{U}$ is small enough, then $h_g$ is close to the identity and $c_g$ is very small.*

This theorem has been stated in [**11**, Theorem 7.1] with the hypothesis '$\mathcal{F}^c$ is *plaque expansive*' instead of '$\mathcal{F}^c$ is $C^1$'. However, [**11**, Theorem 7.2] shows that the central foliation is always plaque expansive if it is a $C^1$ foliation.

Finally, a partially hyperbolic diffeomorphism $f$ is called *dynamically coherent* if there are $f$-invariant foliations $\mathcal{F}^{cs}_f$, $\mathcal{F}^{cu}_f$ an $\mathcal{F}^c_f$ tangent to $E^s_f \oplus E^c_f$, $E^u_f \oplus E^c_f$ and $E^c_f$, respectively.

1.4. *$C^1$-small perturbations of the time-one map of an Anosov flow.* Let $\phi$ be a $C^1$-Anosov flow on a compact manifold $M$.

We will denote by $f_1 : M \to M$, the time-one diffeomorphism of $\phi$ defined as

$$f_1(x) = \phi(x, 1), \quad \text{for all } x \in M.$$

The diffeomorphism $f_1$ has no hyperbolic set because the direction tangent to the flow is invariant but neither expanded nor contracted. However, it is partially hyperbolic, the invariant bundles are those $E^{ss}$, $E^c$ and $E^{uu}$ of the flow $\phi$. Furthermore, the central foliation $\mathcal{F}^c$ of the Anosov flow $\phi$ is a $C^1$ foliation†. Hence one may apply Theorem 3 to $\mathcal{F}^c$.

Hence there is a $C^1$-neighborhood $\mathcal{U}$ of $f_1$ such that any $f \in \mathcal{U}$ satisfies the following properties.

(1) It is a partially hyperbolic diffeomorphism with a splitting $TM = E^{uu}_f \oplus E^c_f \oplus E^{uu}_f$ of the same dimensions as those of $f_1$ (i.e. of $\phi$).

(2) There is a one-dimensional $f$-invariant central foliation $\mathcal{F}^c_f$ tangent to $E^c_f$.

(3) There is a homeomorphism $h_f$ such that $h_f(\mathcal{F}^c_f) = \mathcal{F}^c$.

(4) Furthermore, for every $x \in M$ the point $h_f \circ f \circ h_f^{-1}(x)$ belongs to the same leaf of $\mathcal{F}^c$ as $f_1(x)$, that is the leaf $F^c(x)$ of $x$; as a direct consequence, each leaf of $\mathcal{F}^c_f$ is $f$-invariant.

(5) Finally there is a constant $c_f > 0$ such that the distance between $h_f \circ f \circ h_f^{-1}(x)$ and $f_1(x) = \phi(x, 1)$ in the central leaf is bounded by $c_f$. As a consequence the distance in the leaf $F^c_f(x)$ between $x$ and $f(x)$ is uniformly bounded.

This shows that any diffeomorphism $f \in \mathcal{U}$ belongs to the set $\mathcal{E}_\phi$ defined in the introduction.

1.5. *Perturbations of the time-one map of an Anosov flow along the central foliation.* Let $\phi$ be an Anosov flow on a compact connected manifold $M$ and $\mathcal{E}_\phi$ be the set, defined in the introduction, of partially hyperbolic diffeomorphisms $f$, with $TM = E^s_f \oplus E^c \oplus E^u_f$, satisfying the following properties.

- $f$ has a one-dimensional central foliation $\mathcal{F}^c_f$ conjugated to the central foliation $\mathcal{F}^c$ of $\phi$ by a homeomorphism $h_f$.
- Each leaf $F^c_f(x)$ is invariant by $f$. Furthermore, there is $K_f > 0$ such that for any $x \in M$ there is a path $\gamma \subset F^c_f(x)$ with length $\ell(\gamma) < K_f$ joining $x$ to $f(x)$ in the central leaf $F^c_f(x)$.

LEMMA 1.1. *$h_f \circ f \circ h_f^{-1}$ is a homeomorphism of $M$ of the form $x \mapsto \phi(x, \tau(x))$ where $\tau : M \to \mathbb{R} \setminus \{0\}$ is a continuous function.*

† One also can verify that the foliation $\mathcal{F}^{cs}$ and $\mathcal{F}^{cu}$ are plaque expansive. Theorem 3 may be applied (independently) to each of the foliations $\mathcal{F}^c$, $\mathcal{F}^{cs}$ and $\mathcal{F}^{cu}$.

*Proof.* First notice that $\tau(x)$ is uniquely defined by $h_f \circ f \circ h_f^{-1}(x) = \phi(x, \tau(x))$, on non-compact central leaves. Furthermore, $\tau$ is continuous on the complement of the compact leaves of $\mathcal{F}^c$.

On a compact leaf $F^c(x)$ the equation on $h_f \circ f \circ h_f^{-1} = \phi(x, \tau)$ admits infinitely many solutions $\tau_i$, $i \in \mathbb{Z}$, and the difference $\tau_i - \tau_j$ is precisely $(i - j)$ times the period of the $\phi$-orbit $F^c(x)$. Let us show now that $\tau$ admits a unique extension on $F^c(x)$: the flow $\phi$ has a countable family of periodic orbits and the complement of the union of the periodic orbits is connected (see Remark 1.1). In other words the complement of the compact leaves of $\mathcal{F}^c$ is connected.

As a consequence, the accumulation values of $\tau(y)$ when $y$ tends to $x$ is an interval of $\mathbb{R}$ contained in $\{\tau_i\}_{i\in\mathbb{Z}}$, that is, there is a unique $\tau_i$. We just proved that the function $\tau$ admits a unique continuous extension on $M$.

For ending the proof of Lemma 1.1, it remains to prove that $\tau$ does not vanish. We will use the following lemma.

LEMMA 1.2. *Let $t: M \to \mathbb{R}$ be a continuous function and let $\varphi: M \to M$ be the homeomorphism defined by $\varphi(x) = \phi(x, t(x))$. Assume that $t(x_0) = 0$ for some point $x_0$. Then there is $\delta > 0$ such that the $\delta$-local stable manifold of $x_0$ for $\varphi$ (that is, the set of points $y$ such that the distance $d(\varphi^n(y), \varphi^n(x_0))$ remains smaller than $\delta$ when $n > 0$, and tends to $0$ when $n > 0$ tends to $+\infty$) is included in the $\phi$ orbit of $x_0$.*

*Proof.* Recall that $\phi$ has no fixed point, so there is a flow box at $x_0$. As $t$ is very small in the neighborhood of $x_0$, it follows that if $x$ is close enough to $x_0$ then $\varphi(x)$ remains in the local central leaf through $x$, until it goes out the flow box. Choosing $\delta > 0$ such that the ball of radius $\delta$ centered at $x_0$, $B_\delta(x_0)$, is contained in the flow box, we have that for every $x \in B_\delta(x_0)\backslash F^c(x_0)$ there is $N > 0$ such that $\varphi(x) \notin B_\delta(x_0)$, i.e. $x$ is not in the $\delta$-local stable manifold of $x_0$ for $\varphi$. Hence we get the statement of the lemma. □

The partial hyperbolicity of $f$ ensures the existence of local strong stable and strong unstable manifolds at each point $x$ of $M$ and these manifolds are not contained in the central leaf $F_f^c(x)$. This implies that every point $x \in M$ has a local stable manifold for $h_f \circ f \circ h_f^{-1}$ which is not contained in the $\phi$ orbit of $x$. One concludes that $\tau$ cannot vanish, ending the proof of Lemma 1.1. □

*Notation.* For every $f \in \mathcal{E}_\phi$, we denote $\tau_f = \tau \circ h_f$. Then $h_f \circ f(x) = \phi(h_f(x), \tau_f(x))$.

*Remark 1.3.*

(1) If $f$ is a diffeomorphism which belongs to $\mathcal{E}_\phi$ then its inverse $f^{-1}$ belongs to $\mathcal{E}_\phi$ too; furthermore one can choose $h_{f^{-1}} = h_f$ and $\tau_{f^{-1}} = -\tau_f \circ f^{-1}$. Hence we will now assume (up to replacing $f$ by $f^{-1}$), that $\tau_f > 0$.

(2) If $f$ is a diffeomorphism in $\mathcal{E}_\phi$ then for any $n > 0$ the diffeomorphism $f^n$ belongs to $\mathcal{E}_\phi$, with $h_{f^n} = h_f$ and $\tau_{f^n} = \tau_f + (\tau_f \circ f) + \cdots + (\tau_f \circ f^{n-1})$.

Notice that $\mathcal{F}^c$ is naturally oriented by the flow $\phi$ and that the foliation $\mathcal{F}_f^c = h_f^{-1}(\mathcal{F}^c)$ inherits the image orientation. This orientation coincides, on the non-compact leaves, with the orientation given by the dynamic of $f$, that is the leaf $F_f^c(x)$ is oriented from $x$ to $f(x)$.

From now on, we choose the dynamical orientation for $\mathcal{F}_f^c$.

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A *parametrized central arc* $\gamma\colon [0, 1] \to M$ is an immersion of [0, 1] in a central leaf. Two parametrized central arcs $\gamma_1$ and $\gamma_2$ define the same *oriented central arc* if there is a orientation-preserving homeomorphism $\sigma$ of [0, 1] such that $\gamma_2 = \gamma_1 \circ \sigma$. The $C^0$-topology on the set of parametrized central arcs induces a topology, already called the *$C^0$-topology*, on the set of oriented central arcs. We denote by $\ell(\gamma)$ the length of the arc $\gamma$.

Let $b$ be a point in $\mathcal{F}^c_f(a)$ in the positive direction starting from $a$. We will denote by $C^a_b$ the arc included in $\mathcal{F}^c_f(a)$ between $a$ and $b$.

Let us denote by $D(x)$ the arc of central curve positively oriented and joining $x$ to $f(x)$, and whose image by $h_f$ is the arc $\phi(h_f(x), [0, \tau_f(x)])$. Notice that $D(x)$ and $C^x_{f(x)}$ are equal with a possible exception when $\mathcal{F}^c_f(x)$ is closed. In fact, just in the cases where $W^c(x)$ is closed and $D(x)$ winds around itself more than once we have that $D(x) \neq C^x_{f(x)}$.

*Remark 1.4.* As there are finitely many compact central leaves of length less $K_f$, we get that $D(x) = C^x_{f(x)}$ excepted for $x$ in finitely many closed central leaves.

The family $C^x_{f(x)}$ is not *a priori* continuous. However, as the function $x \mapsto \tau_f(x)$ is continuous, the arcs $D(x)$, $x \in M$ form a continuous family of compact central arcs.

By the continuity of the family $D(x)$, there is $K_f > 0$ such that the length $\ell(D(x))$ is upper bounded by $K_f$ for every $x \in M$. Furthermore, as $\tau_f > 0$ there is a lower bound $c_f > 0$ of $\ell(D(x))$. Finally, $D(f(x)) = f(D(x))$. As a consequence one gets the following properties.

LEMMA 1.3. *Let $\gamma$ be a central arc with $\ell(\gamma) \leq c_f$. Then for any $n \in \mathbb{Z}$, the length $\ell(f^n(\gamma))$ is upper bounded by $K_f$.*

*Proof.* Let us denote $x = \gamma(0)$. Then $\gamma \subset D(x)$. Hence $f^n(\gamma) \subset D(f^n(x))$. So $\ell(f^n(\gamma)) \leq \ell(D(f^n(x))) \leq K_f$. □

LEMMA 1.4. *For any $x \in M$ and any $y \in F^c_f(x)$, there is $n \in \mathbb{Z}$ with $f^n(y) \in D(x)$.*

*Proof.* Just notice that one gets a central curve with infinite length in both positive and negative direction by putting together the segments $D(f^n(x)) = f^n(D(x))$, which are all of length greater than $c_f > 0$. Hence this curve is the whole central leaf $F^c_f(x)$. So $y$ belongs to some $f^n(D(x))$ that is $f^{-n}(y) \in D(x)$. □

1.6. *Dynamical coherence of diffeomorphisms $f \in \mathcal{E}_\phi$.* Two leaves $F^c_f(x)$ and $F^c_f(y)$ will be called *asymptotic at* $+\infty$ if there are parametrizations $\gamma_x\colon \mathbb{R} \to F^c_f(x)$ and $\gamma_y\colon \mathbb{R} \to F^c_f(y)$ preserving the orientation and such that the distance $d(\gamma_x(t), \gamma_y(t))$ tends to 0 when $t \to +\infty$.

The map $f$ is normally hyperbolic (see [**11**]); then there exist strong stable and strong unstable foliations $\mathcal{F}^{ss}_f$, $\mathcal{F}^{uu}_f$.

LEMMA 1.5. *Consider $f \in \mathcal{E}_\phi$ and $h_f$ the homeomorphism associated to $f$, conjugating $\mathcal{F}^c_f$ to $\mathcal{F}^c$. Then we have the following.*

- *There is an $f$-invariant foliation $\mathcal{F}^{cs}_f$ tangent to the bundle $E^{ss}_f \oplus E^c_f$.*
- *There is an $f$-invariant foliation $\mathcal{F}^{cu}_f$ tangent to the bundle $E^{uu}_f \oplus E^c_f$.*
- *For every point $x$ the leaf $F^{cs}_f(x)$ contains the leaves $F^{ss}_f(x)$ and $F^c_f(x)$.*

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- *For every point $x$ the leaf $F_f^{cu}(x)$ contains the leaves $F_f^{uu}(x)$ and $F_f^c(x)$.*
- $\mathcal{F}_f^{cs} = h_f^{-1}(\mathcal{F}^{cs})$ *and* $\mathcal{F}_f^{cu} = h_f^{-1}(\mathcal{F}^{cu})$.

