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Stability analysis for positive solutions for classes of semilinear elliptic boundary-value problems with nonlinear boundary conditions

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We investigate the stability properties of positive steady-state solutions of semilinear initial–boundary-value problems with nonlinear boundary conditions. In particular, we employ a principle of linearized stability for this class of problems to prove sufficient conditions for the stability and instability of such solutions. These results shed some light on the combined effects of the reaction term and the boundary nonlinearity on stability properties. We also discuss various examples satisfying our hypotheses for stability results in dimension 1. In particular, we provide complete bifurcation curves for positive solutions for these examples.

Keywords: nonlinear boundary conditions; semipositone; stability; principle of linearized stability

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1. Introduction

In this paper we analyse the stability properties for positive solutions to the following semilinear elliptic boundary-value problem with nonlinear boundary conditions

$$-\Delta u = \lambda f(u) \quad \text{in } \Omega, \tag{1.1}$$

$$\frac{\partial u}{\partial \eta} = -g(u) \quad \text{on } \partial\Omega, \tag{1.2}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with sufficiently smooth boundary and $n \geqslant 1$, Δ is the Laplace operator, λ is a positive parameter, $\partial u/\partial\eta$ is the outward normal derivative, $f\colon [0,\infty) \to \mathbb{R}$ is a $C^{2+\beta}$ function with $f \not\equiv 0$, $g\colon [0,\infty) \to (0,\infty)$ is a $C^{2+\beta}$ function and $\beta \in (0,1]$. In particular, we are interested in stability properties of classical solutions, i.e. $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$, of (1.1), (1.2). Note that our assumption on g excludes any constant solutions, i.e. $u(x) \equiv C$, $C \in [0,\infty)$, of (1.1), (1.2). It is well known that study of the structure of solutions of (1.1), (1.2) is the consideration of the structure of steady-state solutions for the following initial–boundary-value problem:

$$u_t = \frac{1}{\lambda}\Delta u + f(u), \quad x \in \Omega,\ t > 0, \tag{1.3}$$

$$\frac{\partial u}{\partial \eta} = -g(u), \qquad x \in \partial\Omega,\ t > 0, \tag{1.4}$$

$$u(0,x) = u_0(x), \qquad x \in \Omega. \tag{1.5}$$

In fact, the dynamics of (1.3)–(1.5) are almost completely determined by the structure of its steady-state solutions (see, for example, [8]). Reaction–diffusion equations such as (1.3)–(1.5) have been extensively employed to model a variety of phenomena in several fields, especially combustion theory and population dynamics (see [8, 11, 23, 24, 26]). In a population dynamics context, $u(t,x)$ represents the population density at time t and location x in the patch Ω. The boundary condition (1.4) models the tendency of the population to leave the patch at the boundary in such a way that it depends nonlinearly on the population density itself. In particular, we have recently studied such models with the assumption of negative-density-dependent dispersal of the population on the boundary (see [12–14, 16–18]). In the context of combustion theory, we studied the structure of positive steady-state solutions for (1.1), (1.2) in [15] and for a slightly different boundary condition in [19].

The following definitions of stability and instability presented here come from Lyapunov stability, which is defined with respect to initial perturbations (see, for example, [25]). A solution $u(x)$ of (1.1), (1.2) is said to be *stable* if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\|v(t,\cdot) - u\|_\infty < \varepsilon$ for $t > 0$ whenever $\|u_0 - u\|_\infty < \delta$, where $v(t,x)$ is the solution of (1.3)–(1.5). If, in addition, $\|v(t,\cdot) - u\|_\infty \to 0$ as $t \to \infty$, then u is said to be *asymptotically stable*. The steady state u is said to be *unstable* if it is not stable. Finally, $u(x)$ is said to be an isolated steady state if there exists a neighbourhood N_u of u in $C(\bar{\Omega})$ such that u is the only steady-state solution in N_u.

