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REPLICATOR DYNAMIC LEARNING IN MUTH'S MODEL OF PRICE MOVEMENTS

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We investigate the stability properties of Muth's model of price movements when agents choose a production level using replicator dynamic learning. It turns out that when there is a discrete set of possible production levels, possible stable states and stability conditions differ between adaptive learning and replicator dynamic learning.

Keywords: Asymptotic stability, Replicator dynamics, Cobweb model, E-stability, Nash equilibria

1. INTRODUCTION

Over the past several decades, macroeconomists have spent considerable effort exploring the conditions under which a rational expectations equilibrium (REE) can be learned. This line of research is motivated by the observation that successful identification of the REE in any particular system often requires more information than even an accomplished analyst would have access to. Because it is not likely that a typical population of agents would immediately coordinate on such an equilibrium, it is of interest to determine whether they would eventually settle on it if, over time, they learned about their environment and adjusted their behavior to reflect this learning. If an REE cannot be learned, one cannot feel confident that it will ever be realized.

During this same period, game theorists have constructed an extensive literature that models learning and equilibrium selection in simple strategic environments.[1] However, despite a close conceptual connection to its macroeconomic counterpart, the game-theoretic literature has developed largely in isolation from it. This paper attempts to connect the game-theoretic literature to its macroeconomic counterpart, under a specific model, by comparing the conditions that macroeconomists identify

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as necessary for a REE to be learnable with those that are required when learning is modeled as game theorists often do (replicator dynamics).

Because replicator dynamics inherently involves a population of heterogeneous agents, the game-theoretic model of learning generalizes the approach of most macroeconomic analyses, which typically assume a representative agent or a population of agents that act identically.[2] Thus, this paper contributes to the existing literature on the stability of REEs under learning by checking the robustness of established stability conditions. Of particular interest throughout our analysis is the question of whether stability under replicator dynamic learning is governed by the so-called E-stability condition [Marcet and Sargent (1989); Evans and Honkapohja (1994, 2001)], which is a necessary condition for convergence under simple econometric learning.

So far, the use of replicator dynamics in the field of macroeconomics has mainly focused on predictor selection. Sethi and Franke (1995) consider a model where agents have the choice to pay a resource cost to use rational expectations (RE) or use a costless adaptive rule. They find that under certain conditions, the replicator dynamics will converge to where all agents use RE. Brock and Hommes (1998) and Branch and McGough (2008) use replicator dynamics in a different setting to show the possibility of complex dynamics. Finally, Guse (2010) studies stability properties (under adaptive learning) when agents have the choice of one of two learning rules and predictor selection is dictated by the replicator dynamics. He discovers a result similar to that shown in Sethi and Franke (1995), where only one learning mechanism can exist in the limit.

Our approach is different from those of these papers, as we use the replicator dynamics to model production choice rather than predictor selection. We analyze a game-theoretic version of Muth's "cobweb" model of price movements [Muth (1961)] in which agents choose how much to produce in each period, and replicator dynamics is used to model how agents are learning how much to produce. The cobweb model is appealing for a number of reasons. Its relative simplicity makes its strategic structure easy to describe and analyze. In addition to its simplicity, this model has already been the subject of a number of theoretical studies that examine when its REE will be learnable.[3] Thus, once the requirements for learnability under replicator dynamics have been identified, they can be compared immediately to the E-stability conditions for the cobweb model.

We present a game-theoretic model in which there is a continuum of agents who choose a production level from a discrete choice set. It turns out that the Nash solution corresponding to the REE under the standard Muth model is always locally stable under replicator dynamic learning. This stability condition differs from that under other types of learning for which the REE can be locally unstable under certain conditions. Further to this, we discover a disparity of global stability conditions under adaptive learning[4] and replicator dynamic learning. We also show that for a certain parameter set, under replicator dynamic learning, multiple stable Nash equilibria exist, given a discrete choice set. However, if the discrete choice set is expanded so that the distance between adjacent strategies is very small, then the only possible stable strategies become the REE strategy and the two extreme strategies.

2. LEARNING THE RATIONAL EXPECTATIONS EQUILIBRIUM IN MUTH'S MODEL

In his seminal paper on the role of rational expectations, Muth modeled prices in a perfectly competitive market where, at the time when production decisions are made, firms are unsure about the price they will receive for their goods. The original Muth model can be expressed in a reduced form in which the price level is determined by expected prices. In our presentation of Muth's model, we present the alternative reduced form of the model where output is determined by expected output. This alternative presentation will motivate the strategic version of Muth's model, presented in the next section, where all agents must choose a level of output contained in a discrete choice set.

In this model, there is a continuum of firms, and market demand is assumed to be linear. Letting $f(q)$ denote the probability distribution of firms across available production levels, $q \in (0, \infty)$, the equilibrium price level will satisfy

$$P_t^D = A - B \int_0^\infty q f(q)\, dq. \tag{1}$$

Each firm (indexed by i) is assumed to face the same quadratic cost function:

$$c_{i,t} = \left(\frac{\alpha}{2}\right) q_{i,t}^2;\ \alpha > 0. \tag{2}$$

Given their anticipated level of aggregate output for period t, $E_{i,t}q_t$, each firm maximizes expected profit by producing an amount[5] equal to $\frac{A - BE_{i,t}q_t}{\alpha}$. Thus, the path that equilibrium output actually follows is determined by the expectations of firms. Letting $g_t(q^e)$; $q^e \in (0, \infty)$ represent the probability distribution of firms across expected output levels, the equilibrium output equation can be rewritten as

$$q_t = \frac{A}{\alpha} - \frac{B}{\alpha}\bar{q}_t^e, \tag{3}$$

where $\bar{q}_t^e = \int_0^\infty q^e g_t(q^e)\, dq^e$. If we also assume that output expectations are homogeneous across firms, (3) reduces to[6]

$$q_t = \frac{A}{\alpha} - \frac{B}{\alpha} q_t^e, \tag{4}$$

where q_t^e is the level of output that all firms expect to prevail at time t. In his paper, Muth used a representative firm, effectively making this assumption.

With the equilibrium level of output described by (4), there is a unique level of output, q^{REE}, that can be correctly anticipated by firms. This is the rational expectations equilibrium level of output for the model.[7] Specifically,[8]

$$q^{\mathrm{REE}} = \frac{A}{\alpha + B}. \tag{5}$$

When all firms expect the REE output to prevail, the resulting REE price will be

$$P^{\mathrm{REE}} = \frac{A\alpha}{\alpha + B}. \tag{6}$$

In the rational expectations equilibrium, all firms expect to be able to sell their goods at P^{REE}, and all firms produce the quantity that is optimal given their expectation, q^{REE}.