*Proof.* We first prove that for every $x \in M$ the image $h_f^{-1}(F^{cs}(h_f(x)))$ of the leaf of $\mathcal{F}^{cs}$ through $h_f(x)$ contains the leaves $F_f^{ss}(x)$ and $F_f^c(x)$. Notice that $F^{cs}(h_f(x))$ contains the leaf $F^c(h_f(x))$, and $h_f^{-1}(F^c(h_f(x))) = F_f^c(x)$, by definition of $h_f$. Hence $h_f^{-1}(F^{cs}(h_f(x)))$ contains $F_f^c(x)$.

Consider $y \in F_f^{ss}(x)$. Then the distance $d(f^n(x), f^n(y))$ tends to 0 when $n \to +\infty$. As the central distance $d^c(f^n(x), f^{n+1}(x))$ and $d^c(f^n(y), f^{n+1}(y))$ are uniformly bounded, it follows that the oriented leaves $F_f^c(y)$ and $F_f^c(x)$ are asymptotic (when one follows the foliation in the positive direction). This property persists by conjugacy so that the oriented leaves $F^c(h_f(y))$ and $F^c(h_f(x))$ are asymptotic. This means that $h_f(y) \in F^{cs}(h_f(x))$. Hence $y \in h_f^{-1}(F^{cs}(h_f(x)))$, proving the claim.

This implies that the dimension of $E_f^s \oplus E_f^c = 1 + \dim E_f^s$ is less than (or equal to) the dimension of $E^s \oplus E^c = 1 + \dim E^s$. In the same way one proves that the image $h_f^{-1}(F^{cu}(h_f(x)))$ contains the leaves $F_f^{uu}(x)$ and $F_f^c(x)$, implying $\dim(E_f^u) \leq \dim E^u$. One concludes that $\dim E_f^s = \dim E^s$ and $\dim E_f^u = \dim E^u$.

So we have proved that, for every leaf $F^{cs}$ of $\mathcal{F}^{cs}$, $h_f^{-1}(F^{cs})$ has the same dimension as $E_f^{cs}$; furthermore, for every point $x \in h_f^{-1}(F^{cs})$, the leaves $F_f^c(x)$ and $F_f^{ss}(x)$ are contained in $h_f^{-1}(F^{cs})$. One deduces that $h_f^{-1}(F^{cs})$ is tangent to $E_f^s \oplus E_f^c(x)$, hence it is a $C^1$-immersed submanifold.

Let us denote $\mathcal{F}_f^{cs} = h_f^{-1}(\mathcal{F}^{cs})$. It is a foliation tangent to $E_f^s \oplus E_f^c$ and subfoliated by $\mathcal{F}_f^c$ and $\mathcal{F}_f^{ss}$. As each leaf of $\mathcal{F}_f^c$ is $f$-invariant, each leaf of $\mathcal{F}_f^{cs}$ is invariant.

Analogously it can be proved that $\mathcal{F}_f^{cu} = h_f^{-1}(\mathcal{F}^{cu})$ is a foliation tangent to $E_f^u \oplus E_f^c$ and subfoliated by $\mathcal{F}_f^c$ and $\mathcal{F}_f^{uu}$. □

If $y \in F^{cs}(x)$ we say that $F_f^c(x)$ and $F_f^c(y)$ are *asymptotic at* $+\infty$ *in* $F^{cs}(x)$ if there are parametrizations $\gamma_x : \mathbb{R} \to F_f^c(x)$ and $\gamma_y : \mathbb{R} \to F_f^c(y)$ preserving the orientation and such that the central stable distance $d^{cs}(\gamma_x(t), \gamma_y(t))$ tends to 0 when $t \to +\infty$.

LEMMA 1.6. *For every point $x \in M$, the central stable leaf $F_f^{cs}(x)$ is the union of the strong stable leaves crossing the central leaf $F_f^c(x)$.*

$$F_f^{cs}(x) = \bigcup_{y \in F_f^c(x)} F_f^{ss}(y).$$

*In the same way,*

$$F_f^{cu}(x) = \bigcup_{y \in F_f^c(x)} F_f^{uu}(y).$$

*Proof.* We have proved the inclusion $\bigcup_{y \in F_f^c(x)} F_f^{ss}(y) \subset F_f^{cs}(x)$. It remains to prove the converse inclusion.

Using Lemma 1.5 one can prove that there is $\delta > 0$ such that for every $x \in M$ the ball $B_f^{cs}(x, \delta)$ of radius $\delta$ centered at $x$ in the leaf $F_f^{cs}(x)$ is contained in $\bigcup_{y \in F_f^c(x)} F_f^{ss}(y)$.

Now consider $y \in F^{cs}_f(x)$. By Lemma 1.5 the points $h_f(x)$ and $h_f(y)$ belong to the same central stable leaf of the flow $\phi$. This means that the central leaf of $\phi$ through $h_f(x)$ and $h_f(y)$ are asymptotic at $+\infty$ in $F^{cs}(h_f(x))$.

Let $t_n$ be such that $h_f(f^n(y)) = \phi(h_f(y), t_n)$. Since $t_n \to \infty$ it follows that there exist $t'_n$ such that the distance $d^{cs}(\phi(h_f(y), t_n), \phi(h_f(x), t'_n))$ (in the central stable leaf $F^{cs}(h_f(x))$) tends to 0. Let $x_n = h_f^{-1}(\phi(h_f(x), t'_n))$. As a consequence $x_n \in F^c_f(x)$ and one can prove that for $n \to +\infty$, the distance $d^{cs}_f(f^n(y), x_n)$ (in the leaf $F^{cs}_f(x) = h_f^{-1}(F^{cs}(h_f(x)))$ tends to 0. In particular this distance is less than $\delta$ for large $n$. As a consequence, $f^n(y)$ belongs to the strong stable leaf through a point $x'_n \in F^c_f(x)$. One conclude: $y \in F^{ss}_f(f^{-n}(x'_n)) \subset \bigcup_{z \in F^c_f(x)} F^{ss}_f(z)$, proving the converse inclusion. □

LEMMA 1.7. *Let $x$ be a point of $M$ and $\gamma : [0, 1] \to F^c_f(x)$ be a path in the central leaf trough $x$, such that $\gamma(0) = x$. Let $y$ be a point of $F^{ss}_f(x)$. Then there is a unique path $\sigma : [0, 1] \to F^c_f(y)$ such that $\sigma(0) = y$ and for every $t \in [0, 1]$ one has $\sigma(t) \in F^{ss}_f(\gamma(t))$.*

*Proof.* First notice that it is enough to prove Lemma 1.7 for central paths whose length is upper bounded by some constant $c$: one deduces the general case by cutting $\gamma$ in pieces of length smaller that $c$.

Now, given some fixed constant $c$, there is $\varepsilon(c) > 0$ such that the conclusion of Lemma 1.7 holds with the following hypotheses.

- $\gamma$ is a central path with length $\ell(\gamma) < c$, $\gamma(0) = x$.
- $y$ is a point in $F^{ss}_f(x)$ such that the distance $d^{ss}(x, y)$ (in $F^{ss}_f(x)$) is less than $\varepsilon(c)$.

Let $\gamma$ be a parametrized arc of length less than $c_f$, where $c_f$ is a lower bound of the length $\ell(D(x))$ defined before Lemma 1.3. Let $x = \gamma(0)$ and let $y$ be a point of $F^{ss}_f(x)$. By Lemma 1.3 we have that for any $n \in \mathbb{N}$, $\ell(f^n(\gamma)) \leq K_f$. There is $n > 0$ such that the distance $d^{ss}(f^n(x), f^n(y))$ (in the leaf $F^{ss}_f(f^n(x))$ is less that $\varepsilon(K_f)$. So there is a central arc $\tilde{\sigma} \subset F^c_f(f^n(y))$ such that $\tilde{\sigma}(0) = f^n(y)$ and $\tilde{\sigma}(t) \in F^{ss}_f(f^n(\gamma(t)))$. One denotes $\sigma = f^{-n}(\tilde{\sigma})$. It is a central arc in $F^s_f(y)$ starting at $\sigma(0) = y$ and $\sigma(t) \in F^{ss}_f(\gamma(t))$. □

In Lemma 1.7, we say that $\sigma$ is the image of $\gamma$ by the holonomy of the foliation $\mathcal{F}^{ss}_f$ from $x$ to $y$ and we denote $\sigma = \mathcal{H}^{ss}_f(\gamma, y)$. The next lemma asserts that the action of the holonomy of $\mathcal{F}^{ss}_f$ on central arcs is continuous.

LEMMA 1.8. *Given $K_1, K_2 > 0$, the map $(\gamma, y) \mapsto \mathcal{H}^{ss}_f(\gamma, y)$ is a continuous map, from the space of pairs $(\gamma, y)$ where $\gamma$ is a central arc with $\ell(\gamma) \leq K_1$ and $y \in F^{ss}_f(\gamma(0))$ satisfies $d^{ss}(\gamma(0), y) \leq K_2$, to the space of central arcs.*

*Proof.* As for Lemma 1.7, cutting $\gamma$ in pieces of length less that $c_f$ and iterating positively by $f$, it is enough to show Lemma 1.8 for segment $\gamma$ of length bounded by $K_f$, and with $d^{ss}(\gamma(0), y)$ less that an arbitrarily small $\varepsilon > 0$. Then the statement follows by working in foliated charts of the foliation $\mathcal{F}^c_f$. □

Assume now that the Anosov flow $\phi$ is transitive. As we have seen, periodic orbits are dense, orbits of generic points are dense, and all central stable or unstable leaves are dense.

All these properties holds for any $f \in \mathcal{E}_\phi$:

$$\{x | F_f^c(x) \text{ is a closed set}\} \text{ is dense in } M,$$
$$\{x | F_f^c(x) \text{ is dense in } M\} \text{ is a residual set},$$

and for any $x \in M$, $F_f^{cs}(x)$ and $F_f^{cu}(x)$ are dense.

## 2. *Periodic orbits and compact central leaves*

From now on, $\phi$ is a transitive Anosov flow on a compact manifold, $f_1$ denotes the time-one map of $\phi$, and $f \in \mathcal{E}_\phi$ is an Axiom-A diffeomorphism, with $\tau_f > 0$. Let us denote by $F_f^c(x)$ or by $W^c(x)$ the leaf of the central foliation through the point $x$. We choose the dynamical orientation for $\mathcal{F}_f^c$.

We denote by $k$ the dimension of $E_f^{uu}$ and by per$(f)$ the set of periodic points of $f$.

The metric induced by the Riemannian metric on the leaves of $\mathcal{F}_f^c$ will be denoted $d^c$. Analogously we define $d^s$ and $d^u$.

Consider $x \in \Omega(f)$. As $\Omega(f)$ is hyperbolic, the central direction $E_f^c(x)$ is contained either in the unstable or in the stable space at $x$. In the first case, there is a neighborhood $F_{f,\text{loc}}^{cu}(x)$ of $x$ in $F_f^{cu}(x)$ which coincides with the local unstable manifold $W_{\text{loc}}^u(x)$. In the second case, $F_{f,\text{loc}}^{cs}(x)$ coincides with the local stable manifold $W_{\text{loc}}^s(x)$.

*Remark 2.1.* If $x$ is a periodic point of $f$ then the central leaf $E_f^c(x)$ is compact, because each central leaf is $f$-invariant and $\tau_f$ is strictly positive. Indeed $h_f(x)$ is a periodic point of $\phi$; a period of $h_f(x)$ is the sum of the $\tau_f(y)$ for $y$ in the $f$-orbit of $x$.

The next proposition asserts that, conversely, every compact central leaf contains periodic orbits of $f$.

PROPOSITION 2.1. *If $\mathcal{O} = F_f^c(x)$ is a closed curve then we have the following.*

- *The rotation number of $f|\mathcal{O}$ is rational.*
- *There exists at least two periodic points in $\mathcal{O}$ with different indices.*
- *The points in $\Omega(f) \cap \mathcal{O}$ are the periodic ones.*

*Proof.* Let $\mathcal{O} = F_f^c(x)$ be a closed curve. Let us prove that the rotation number of $f|\mathcal{O}$ is rational. Assume, by contradiction that it is irrational. Then there exists a unique minimal set $I \subset \mathcal{O}$ which is not periodic. As $I \subset \Omega(f)$ and $f$ is Axiom A, $I$ is hyperbolic and included in a basic set $\Lambda$.

Besides, for all $y \in \mathcal{O}$, $\alpha(y) = \omega(y) = I$, where $\alpha(y)(\omega(y))$ is the set of limit points of $\{f^n(y)\}$ when $n \to -\infty$ ($n \to +\infty$) (see, for example, [**6**, p. 34]). Hence

$$y \in W^s(I) \cap W^u(I) \subset W^s(\Lambda) \cap W^u(\Lambda) \subset \Lambda,$$

therefore $y \in \Omega(f)$. Then $\mathcal{O} \subset \Lambda \subset \Omega(f)$; from the hyperbolicity of $f|_{\Omega(f)}$ we have that $f|_{\mathcal{O}}$ is hyperbolic and therefore $f|_{\mathcal{O}}$ is expansive which leads to a contradiction with the non-existence of one-dimensional expansive diffeomorphisms (see [**13**]).