In [7], the instability of positive solutions for (1.1) is studied in conjunction with the linear boundary condition

$$\alpha h u + (1-\alpha)\frac{\partial u}{\partial \eta} = 0 \quad \text{on } \partial\Omega, \tag{1.6}$$

where $\alpha \in [0,1]$ is a constant and $h\colon \partial\Omega \to (0,\infty)$ is a smooth function with $h \equiv 1$ when $\alpha = 1$, i.e. the boundary condition may be of Dirichlet, Neumann or mixed type. They proved that if $f(u)$ is a smooth function satisfying

$$f(0) < 0, \tag{1.7}$$

$$f'(0) > 0, \quad u > 0, \tag{1.8}$$

$$f''(u) \geqslant 0, \quad u > 0, \tag{1.9}$$

then every positive solution of (1.1) and (1.6) is unstable. Note that, under the above assumption, (1.7), (1.1) and (1.6) give rise to what is known as a semipositone problem. Boundary-value problems with semipositone structure pose challenging mathematical problems, as indicated in the celebrated paper by Lions [22]. This was recently confirmed by the remarkable work of Berestycki *et al*. [2–6] on the qualitative behaviour of positive solutions to semilinear equations on unbounded domains. A review of earlier results for semipositone problems can be found in [9]. This stability result was later extended to the case when (1.8) was relaxed, first by Tertikas [27] using sub- and supersolutions and then by Chhetri and Shivaji [10] by reducing the problem to the monotone case through a decomposition of f to a monotone and linear function. Tertikas [27] also showed that if $f(u)$ satisfies either

$$f(0) = 0, \qquad f''(u) < 0, \quad u > 0, \tag{1.10}$$

or

$$f(0) > 0, \qquad f''(u) \leqslant 0, \quad u > 0, \tag{1.11}$$

then every positive solution of (1.1) and (1.6) is stable and unique. Karatson and Simon [20] later gave a more direct proof of these results. Each of the previous works made use of the well-known principle of linearized stability (see, for example, [25]) to prove their results. Particularly, stability of a positive solution u of (1.1) and (1.6) is determined by ascertaining the sign of the principal eigenvalue, σ_1, of the linearized equation associated with (1.1) and (1.6), namely

$$-\Delta\phi - \lambda f_u(u)\phi = \sigma\phi \quad \text{in } \Omega, \tag{1.12}$$

$$\alpha h\phi + (1-\alpha)\frac{\partial\phi}{\partial\eta} = 0 \qquad \text{on } \partial\Omega, \tag{1.13}$$

with corresponding eigenfunction $\phi(x) > 0$ in Ω. If $\sigma_1 > 0$, then the positive solution u of (1.1) and (1.6) is stable, whereas if $\sigma_1 < 0$, then u is unstable.

The major aim of this work is to prove stability and instability results for (1.1), (1.2) under analogous hypothesises on $f(u)$ and $g(u)$. To this end, we first present a principal of linearized stability for (1.1), (1.2).

Theorem 1.1. *Let σ_1 be the principal eigenvalue of the linearized equation associated with (1.1), (1.2), namely*

$$-\Delta\phi - \lambda f_u(u)\phi = \sigma\phi \quad \textit{in } \Omega, \tag{1.14}$$

$$\frac{\partial\phi}{\partial\eta} + g_u(u)\phi = \sigma\phi \quad \textit{on } \partial\Omega, \tag{1.15}$$

with corresponding eigenfunction ϕ chosen such that $\phi(x) > 0$ in $\bar{\Omega}$, where u is any solution of (1.1), (1.2). Then the following hold.

(a) *If $\sigma_1 > 0$, then u is stable. Moreover, if u is isolated from other steady-state solutions, then u is asymptotically stable.*

(b) *If $\sigma_1 < 0$, then u is unstable.*

REMARK 1.2.

(a) The eigenvalue problem (1.14), (1.15) has a smallest (principal) eigenvalue due to [1, theorem 2.2]. The corresponding eigenfunction can also be chosen to be positive in $\bar{\Omega}$.