Much of the literature about learning in macroeconomic rational expectations models has centered on whether agents using econometric learning rules will eventually be led to an REE.[9] These papers typically posit a representative agent who uses ordinary least squares (OLS) to estimate an incomplete econometric model of the law of motion (often called the perceived law of motion) for a variable or vector of variables he is trying to anticipate. The question of interest is whether OLS will give parameter estimates for the perceived law of motion that converge over time to the parameters of the actual law of motion (the relationship between the estimated parameters of the perceived law of motion and the state variables).

The condition for an equilibrium to be locally stable under a simple learning rule such as OLS is known as expectational stability, or E-stability. It is well known [Marcet and Sargent (1989); Evans and Honkapohja (2001)] that, under fairly general conditions, local convergence of OLS learning and fairly related learning rules in linear models is governed by the so-called E-stability condition presented by Evans (1989). Letting ϕ denote the vector of parameters in the perceived law of motion that the agent estimates and $T(\phi)$ denote the vector of parameters in the actual law of motion when expectations are based on ϕ, the E-stability condition for any linear system is met when the REE is locally asymptotically stable under the differential equation

$$\frac{d\phi}{d\tau} = T(\phi) - \phi. \tag{7}$$

For the nonstochastic version of Muth's model considered in this paper, the analysis is particularly simple. We suppose that firms incorrectly assume that production is constant. Thus, firms base their expectations on a perceived law of motion given by

$$q_t = \psi. \tag{8}$$

In view of (8), each firm will expect the period-t level of output to equal its estimate of ψ based on the history of output through period $t-1$, which we will denote by $\hat{\psi}_{t-1}$. This makes the actual law of motion follow

$$q_t = T(\hat{\psi}_{t-1}) = \frac{A}{\alpha} - \frac{B}{\alpha}\hat{\psi}_{t-1}.$$

Under the actual law of motion, output will coincide with firm forecasts when firms expect output to equal its REE level, $\frac{A}{\alpha+B}$. From (7), the E-stability condition will be met whenever the differential equation $d\hat{\psi}/d\tau = A/B - \left(1+\frac{B}{\alpha}\right)\hat{\psi}$ is locally asymptotically stable at $\hat{\psi} = q^{\text{REE}}$. For the cobweb model, it turns out[10] that the REE is globally E-stable if $B/\alpha > -1$ and E-unstable for $B/\alpha < -1$. This means that if $\frac{B}{\alpha} > -1$, then no matter what the initial beliefs of agents concerning the expected level of output, agents will coordinate through time to the REE. In the following, we show that the stability condition of the cobweb model will be different under replicator dynamic learning: learning where firms tend to imitate the production decisions of the firms receiving higher-than-average profits.

3. A STRATEGIC VERSION OF MUTH'S MODEL

To study the cobweb model under evolutionary learning, we first present the cobweb model as a game of production choice. In this section, we formulate a strategic version of Muth's model in which evolutionary learning can be applied. The REE quantity is shown to be a symmetric Nash equilibrium strategy of this game. We then model evolutionary learning using replicator dynamics (sometimes referred to as imitation dynamics) in which agents tend to "evolve" or "replicate" to strategies with higher-than-average payoffs.

To enable meaningful comparison between the stability requirements of a strategic version of Muth's model and the stability requirements of the dynamic formulation in (1), (2), and (3), we have retained virtually all of the structure from (1), (2), and (3) in the strategic formulation. The game consists of an infinite number of firms that select production levels. All firms face quadratic costs as in (2). In addition, the market demand curve continues to be linear, so that market-clearing prices satisfy (1).

The game that we analyze does differ from the Muth model in one important respect. In the game we study, firms select from a *finite* list of production levels in the neighborhood of the REE quantity. In the model presented earlier, firms can produce any positive amount; however, we assume a finite strategy set to enable a tractable application of replicator dynamics to the game.

We set up a *cobweb game* that will be expressed in normal form as $G = (I, S, \pi)$, where I is the set of players, S is its pure-strategy space, and π is the pure-strategy payoff function. Assume that there exists a continuum of firms, i.e., $I = [0, 1]$. Each firm chooses a level of production under a common strategy set consisting of n strategies, where

$$s = \left\{s_1, s_2, \ldots, s_n\right\},$$

$s_1 \geq 0$, the kth element is the REE quantity, so that[11] $s_k = s^* = \frac{A}{B+\alpha}$,

$$s_j = s^* + (j-k)\,\varepsilon,$$

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and

$$\varepsilon = \frac{s^* - s_1}{k - 1}.$$

We will assume that $k \notin \{1, n\}$ so that each agent has the opportunity to produce more than or less than the REE quantity.[12] The pure strategy profile, $x \in S$ (S defined later), denotes the distribution of players across the available strategies. Therefore, the pure strategy space is the unit simplex

$$S = \Delta = \left\{ x \in \mathcal{R}^n_+ : \sum_{h=1}^{n} x_h = 1 \right\}.$$

We also denote the interior of the unit simplex as follows:

$$\text{int}\,(\Delta) = \{x \in \Delta : x_h > 0\ \forall h\}.$$

For a given x, the payoff function for using strategy h, $\pi_h : S \to \mathcal{R}$, is defined by

$$\pi\,(s_h, x) = [A - B\,(s \bullet x)]\, s_h - \frac{\alpha}{2} s_h^2.$$

This is the profit for producing s_h given that total supply is equal to $s \bullet x$. The pure-strategy payoff of the game $\pi : S \to \mathcal{R}^n$ is thus the following:

$$\pi(x) = [\pi(s_1, x), \pi(s_2, x), \ldots, \pi(s_n, x)].$$

In a cobweb game, a firm's best strategy depends upon how its competitors behave. Thus, aggregate production and the price level implicitly depend upon firm expectations. However, because expectations do not formally enter into the game, there is no REE per se. Instead, the game's equilibria must be characterized in terms of the Nash equilibrium concept and refinements of it. It turns out that, in the cobweb game, the REE quantity in the corresponding cobweb model, q^{REE}, is always a strict symmetric Nash equilibrium strategy.