Hence, $f|\mathcal{O}$ has periodic points, since the rotation number is rational. Since $f$ is Axiom A, all the periodic points are hyperbolic and (restricted to $\mathcal{O}$) they are alternately attractors or repellers. An attractor (respectively repeller) point corresponds in $M$ to a periodic saddle whose stable manifold has dimension equal to dim $E^s + 1$ (respectively dim $E^s$): hence $\mathcal{O}$ contains periodic orbits of different indices.

All the points in $\Omega(f) \cap \mathcal{O}$ must be periodic, otherwise, if there were a non-periodic point, $x \in \Omega(f) \cap \mathcal{O}$ then the invariance of $\Omega(f) \cap \mathcal{O}$ implies that $\alpha(x)$ and $\omega(x)$ would be periodic points of different indices so they would be in different basic sets. This is a contradiction with the fact that a non-wandering point of an Axiom-A diffeomorphism must have the $\alpha$ and the $\omega$ limit set in the same basic set. □

Let $\mathcal{O}$ be a compact leaf of $\mathcal{F}^c_f$. Then the *stable manifold* $W^s(\mathcal{O})$ *of* $\mathcal{O}$ is the set of points whose $\omega$-limit set is included in $\mathcal{O}$. Let us state some properties of $W^s(\mathcal{O})$.

*Remark 2.2.*

- The leaf $\mathcal{O}$ is a normally hyperbolic invariant compact manifold. As a consequence $W^s(\mathcal{O})$ is the union of the strong stable leaves through $\mathcal{O}$.
- Hence, according to Lemma 1.6, $W^s(\mathcal{O})$ is the leaf of $\mathcal{F}^{cs}_f$ containing $\mathcal{O}$.
- We have seen that every leaf of $\mathcal{F}^{cs}_f$ is dense in $M$; so the stable manifold $W^s(\mathcal{O})$ is dense in $M$.
- The stable manifold $W^s(\mathcal{O})$ is the union of the stable manifolds of the periodic points in $\mathcal{O}$. More precisely, let per($\mathcal{O}$) denote per$(f) \cap \mathcal{O}$, the set of periodic points contained in $\mathcal{O}$. Then
$$W^s(\mathcal{O}) = \bigcup_{x \in \mathrm{per}(\mathcal{O})} W^s(x).$$

One defines analogously the invariant manifold $W^u(\mathcal{O})$ of $\mathcal{O}$ and it satisfies the corresponding properties.

## 3. *Properties of attracting and repelling basic sets*

We include some statements in the present section that have been published in [**10**], assuming some extra hypotheses. Their proofs include simple ideas that are essential in the proof of the main theorem.

Let us recall that, as $f$ is an Axiom-A diffeomorphism, there is a finite number of attractor and repeller basic sets. We will show here that each attractor and each repeller meets every central leaf (so it looks like a complete cross section of the Anosov flow).

Let $A$ denote an attractor basic set of the spectral decomposition of $f$. Notice that $A \neq M$ because $f$ is not an Anosov diffeomorphism. According to Remarks 1.2 and 1.3, every positive iterate $f^n$ is an Axiom-A diffeomorphism in $\mathcal{E}_\phi$ and there is $n > 0$ such that every transitive attractor and repeller is connected. Hence, up to replacing $f$ by $f^n$ we may assume that $A$ is connected.

Recall that we denote dim $E^{uu}_f = k$.

LEMMA 3.1. *$\dim(W^s(x)) = n - k$, for all $x \in A$.*

*Proof.* By hypothesis $\dim(E^{ss}_\phi) = n - k - 1$. For every $x \in A$, either its local central leaf $F^c_{f,\mathrm{loc}}(x)$ is expanding (i.e. $\dim(W^s(x)) = n - k - 1$), or its local central leaf is contracting (i.e. $\dim(W^s(x)) = n - k$).

Consider a periodic point $x \in A \cap \mathrm{per}(f)$. We have seen that its central leaf $F^c_f(x)$ is a closed curve.

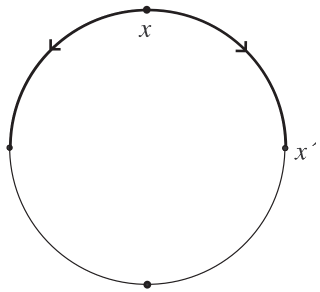


FIGURE 1. The points of different index $x, x' \in W^u(x) \subset A$.

We argue by contradiction, assuming that the central direction $E^c_f(x)$ is unstable. Since $A$ is an attractor it contains its unstable manifold; in particular one has $W^u(x) \subset A$; hence the local central leaf $F^c_{f,\mathrm{loc}}(x)$ is contained in $A$ (see Figure 1).

Consider now the dynamic of $f$ restricted to the circle $F^c_f(x)$. The point $x$ is a repeller for this dynamic. A point $y \in F^c_{f,\mathrm{loc}}(x)$ has positive iterates that converge to an attracting point $x'$ of $f|_{F^c_f(x)}$. The set $A$ is closed and $f$-invariant then $x'$ belongs to $A$. But $\dim(W^s(x')) = n - k$ (because the central direction $E^c_f(x')$ is contracting on the orbit of $x'$). It follows that there exist two periodic points of different indices in $A$, which contradicts the hyperbolicity of $A$.

Then the local central leaf is stable for every point in $A$ and the claim follows. □

Recall that $K_f > 0$ is an upper bound of the length of $D(x)$, for $x \in M$.

LEMMA 3.2. *We have the following.*

- *For every closed curve $\mathcal{O}$ in $\mathcal{F}^c_f$ there exists a periodic point $p \in A \cap \mathcal{O}$.*
- *In every central arc $\gamma$ with $length(\gamma) \geq K_f$, there exists a point $p \in \gamma \cap A$.*

*In particular, every leaf of $\mathcal{F}^c_f$ intersects $A$.*

*Proof.* According to Remark 2.2 the stable manifold $W^s(\mathcal{O})$ is dense in $M$. Since $W^s(A)$ is an open set, there exists $y$ in the intersection $W^s(\mathcal{O}) \cap W^s(A)$. By Remark 2.2 there is a periodic point $x \in \mathcal{O}$ such that $y \in W^s(x) \cap W^s(A)$. Hence $x \in A$, proving $A \cap \mathcal{O} \neq \emptyset$.

As $A$ is $f$-invariant, the whole orbit of $x$ is contained in $A$. By Lemma 1.4, for every $y \in \mathcal{O}$, the arc $D(y)$ contains at least one point of the orbit of $x$: hence, for every point $y$ in a compact central leaf, the segment $D(y)$ meets $A$.

Then $A$ meets $D(y)$ for every $y$ in the dense subset of $M$ equal to the union of the closed central leaves. Since the family of arcs $D(z)$, $z \in M$ is a continuous family of compact arcs and $A$ is compact, it follows that $A \cap D(z) \neq \emptyset$ for every $z \in M$. □

*Remark 3.1.* Analogously we can show that there exists $q \in \mathcal{O} \cap \Lambda$, where $\Lambda$ is a repeller set. Moreover, in every segment $\gamma$ of a central curve with $\mathrm{length}(\gamma) \geq K_f$, there exists a periodic point $q \in \gamma \cap \Lambda$.

As a direct consequence of Remark 3.1 one gets that every leaf of the central foliation 'goes away' from the basin of attraction of any attractor.

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COROLLARY 3.1. *In every leaf of $\mathcal{F}^c_f$ there exists at least one point outside of $W^s(A)$.*

LEMMA 3.3. *For every attractor basic set $A$, every repeller basic set $\Lambda$, and every compact central segment $\gamma$, the intersections $\gamma \cap A$ and $\gamma \cap \Lambda$ are finite sets.*

*Proof.* Each central leaf is contracting at each point of the attracting basic sets. One deduces that there is $\delta > 0$ such that, for any two distinct points $x, y \in A$ in the same central leaf, the central distance $d^c(f^{-n}(x), f^{-n}(y))$ is larger than $\delta$ for every $n > 0$ large enough.

We will show that for any $x \in M$ the intersection $A \cap D(x)$ is finite.

Consider $x \in M$ and $\{x_i\}$ in $A \cap D(x)$, such that $x_0 < x_1 < x_2 < \cdots < x_l$ in the given orientation of $D(x)$.

Then there exists $n_i \in \mathbb{N}$, $i = 1, \ldots, l$ verifying that $\ell(f^{-n_i}(C^{x_{i-1}}_{x_i})) > \delta$, for all $n \geq n_i$.

So for $n \geq \sup_i n_i$ one gets $\ell(f^{-n}(D(x))) = \ell(D(f^{-n}(x)))$ is larger than $l \cdot \delta$. However $\ell(D(f^{-n}(x))) \leq K_f$ by definition of $K_f$. So $l \leq K_f/\delta$, ending the proof of the claim.

Consider now a compact central segment $\gamma$, and let $x$ be the origin of $\gamma$. Then there is $i > 0$ such that $\gamma$ is contained in $\bigcup_1^i D(f^j(x))$. Hence $\gamma \cap A$ is finite. One proves in the same way (using positive iterations instead of negative ones) that $\gamma \cap \Lambda$ is finite. □

## 4. *Properties of the central foliation in the basin of A*

Let $A$ be an attractor of $f$. The aim of this section is to show the following.

- For $x$ in an open and dense subset of the basin $W^s(A)$, the connected component of $x$ in $W^c(x) \cap W^s(A)$ contains at least a point $y_x \in A$ (Lemma 4.5).
- For generic points in $W^s(A)$ the connected component of $x$ in $W^c(x) \cap W^s(A)$ contains exactly one point in $A$: the set

$$\{x \in W^s(A) | \sharp\{\text{conn. comp. of } x \text{ in } (W^c(x) \cap W^s(A)) \cap A\} = 1\}$$

is a residual set of $W^s(A)$ (Lemma 4.6).

In order to get these properties we introduce, for every $x \in W^s(A)$, the *entering point* $\tilde{S}_A(x)$ and the *exit point* $S_A(x)$ of its center leaf $F^c_f(x)$ in $W^s(A)$, as follows.

*Definition 4.1.* We denote by $S_A : W^s(A) \to \partial W^s(A)$ the map defined by: $S_A(x)$ is the nearest point of the central leaf of $x \in W^s(A)$ in the positive direction which is not in $W^s(A)$, i.e.,

$$S_A(x) = \sup\{y \in F^c_f(x) \mid C^x_y \subset W^s(A)\}.$$

In the same way $\tilde{S}_A : W^s(A) \to \partial W^s(A)$ is the map defined by: $\tilde{S}_A(x)$ is the nearest point of the central leaf of $x \in W^s(A)$ in the negative direction which is not in $W^s(A)$.

By Corollary 3.1, every segment of central leaf with length greater than $K_f$ contains points out of $W^s(A)$ so that the maps $S_A$ and $\tilde{S}_A$ are well defined.

We denote by $\widehat{W^c(x)} = C^{\tilde{S}_A(x)}_{S_A(x)}$ the arc of central curve which is the closure of the connected component of $W^c(x) \cap W^s(A)$ which contains $x$. Notice that the interior of this segment is contained, by definition, in $W^s(A)$, so that its length is bounded by $K_f$.

LEMMA 4.1. *If $x \in W^s(A)$ belongs to a compact central leaf, then $\widehat{W^c(x)}$ cuts $A$ in exactly one point. Furthermore $S_A(x)$ and $\tilde{S}_A(x)$ are periodic points for which the central direction is expanding.*

*Proof.* If $x \in W^s(A)$ is periodic, then $x \in A$; furthermore, by the proof of Lemma 3.1 the intersection $W^s(x) \cap F^c_f(x)$ is an interval in $W^s(A)$. The end points of this interval are periodic ones of different index from the index of $x$, hence out of $W^s(A)$. Finally, by Proposition 2.1 $x$ is the unique non-wandering point in that open interval, so that $\widehat{W^c(x)}$ is the closure of $W^s(x) \cap F^c_f(x)$.

If $F^c_f(x)$ is compact but $x$ is not periodic, it belongs to the stable manifold of a periodic point $y \in F^c_f(x) = F^c_f(y)$, and $\widehat{W^c(x)}$ is the closure of $W^s(y) \cap F^c_f(y)$. □

LEMMA 4.2. *For $x \in W^s(A)$, let $\ell_+(x)$ and $\ell_-(x)$ be the length of the central arcs $C^x_{S_A(x)}$ and $C_x^{\tilde{S}_A(x)}$ respectively. The maps $x \mapsto \ell_+(x)$ and $x \mapsto \ell_-(x)$ are lower semi-continuous.*

*Proof.* This is a classical consequence of the compactness of the complement of $W^s(A)$. Let $x_n \in W^s(A)$ be a sequence converging to $x$. As the length $\ell_+$ is uniformly bounded by $K_f$, by considering a subsequence, one may assume that the central arcs $C^{x_n}_{S_A(x_n)}$ converge to a central arc $C^x_y$, with $\ell(C^x_y) = \lim \ell_+(x_n)$, where $\ell(C)$ is the length of the arc $C$. Moreover, $y \notin W^s(A)$. Hence $C^x_{S_A(x)}$ is a subarc of $C^x_y$, proving $\ell_+(x) \leq \lim \ell_+(x_n)$. This proves that $\ell_+$ is lower semi-continuous, and the proof of the semi-continuity of $\ell_-$ is identical. □

We denote by $Q \subset W^s(A)$ the set of points such that the map $x \mapsto \widehat{W^c(x)}$ is continuous at $x$; this is equivalent to the fact that both $\ell_-$ and $\ell_+$ are continuous at $x$. Let us state some properties of the set $Q$.