(b) Umezu first proved this principle of linearized stability in the context of a specific population model where $f(x,u)$ is a spatially heterogenous logistic growth function and (1.2) is replaced with a different nonlinear boundary condition modelling an influx of the population into the patch Ω (see [29]).

For completeness, we provide a proof of theorem 1.1 in § 2. We now present the main stability/instability results. First, we list several essential hypotheses dealing with the relationship between the reaction term $f(u)$ and the boundary nonlinearity $g(u)$. In the following theorems we will always assume one of the following hypotheses:

(FG1) $\dfrac{\mathrm{d}}{\mathrm{d}u}\left[\dfrac{f(u)}{g(u)}\right] < 0$ for $u > 0$,

(FG2) $\dfrac{\mathrm{d}}{\mathrm{d}u}\left[\dfrac{f(u)}{g(u)}\right] > 0$ for $u > 0$,

(FG3) $\dfrac{\mathrm{d}}{\mathrm{d}u}\left[\dfrac{f(u)}{g(u)}\right] = 0$ for $u > 0$.

Note that

$$\frac{\mathrm{d}}{\mathrm{d}u}\left[\frac{f(u)}{g(u)}\right] = \frac{f_u(u)g(u) - f(u)g_u(u)}{[g(u)]^2}. \tag{1.16}$$

Thus, (FG1) implies that $f_u(u)g(u) - f(u)g_u(u) < 0$ for $u > 0$ and (FG2) implies $f_u(u)g(u) - f(u)g_u(u) > 0$ for $u > 0$. Also, (FG3) is equivalent to the fact that there exists a $C \in \mathbb{R}$ such that $f(u) = Cg(u)$ for $u > 0$.

We now present results on the stability properties of positive solutions for (1.1), (1.2), making explicit assumptions only on the concavity/convexity of the boundary nonlinearity $g(u)$. Given the hypotheses

(G1) $g_{uu}(u) < 0$ for $u > 0$,

(G2) $g_{uu}(u) > 0$ for $u > 0$,

(G3) $g_{uu}(u) = 0$ for $u > 0$,

we prove the following theorems. Note that (G3) is equivalent to the fact that $g(u)$ is linear.

THEOREM 1.3. *The following hold.*

(a) *If (G1) and (FG1) both hold, then every positive solution of (1.1), (1.2) is stable. Moreover, if u is an isolated positive solution of (1.1), (1.2), then u is asymptotically stable.*

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(b) *If (G2) and (FG2) both hold, then every positive solution of (1.1), (1.2) is unstable.*

THEOREM 1.4. *Suppose that (G3) holds. Then we have the following.*

(a) *If (FG1) also holds, then every positive solution of (1.1), (1.2) is stable. Moreover, if u is an isolated positive solution of (1.1), (1.2), then u is asymptotically stable.*

(b) *If (FG2) also holds, then every positive solution of (1.1), (1.2) is unstable.*

REMARK 1.5. These results require neither an explicit sign condition on $f(0)$ nor a monotonicity or even concavity/convexity condition on the reaction term $f(u)$, other than what is implied by (FG1) or (FG2). This is in stark contrast with previous results such as [20], where the sign condition on $f(0)$ was shown to be necessary for a similar result to hold in the case of the linear boundary condition (1.6).

Theorems 1.3 and 1.4 are proved in § 3. We now list results on the stability properties of positive solutions of (1.1), (1.2) by making assumptions on the reaction term $f(u)$.

(F1) $f(u) > 0$ for $u > 0$; $f_{uu}(u) < 0$ for $u > 0$.

(F2) $f(u) > 0$ for $u > 0$; $f_{uu}(u) > 0$ for $u > 0$,

(F2A) $f(u) < 0$ for $u \in [0, \beta)$ for some $\beta > 0$; $f(u) > 0$ for $u > \beta$; $f_{uu}(u) > 0$ for $u > 0$, or

(F3) $f(u) > 0$ for $u > 0$; $f_{uu}(u) = 0$ for $u > 0$.