PROPOSITION 1. *In any cobweb game,* $s^* = \frac{A}{B+\alpha}$ *is a strict symmetric Nash equilibrium strategy.*

Proof. In a cobweb game, players share a best response function that varies with the distribution of strategies across the population, x. Specifically, this function is given by $R(x) = \arg\max_{\{s_h \in s\}}\{[A - B(s \bullet x)]s_h - \frac{\alpha}{2}s_h^2\}$. If s included all positive real numbers, $R\,(x)$ would be $\frac{A-B(s\bullet x)}{\alpha}$, which we will refer to as the ideal response. Because s may not contain the ideal response and the payoff function is symmetric around the ideal response, $R\,(x)$ is instead the strategy in s that is closest in magnitude to $\frac{A-B(s\bullet x)}{\alpha}$. By definition, a strict symmetric Nash equilibrium is a population distribution that places all probability on some strategy z and to which the unique best response is z. In a cobweb game, when all players choose z, $(s \bullet x) = z$. Thus, a strict symmetric Nash equilibrium strategy in a cobweb game is a strategy z, which is closer to $\frac{A-Bz}{\alpha}$ than any other strategy

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in s. Because s^* satisfies $s^* = \frac{A-Bs^*}{\alpha}$, it is a strict symmetric Nash equilibrium strategy. ■

4. STABILITY RESULTS

Our model of learning will focus on firms attempting to learn which strategy will maximize their profits. We could model this in terms of learning expected prices or learning quantity. From equation (3), if an agent chooses strategy s_i, then the agent's expectation of output is the following:

$$E_i q = \frac{A}{B} - \left(\frac{B}{\alpha}\right)^{-1} s_i.$$

Therefore, it follows that the expected price from using strategy s_i is

$$E_i P = \alpha E_i q.$$

As the transformation from a production strategy set to a strategy set of expected prices is one-to-one, the stability properties of the game will be the same under either strategy set. We choose to model learning through output rather than expected price, as we assume that firms may choose not to reveal expectations, but they cannot avoid revealing their production choices.

To determine if firms will settle on the REE quantity over time, we examine its stability properties in a dynamic system in which the population density for each strategy is continuously adjusted to reflect how well that strategy performs relative to other available strategies. This population dynamics is the well-known replicator dynamics [Taylor and Jonker (1978)], sometimes referred to as the imitation dynamics. We define replicator dynamic learning as follows:[13]

DEFINITION 1. *Under replicator dynamic learning, population proportions evolve in continuous time according to*

$$\dot{x}_h = x_h \left[\pi\,(s_h, x) - \left(\sum_{j=1}^{n} x_j \pi(s_j, x) \right) \right], \tag{9}$$

where $h = 1, 2, \ldots, n$.

Applied to a cobweb game, replicator dynamics models a scenario in which firms play the game repeatedly and, from time to time, compare the relative profitability of their production level to other levels. Relatively unprofitable firms are assumed to gradually change their production levels to mimic the relatively profitable firms. Thus, over time, the proportion of the population that produces at levels generating less than average profit will decline, whereas the proportion of the population that selects the relatively profitable production levels will increase. Replicator dynamic learning is different from adaptive learning, as firms are not concerned with prices but with relative profits.

Before we proceed, it should be noted that under replicator dynamics the population distributions are attractive (or not), rather than the strategies themselves. Thus, precisely put, the question throughout our analysis is to what extent the population distribution that places all probability on a particular strategy (such as s^*) is dynamically stable.

DEFINITION 2. *In a given cobweb game, the REE population distribution, x^*, will be the vector in Δ satisfying $x^*_k = 1$ and $x^*_h = 0\ \forall h \neq k$.*

For the sake of convenience, however, we will continue to refer to strategies as being stable or unstable. Hereafter, we will follow the convention of calling a strategy “stable” or “attractive” when the distribution that places all probability on this strategy is asymptotically stable under the replicator dynamics. We will discuss situations in which a strategy is locally or globally stable under the replicator dynamics.[14]

DEFINITION 3. *A strict symmetric Nash population distribution, $x^*_j \in \Delta$, where $x_j = 1$ and $x_i = 0\ \forall i \neq j$, is* locally stable *under the replicator dynamic if there exists some $\tilde{\Delta} \subset \Delta$, where $\tilde{\Delta}$ is a neighborhood of x^*_j within Δ, such that $x \to x^*_j$ under the replicator dynamics for any $x \in \tilde{\Delta}$. x^*_j is said to be* globally stable *if $x \to x^*_j$ under the replicator dynamics for any $x \in D$ where $D = \{x \in \mathcal{R}^n_+ : \sum_{h=1}^n x_h = 1, x_j \neq 0\}$.*

Note that only Nash population distributions can be stable under replicator dynamics, as they are optimal against themselves. All non-Nash population distributions are not optimal against themselves, so that some other strategy would return higher profits. Therefore, the replicator dynamic would always direct the population away from all non-Nash population distributions.

Based on the stability results in other models of learning, one might expect that for some range of parameters, the REE quantity, s^*, is a locally unstable equilibrium strategy under the replicator dynamics. In particular, you might expect this to be true when $B/\alpha < -1$, which violates the E-stability condition.[15] As it turns out, however, s^* will be locally stable under replicator dynamic learning for any $B \in (-\infty, \infty)$ and $\alpha > 0$.

Figure 1 depicts a projection of the phase space for a three-strategy game such that the REE is not E-stable. In the game depicted, $A = -0.28$, $B = -0.04$, and $\alpha = 0.032$. Firms select production levels from {34, 35, 36}, where the REE quantity, 35, is represented by the origin. In this example, although the REE is locally unstable under adaptive learning, the REE strategy, s^*, is attractive within a large local neighborhood under replicator dynamic learning. Note that the other strategies are locally stable as well. This result is consistent with our findings in the following.

Ultimately, the robust local stability of s^* derives from the fact that s^* is always a strict symmetric Nash equilibrium. As shown later, local asymptotic stability is a general feature of strict symmetric Nash equilibria in cobweb games. Because x^* is always a strict symmetric Nash equilibrium, it will always be locally stable.

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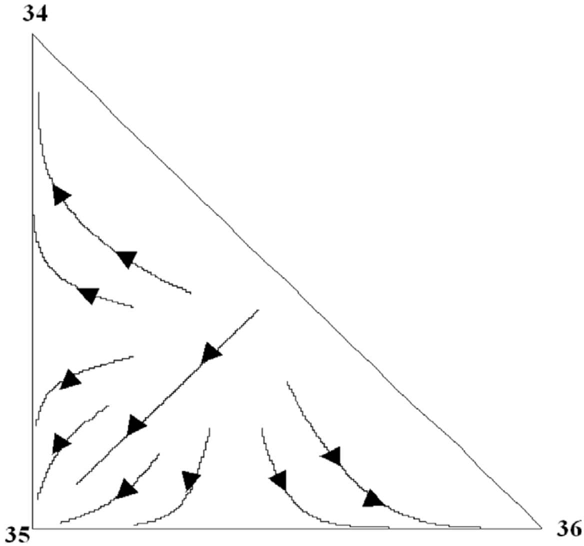

FIGURE 1. Phase diagram for a cobweb game that fails the E-stability condition.