*Remark 4.1.*
(1) $Q$ is invariant by $f$.
(2) As semi-continuous maps are continuous on generic points, the set $Q$ is residual in $W^s(A)$.
(3) If $x \in Q$ then the open arc $\widehat{W^c(x)} \setminus \{\tilde{S}_A(x), S_A(x)\}$ is contained in $Q$.
(4) If $x, y \in W^s(A)$ belong to the same strong stable leaf, then $\widehat{W^c(y)}$ is the image of $\widehat{W^c(x)}$ by the holonomy map of the foliation $\mathcal{F}^{ss}_f$ from $F^c_f(x)$ to $F^c_f(y)$.
(5) If $x \in Q$ then the strong stable leaf $F^{ss}_f(x)$ is contained in $Q$ (this claim is a consequence of the continuity of the holonomy map (see Lemma 1.8)).

*Remark 4.2.* For any $\alpha > 0$ let us denote by $\widetilde{U}_\alpha \subset W^s(A)$ the set of points $x$ such that there is a neighborhood $V_{x,\alpha}$ of $x$ verifying $\ell_+(y) < \ell_+(x) + \alpha$ and $\ell_-(y) < \ell_-(x) + \alpha$ for every $y \in V_{x,\alpha}$. Since the functions $\ell_-$ and $\ell_+$ are lower semi-continuous, positive and upper bounded by $K_f$ it follows that $\widetilde{U}_\alpha$ is dense in $W^s(A)$ for any $\alpha > 0$.

We denote by $U_\alpha$ the dense subset of $W^s(A)$ defined by $U_\alpha = \bigcup_{\alpha' < \alpha} \widetilde{U}_{\alpha'}$.

LEMMA 4.3. *For any $\alpha > 0$ the set $U_\alpha$ is an open and dense subset of $W^s(A)$.*

*Proof.* Let $x$ be a point of $U_\alpha$. By definition of $U_\alpha$ there is $0 < \alpha' < \alpha$ such that $x$ belongs to $\widetilde{U}_{\alpha'}$. Fixe $0 < \epsilon < \alpha - \alpha'$. Since the functions $\ell_-$ and $\ell_+$ are lower semi-continuous, it follows that there exists an open neighborhood of $x$, $\widetilde{U}^\epsilon$, such that if $y \in \widetilde{U}^\epsilon$ then $\ell_+(y) > \ell_+(x) - \epsilon$ and $\ell_-(y) > \ell_-(x) - \epsilon$. Since $x$ belongs to $\widetilde{U}_{\alpha'}$, there is a neighborhood $V_{x,\alpha'}$ of $x$ verifying $\ell_+(y) < \ell_+(x) + \alpha'$ and $\ell_-(y) < \ell_-(x) + \alpha'$ for every $y \in V_{x,\alpha'}$.

Let $y \in V_{x,\alpha'} \cap U^\epsilon$. We will show that $y \in U_\alpha$. Let $z \in V_{x,\alpha'}$, then $\ell_+(z) < \ell_+(x) + \alpha'$ and $\ell_-(z) < \ell_-(x) + \alpha'$. Since $y \in U^\epsilon$ we have that $\ell_+(z) < \ell_+(y) + \epsilon + \alpha' < \ell_+(y) + \alpha$ and $\ell_-(z) < \ell_-(y) + \epsilon + \alpha' < \ell_-(y) + \alpha$, hence $y \in U_\alpha$ and $U_\alpha$ is an open subset of $W^s(A)$. □

*Remark 4.3.* Clearly, $Q$ is contained in $U_\alpha$ for all $\alpha > 0$. More precisely, $Q = \bigcap_\alpha U_\alpha$.

LEMMA 4.4. *There is a dense open subset $U$ of $W^s(A)$ such that for every closed central leaf $\mathcal{O}$ the intersection $\mathcal{O} \cap U$ is contained in $Q$.*

*Proof.* We just sketch the proof which is done in detail in [**10**, Lemma 2.4]. Let $\delta_f > 0$ such that $2\delta_f$ is less than the infimum distance between two different basic sets of $f$, and we denote $U = U_{\delta_f}$. Consider a closed leaf $\mathcal{O}$ and $x \in \mathcal{O} \cap U$. We argue by contradiction assuming that $x$ is not a continuity point of $\ell_+$.

Hence there is a sequence of points $x_n$ converging to $x$ such that $\ell_+(x_n)$ converge but $\lim \ell_+(x_n) \neq \ell_+(x)$; as $\ell_+$ is lower semi-continuous this implies that $\lim \ell_+(x_n) > \ell_+(x)$. The closed central curves are dense in $M$ and $\ell_+$ is lower semi-continuous, hence there is $y_n$ close to $x_n$ such that $F^c_f(y_n)$ is closed and $\ell_+(y_n) \geq \ell_+(x_n) - (1/n)$. From Lemma 4.1 above, the point $z_n = S_A(y_n)$ is a periodic point for which the central direction is expanding; in particular it belongs to $\Omega(f)$. Up to choosing a subsequence, one may assume that the arcs $C^{y_n}_{z_n}$ converge to some arc $\gamma$ strictly larger than $C^x_{S_A(x)}$, and joining $x$ to a point $z = \lim z_n$. As $\Omega(f)$ is compact, the point $z$ is non-wandering. As $z$ belongs to the closed leaf $\mathcal{O}$ this implies that $z$ is a periodic point; furthermore the central direction is expanding along the orbit of $z$. So $z$ has the same index as the periodic point $S_A(x)$. Let $\sigma$ be the arc joining $S_A(x)$ to $z$ and obtained by removing $C^x_{S_A(x)}$ to $\gamma$. The central arc $\sigma$ is joining two periodic points with the same index in $\mathcal{O}$ (and its length is not 0). As the periodic points in $\mathcal{O}$ are alternately attracting and repelling, $\sigma$ contains a periodic point with different index, which implies that $\ell(\sigma) > \delta_f$. So we have proved $\lim \ell_+(y_n) > \ell_+(x) + \delta_f$ and $\lim y_n = x$, which contradicts $x \in U$. □

LEMMA 4.5. *For every point $x \in U = U_{\delta_f}$ the arc $\widehat{W^c(x)}$ meets $A$:*

$$\widehat{W^c(x)} \cap A \neq \emptyset.$$

*Proof.* Consider $x \in U$ and $x_n \to x$ a sequence of points converging to $x$ and whose central leaf is compact. According to Lemma 4.1, $\widehat{W^c(x_n)}$ contains a unique periodic point $y_n$ in $A$ and its extremities are periodic points $z_n^-$ and $z_n^+$ for which the central direction is expanding. In particular, by definition of $\delta_f$ (see the proof of the previous lemma), the distances $d(y_n, z_n^+)$ and $d(y_n, z_n^-)$ are larger that $2\delta_f$.

Since the lengths of $C^{y_n}_{z_n+}$ and $C^{z_n-}_{y_n}$ are bounded, up to considering a subsequence we may assume that the points $z_n^-$, $y_n$ and $z_n^+$ converge to points $z^-$, $y$ and $z^+$, such that the distance $d(y, z^+)$ and $d(y, z^-)$ are larger that $2\delta_f$. The points $z^-$ and $z^+$ are on the central leaf of $x$, and by definition of the set $U$ the central distance between them and the extremities $\tilde{S}_A(x)$ and $S_A(x)$ is less than $\delta_f$, respectively. Furthermore, the point $y$

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belongs to the arc $C_{z^-}^{z^+}$ and it is at a distance larger than $2\delta_f$ from the extremities, so that $y \in \widehat{W^c(x)}$. Finally $y$ belongs to $A$ by compactness of $A$, ending the proof. □

COROLLARY 4.1. *For $x \in Q$, $\widehat{W^c(x)} \cap A \neq \emptyset$. Furthermore, the extremities $\tilde{S}_A$ and $S_A$ are non-wandering points for which the central direction is expanding.*

*Proof.* The first part follows from $Q \subset U$. The second part comes from the proof of Lemma 4.5: as $x$ is a continuity point of the map $z \mapsto \widehat{W^c(z)}$, the points $z^- = \lim z_n^-$ and $z^+ = \lim z_n^+$ (in the notation of the proof of Lemma 4.5) coincide with $\tilde{S}_A(x)$ and $S_A(x)$, respectively. Furthermore, $z^-$ and $z^+$ are limits of periodic points and hence are non-wandering. As $f$ is axiom $A$, the points $z^-$ and $z^+$ belong to the same basic sets as $z_n^-$ and $z_n^+$, respectively, for large $n$. Hence the central direction is expanding along the orbit of $z^-$ and $z^+$, ending the proof. □

LEMMA 4.6. *Let $D_A$ be the subset of points of $Q$ verifying that the intersection $\widehat{W^c(x)} \cap A$ is exactly one point. Then $D_A \subset Q$ is residual in $Q$ (hence in $U$ and in $W^s(A)$).*

*Proof.* According to Lemma 4.5, for every $x \in U$ the intersection $\widehat{W^c(x)} \cap A$ is not empty.

Let $\alpha_x$ denote the smallest arc in $\widehat{W^c(x)}$ containing $\widehat{W^c(x)} \cap A$, and let $a(x)$ denote the length $a(x) = \ell(\alpha_x)$. We will show that the restriction of the function $x \mapsto a(x)$ to $Q$ is upper semi-continuous.

Fix $x \in Q$ and consider a sequence $x_n \in Q$ converging to $x$. Then the points $z_n^- = \tilde{S}_A(x_n)$ and $z_n^+ = S_A(x_n)$ are non-wandering and the central direction is expanding along their orbits. As a consequence, the distance between the extremities of $\alpha_n = \alpha(x_n)$ and the points $z_n^+$ and $z_n^-$ is larger that $2\delta_f$. Up to considering a subsequence one may assume that the central arcs $\alpha_n$ converge to a central arc $\alpha$. Notice that the extremities of $\alpha$ belong to $A$, by compactness of $A$.

By definition of $Q$ the segments $\widehat{W^c(x_n)}$ converge to $\widehat{W^c(x)}$, so that $\alpha \subset \widehat{W^c(x)}$. As the extremities of $\alpha$ belong to $A$ one deduces $\alpha \subset \alpha(x)$. This proves $\lim a(x_n) \leq a(x)$ and so the upper semi-continuity of the function $a$ restricted to $Q$.

Now Lemma 4.4 asserts that the points in $U$ whose central leaf is closed belong to $Q$. The union of the closed central leaf is dense in $M$, hence in the open set $U$. Furthermore Lemma 4.1 asserts that, for $x \in U$ on a closed central leaf, $\widehat{W^c(x)} \cap A$ is exactly one point, so that $a(x) = 0$. The restriction of the function $a$ to $Q$ is an upper continuous function which vanishes on a dense subset, so there is a residual subset $D_A$ of $Q$ (hence of $W^s(A)$ because $Q$ is residual in $W^s(A)$), on which $a$ vanishes. This means that for every $x \in D_A$ the intersection $\widehat{W^c(x)} \cap A$ is exactly one point. □

*Remark 4.4.* Recall that, if $x \in Q$ then the interior of the arc $\widehat{W^c(x)}$ is contained in $Q$. Hence, by definition of $D_A$, if $x \in D_A$ then the interior of the arc $\widehat{W^c(x)}$ is contained in $D_A$.

LEMMA 4.7. *The set of periodic points $x \in D_A$ is dense in the attractor $A$.*

*Proof.* Recall that every periodic point in $Q$ belongs to $A$. Consider a periodic point $x$ in the open set $U$. Then $A \cap U$ contains a neighborhood of $x$ in $A$, hence $A \cap U$ is a non-empty open subset of $A$. The periodic points are dense in $A$, so they are dense in the open set $A \cap U$ of $A$. Since the periodic points in $A \cap U$ are included in $Q$, the periodic points of $Q$ are dense in $A \cap U$.

Let $V_A$ be the set of periodic points in $D_A$. Since $V_A = D_A \cap \mathrm{per}(f) = Q \cap \mathrm{per}(f)$, we get that $V_A$ is dense in $A \cap U$. Notice that $V_A$ is invariant by $f$. So $V_A$ is dense in the union of the iterates $f^i(A \cap U)$, $i \in \mathbb{Z}$.