Also, note that, under (F3), $f(x)$ is necessarily linear.

THEOREM 1.6. *The following hold.*

(a) *If (F1) and (FG1) both hold, then every positive solution of (1.1), (1.2) is stable. Moreover, if u is an isolated positive solution of (1.1), (1.2), then u is asymptotically stable.*

(b) *If (F2) and (FG2) both hold, then every positive solution of (1.1), (1.2) is unstable.*

THEOREM 1.7. *Suppose that (F2A) and (FG2) hold. Then every positive solution of (1.1), (1.2) is unstable.*

THEOREM 1.8. *Suppose that (F3) holds. Then we have the following.*

(a) *If (FG1) also holds, then every positive solution of (1.1), (1.2) is stable. Moreover, if u is an isolated positive solution of (1.1), (1.2), then u is asymptotically stable.*

(b) *If (FG2) also holds, then every positive solution of (1.1), (1.2) is unstable.*

Theorems 1.6–1.8 are proved in § 4, along with the following result in the case when (FG3) holds.

THEOREM 1.9. *Suppose that (FG3) holds. Then we have the following.*

(a) *If either (F1) or (G1) also holds, then every positive solution of (1.1), (1.2) is stable. Moreover, if u is an isolated positive solution of (1.1), (1.2), then u is asymptotically stable.*

(b) *If either (F2A), (F2), or (G2) also holds, then every positive solution of (1.1), (1.2) is unstable.*

In § 5, we discuss various examples satisfying our various hypotheses for stability results in dimension 1, i.e. $n = 1$ and $\Omega = (0, 1)$. In particular, for these examples we provide complete bifurcation curves for positive solutions via a time map analysis (see [21], in which such an analysis was introduced for boundary-value problems with Dirichlet boundary conditions).

2. Proof of theorem 1.1

To prove theorem 1.1, assume that $\tilde{u}$ is a positive solution of (1.1), (1.2) and that σ_1 is the principal eigenvalue of (1.14), (1.15) with corresponding eigenfunction ϕ chosen to be positive in $\bar{\Omega}$ and scaled such that $\|\phi\|_\infty = 1$. Note that by Taylor's theorem we have

$$f(\tilde{u} + v) = f(\tilde{u}) + f_u(\tilde{u})v + \tfrac{1}{2} f_{uu}(C_v)v^2, \tag{2.1}$$

$$g(\tilde{u} + v) = g(\tilde{u}) + g_u(\tilde{u})v + \tfrac{1}{2} g_{uu}(C_v^*)v^2 \tag{2.2}$$

for some C_v, C_v^* between $\tilde{u}$ and $\tilde{u} + v$. Let $\varepsilon > 0$ and define $\psi_1 := \tilde{u} - \varepsilon\phi$ and $\psi_2 := \tilde{u} + \varepsilon\phi$. We now show that ψ_i, $i = 1, 2$, are sub- or supersolutions of (1.1), (1.2) depending on the sign of σ_1.

For $x \in \Omega$,

$$\begin{aligned} -\Delta\psi_1 - \lambda f(\psi_1) &= -\Delta\tilde{u} + \varepsilon\Delta\phi - \lambda f(\tilde{u} - \varepsilon\phi) \\ &= -\Delta\tilde{u} - \lambda f(\tilde{u}) - \varepsilon[-\Delta\phi - \lambda f_u(\tilde{u})\phi] \\ &\qquad - \lambda[f(\tilde{u} - \varepsilon\phi) - f(\tilde{u}) - f_u(\tilde{u})(-\varepsilon\phi)] \\ &= \varepsilon\phi[-\sigma_1 - \tfrac{1}{2}\lambda\varepsilon\phi f_{uu}(C_{\varepsilon\phi})], \end{aligned} \tag{2.3}$$

since $\tilde{u}$ is a solution of (1.1), (1.2), ϕ is a solution of (1.14), (1.15), and by (2.1). Thus, there exists an $\varepsilon_1 > 0$ such that for $\varepsilon < \varepsilon_1$ we have that $\operatorname{sgn}(-\Delta\psi_1 - \lambda f(\psi_1)) = \operatorname{sgn}(-\sigma_1)$.