PROPOSITION 2. *Any strict symmetric Nash equilibrium in a cobweb game will be locally asymptotically stable under replicator dynamic learning.*[16]

The proof is given in Appendix A. Although the local stability of s^* under all possible choices of B and α seems at first surprising, upon reflection it has a simple intuitive explanation.[17] Suppose $B/\alpha < -1$, so that the E-stability condition is not met in the Muth model. Suppose, in addition, that a proportion of firms $(1 - \mu)$ are producing some $s_i \in S$ such that $s_i = s^* + \omega > s^*$, whereas the remaining firms produce s^*, so that the the average production level $(s^* + (1 - \mu)\,\omega)$ initially exceeds s^*. In Muth's model, this will generate a divergent price path when firms use least-squares learning. Above-equilibrium production corresponds to above-equilibrium expected prices. But this leads to even higher actual prices, which increases expected prices and production levels and ultimately leads to divergence.

Under the replicator dynamics, as we are only discussing two different strategies in this example, we will find that s^* is asymptotically stable if the payoff from using the REE strategy is larger than the payoff for producing s_i. The difference in payoffs from the two strategies can be expressed in the following function:

$$\pi(s^*, x) - \pi\,(s_i, x) = \left[\frac{\alpha}{2} + (1 - \mu)\,B\right]\omega^2.$$

As $\alpha > 0$, we can see that this expression is always greater than zero when $B/\alpha > -0.5$. Further to this, there exists some $\mu \in (0, 1)$ such that the expression is greater than zero for any $B/\alpha \in (-\infty, -0.5)$. This means that as long as the proportion of firms using s^* is large enough, it will still be in each firm's best interest to choose the Nash production level, s^*. The size of ω and the location of s^* (determined by k) do not affect this result.

According to Proposition 2, the REE distribution in a cobweb game, x^*, will be a limit state under replicator dynamics learning whenever the initial population composition is sufficiently close to the equilibrium composition. As long as production levels are initially in the neighborhood of the REE level, all firms will eventually learn to produce the REE quantity. Of course, that does not rule out the possibility that firms initially select a wide variety of production levels and ultimately learn to produce something other than s^*.

In what follows, we show that in many cobweb games, s^* will be the unique limit state. Specifically, in any cobweb game for which $B/\alpha > -0.5$, replicator dynamics learning will eventually lead all firms to produce the REE quantity, as long as a positive fraction of firms are initially producing it.

PROPOSITION 3. *The REE distribution, x^*, is globally stable under replicator dynamics in any cobweb game for which α, $B > 0$ or $B < 0 < \alpha$ and $B/\alpha > -0.5$.*[18]

The proof is given in Appendix B. Under the standard setup in which $B/\alpha > 0$, Proposition 3 reveals a close link between the stability condition under replicator dynamic learning and the E-stability condition. In Muth's model, the (global) E-stability condition is always met when demand is downward sloping and supply is upward sloping. According to Proposition 3, replicator dynamic learning in a cobweb game will also lead to a state in which all firms produce the REE quantity, given the initial $x_k \in (0, 1]$, when demand slopes down and supply slopes up. Among cobweb models with upward-sloping demand[19] ($B < 0$), those in which $B/\alpha > -1$ will satisfy the (global) E-stability condition. Similarly, the REE distribution in a cobweb game remains attractive, within D, under replicator dynamic learning for a range of (α, B)-pairs when B is allowed to be negative, though in cobweb games the range is smaller. In the next section, we show that the reason s^* is locally stable for $B/\alpha \in (-1, -0.5)$ rather than globally stable is that there may exist other locally stable strict symmetric Nash equilibria for this range of B/α.

5. MULTIPLE LOCALLY ATTRACTIVE OUTCOMES

Proposition 3 establishes that in cobweb games satisfying $B/\alpha > -0.5$, firms will invariably end up producing the REE quantity of the associated cobweb model. As long as $B/\alpha > -0.5$ and $x \in D$, s^* will be attractive under replicator dynamics learning. In this section, we consider cobweb games in which $B/\alpha < -0.5$, and show that, in such games, replicator dynamics learning will not generally lead all firms to produce s^*. Note that although such games can be interpreted as cobweb models in which demand is upward-sloping, they can also be shown to model more economically palatable scenarios.[20]

For $B/\alpha < -0.5$, as s^* is not asymptotically stable for some $x \in D$, we consider other limit points for the replicator dynamics when the initial population distribution is outside the neighborhood of local stability for s^*. It turns out that a cobweb game can have a number of locally attractive strategies when $B/\alpha < -0.5$, as we show hereafter.

Let $s_j(B, \alpha)$ denote the strategy $s^*(B, \alpha) + (j-k)\varepsilon$, $j \neq k$, and consider the ideal response to the population distribution that places all probability on s_j. Because the ideal response equals $s^* - \frac{B}{\alpha}(j-k)\varepsilon$, it will approach s_j as B/α approaches -1. Although the ideal response will not equal s_j, the best available response may still be s_j due to the discrete production choice set. In this case, s_j will be a strict symmetric Nash equilibrium strategy and, therefore, locally attractive. We begin with the sufficient condition for a strategy s_j (where $j \neq k$) to be locally asymptotically stable and then present the necessary conditions for local stability. For the discussion that follows, we will define

$$M_j^L = -1 - \frac{1}{2|j-k|},$$
$$M_j^H = -1 + \frac{1}{2|j-k|}.$$

PROPOSITION 4. *If*

$$\frac{B}{\alpha} \in \left(M_j^L, M_j^H\right), \tag{10}$$

then the strategy $s_j \in s$ $(j \neq k)$ will be locally asymptotically stable under replicator dynamic learning.

The proof is given in Appendix C. Proposition 4 establishes the result advertised at the beginning of this section. When $B/\alpha < -0.5$, replicator dynamic learning will not, in general, lead all firms to produce the REE quantity. Depending upon the initial distribution of production levels, firms may end up producing more or less than s^*.

Proposition 4 paints an incomplete picture of replicator dynamic learning when $B/\alpha < -0.5$. If x^* is not the only possible outcome in this circumstance, one would like to know the complete set of possible outcomes and be able to identify the conditions under which each can occur. To complete the picture of replicator dynamic learning in cobweb games, we need to know whether firms can learn to produce s_j if (10) is not satisfied, and if so under what conditions.[21]

As it turns out, these questions can be answered with relative ease. Except for the extreme strategies, s_1 and s_n, it can be shown that the necessary condition for local attractiveness is virtually identical to the sufficient condition described by (10).