As $A$ is transitive, $\bigcup_{i \in \mathbb{Z}} f^i(A \cap U)$ is a dense open subset of $A$, proving that $V_A$ is dense in $A$. □

## 5. *Predecessor and successor of an attractor*

LEMMA 5.1. *Let $A$ be a transitive attractor of $f$. There are transitive repellers $\Lambda_-$ and $\Lambda_+$ such that $\tilde{S}_A(x) \in \Lambda_-$ and $S_A(x) \in \Lambda_+$ for every $x \in Q$ (where $Q \subset W^s(A)$ is the residual subset of continuity points of the function $x \mapsto \widehat{W^c(x)}$).*

*Proof.* Fix $x \in Q$. Corollary 4.1 asserts that $S_A(x)$ and $\tilde{S}_A(x)$ are non-wandering points. Then, the closure $\overline{\mathrm{Im}(S_A|Q)}$ of the image of $S_A$ restricted to $Q$, is a compact invariant set included in the non-wandering set.

By Lemma 4.5, we know that the image $\mathrm{Im}(S_A|Q)$ of $S_A$ restricted to $Q$ is equal to the image of $S_A$ restricted to $Q \cap A$.

Since the set of dense orbits is a residual set of $A$, it follows that there exists $x \in A \cap Q$ such that $\{f^n(x)\}_{n \in \mathbb{N}}$ is dense in $A$. Then $\{f^n(x)\}_{n \in \mathbb{N}}$ is dense in $A \cap Q$. As the map $S_A$ is continuous restricted to $Q$, the sequence $\{S_A(f^n(x))\}_{n \in \mathbb{N}}$ is dense not only in $S_A(A \cap Q)$ but in $\overline{S_A(A \cap Q)}$ as well. As $S_A(f^n(x)) = f^n(S_A(x))$, we have that there exists a dense orbit in $\overline{\mathrm{Im}(S_A|A \cap Q)} = \overline{\mathrm{Im}(S_A|Q)}$. Hence, there exists a basic set $\Lambda_+$ such that $\overline{\mathrm{Im}(S_A|Q)} \subset \Lambda_+$.

In the same way there is a basic set $\Lambda_-$ containing $\overline{\mathrm{Im}(\tilde{S}_A|Q)}$. It remains to prove that $\Lambda_+$ and $\Lambda_-$ are repellers.

Fix a point $x \in Q$. Remark 4.1 claims that $Q$ is invariant by the foliation $\mathcal{F}^{ss}_f$ and that for $y \in F^{ss}_f(x)$ the arc $\widehat{W^c(y)}$ is the image by holonomy of $\mathcal{F}^{ss}_f$ of the arc $\widehat{W^c(x)}$. Conversely, a point $z \in F^{ss}_f(S_A(x))$ is the end point of an arc which is the image by holonomy of $\mathcal{F}^{ss}_f$ of $\widehat{W^c(x)}$. This arc is of the form $\widehat{W^c(y)}$ for some $y \in F^{ss}_f(x)$. So $z$ is the image $S_A(y)$ with $y \in Q$. Hence $z \in \Lambda_+$.

As a consequence, $F^{ss}_f(S_A(x))$ is contained in $\Lambda_+$. This implies that $\Lambda_+$ is a repeller. One proves in the same way that $\Lambda_-$ is a repeller. □

Putting together Lemma 5.1 and Lemma 4.6 one gets the following.

COROLLARY 5.1. *For every transitive attractor $A$ there are two transitive repellers $\Lambda_-$ and $\Lambda_+$ with the following property.*

*For $x \in D_A$ the arc $\widehat{W^c(x)}$ meets $\Omega(f)$ in exactly $\tilde{S}_A(x) \in \Lambda_-$, $S_A(x) \in \Lambda_+$ and in only one point in the interior of $\widehat{W^c(x)}$. This point belongs to $A$.*

*Definition 5.1.* Following the notation of the previous lemma, $\Lambda_-$ and $\Lambda_+$ are called the *predecessor* and the *successor* of $A$ respectively.

*Remark 5.1.* Analogously we can prove the following.

Let $\Lambda$ be a transitive repeller. There are two transitive attractors $A_-$ and $A_+$ and a residual set $D_\Lambda$ of $W^u(\Lambda)$, such that for every $x \in D_\Lambda$ we have the following.

- The connected component of $(W^u(\Lambda) \cap W^c(x))$ that contains $x$ intersects $\Omega(f)$ in just a point which belongs to $\Lambda$.
- Let $C_{b(x)}^{a(x)}$ be the closure of the connected component of $W^u(\Lambda) \cap W^c(x)$ that contains $x$, then $a(x) \in A_-$, $b(x) \in A_+$.

We call $A_-$ and $A_+$ *the predecessor* and *the successor of* $\Lambda$, respectively.

The next lemma proves that our definitions of successor and predecessor of attractors and repellers are coherent.

LEMMA 5.2. *A repeller $\Lambda$ is the successor (respectively the predecessor) of an attractor $A$ if and only if $A$ is the predecessor (respectively the successor) of $\Lambda$.*

*Proof.* Consider a repeller $\Lambda$ and its predecessor $A$. Let $x_0$ be a point of $D_\Lambda$. By definition of the predecessor of $\Lambda$, the point $y_0 = \tilde{S}_\Lambda(x)$ belongs to $A$. Let $\delta > 0$ such that $\delta < \delta_f$ (recall that $2\delta_f$ is the infimum distance between two basic sets) and such that the ball $B(y_0, \delta)$ is contained in $W^s(A)$. We fix a point $x_1$ in the interior of $C_{x_0}^{y_0}$ at distance less than $\delta/2$ of $y_0$: more precisely we require $\ell(C_{x_1}^{y_0}) < \delta/2$. As $x_1$ belongs to the interior of the connected component of $(W^u(\Lambda) \cap W^c(x_0))$ that contains $x_0$ one gets the following.

- $x_1$ belongs to $W^u(\Lambda)$.
- $x_1$ is a continuity point of the function $x \mapsto \ell(C_x^{\tilde{S}_\Lambda(x)})$.
- $\tilde{S}_\Lambda(x_1) = y_0$.

As a consequence, there is an open neighborhood $V$ of $x_1$ such that, for every $x \in V$ we have the following.

- $x \in W^u(\Lambda) \cap W^s(A)$.
- The arc $C_x^{\tilde{S}_\Lambda(x)}$ is contained in $B(y_0, \delta) \subset W^s(A)$.
- $\ell(C_x^{\tilde{S}_\Lambda(x)}) < (3/4)\delta$.

As the sets $D_A$ and $D_\Lambda$ are residual in $W^s(A)$ and $W^u(\Lambda)$, respectively, they are both residual in $V$. Hence $D_A \cap D_\Lambda \cap V \neq \emptyset$. Choose $x \in D_A \cap D_\Lambda \cap V$, and let us denote $y_- = \tilde{S}_\Lambda(x) \in A$, $y_+ = S_\Lambda(x)$. By definition of $D_\Lambda$ the interior of the arc $C_{y_+}^{y_-}$ meets $\Omega(f)$ in an unique point $z \in \Lambda$.

By definition of $V$, the arc $C_x^{y_-}$ is contained in $W^s(A)$ then it is disjoint from $\Lambda$. Hence the point $x$ belongs to the interior of the central arc $C_z^{y_-}$; furthermore the interior of this arc is disjoint from $\Omega(f)$. As $x$ belongs to $D_A$, this implies that $z = S_A(x)$; therefore $\Lambda$ is the successor of $A$. □

Denote by $X_\mathcal{A}$ and $X_\mathcal{R}$ the sets of transitive attractors and repellers of $f$, respectively. These sets are finite, and the function which maps an attractor to its successor induces a bijection between this two sets. Furthermore, the function on $X_\mathcal{A} \cup X_\mathcal{R}$ which maps any element to its successor is a permutation of $X_\mathcal{A} \cup X_\mathcal{R}$. An orbit of this permutation will be called a *cycle of attractors and repellers*. The cycles of attractors and repellers form a partition of $X_\mathcal{A} \cup X_\mathcal{R}$ (we will see at Lemma 5.4 that there is a unique cycle).

*Remark 5.2.* Let $A$ and $\Lambda$ be a transitive attractor and a transitive repeller of $f$. Then

$$W^s(A) \cap W^u(\Lambda) \neq \emptyset \iff A \text{ is the predecessor or the successor of } \Lambda.$$

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LEMMA 5.3. *There is a residual set $D$ of $M$ such that, for all $x \in D$, we have the following.*

(1) *The intersection $F_f^c(x) \cap \Omega(f)$ is contained in the union of transitive attractors and repellers of $f$.*
(2) *Furthermore, considering an orientation preserving parametrization of $F_f^c(x)$ by $\mathbb{R}$, then $F_f^c(x) \cap \Omega(f)$ is an increasing sequence $\{x_i, i \in \mathbb{Z}\}$ such that $\lim_{i\to-\infty} x_i = -\infty$ and $\lim_{i\to+\infty} x_i = +\infty$.*
(3) *The points $x_i$ alternately belong to an attractor or to a repeller; for fixing the idea, we can choose an indexation such that $a_i = x_{2i}$ belongs to an attractor $A_i$ and $r_i = x_{2i+1}$ belongs to a repeller $\Lambda_i$, for all $i \in \mathbb{Z}$.*
(4) *With the notation above, the attractor $A_i$ is the predecessor of the repeller $\Lambda_i$ and the successor of the repeller $\Lambda_{i-1}$.*
(5) *Moreover, the interior of the central arc $C_{r_i}^{a_i}$ is included in $W^u(\Lambda_i) \cap W^s(A_i)$, and the interior of $C_{a_{i+1}}^{r_i}$ is included in $W^u(\Lambda_i) \cap W^s(A_{i+1})$.*

*Proof.* Let $A$ be an attractor and $D_A$ be the set defined in Lemma 4.6. Let $M_A = \{x \in W^s(A) | x \notin D_A\}$. Since $D_A$ is a residual set in $W^s(A)$ by Lemma 4.6, we have that $M_A$ is a meagre set (First Baire category). For every $x \in D_A$ we have that the interior of the arc $\widehat{W^c(x)}$ is contained in $D_A$, therefore for every $x \in M_A$ we have that the interior of $\widehat{W^c(x)}$ is contained in $M_A$.

Let $\mathbb{M}_A = \bigcup_{x \in M_A} F_f^c(x)$ be the union of the whole central leaves through $M_A$. We will show that $\mathbb{M}_A$ is meagre. For that, consider the homeomorphism $h_f$ conjugating $\mathcal{F}_f^c$ to the central foliation $\mathcal{F}^c$ of the flow $\phi$. Let us denote by $\varphi_t$, for $t \in \mathbb{R}$, the homeomorphism defined by $\varphi_t(x) = h_f^{-1}\phi(h_f(x), t)$, that is the conjugation by $h_f$ of the time $t$ of the flow $\phi$. Notice that $\varphi_t$ is a topological flow whose orbits are the central leaves of $\mathcal{F}_f^c$.

As $\varphi_t$ is a homeomorphism, one gets that $\varphi_t(M_A)$ is meagre for all $t$. As the union of countably many meagre sets is a meagre set, one gets that $\bigcup_{t \in \mathbb{Q}} \varphi_t(M_A)$ is a meagre set. One concludes the claim by noticing that $\mathbb{M}_A = \bigcup_{t \in \mathbb{Q}} \varphi_t(M_A)$: in fact for every $x \in M$ the set $\{\varphi_t(x), t \in \mathbb{Q}\}$ is dense in the leaf $F_f^c(x)$. However, if $x$ belongs to $\mathbb{M}_A$ then, by definition, $F_f^c(x)$ contains a point $y \in M_A$; hence it contains the open-arc interior of $\widehat{W^c(y)}$. Notice that there is $t \in \mathbb{Q}$ such that $\varphi_t(x)$ belongs to this open arc. This means that $x$ belongs to $\varphi_{-t}(M_A)$ hence to $\bigcup_{t \in \mathbb{Q}} \varphi_t(M_A)$. This proves $\mathbb{M}_A \subset \bigcup_{t \in \mathbb{Q}} \varphi_t(M_A)$, implying that $\mathbb{M}_A$ is meagre (the other inclusion is straightforward, and we will not use it).

Analogously we construct the sets $\mathbb{M}_{A_i}$ for every attractor $A_i$ and $\mathbb{M}_{\Lambda_i}$ for every repeller $\Lambda_i$. Since there exist finitely many attractors and repellers we have that

$$\Upsilon = \bigcup_{i=1,\ldots,n} \mathbb{M}_{\Lambda_i} \cup \mathbb{M}_{A_i}$$

is meagre.

Notice that the union $\bigcup_{A \text{ attractor of } f} W^s(A)$ of the basins of attractors of $f$ is a dense open subset of $M$ and in the same way the union $\bigcup_{\Lambda \text{ repeller of } f} W^u(\Lambda)$ of the basins of the repellers is a dense open set.

Let us denote

$$D = \Big(\bigcup_{A \text{ attractor of } f} W^s(A) \cup \bigcup_{\Lambda \text{ repeller of } f} W^u(\Lambda)\Big) \Big\backslash \Upsilon.$$

It follows that $D$ is a residual set of $M$.