For $x \in \partial\Omega$,

$$\begin{aligned} \frac{\partial\psi_1}{\partial\eta} + g(\psi_1) &= \frac{\partial\tilde{u}}{\partial\eta} - \varepsilon\frac{\partial\phi}{\partial\eta} + g(\tilde{u} - \varepsilon\phi) \\ &= \frac{\partial\tilde{u}}{\partial\eta} + g(\tilde{u}) - \varepsilon\left[\frac{\partial\phi}{\partial\eta} + g_u(\tilde{u})\phi\right] \\ &\qquad + g(\tilde{u} - \varepsilon\phi) - g(\tilde{u}) - g_u(\tilde{u})(-\varepsilon\phi) \\ &= \varepsilon\phi[-\sigma_1 + \tfrac{1}{2}\varepsilon\phi g_{uu}(C^*_{\varepsilon\phi})] \end{aligned} \tag{2.4}$$

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again since $\tilde{u}$ is a solution of (1.1), (1.2), ϕ is a solution of (1.14), (1.15), and by (2.2). Hence, there exists an $\varepsilon_2 > 0$ such that for $\varepsilon < \varepsilon_2$ we have that

$$\operatorname{sgn}\left(\frac{\partial \psi_1}{\partial \eta} + g(\psi_1)\right) = \operatorname{sgn}(-\sigma_1).$$

A similar argument gives that there exists an $\varepsilon_3 > 0$ such that for $\varepsilon < \varepsilon_3$ we have that

$$\operatorname{sgn}(-\Delta\psi_2 - \lambda f(\psi_2)) = \operatorname{sgn}(\sigma_1) \quad \text{and} \quad \operatorname{sgn}\left(\frac{\partial \psi_2}{\partial \eta} + g(\psi_2)\right) = \operatorname{sgn}(\sigma_1).$$

Now, let $\rho \in (0, \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\})$ and denote by U_1, U_2 the solutions of (1.3)–(1.5) with $U_1(0,x) = \psi_1 = \tilde{u} - \rho\phi$ and $U_2(0,x) = \psi_2 = \tilde{u} + \rho\phi$. In the case when $\sigma_1 > 0$, we have that ψ_1 and ψ_2 are strict sub- and supersolutions of (1.1), (1.2), respectively. From [25, § 5.6, lemma 6.1], we have that U_1 is increasing in t, U_2 is decreasing in t and

$$\tilde{u} - \rho\phi \leqslant U_1(t,x) \leqslant \tilde{u} \leqslant U_2(t,x) \leqslant \tilde{u} + \rho\phi, \quad t > 0,\ x \in \Omega.$$

Therefore, for $\varepsilon < \rho$, if u is the solution of (1.14), (1.15) with $u(0,x) = u_0(x)$ in Ω and u_0 is such that $\|u_0 - \tilde{u}\|_\infty < \varepsilon$, then $\tilde{u} - \varepsilon\phi \leqslant u_0 \leqslant \tilde{u} + \varepsilon\phi$, and thus $\|u(t,\cdot) - \tilde{u}\|_\infty < \varepsilon$ for $t > 0$. Hence, $\tilde{u}$ is stable. Moreover, if $\tilde{u}$ is isolated from other steady-state solutions, u is the solution of (1.14), (1.15) with $u(0,x) = u_0(x)$ in Ω and u_0 is such that $\|u_0 - \tilde{u}\|_\infty < \varepsilon$, then $\|u(t,\cdot) - \tilde{u}\|_\infty \to 0$ as $t \to \infty$.