PROPOSITION 5. *For all $j \notin \{1, k, n\}$, local asymptotic stability of strategy $s_j \in s$ requires that*

$$\frac{B}{\alpha} \in \left[M_j^L, M_j^H\right].$$

Extreme strategies, on the other hand, will continue to be locally attractive as $B/\alpha \to -\infty$, as shown in Proposition 6:

PROPOSITION 6. *For* $j \in \{1, n\}$ *local asymptotic stability of strategy* $s_j \in s$ *requires that*

$$\frac{B}{\alpha} \in \left(-\infty, M_j^H\right].$$

See Appendix D for proofs of both propositions. These propositions show that every strategy may be locally attractive when B/α is contained in some subset of $(-1.5, -0.5)$; however, the set of locally attractive symmetric Nash equilibria will be confined to $\{s_1, s^*, s_n\}$ when $B/\alpha < -1.5$.

One final consequence of Propositions 5 and 6 is noteworthy. Consider the effect of dividing up a strategy set more finely. Take a strategy set from any cobweb game and create a new strategy set by including all production levels that were available strategies in the first set plus all production levels that lie midway between adjacent strategies in the first set. Repeating this procedure again and again generates a sequence of strategy sets. Each set in the sequence contains more strategies than its predecessor, which, moreover, lie closer to each other than in its predecessor. Under this setup, any jth element of the original strategy set can be defined as

$$s_j = s^* + (j-k)\, 2^r \varepsilon_r,$$

where $r \in \mathcal{Z}^+$ refers to the number of times the strategies have been split and

$$\varepsilon_r = \frac{(s^* - s_1)}{(k-1)\, 2^r}.$$

Now the distance between a strategy and its adjacent strategies can be reduced to any desired level to consider the effect a discrete set has on potential stable strategies.

The purpose of this exercise is to show that the discrete strategy set is responsible for results shown in Propositions 4–6 for $B/\alpha \in (-1.5, -\frac{1}{2}]$. We show in the following that if the distance between two adjacent strategies becomes infinitesimal, then s^* is the only strict Nash equilibrium for $B/\alpha \in (-1, -\frac{1}{2}]$. First, we show that any pure strategy, other than the Nash solution (s^*) and the extreme strategies (s_1 and s_n), is never stable for a sufficiently small ε_r (a large r).

PROPOSITION 7. *For a sufficiently large* r*, any strategy* $s_j \in s$*, where* $j \notin \{1, k, n\}$*, is not locally asymptotically stable under replicator dynamics.*

Proof. A population distribution can be locally asymptotically stable only if it is a Nash equilibrium. We can show that for a sufficiently large r, any $s_j \in s$, where $j \notin \{1, k, n\}$, will not be a Nash equilibrium and thus cannot be locally asymptotically stable. The ideal best response to any s_j is

$$s^* - \frac{B}{\alpha}(j-k)\, 2^r \varepsilon_r.$$

Therefore, s_j will be a Nash equilibrium if

$$\left|(j-k)\,2^r + \frac{B}{\alpha}\,(j-k)\,2^r\right| < \frac{1}{2}.$$

This is true if and only if

$$\frac{B}{\alpha} \in H,$$

where $H = (-1 - \frac{1}{2^{r+1}|j-k|}, -1 + \frac{1}{2^{r+1}|j-k|})$. Because

$$\lim_{r\to\infty} H = -1,$$

then,[22] for a sufficiently large r, $B/\alpha \notin H$ and s_j, where $j \in \{1, k, n\}$, does not represent a Nash equilibrium. ■

We have shown in the preceding that no pure strategy other than s^* and the extreme strategies can ever be a strict symmetric Nash equilibrium with a sufficiently large r. This intuition can be extended to the cobweb game with a continuous strategy set. Therefore, for a continuous strategy set, the only potential (pure strategy) rest points will be the Nash solution, s^*, or the extreme strategies, s_1 or s_n. We next show that if the strategy set is continuous, then the extreme points can only be locally stable when demand is steeper than supply, i.e., $B/\alpha < -1$.

PROPOSITION 8. *For a sufficiently large r, a strategy $s_j \in$, where $j \in \{1, n\}$, is locally asymptotically stable under replicator dynamics only when $B/\alpha < -1$.*

Proof. Following the proof for Proposition 7, the extreme strategies are strict symmetric Nash equilibria only if

$$\frac{B}{\alpha} < -1 + \frac{1}{2^{r+1}\,(j-k)}.$$

■

Under normal conditions (i.e., $B/\alpha > 0$), we have shown that s^* is globally stable for both OLS learning and replicator dynamic learning. However, for $B/\alpha \in (-\infty, -\frac{1}{2}]$, we have also shown that there exists a stability disparity in the cobweb model between OLS adaptive learning and replicator dynamic learning. The first difference is that the REE solution is globally E-stable for $B/\alpha > -1$ and it is globally stable under replicator dynamic learning for $B/\alpha > -\frac{1}{2}$. The second difference is that the REE strategy, s^*, remains locally stable under replicator dynamic learning for any $B/\alpha \in (-\infty, \infty)$, whereas it is unstable (E-unstable) under OLS learning for $B/\alpha < -1$. It appears that the source of the stability disparities may be the strategy set. In our analysis, we have assumed a discrete strategy set, whereas it is commonly assumed in other papers that the strategy set is continuous, usually the set of positive real numbers.

We present these stability disparities, but do not attempt to reconcile them, as this is beyond the scope of this paper. We have shown in this paper that, for $B/\alpha \in (-1, -\frac{1}{2}]$, the possibility of strict symmetric Nash solutions other than s^* exists because of the discrete strategy set. From this, we can conclude that s^* is the only possible pure Nash strategy for $B/\alpha \in (-1, -\frac{1}{2}]$. However, this does not prove global stability of s^* under replicator dynamics for this parameter set, as it does not exclude other possibilities for the dynamics. Other possible situations exists, such as a locally stable nonsymmetric Nash equilibrium in which multiple quantities are produced, or some kind of cycle. In future work, we plan on addressing both of these stability disparities in a cobweb game with a continuous strategy set and considering the E-stability properties under a discrete production choice set.

6. CONCLUSION

This paper considers a question, familiar to macroeconomists, with a tool ordinarily used by game theorists. For many years, macroeconomists have inquired about the conditions under which an REE could be learned by forward-looking agents who act reasonably, but without perfect information. In typical treatments, agents are assumed to have homogeneous expectations and, thus, to act identically. Often, the answer depends on whether the E-stability condition is met.

In this paper, we ask the same question in a game-theoretic setting. Because Muth's rational expectations model of price movements can be recast as a game in which the REE is a Nash equilibrium, we have chosen to study the requirements for convergence to this equilibrium when learning is modeled using replicator dynamics. In contrast to typical macroeconomic studies, this formulation of the problem does not rely on the assumption that all agents act identically. In addition, it is of general interest to determine how popular game-based models of learning compare to those employed in macroeconomic modeling.