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Furthermore, for every $x \in D$ we have that each connected component of $W^c(x) \cap W^s(A_i)$ is included in $D_{A_i}$ for every attractor $A_i$; in the same way, each connected component of $W^c(x) \cap W^u(\Lambda_j)$ is included in $D_{\Lambda_j}$ for every attractor $\Lambda_j$. Moreover there is an attractor or a repeller such that $x$ belongs to the basin of it. For instance $x \in W^s(A_0)$. As $x \in D_{A_0}$ the connected component of $x$ in $W^c(x) \cap W^s(A_0)$ is an open central arc which meets $A_0$ in a (unique) point $x_0 = a_0$; furthermore the origin $x_{-1} = r_{-1}$ of the oriented arc belongs the predecessor $\Lambda_{-1}$ of $A_0$ and its end point is a point $x_1 = r_0$ in the successor $\Lambda_0$ of $A_0$. Now, $r_{-1}$ belongs to $D_{\Lambda_{-1}}$ and $r_0$ belongs to $D_{\Lambda_0}$; this allows us to build inductively the sequence $(a_i, r_i)$, $a_i$ in the successor of $r_{i-1}$ and $r_i$ in the successor of $a_i$, and by construction the open central arc joining $a_i$ to $r_i$ is contained in $W^u(\Lambda_i) \cap W^s(A_i)$ and the open central arc joining $a_{i-1}$ to $r_i$ is contained in $W^u(\Lambda_i) \cap W^s(A_{i-1})$.

For ending the proof it remains to remark that the central distance between $x_i$ and $x_{i+1}$ is larger than $\delta_f$ so that the union of the arc $C^{x_i}_{x_i+1}$ covers the whole central leaf $F^c_f(x)$. $\square$

LEMMA 5.4. *There exists an unique cycle of attractors and repellers.*

*Proof.* Notice that, in Lemma 5.3, for any $x \in D$ the central leaf $F^c_f(x)$ meets $\Omega(f)$ along the sequence $\{a_i, r_i\}$ and the corresponding sequence of attractors repellers $A_i$, $\Lambda_i$ is exactly a cycle of attractors and repellers.

However, according to Lemma 3.2, the central leaf $F^c_f(x)$ meets every attractor in $X_{\mathcal{A}}$ and every repeller in $X_\Lambda$: as a consequence, there is an unique cycle of attractors and repellers. In other words, the notion of successor induces a cyclic order on the set of attractors and repellers of $f$.

Let $k$ denote the number of attractors of $f$. There is an indexation $A_i$, $\Lambda_i$, $i \in \mathbb{Z}/k\mathbb{Z}$, of the attractors and repellers of $f$ such that $\Lambda_i$ is the successor of $A_i$ and the predecessor of $A_{i+1}$. $\square$

So, the central leaves through the residual set $D$ visit all the attractors and repellers, following the cyclic order on $X_{\mathcal{A}} \cup X_{\mathcal{R}}$ given by the notion of successor. The next lemma shows that this property holds for any central leaf, if we allow repetition (i.e. a central leaf may cross an attractor or a repeller in more than one point before crossing its successor).

PROPOSITION 5.1. *Let $A_0, \Lambda_0, \ldots, A_{k-1}, \Lambda_{k-1}$ be the sequence of attractors and repellers with the indexation compatible with the cycle.*

*Then for every $x \in M$, $F^c_f(x) \cap (\bigcup_{i \in \mathbb{Z}/k\mathbb{Z}} A_i \cup \Lambda_i)$ is a sequence $\ldots a^i_1, \ldots, a^i_{n_i}$, $r^i_1, \ldots, r^i_{m_i}, a^{i+1}_1 \ldots$ such that $\{a^i_1, \ldots, a^i_{n_i}\} \subset A_i$, $\{r^i_1, \ldots, r^i_{m_i}\} \subset \Lambda_i$.*

*Proof.* Consider two points $x$, $y$ in the same central leaf $F^c_f(x)$ such that the segment $C^x_y$ is positively oriented. Assume that $x$ belongs to an attractor $A_i$, and $y$ belongs to an attractor or a repeller $K$ which is neither $A_i$ nor the successor $\Lambda_i$ of $A_i$. We will prove that $C^x_y \cap \Lambda_i \neq \emptyset$.

According to Lemma 4.7 there is a sequence of periodic points $x_n \in D_{A_i}$ converging to $x$ and a sequence of points $y_n \in F^c_f(x_n)$ such that the arcs $C^{x_n}_{y_n}$ converge to $C^x_y$. For $n$ large enough, the points $y_n$ belongs to the basin of $K$. However, according to Lemma 4.7, the point $z_n = S_{A_i}(x_n)$ belongs to the successor $\Lambda_i$ of $A_i$ and the interior of the arc

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$C_{z_n}^{x_n}$ is contained in $W^s(A_i) \cap W^u(\Lambda_i)$, hence it is disjoint from the basin of $K$. As a consequence, $y_n$ does not belong to $C_{z_n}^{x_n}$, so that $z_n$ belongs to the arc $C_{y_n}^{x_n}$. Now, any accumulation point $z$ of the sequence $z_n$ is a point of $\Lambda_i$ in $C_y^x$.

Let $\gamma$ be the connected component of the intersection of any central leaf with any basin of an attractor (or repeller). Since $l(\gamma)$ is bounded by $K_f$, Lemma 3.3 implies that there is finitely many points in the intersection of $\gamma$ with the attractor (or the repeller). One concludes that every compact central arc meets $(\bigcup_{i\in\mathbb{Z}/k\mathbb{Z}} A_i \cup \Lambda_i)$ on a finite set, so that the intersection $F_f^c(x) \cap (\bigcup_{i\in\mathbb{Z}/k\mathbb{Z}} A_i \cup \Lambda_i)$ is a sequence of points going from $-\infty$ to $\infty$. □

## 6. *The basins of the attractors and the repellers*

In what follows, we will look at the relative positions of the basins of the attractors and repellers, the closures of these basins, and the interiors of these closures.

All the results in this section related to attractors admit analogous version for repellers. For instance we have the following.

*Remark 6.1.* The basins of two different attractors $A_i$, $A_j$ are disjoint open sets. As a direct consequence, the closure $\overline{W^s(A_i)}$ is disjoint from the interior of the closure $\mathrm{Int}\overline{W^s(A_j)}$. On the other hand, the union of the closures of the basins cover $M$ (in formula: $M \subset \bigcup_{i=0}^k \overline{W^s(A_i)}$). As a consequence one gets

$$\mathrm{Int}(\overline{W^s(A_i)}) = M \Big\backslash \bigcup_{j\neq i} \overline{W^s(A_j)}, \quad \text{and}$$

$$\overline{W^s(A_i)} = M \setminus \mathrm{Int}\bigg(\bigcup_{j\neq i} \overline{W^s(A_j)}\bigg).$$

Let $\alpha, \beta : M \to M$ be the maps defined as follows: for every $x$ in $M$ such that $x$ is not included in any attractor or repeller set, $\alpha(x)$ is the first point in its central leaf in the negative direction verifying that it is in any attractor or in any repeller and $\beta(x)$ is the first point in its central leaf in the positive direction verifying that it is in any attractor or in any repeller. In the case that $x$ belongs to an attractor or a repeller we define $\alpha(x) = \beta(x) = x$.

According to Proposition 5.1, either $\alpha(x)$ and $\beta(x)$ belong to the same (attracting or repelling) basic set, or $\beta(x)$ belongs to the successor of the basic set containing $\alpha(x)$.

LEMMA 6.1. *Assume that $\alpha(x)$ or $\beta(x)$ belong to an attractor $A$. Then $x \in \mathrm{Int}(\overline{W^s(A)})$. In the same way, if $\alpha(x)$ or $\beta(x)$ belong to a repeller $\Lambda$, then $x \in \mathrm{Int}(\overline{W^u(\Lambda)})$.*

*Proof.* Let us prove the Lemma for $\alpha(x) \in A$. The other cases are analogous. If $x = \alpha(x)$ that is $x \in A$, the point $x$ admits a neighborhood contained in $W^s(A)$, ending the proof. Let us now assume that $x \neq \alpha(x)$, and then $x \neq \beta(x)$. Let $K$ be the attractor or repeller containing $\beta(x)$. We fix two points $y_0$, $z_0$ in the interior of the arc $C_{\beta(x)}^{\alpha(x)}$ in such a way that the arc $C_{z_0}^{y_0}$ is positively oriented, the point $x$ belongs to $C_{z_0}^{y_0}$, the point $y_0$ belongs to $W^s(A)$ and $z_0$ belongs to the basin of $K$.

Consider two disks, $\Delta_{y_0}$ and $\Delta_{z_0}$, transverse to the foliation $\mathcal{F}_f^c$ and centered at $y_0$ and $z_0$, respectively. Up to shrinking the disks $\Delta_{y_0}$ and $\Delta_{z_0}$, one may assume that the holonomy

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map of the foliation $\mathcal{F}^c_f$ along the path $C^{y_0}_{z_0}$ is well defined and it is a homeomorphism $h\colon \Delta_{y_0} \to \Delta_{z_0}$. For $y \in \Delta_{y_0}$ we denote by $\gamma_y$ the central arc $C^y_{h(y)}$. The family $\gamma_y$ is a continuous family of central arcs and $\gamma_{y_0} = C^{y_0}_{z_0}$. Notice that the union $V_x = \bigcup_{y\in\Delta_{y_0}} \gamma_y$ is a neighborhood of $x$.

By construction, the segment $\gamma_{y_0}$ is disjoint from the compact set obtained as the union of all the attracting and repelling basic sets of $f$, hence, up to shrinking once more $\Delta_{y_0}$ one may assume that every arc $\gamma_y$, $y \in \Delta_{y_0}$, is disjoint from the union of the transitive attractors and repellers of $f$. Recall that the set $D$ is residual and saturated for the central foliation $\mathcal{F}^c_f$; as a direct consequence, $D$ meets any transverse section $\Delta$ of $\mathcal{F}^c_f$ in a residual subset of $\Delta$. In particular, generic points $y \in \Delta_{y_0}$ belong to $D$. Lemma 5.3 implies that for $y \in D \cap \Delta_{y_0}$, the arc $\gamma_y$ is contained in $W^s(A)$. As $W^s(A)$ is open, and the union of $\gamma_y$, $y \in D \cap \Delta_{y_0}$ is dense in $V_x$, one gets that $W^s(A) \cap V_x$ is a dense open subset of $V_x$. This implies that $V_x \subset \overline{W^s(A)}$ and then $x \in \mathrm{Int}\overline{W^s(A)}$, concluding the proof of the lemma. □

LEMMA 6.2. *Assume that both $\alpha(x)$ and $\beta(x)$ belong to an attractor $A$, and let $\Lambda_-$ and $\Lambda_+$ be the predecessor and the successor of $A$. Then we have the following.*

(1) *For any transitive repeller $\Lambda$ of $f$,*

$$x \in \overline{W^u(\Lambda)} \Longleftrightarrow \Lambda \in \{\Lambda_-, \Lambda_+\}.$$

(2) *Hence*

$$x \in \overline{W^u(\Lambda_-)} \cap \overline{W^u(\Lambda_+)} \cap \mathrm{Int}(\overline{W^u(\Lambda_-)} \cup \overline{W^u(\Lambda_+)}).$$

*Proof.* As $\alpha(x) \in A$ there is a sequence of periodic points $y_n \in D_A$ converging to $\alpha(x)$. Then the points $z_n = S_A(y_n)$ belong to $\Lambda_+$ and the interior of the arc $\gamma_n = C^{y_n}_{z_n}$ is contained in $W^u(\Lambda_+)$. Up to considering a subsequence, one may assume that the arc $\gamma_n$ converges to a central arc $\gamma = C^y_z$ beginning at $y = \alpha(x)$. Notice that $z \in \Lambda_+$ and $\gamma \subset \overline{W^u(\Lambda_+)}$. As there are no points of $\Lambda_+$ in the arc $C^{\alpha(x)}_{\beta(x)}$ one gets that $C^{\alpha(x)}_{\beta(x)} \subset \gamma \subset \overline{W^u(\Lambda_+)}$. So $x \in \overline{W^u(\Lambda_+)}$. Using $\beta(x)$ instead of $\alpha(x)$ one proves in the same way that $x \in \overline{W^u(\Lambda_-)}$. For concluding the proof of the lemma, it remains to show $x \in \mathrm{Int}(\overline{W^u(\Lambda_-)} \cup \overline{W^u(\Lambda_+)})$. Notice that $M$ is the union of the closure of the basins of the transitive repellers of $f$. Hence

$$M \Big\backslash \bigcup_{\Lambda\in X_R\setminus\{\Lambda_-,\Lambda_+\}} \overline{W^u(\Lambda)} \subset \mathrm{Int}(\overline{W^u(\Lambda_-)} \cup \overline{W^u(\Lambda_+)}).$$

For concluding the proof of the lemma, it is enough to prove that $x \notin \overline{W^u(\Lambda)}$, for every repeller $\Lambda$ different from $\Lambda_+$ and $\Lambda_-$.