In the case when $\sigma_1 < 0$, we have that ψ_1 and ψ_2 are *strict* super- and subsolutions of (1.1), (1.2), respectively. Again from [25], we have that U_1 is decreasing in t and U_2 is increasing in t. For any $\varepsilon > 0$ arbitrarily small we have that for all $\delta > 0$ with $\|U_i(0,\cdot) - \tilde{u}\|_\infty < \delta$, $i = 1, 2$, there exists a $t^* > 0$ such that $\|U_i(t,\cdot) - \tilde{u}\|_\infty > \varepsilon$ for $t > t^*$ for $i = 1, 2$. Hence, $\tilde{u}$ is unstable.

3. Proof of theorems 1.3 and 1.4

Before providing a proof of theorems 1.3 and 1.4, we first present a necessary lemma that provides a connection between the boundary nonlinearity g, conditions (FG1)–(FG3) and σ_1.

LEMMA 3.1. *If u is a positive solution of (1.1), (1.2), then*

$$-\sigma_1\left[\int_{\partial\Omega} g(u)\phi \,\mathrm{d}s + \int_\Omega g(u)\phi \,\mathrm{d}x\right] = \int_\Omega g_{uu}(u)|\nabla u|^2\phi \,\mathrm{d}x + \int_\Omega \lambda[f_u(u)g(u) - f(u)g_u(u)]\phi \,\mathrm{d}x, \tag{3.1}$$

where σ_1 is the principal eigenvalue of (1.14), (1.15) with corresponding eigenfunction ϕ.

Proof of lemma 3.1. Suppose that u is a positive solution of (1.1), (1.2) and σ_1 is the principal eigenvalue of (1.14), (1.15) with corresponding eigenfunction ϕ chosen to be positive on $\bar{\Omega}$. Note that

$$\Delta g(u) = \operatorname{div}(g_u(u)\nabla u) \tag{3.2}$$

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and

$$\Delta g(u) = g_{uu}(u)|\nabla u|^2 + g_u(u)\Delta u. \tag{3.3}$$

Now, by (3.2) and Green's second identity we have

$$\begin{aligned}\int_{\Omega}[(-\Delta\phi)g(u) + (\Delta g(u))\phi]\,\mathrm{d}x &= \int_{\partial\Omega} \frac{-\partial\phi}{\partial\eta} g(u) + g_u(u)\frac{\partial u}{\partial\eta}\phi\,\mathrm{d}s \\ &= \int_{\partial\Omega}[g_u(u) - \sigma_1]g(u)\phi - g(u)g_u(u)\phi\,\mathrm{d}s \\ &= -\sigma_1\int_{\partial\Omega} g(u)\phi\,\mathrm{d}s, \end{aligned} \tag{3.4}$$

since ϕ is a positive eigenfunction to σ_1 of (1.14), (1.15). But, using (3.3) and the fact that ϕ is a positive eigenfunction to σ_1 of (1.14), (1.15) and u is a positive solution of (1.1), (1.2), we have that

$$\begin{aligned}&\int_{\Omega}[(-\Delta\phi)g(u) + (\Delta g(u))\phi]\,\mathrm{d}x \\ &\quad = \int_{\Omega}[\lambda f_u(u) + \sigma_1]g(u)\phi\,\mathrm{d}x + \int_{\Omega}[g_{uu}(u)|\nabla u|^2 + g_u(u)\Delta u]\phi\,\mathrm{d}x \\ &\quad = \int_{\Omega}\lambda[f_u(u)g(u) - f(u)g_u(u)]\phi\,\mathrm{d}x + \sigma_1\int_{\Omega} g(u)\phi\,\mathrm{d}x \\ &\qquad + \int_{\Omega} g_{uu}(u)|\nabla u|^2\phi\,\mathrm{d}x. \end{aligned} \tag{3.5}$$

Finally, combining (3.4) and (3.5) gives the desired result, namely

$$\begin{aligned}-\sigma_1\left[\int_{\partial\Omega} g(u)\phi\,\mathrm{d}s + \int_{\Omega} g(u)\phi\,\mathrm{d}x\right] &= \int_{\Omega} g_{uu}(u)|\nabla u|^2\phi\,\mathrm{d}x \\ &\quad + \int_{\Omega}\lambda[f_u(u)g(u) - f(u)g_u(u)]\phi\,\mathrm{d}x. \end{aligned} \tag{3.6}$$

□

Using lemma 3.1, we now present the proofs of theorems 1.3 and 1.4.