We show that, under regular conditions (i.e., $B/\alpha > 0$), convergence to the REE outcome under replicator dynamic learning is closely linked to the E-stability condition, which has been proven elsewhere to govern the convergence of least-squares learning to the REE in linear rational expectations models. In addition, we show that, for $B/\alpha < -\frac{1}{2}$, when the strategy set is restricted to a finite number of options, replicator dynamic learning may differ substantially in its outcome from other macroeconomic models of learning. In the case of a finite strategy set, the REE quantity will be locally attractive under replicator dynamic learning regardless of the demand and cost parameters. In addition, under a well-defined set of parameters, the outcome of replicator dynamic learning may depend on the initial production levels of firms, because many strategies may be locally attractive. Finally, we discuss the possibility that the stability disparities presented in the paper may result from the discrete choice set used in the game; however, further investigation is required.

NOTES

1. Fudenberg and Levine (1998) provide a comprehensive overview of this literature.

2. Although the main focus has been on learning under homogeneous expectations, there are a growing number of papers that study learning with heterogeneous expectations. A nonexhausting list consists of Evans and Honkapohja (1997), Evans et al. (2001), Giannitsarou (2003), and Guse (2005).

3. See, for example, Bray and Savin (1986); Fourgeaud et al. (1986); Guesnerie (1992); and Arifovic (1994).

4. See Evans and Honkapohja (2001) for a extensive discussion of adaptive learning.

5. Note that this is equivalent to producing an amount equal to $E_{i,t} P_t/\alpha$. Our setup follows expected quantity rather than expected price, as agents are going to learn quantity rather than price. As expected quantity is just a monotonic transformation of expected prices, it follows that the E-stability properties under this setup are equivalent to the setup focusing on expected prices. These conditions are given hereafter.

6. The reduced form is usually presented as $P_t = A - \frac{B}{\alpha} P_t^e$. It can be easily shown that these two forms are identical, given that it is optimal for a firm to produce at $q_{it} = \frac{E_{i,t} P_t}{\alpha} = \frac{A - B E_{t,t} q_t}{\alpha}$.

7. Because this version of Muth's model is nonstochastic, the REE output is the perfect foresight solution. In general, this will not be the case.

8. To ensure the existence of a REE, we will assume that $A \geq 0$ if $\alpha + B > 0$ and $A < 0$ if $\alpha + B < 0$.

9. See Evans and Honkapohja (2001) for a thorough discussion of this literature.

10. See Evans and Honkapohja (2001) for a discussion of the E-stability of the cobweb model.

11. If $B/\alpha = -1$, then s^* is undefined. We assume that $B/\alpha \neq -1$ throughout this paper to avoid this issue.

12. Note that we make no assumption of the location of the REE quantity other than that it is included in the stategy set and it is not an extreme strategy.

13. Note that there are several ways to present replicator dynamic learning. For more information on replicator dynamics, see Weibull (1997).

14. We restrict our analysis to strict symmetric Nash equilibria. It turns out that nonsymmetric Nash equilibria do exist when $B/\alpha < -0.5$.

15. DeCanio's model of adaptive learning (1979) and Guesnerie's eductive learning (1992) would also lead to divergence in this case.

16. This result parallels a more general property of strict Nash equilibria in finite player games. Among others, Weibull (1997) proves that strict Nash equilibria in finite player games are locally asymptotically stable under a large class of selection dynamics.

17. We are not the first to show local stability of an equilibrium in the cobweb model when the REE is E-unstable. Arifovic (1994) finds an expanded range for stability when agents use a genetic algorithm. Granato et al. (2008) show possible stability when some agents are learning using OLS and others are learning from the expectations from those using OLS.

18. We do not consider the case when α is negative, because a negative α means that firms no longer maximize profits by producing $P_{i,t}^e/\alpha$, in which case (3), (4), (5), and (6) would no longer apply. In their seminal analysis of least squares learning in the Muth model, Bray and Savin (1986) also ignore this case.

19. In several variants of the Muth model, $B < 0$ does not suggest that demand is upward-sloping. One version, presented by Evans and Guesnerie (1993), is discussed in Appendix E.

20. For example, we discuss a model presented in Appendix E, originally presented by Evans and Guesnerie (1993). Along with the standard quadratic cost function, this model includes a postitive externality causing a decreasing marginal cost with aggregate production. The appendix of Honkapohja and Mitra (2003) also presents another variant of the Muth model with $B/\alpha < 0$. This is a model with two interrelated markets in which the supply of one of the goods is affected by a production lag and the supply of the other good is not.

21. One other possibility is that nonsymmetric Nash equilibria in which multiple quantities are produced could be limit states. When $B/\alpha < -0.5$, such equilibria can exist.

22. Recall that we have assumed for the paper that $B/\alpha \neq -1$, so that the REE is well defined.

23. The following is a linearization on the boundary of the simplex. We only need to consider initial points in the interior of the simplex as the trajectories of the ordinary differential equation are naturally constrained to remain in the simplex.

24. For a complete discussion of stability in homogeneous linear systems see Boyce and DiPrima (1992).

REFERENCES

Arifovic, Jasmina (1994) Genetic algorithm learning and the cobweb model. *Journal of Economic Dynamics and Control* 18, 3–28.

Boyce, W.E. and R.C. DiPrima (1992) *Elementary Differential Equations*. New York: Wiley.

Branch, William and Bruce McGough (2008) Replicator dynamics in a cobweb model with rationally heterogeneous expectations. *Journal of Economic Behavior and Organization* 65, 224–244.

Bray, Margaret and Nathan E. Savin (1986) Rational expectations equilibria, learning, and model specification. *Econometrica* 54, 1129–1160.

Brock, William and Cars Hommes (1998) Heterogeneous beliefs and routes to chaos in a sample asset pricing model. *Journal of Economic Dynamics and Control* 22, 1235–1274.

DeCanio, Stephen J. (1979) Rational expectations and learning from experience. *Quarterly Journal of Economics* 93, 47 –57.

Evans, George (1989) The fragility of sunspots and bubbles. *Journal of Monetary Economics* 23, 297–317.

Evans, George and Roger Guesnerie (1993) Rationalizability, strong rationality, and expectational stability. *Games and Economic Behavior* 5, 632–646.

Evans, George and Seppo Honkapohja (1994) Learning, convergence and stability with multiple rational expectations equilibria. *European Economic Review* 38, 1071–1098.

Evans, George and Seppo Honkapohja (1997) Least squares learning with heterogeneous expectations. *Economic Letters* 52, 197–201.

Evans, George and Seppo Honkapohja (2001) *Learning and Expectations in Macroeconomics*. Princeton, NJ: Princeton University Press.

Evans, George, Seppo Honkapohja, and Ramon Marimon (2001) Convergence in monetary inflation models with heterogeneous learning rules. *Macroeconomic Dynamics* 5, 1–31.