Let $\Lambda$ be a repeller such that $x \in \overline{W^u(\Lambda)}$. According to Lemma 6.1 the point $x$ belongs to the interior of the closure of $W^s(A)$. One deduces that $W^u(\Lambda)$ meets the interior of the closure of $W^s(A)$. As $W^u(\Lambda)$ is an open set, this implies that $W^u(\Lambda) \cap W^s(A) \neq \emptyset$. Remark 5.2 implies that $\Lambda \in \{\Lambda_+, \Lambda_-\}$ ending the proof. □

COROLLARY 6.1. *Let $A$ be a transitive attractor. Then a point $x$ belongs to $\overline{W^s(A)}$ if and only if $\{\alpha(x), \beta(x)\} \subset \Lambda_- \cup A \cup \Lambda_+$.*

*Proof.* First assume that $\{\alpha(x), \beta(x)\} \subset \Lambda_- \cup A \cup \Lambda_+$. If $\alpha(x)$ or $\beta(x)$ belong to $A$, Lemma 6.1 asserts that $x$ belongs to the interior of $\overline{W^s(A)}$. Otherwise, $\alpha(x), \beta(x) \in$

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$\Lambda_- \cup \Lambda_+$; assume for instance $\alpha(x), \beta(x) \in \Lambda_-$. The version of Lemma 6.2 for repellers implies that $x$ belongs to the intersection of the closures of the basins of the predecessor and of the successor of $\Lambda_-$. In particular, $x \in \overline{W^s(A)}$.

Conversely, consider a point $x$ such that $\{\alpha(x), \beta(x)\} \not\subset \Lambda_- \cup A \cup \Lambda_+$. If $\alpha(x)$ or $\beta(x)$ belong to an attractor $A_i \neq A$ then Lemma 6.1 implies that $x \in \mathrm{Int}(\overline{W^s(A_i)})$ which is disjoint from $\overline{W^s(A)}$. In the other case there is $\Lambda_i \notin \{\Lambda_-, \Lambda_+\}$ such that $\alpha(x), \beta(x) \in \Lambda_i$. Then the version of Lemma 6.2 for repellers implies that $x$ belongs to $\mathrm{Int}(\overline{W^s(A_i) \cup W^s(A_{i+1})})$ which is disjoint from $\overline{W^s(A)}$ by remark 6.1, because $A \notin \{A_i, A_{i+1}\}$. □

COROLLARY 6.2. *Let $A$ be a transitive attractor, $\Lambda_-$ its predecessor and $\Lambda_+$ its successor. Then*

$$\overline{W^s(A)} \subset \mathrm{Int}(\overline{W^u(\Lambda_-) \cup W^u(\Lambda_+)}).$$

*Proof.* Consider $x \in \overline{W^s(A)}$. We know from Corollary 6.1 that $\{\alpha(x), \beta(x)\} \subset \Lambda_- \cup A \cup \Lambda_+$.

If $\alpha(x)$ or $\beta(x)$ belongs to $\Lambda_- \cup \Lambda_+$ then Lemma 6.1 implies that $x \in \mathrm{Int}(\overline{W^u(\Lambda_-)}) \cup \mathrm{Int}(\overline{W^u(\Lambda_+)}) \subset \mathrm{Int}(\overline{W^u(\Lambda_-) \cup W^u(\Lambda_+)})$. Otherwise, $\alpha(x)$ and $\beta(x)$ belong to $A$; then Lemma 6.2 claims that $x \in \mathrm{Int}(\overline{W^u(\Lambda_-) \cup W^u(\Lambda_+)})$, which concludes the proof. □

LEMMA 6.3. *It holds that $\overline{W^s(A_i)} \cap \overline{W^s(A_j)} \neq \emptyset$ if and only if either $A_i = A_j$ or there is a repeller set $\Lambda$ such that $\{A_i, A_j\} = \{A_-, A_+\}$ where $A_-$ is the predecessor of $\Lambda$ and $A_+$ is its successor. In other words,*

$$\overline{W^s(A_i)} \cap \overline{W^s(A_j)} \neq \emptyset \Longleftrightarrow |i - j| \leq 1.$$

*Proof.* Assume that $A_i \neq A_j$ and consider $x \in \overline{W^s(A_i)} \cap \overline{W^s(A_j)}$. Then Lemma 6.1 implies that neither $\alpha(x)$ nor $\beta(x)$ can belong to $A_i \cup A_j$. So Corollary 6.1 implies that $\alpha(x), \beta(x)$ belong to a repeller $\Lambda$ which needs to be not only the successor or the predecessor of $A_i$ but the successor or the predecessor of $A_j$, as well. It follows that $\{A_i, A_j\} = \{A_-, A_+\}$ where $A_-$ is the predecessor of $\Lambda$ and $A_+$ is its successor.

Conversely, if $A_i$ and $A_j$ are the predecessor and the successor of $\Lambda$, then $\overline{W^s(A_i)} \cap \overline{W^s(A_j)}$ contains $\Lambda$. □

LEMMA 6.4. *Given any attractor $A$ and any repeller $\Lambda$,*

$$\overline{W^s(A)} \cap \overline{W^u(\Lambda)} \neq \emptyset \Longleftrightarrow \Lambda \text{ is the successor or the predecessor of } A.$$

*Proof.* Let $\Lambda_-$ and $\Lambda_+$ be the predecessor and the successor of $A$, respectively. According to Corollary 6.2 one has $\overline{W^s(A)} \subset \mathrm{Int}(\overline{W^u(\Lambda_-) \cup W^u(\Lambda_+)})$. If $\overline{W^s(A)} \cap \overline{W^u(\Lambda)} \neq \emptyset$ then $\mathrm{Int}(\overline{W^u(\Lambda_-) \cup W^u(\Lambda_+)} \cap \overline{W^u(\Lambda)} \neq \emptyset$, implying that $\Lambda \in \{\Lambda_-, \Lambda_+\}$.

The converse implication is a direct consequence of the definition of successor or predecessor. □

## 7. *Axiom-A diffeomorphisms in $\mathcal{E}_\phi$ with more than one attractor and repeller*

We now assume that $f \in \mathcal{E}_\phi$ is an Axiom-A diffeomorphism having at least two transitive attractors. We consider the attractors $A_i$ and the repellers $\Lambda_i$, $i \in \mathbb{Z}/k\mathbb{Z}$, $k > 1$, where $\Lambda_i$ is the successor or $A_i$ and it is the predecessor of $A_{i+1}$.

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LEMMA 7.1. *For every $i$, the boundary $\partial\overline{W^u(\Lambda_i)}$ is*

$$\partial\overline{W^u(\Lambda_i)} = (\overline{W^u(\Lambda_{i-1})} \cap \overline{W^u(\Lambda_i)}) \cup (\overline{W^u(\Lambda_i)} \cap \overline{W^u(\Lambda_{i+1})}).$$

*Proof.* Just notice that for every $j$, $\overline{W^u(\Lambda_j)}$ is a compact set; it coincides with the closure of its interior. Furthermore these interiors are pairwise disjoints, and the union of the $\overline{W^u(\Lambda_j)}$ is $M$. Hence $\mathrm{Int}(\overline{W^u(\Lambda_i)})$ is the complement of $\bigcup_{j\neq i}\overline{W^u(\Lambda_j)}$, and $\partial\overline{W^u(\Lambda_i)} = \overline{W^u(\Lambda_i)} \cap (\bigcup_{j\neq i}\overline{W^u(\Lambda_j)})$. The version of Lemma 6.3 for repeller sets implies that $\overline{W^u(\Lambda_i)} \cap (\bigcup_{j\neq i}\overline{W^u(\Lambda_j)}) = \overline{W^u(\Lambda_i)} \cap (\overline{W^u(\Lambda_{i-1})} \cup \overline{W^u(\Lambda_{i+1})})$ (because $\overline{W^u(\Lambda_i)} \cap \overline{W^u(\Lambda_j)} = \emptyset$ if $j \notin \{i-1, i, i+1\}$), concluding the proof. □

*Remark 7.1.* If $f$ has a unique repeller $\Lambda$, then $\overline{W^u(\Lambda)} = M$ so that its boundary is empty: the hypothesis $k > 1$ is actually necessary for Lemma 7.1 and for all the results in this section.

The next lemma states some properties of the boundaries of the closures of the basins of the attractors and repellers.

LEMMA 7.2.

(1) *The boundary $\partial\overline{W^u(\Lambda_i)}$ is contained in $\overline{W^s(A_i)} \cup \overline{W^s(A_{i+1})}$.*

(2) *Let $K_i = (\overline{W^u(\Lambda_{i-1})} \cap \overline{W^u(\Lambda_i)}) \cap \overline{W^s(A_i)}$. It holds that $\partial\overline{W^u(\Lambda_i)} \cap \overline{W^s(A_i)} = K_i$.*

(3) *Let $K_{i+1} = (\overline{W^u(\Lambda_i)} \cap \overline{W^u(\Lambda_{i+1})}) \cap \overline{W^s(A_{i+1})}$. It holds that $K_{i+1} = \partial\overline{W^u(\Lambda_i)} \cap \overline{W^s(A_{i+1})}$.*

(4) *For every $i$, the compact set $K_i$ is contained in* $\mathrm{Int}(\overline{W^s(A_i)})$.

(5) *The compact set $K_i$ is characterized by*

$$x \in K_i \iff \alpha(x), \beta(x) \in A_i.$$

*Proof.* The version of Corollary 6.1 for repeller sets asserts that $x \in \overline{W^u(\Lambda_i)}$ if and only if $\alpha(x)$ and $\beta(x)$ belong to $A_i \cup \Lambda_i \cup A_{i+1}$. As a consequence, $x \in \overline{W^u(\Lambda_{i-1})} \cap \overline{W^u(\Lambda_i)}$ if and only if $\{\alpha(x), \beta(x)\} \subset (A_{i-1} \cup \Lambda_{i-1} \cup A_i) \cap (A_i \cup \Lambda_i \cap A_{i+1})$.

*First case: if $k \neq 2$.* In this case, $(A_{i-1} \cup \Lambda_{i-1} \cup A_i) \cap (A_i \cup \Lambda_i \cap A_{i+1}) = A_i$. So

$$x \in \overline{W^u(\Lambda_{i-1})} \cap \overline{W^u(\Lambda_i)} \iff \{\alpha(x), \beta(x)\} \subset A_i.$$

Then, Lemma 6.1 implies that, for all $i \in \mathbb{Z}/k\mathbb{Z}$, $\overline{W^u(\Lambda_{i-1})} \cap \overline{W^u(\Lambda_i)}$ is contained in $\mathrm{Int}\overline{W^s(A_i)}$. Analogously $\overline{W^u(\Lambda_i)} \cap \overline{W^u(\Lambda_{i+1})}$ is contained in $\mathrm{Int}\overline{W^s(A_{i+1})}$. Then Lemma 7.1 implies (1).

From Lemma 6.4, we have that $\overline{W^u(\Lambda_{i-1})} \cap \overline{W^s(A_{i+1})} = \emptyset$ and $\overline{W^u(\Lambda_{i+1})} \cap \overline{W^s(A_i)} = \emptyset$, then (2), (3) and (4) hold. To prove (5) is enough to show that if $\alpha(x), \beta(x) \in A_i$ then $x \in K_i$, but this is a consequence of Lemmas 6.1 and 6.2.

*Second case: if $k = 2$.* In this case, $(A_{i-1} \cup \Lambda_{i-1} \cup A_i) \cap (A_i \cup \Lambda_i \cap A_{i+1}) = A_i \cup A_{i+1} = A_0 \cup A_1$. So

$$x \in \overline{W^u(\Lambda_{i-1})} \cap \overline{W^u(\Lambda_i)} \iff \{\alpha(x), \beta(x)\} \subset A_i \cup A_{i+1}.$$

Notice that as a consequence of Lemma 6.1 $\{\alpha(x), \beta(x)\} \subset A_i \cup A_{i+1}$ if and only if $\{\alpha(x), \beta(x)\} \subset A_i$ or $\{\alpha(x), \beta(x)\} \subset A_{i+1}$. Then, Lemma 6.1 implies that, for all $i \in \mathbb{Z}/2\mathbb{Z}$, $\overline{W^u(\Lambda_{i-1})} \cap \overline{W^u(\Lambda_i)}$ is contained in $\mathrm{Int}\overline{W^s(A_i)} \cup \mathrm{Int}\overline{W^s(A_{i+1})}$. All the items of the lemma follow immediately. □

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By claim (5) of the previous lemma and by definition of $K_i$, we obtain the following.

*Remark 7.2.* Any connected component of the intersection of a central leaf with $K_i$ is either a point in $A_i$ or it is a central arc whose both extremities belong to $A_i$.

As a direct consequence of Lemma 7.2 one has the following.

COROLLARY 7.1. *For all different $i$, $j$ in $\mathbb{Z}/k\mathbb{Z}$, $K_i$ and $K_j$ are disjoint compact sets and $\partial\overline{W^u(\Lambda_i)} = K_i \cup K_{i+1}$. Furthermore*

$$K_0 \cup K_1 = \partial\overline{W^u(\Lambda_0)} = \partial\bigg(\bigcup_{i\neq 0} \overline{W^u(\Lambda_i)}\bigg).$$

Let $\theta\colon M \to [0, 1]$ denote a continuous function such that $\theta^{-1}(0) = K_0$ and $\theta^{-1}(1) = K_1$: for instance one can choose $\theta$ defined by $\theta(x) = ((d(x, K_0))/(\sup\{d(x, K_0), d(x, K_1)\}))$. We denote by $\rho\colon M \to S^1 = \mathbb{R}/\mathbb{Z}$ the map defined as follows.

- For $x \in \overline{W^u(\Lambda_0)}$, $\rho(x)$ is the class modulo $\mathbb{Z}$ of $(1/2)\theta(x)$.
- For $x \in \bigcup_{i\neq 0} \overline{W^u(\Lambda_i)}$, $\rho(x)$ is the class modulo $\mathbb{Z}$ of $1 - (1/2)\theta(x)$.