Fourgeaud, Claude, Christian Gourieroux, and Jacqueline Pradel (1986) Learning procedures and convergence to rationality. *Econometrica* 54, 845–868.

Fudenberg, Drew and David Levine (1998) *The Theory of Learning in Games*. Cambridge, MA: MIT Press.

Giannitsarou, Chryssi (2003) Heterogeneous learning. *Review of Economic Dynamics* 6, 885–906.

Granato, James, Eran Guse, and M.C. Sunny Wong (2008) Learning from the expectations of others. *Macroeconomic Dynamics* 12, 345–377.

Guesnerie, Roger (1992) An exploration of the eductive justifications for the rational expectations hypothesis. *American Economic Review* 82, 1254–1278.

Guse, Eran (2005) Stability properties for learning with heterogeneous expectations and multiple equilibria. *Journal of Economic Dynamics and Control* 29, 1623–1642.

Guse, Eran (2010) Heterogeneous expectations, adaptive learning, and evolutionary dynamics. *Journal of Economic Behavior and Organization* 74, 1623–1642.

Honkapohja, Seppo and Kaushik Mitra (2003) Learning with bounded memory in stochastic models. *Journal of Economic Dynamics and Control* 27, 1437–1457.

Marcet, Albert and Thomas J. Sargent (1989) Convergence of least-squares learning mechanisms in self-referential linear stochastic models. *Journal of Economic Theory* 48, 337–368.

Muth, John F. (1961) Rational expectations and the theory of price movements. *Econometrica* 29, 315–335.
Sethi, Rajiv and Reiner Franke (1995) Behavioral heterogeneity under evolutionary pressure: Macroeconomic implications of costly optimisation. *Economic Journal* 105, 583–600.
Taylor, Peter and Leo Jonker (1978) Evolutionarily stable strategies and game dynamics. *Mathematical Biosciences* 40, 145–156.
Weibull, Jorgen W. (1997) *Evolutionary Game Theory*. Cambridge, MA: MIT Press.

APPENDIX A: PROOF OF PROPOSITION 2

Because $\sum_{h=1}^{n} x_h = 1$ by definition, the system of n differential equations given in (9) can be regarded as an identity plus a system of $n-1$ differential equations that can be linearized around any point in the simplex of $\mathbf{R}^{n-1}$:

$$\dot{x}_h = x_h \left[\pi\left(s_h, x\right) - \left(\sum_{j=1}^{n} x_j \pi\left(s_j, x\right) \right) \right], h \neq m \qquad \textbf{(A.1)}$$

$$\dot{x}_m = -\sum_{h \neq m} \dot{x}_h. \qquad \textbf{(A.2)}$$

For notational purposes, let e denote the index of some strict symmetric Nash equilibrium strategy and $\hat{x}$ denote the state vector when $x_e = 1$ and $x_h = 0\ \forall h \neq e$. Also, let

$$\varphi_h\left(x\right) = x_h \left[\pi\left(s_h, x\right) - \left(\sum_{j=1}^{n} x_j \pi\left(s_j, x\right) \right) \right].$$

Totally differentiating the equations of (A.1) and evaluating the results at $\hat{x}$, we find that for all $\varphi_h\left(x\right)$ where $h \neq e$,

$$\frac{\partial \varphi_h\left(\hat{x}\right)}{\partial x_h} = \left[\pi\left(s_h, \hat{x}\right) - \pi\left(s_e, \hat{x}\right)\right]$$

and

$$\frac{\partial \varphi_h\left(\hat{x}\right)}{\partial x_j} = 0$$

$\forall j \neq h$; whereas

$$\frac{\partial \varphi_e\left(\hat{x}\right)}{\partial x_e} = -\pi\left(s_e, \hat{x}\right)$$

and

$$\frac{\partial \varphi_e\left(\hat{x}\right)}{\partial x_j} = -\pi\left(s_j, \hat{x}\right).$$

Along with the identity in (A.2), these partial derivatives imply that the linearized dynamics[23] satisfies

$$\dot{x}_h = x_h \left[\pi\left(s_h, \hat{x}\right) - \pi\left(s_e, \hat{x}\right)\right] \qquad \textbf{(A.3)}$$

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$\forall h \neq e$ and

$$\dot{x}_e = -\pi\left(s_e, \hat{x}\right)\left(x_e - 1\right) - \sum_{j \neq e} \pi\left(s_j, \hat{x}\right) x_j, \quad \textbf{(A.4)}$$

or more succinctly,

$$\dot{y} = \Omega y,$$

where $y = x - \hat{x}$. Using standard results, $\hat{x}$ is asymptotically stable under these dynamics in case all eigenvalues of Ω are negative.[24] From (A.3) and (A.4), it follows that the eigenvalues of Ω are the solutions of $(-\pi\left(s_e, \hat{x}\right) - \lambda) \prod [(\pi\left(s_h, \hat{x}\right) - \pi\left(s_e, \hat{x}\right)) - \lambda] = 0$, which are all negative because $\pi\left(s_e, \hat{x}\right) > \pi\left(s_h, \hat{x}\right)\ \forall h \neq e$. ■

APPENDIX B: PROOF OF PROPOSITION 3

By Lyapunov's second theorem, a point, x^*, will be asymptotically stable in some neighborhood, D, under dynamics $\dot{x} = \varphi\left(x\right)$ whenever there exists a function $v : D \to \mathbf{R}_+$ that satisfies

$$v\left(x^*\right) = 0 \quad \textbf{(B.1)}$$

$$v\left(x\right) > 0, \forall x \in D; x \neq x^* \quad \textbf{(B.2)}$$

$$\frac{dv\left[x\left(t\right)\right]}{dt} < 0\ \forall x \in D; x \neq x^*. \quad \textbf{(B.3)}$$

Let $D = \{x \in \mathcal{R}^n_+ : \sum_{h=1}^n x_h = 1, x_k \neq 0\}$, and let $v\left(x\right) = \log\left(1/x_k\right)$. Note that (B.1) and (B.2) are satisfied. Differentiating $v\left(x\right)$ with respect to time gives

$$\frac{dv\left[x\left(t\right)\right]}{dt} = -\left[\pi\left(s^*, x\right) - \sum_{h=1}^{n} x_h \pi\left(s_h, x\right)\right]. \quad \textbf{(B.4)}$$

Let δ denote the normalized strategy vector, $\delta = (s_1 - s^*, s_2 - s^*, \ldots, s_n - s^*)$, so that (B.4) can be rewritten as

$$\frac{dv\left[x\left(t\right)\right]}{dt} = -\left[B\left(x \bullet \delta\right)^2 + \frac{\alpha}{2}\left(x \bullet \delta^2\right)\right], \quad \textbf{(B.5)}$$

where $\delta^2 = (\delta_1^2, \delta_2^2, \ldots, \delta_n^2)$. Because $\left(x \bullet \delta\right)^2$ and $x \bullet \delta^2$ are strictly positive, this will be negative when B and α are positive. Because $\left(x \bullet \delta\right)^2 < x \bullet \delta^2$ for all $x \in D$, this will also be negative when $B < 0 < \alpha$ and $B/\alpha > -0.5$. ■

APPENDIX C: PROOF OF PROPOSITION 4

By Proposition 2, a population distribution will be locally asymptotically stable whenever it is a strict symmetric Nash equilibrium. Consequently, we can ensure that it is locally asymptotically stable by picking B/α to make s_j the unique best response to a population

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distribution of x_j. Because $s^* - \frac{B}{\alpha}(j-k)\varepsilon$ is the ideal response to x_j, this will be the case whenever

$$\left|(j-k) + \left(\frac{B}{\alpha}\right)(j-k)\right| < \frac{1}{2}. \qquad \textbf{(C.1)}$$

But this can be true if and only if $B/\alpha \in (-1 - \frac{1}{2|j-k|}, -1 + \frac{1}{2|j-k|})$. ■

APPENDIX D: PROOFS OF PROPOSITIONS 5 AND 6

We begin by establishing that limit states of solution trajectories to (9) must be Nash equilibrium population distributions, because this is required to establish the other results. Let $\phi(t, x_0)$ denote the solution path of the population distribution as a function of time when the initial distribution is x_0, and let $\phi_i(t, x_0)$ denote the ith element of this vector. Recall that for a game with n strategies, $\phi(t, x_0)$ will be a vector in the unit simplex of n-space, which we will denote by Δ. Define $C(x)$ to satisfy $C(x) \equiv \{h | x_h > 0\}$. Thus, $C(x)$ is the set of strategies that receive strictly positive probability in the distribution described by x. Finally, let Δ^{NE} denote the subset of points in Δ that are Nash equilibria in the game under consideration.

LEMMA 1. *If* $\lim_{t\to\infty} \phi(t, x_0) = x$, *then* $x \in \Delta^{\text{NE}}$.

Proof. Suppose $\lim_{t\to\infty} \phi(t, x_0) = x$ and $x \notin \Delta^{\text{NE}}$. By definition of Nash equilibrium, we know that one of the following must hold:

(i) $\exists (i, j) \in C(x)$ such that $\pi(s_i, x) \neq \pi(s_j, x)$.
(ii) $\exists s_h \in S, h \notin C(x)$ such that $\pi(s_h, x) > \pi(s_i, x)\ \forall i \in C(x)$.

In case (i), x will not be a stationary point of (10). Plainly, in this case, it cannot be the limit of a solution trajectory. Suppose instead that (ii) is true. From continuity of $\pi()$, it follows that for some time T

(iii) $\pi[s_h, \phi(t, x_0)] > \pi[s_i, \phi(t, x_0)]\ \forall i \in C(x), \forall t > T$.

But, given (iii), (10) implies that $\frac{\dot{x}_h}{x_h} > \frac{\dot{x}_i}{x_i}\ \forall i \in C(x)$ at all points in time beyond T. Because this cannot be true if the population distribution eventually converges to x, the proof is complete. ■

Proof of Proposition 5. Suppose $j \notin \{1, k, n\}$ and x_j is locally asymptotically stable. By Lemma 1, we know that x_j must be a symmetric Nash equilibrium. Following the logic presented in the proof of Proposition 4, this will be true if and only if $\left|(j-k) + \frac{B}{\alpha}(j-k)\right| \leq \frac{1}{2}$. It follows that x_j will be a (possibly nonstrict) symmetric Nash equilibrium if and only if

$$\frac{B}{\alpha} \in \left[-1 - \frac{1}{2|j-k|}, -1 + \frac{1}{2|j-k|}\right].$$

■

Proof of Proposition 6. Suppose $j \in \{1, n\}$ and x_j is locally asymptotically stable. Again, Lemma 1 ensures that x_j is a symmetric Nash equilibrium. For $j \in \{1, n\}$, s_j will be the unique best response to x_j whenever either $|(j-k) + B/\alpha\,(j-k)| \leq \frac{1}{2}$ or $B/\alpha < -1$.

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But this will be true if and only if

$$\frac{B}{\alpha} \in \left[-\infty, -1 + \frac{1}{2|j-k|}\right].$$

■

APPENDIX E: A PLAUSIBLE SETTING WITH MULTIPLE STABLE STRATEGIES

As illustrated by Proposition 4, when the parametric conditions for global stability are not met, cobweb games can have multiple strategies that are locally attractive under replicator dynamic learning. Of course, how interesting this result is depends in large part upon whether one can identify economically plausible scenarios that correspond to this class of games. Based on the standard form of the cobweb model, one might be tempted to argue that Proposition 4 is not a very interesting result, because the parametric requirements are only satisfied in the presence of upward-sloping aggregate demand. The purpose here is to provide an economically palatable example that satisfies the requirement for multiple, locally attractive Nash equilibria.

Consider the generalization of Muth's basic model discussed by Evans and Guesnerie (1993). In their study, Evans and Guesnerie augment the structure of Muth's model to include a production externality. Specifically, they continue to model marginal cost as a linearly increasing function of individual firm production, but include in the cost function a term that makes marginal cost linearly decreasing in aggregate production. In this setting, the cost function faced by each individual firm can be represented as

$$c_{i,t} = \beta_0 q_{i,t} + \beta_1 q_{i,t}^2 - \beta_2 q_{i,t} \int_0^\infty q f(q)\, dq. \qquad \textbf{(E.1)}$$

Assume, as in Section 3, that the set of firms is infinite and the set of production levels from which firms can select is finite, and continue to assume that aggregate demand is linear. Using the notation of Section 3, each firm selects its production level to maximize

$$\tilde{\pi}(s_h, x) = [A - B(s \bullet x)] - \left[\beta_0 s_h + \beta_1 s_h^2 - \beta_2 (s \bullet x) s_h\right] \qquad \textbf{(E.2)}$$

when cost functions are given by (E.1). Notice that (E.2) can be rewritten in the form of firm payoff functions in a cobweb game,

$$\tilde{\pi}(s_h, x) = \left[\tilde{A} - \tilde{B}(s \bullet x)\right] s_h - \frac{\tilde{\alpha}}{2} s_h^2,$$

where $\tilde{A} = A - \beta_0$, $\tilde{B} = B - \beta_2$, and $\tilde{\alpha} = 2\beta_1$. Then, because $\tilde{B}$ can be negative despite B being positive, the requirements for multiple, locally attractive equilibria can be met with downward-sloping demand. In case the production externality is strong enough, firms may learn to produce levels other than the REE quantity.