The map $\rho(x)$ is well defined: if $x \in \overline{W^u(\Lambda_0)} \cap \bigcup_{i\neq 0} \overline{W^u(\Lambda_i)}$ then either $x \in K_0$ and $\rho(x) = 0 = 1 \in S^1$ or $x \in K_1$ and $\rho(x) = 1/2$. The map $\rho$ is continuous restricted to both compact sets $\overline{W^u(\Lambda_0)}$ and $\bigcup_{i\neq 0} \overline{W^u(\Lambda_i)}$ whose union is $M$, hence is continuous on $M$.

Let us consider the universal cover $\pi : \mathbb{R} \to S^1$. Let $\Pi\colon \tilde{M} \to M$ be the pull-back of the covering $\pi$ by $\rho$. Recall that there is a commutative diagram.

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\ \tilde{\rho}\ } & \mathbb{R} \\ \Pi\big\downarrow & & \big\downarrow\pi \\ M & \xrightarrow{\ \rho\ } & S^1 \end{array}$$

LEMMA 7.3. *Let $\gamma\colon [0, 1] \to M$ be a central arc such that $x = \gamma(0)$ belongs to $A_0$, $y = \gamma(1)$ belongs to $A_1$, and $\gamma$ is disjoint from $\Lambda_i$ for $i \neq 0$. Let $\tilde{\gamma}$ be a lift of $\gamma$ on $\tilde{M}$ and let denote $\tilde{x} = \tilde{\gamma}(0) \in \Pi^{-1}(x)$ and $\tilde{y} = \tilde{\gamma}(1) \in \Pi^{-1}(y)$. Then $\tilde{\rho}(\tilde{y}) - \tilde{\rho}(\tilde{x}) = 1/2$.*

*Proof.* By hypotheses, $\rho(x) = 0$ and $\rho(y) = 1/2$. We just need to prove that $\rho(\gamma)$ is contained in the arc $[0, 1/2]$ of $S^1 = \mathbb{R}/\mathbb{Z}$ (hence is equal to that arc). For that, it is enough to see that $\gamma$ is contained in $\overline{W^u(\Lambda_0)}$. Proposition 5.1 and the fact that $\gamma$ joins the point $x \in A_0$ to $y \in A_1$ without crossing $\Lambda_i$ for $i \neq 0$ imply that, for any $z \in \gamma$ one has $\{\alpha(z), \beta(z)\} \subset A_0 \cup \Lambda_0 \cup A_1$. We conclude using Corollary 6.1 that $z \in \overline{W^u(\Lambda_0)}$, ending the proof. □

LEMMA 7.4. *Let $\gamma\colon [0, 1] \to M$ be a central arc such that $x = \gamma(0)$ belongs to $A_1$, $y = \gamma(1)$ belongs to $A_0$, and $\gamma$ is disjoint from $\Lambda_0$. Let $\tilde{\gamma}$ be a lift of $\gamma$ on $\tilde{M}$ and let denote $\tilde{x} = \tilde{\gamma}(0) \in \Pi^{-1}(x)$ and $\tilde{y} = \tilde{\gamma}(1) \in \Pi^{-1}(y)$. Then $\tilde{\rho}(\tilde{y}) - \tilde{\rho}(\tilde{x}) = 1/2$.*

*Proof.* This time, $\rho(x) = 1/2$ and $\rho(y) = 1 = 0 \in S^1$. We just need to prove that $\rho(\gamma)$ is contained in the arc $[1/2, 1]$ of $S^1 = \mathbb{R}/\mathbb{Z}$. In other words, we have to prove that $\gamma$ is disjoint from $\mathrm{Int}(\overline{W^u(\Lambda_0)})$, that is, it is included in $\bigcup_{i\neq 0} \overline{W^u(\Lambda_i)}$. Proposition 5.1 and the fact that $\gamma$ joins the point $x \in A_1$ to $y \in A_0$ without crossing $\Lambda_0$ imply that,

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for any $z \in \gamma$, $\alpha(z) \notin \Lambda_0$ and $\beta(z) \notin \Lambda_0$. Using the fact that $\beta(z)$ belongs either to the basic set containing $\alpha(z)$ or to its successor, one shows that there is $i \neq 0$ in $\mathbb{Z}/k\mathbb{Z}$ such that $\{\alpha(z), \beta(z)\} \subset A_i \cup \Lambda_i \cup A_{i+1}$. We conclude using the version of Corollary 6.1 for repeller sets that $z \in \overline{W^u(\Lambda_i)}$ (with $i \neq 0$), ending the proof. □

LEMMA 7.5. *Let $\gamma : [0, 1] \to M$ be a central arc meeting $A_0$ and $A_1$ at most at its extremities (in formula: $\gamma \cap (A_0 \cup A_1) \subset \{\gamma(0), \gamma(1)\}$). Let $\tilde{\gamma}$ be a lift of $\gamma$ on $\tilde{M}$ and let $\tilde{x} = \tilde{\gamma}(0) \in \Pi^{-1}(x)$ and $\tilde{y} = \tilde{\gamma}(1) \in \Pi^{-1}(y)$. Then $|\tilde{\rho}(\tilde{y}) - \tilde{\rho}(\tilde{x})| \leq 1/2$.*

*Proof.* We will prove that, either $\gamma \subset \overline{W^u(\Lambda_0)}$ or $\gamma \cap \mathrm{Int}\overline{W^u(\Lambda_0)} = \emptyset$. Recall that the boundary $\partial \overline{W^u(\Lambda_0)}$ is the union $K_0 \cup K_1$. According to Remark 7.2, we have that any connected component of the intersection of a central leaf with $K_0$ is either a point in $A_0$ or it is a central arc whose both extremities belong to $A_0$. As the interior of $\gamma$ is disjoint from $A_0$ we get that $\gamma$ is either contained in $K_0$ or its interior is disjoint from $K_0$. The same holds for $K_1$. So either $\gamma$ is contained in $\partial \overline{W^u(\Lambda_0)}$ (hence in $\overline{W^u(\Lambda_0)}$) or the interior of $\gamma$ is disjoint from $\partial \overline{W^u(\Lambda_0)}$; in this case the interior of $\gamma$ is either contained in $\mathrm{Int}\overline{W^u(\Lambda_0)}$ or it is disjoint from $\overline{W^u(\Lambda_0)}$, ending the proof. □

We denote by $\tilde{\mathcal{F}}^c_c$ the lift on $\tilde{M}$ of the foliation $\mathcal{F}^c_f$. Given a point $x \in M$ we denote by $\gamma_x : \mathbb{R} \to M$ the infinite positively oriented central arc, parametrized by the arc length, such that $\gamma_x(0) = x$. Consider $\tilde{x} \in \Pi^{-1}(x)$. We denote by $\gamma_{\tilde{x}}$ the lift of $\gamma_x$ on $\tilde{M}$ with $\gamma_{\tilde{x}}(0) = \tilde{x}$. Recall that $K_f$ is an upper bound of the length of the central arc $D(x)$, for every $x \in M$.

LEMMA 7.6. *For every point $\tilde{x} \in \tilde{M}$ and for any $\ell > 4K_f$, the map $\varphi_x = \tilde{\rho} \circ \gamma_{\tilde{x}} : \mathbb{R} \to \mathbb{R}$ satisfies*

$$\varphi_x(t + \ell) \geq \varphi_x(t) + 1.$$

*In particular,* $\lim_{t\to-\infty} \varphi_x(t) = -\infty$ *and* $\lim_{t\to+\infty} \varphi_x(t) = +\infty$.

*Proof.* Let $x$ be the projection of $\tilde{x}$ on $M$. Let $\gamma$ be the restriction of $\gamma_x$ to $[0, 4K_f]$, and let $\tilde{\gamma}$ be the lift of $\gamma$ on $\tilde{M}$ starting at $\tilde{x}$. We want to prove $\tilde{\rho}(\tilde{\gamma}(4K_f)) - \tilde{\rho}(\tilde{\gamma}(0)) \geq 1$. For that, using Proposition 5.1 we write $\gamma$ as being the concatenation $\gamma_0 \cdot \gamma_1 \cdots \gamma_m \cdot \gamma_{m+1}$ where we have the following.

- $\gamma_0$ is a central arc starting at $x$ and joining $x$ to the first point of $\gamma$ in $A_0 \cup A_1$.
- By the proof of Lemma 7.5 for $i \in \{1, \ldots, m\}$, $\gamma_i$ is either
  - a central arc joining a point of $A_0$ to a point of $A_1$ included in $\overline{W^u(\Lambda_0)}$ and then it is disjoint of $\Lambda_i$, for $i \neq 0$; or
  - a central arc joining a point of $A_1$ to a point of $A_0$ included in the complement of $\mathrm{Int}\overline{W^u(\Lambda_0)}$ and then it is disjoint of $\Lambda_0$.
- $\gamma_{m+1}$ is the arc in $\gamma$ joining the last point of $\gamma$ in $A_0 \cup A_1$ to $\gamma(4K_f)$.

Lemmas 7.3, 7.4 and 7.5 imply that

$$\tilde{\rho}(\tilde{\gamma}(4K_f)) - \tilde{\rho}(\tilde{\gamma}(0)) \geq \frac{m}{2} - 1.$$

Recall that each arc $D(z)$, $z \in M$ meets any attractor and any repeller and its length is less than $K_f$. So every arc $D(z)$ contains at least one segment either joining $A_0$ to $A_1$ or joining $A_1$ to $A_0$. As the length of $\gamma$ is larger than $4K_f$, it contains at least four disjoint arcs of the form $D(z)$. Hence $m \geq 4$, ending the proof. □

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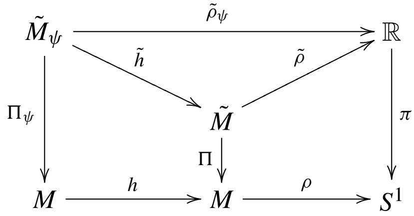
Let $\psi\colon M \times \mathbb{R} \to M$ be a smooth flow, $C^1$-close to $\phi$ so that every non-zero time map of the flow $\psi$ belongs to $\mathcal{E}_\phi$. Let $h\colon M \to M$ be an homeomorphism such that $h(\mathcal{F}^c_\psi) = \mathcal{F}^c_f$. Let $\rho_\psi = \rho \circ h\colon M \to S^1$. Let $\Pi_\psi\colon \tilde{M}_\psi \to M$ be the pull-back by $\rho_\psi$ of the universal cover $\pi : \mathbb{R} \to S^1$, and $\tilde{\rho}_\psi : \tilde{M}_\psi \to \mathbb{R}$ be the lift of $\rho_\psi$. Notice that $\tilde{\rho}_\psi$ splits in a product $\tilde{\rho}_\psi = \tilde{\rho} \circ \tilde{h}$, where $\tilde{h}\colon \tilde{M}_\psi \to \tilde{M}$ is a lift of $h$. One has the following Abelian diagram.




Let $\tilde{\psi}$ be the lift of the flow $\psi$ on $\tilde{M}_\psi$. Then one gets the following.

COROLLARY 7.2. *There is $L > 0$ such that for every $x \in \tilde{M}_\psi$ one has:*

$$\tilde{\rho}_\psi(\tilde{\psi}(x, L)) - \tilde{\rho}_\psi(x) > 1.$$

Then a nice argument of Schwartzman (see [**18**] or [**25**, Teorema 5.1]) allows us to conclude the proof of Theorem 2. We reproduce here this argument for completeness.

*Proof of Theorem 2.* Let $\mu\colon M \to S^1$ be a smooth map $C^0$-close to the map $\rho_\psi$, and $\tilde{\mu}$ be the lift of $\mu$ on $\tilde{M}_\psi$. If $\mu$ is close enough to $\rho_\psi$, for every $x \in \tilde{M}_\psi$ one has

$$\tilde{\mu}(\tilde{\psi}(x, L)) - \tilde{\mu}(x) > 1.$$

For $x \in \tilde{M}_\psi$ let us denote

$$\tilde{\lambda}(x) = \frac{1}{L}\int_0^L \tilde{\mu}(\tilde{\psi}(x, t))\, dt.$$

Notice that $\tilde{\lambda}\colon \tilde{M}_\psi \to \mathbb{R}$ is a smooth function which projects on $M$ in a map $\lambda\colon M \to S^1$. Furthermore, for any $x \in \tilde{M}_\psi$ the derivative of $\tilde{\lambda}$ along the $\tilde{\psi}$-orbit is

$$\frac{\partial}{\partial t}\tilde{\lambda}(\psi(x, t))|_{t=0} = \frac{1}{L}(\tilde{\mu}(\tilde{\psi}(x, L)) - \tilde{\mu}(x)) > \frac{1}{L} > 0.$$

This proves that the map $\lambda\colon M \to S^1$ is a submersion and that the orbits of $\psi$ are transverse to the fibers; let us denote by $N$ the fiber of this fibration. So, $\psi$ (hence also $\phi$) is topologically equivalent to a suspension of a diffeomorphism $g$ of $N$, which inherits the hyperbolic structure of the flow (the stable and unstable bundles of $g$ are the intersections of the corresponding bundles for $\psi$ with $TN$); so $g$ is an Anosov diffeomorphism and the transitivity of $\psi$ implies the transitivity of $g$. □

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