Proof of theorem 1.3. Assume that u is a positive solution of (1.1), (1.2) and σ_1 is the principal eigenvalue of (1.14), (1.15) with corresponding eigenfunction ϕ chosen to be positive on $\bar{\Omega}$.

Suppose that (G1) and (FG1) both hold. Then by lemma 3.1 we have

$$\begin{aligned}&-\sigma_1\left[\int_{\partial\Omega} g(u)\phi\,\mathrm{d}s + \int_{\Omega} g(u)\phi\,\mathrm{d}x\right] \\ &\quad = \int_{\Omega} g_{uu}(u)|\nabla u|^2\phi\,\mathrm{d}x + \int_{\Omega}\lambda[f_u(u)g(u) - f(u)g_u(u)]\phi\,\mathrm{d}x < 0, \end{aligned} \tag{3.7}$$

since $g(u) > 0$ for $u > 0$, $g_{uu}(u) < 0$ for $u > 0$, $\phi > 0$ in $\bar{\Omega}$ and $f_u(u)g(u) - f(u)g_u(u) < 0$ for $u > 0$ from (FG1). Thus, $\sigma_1 > 0$ and the result follows from theorem 1.1.

On the other hand, if we suppose that (G2) and (FG2) both hold, then lemma 3.1 again gives that

$$-\sigma_1\left[\int_{\partial\Omega} g(u)\phi\,\mathrm{d}s + \int_{\Omega} g(u)\phi\,\mathrm{d}x\right] = \int_{\Omega} g_{uu}(u)|\nabla u|^2\phi\,\mathrm{d}x + \int_{\Omega} \lambda[f_u(u)g(u) - f(u)g_u(u)]\phi\,\mathrm{d}x > 0, \quad (3.8)$$

since $g(u) > 0$ for $u > 0$, $g_{uu}(u) > 0$ for $u > 0$, $\phi > 0$ in $\bar{\Omega}$ and $f_u(u)g(u) - f(u)g_u(u) > 0$ for $u > 0$ from (FG2). Thus, $\sigma_1 < 0$ and the result follows from theorem 1.1. □

Proof of theorem 1.4. Suppose that (G3) holds, u is a positive solution of (1.1), (1.2) and σ_1 is the principal eigenvalue of (1.14), (1.15) with corresponding eigenfunction ϕ chosen to be positive on $\bar{\Omega}$.

By lemma 3.1 and (G3) we have

$$-\sigma_1\left[\int_{\partial\Omega} g(u)\phi\,\mathrm{d}s + \int_{\Omega} g(u)\phi\,\mathrm{d}x\right] = \int_{\Omega} \lambda[f_u(u)g(u) - f(u)g_u(u)]\phi\,\mathrm{d}x. \quad (3.9)$$

Since $g(u) > 0$ for $u \geqslant 0$ and $\phi > 0$ in $\bar{\Omega}$, if (FG1) holds, then $f_u(u)g(u) - f(u)g_u(u) < 0$ for $u > 0$, and thus (3.9) implies that $\sigma_1 > 0$. On the other hand, if (FG2) holds, then $f_u(u)g(u) - f(u)g_u(u) > 0$ for $u > 0$, and thus (3.9) implies that $\sigma_1 > 0$. The result now follows from theorem 1.1. □

4. Proof of theorems 1.6–1.9

We begin by presenting a lemma that provides a connection between the reaction nonlinearity f, conditions (FG1), (FG2) and (FG3) and σ_1. Note that this result was first proved in [28, lemma 2.1].

LEMMA 4.1. *If u is a positive solution of (1.1), (1.2), then*

$$-\sigma_1\left[\int_{\partial\Omega} f(u)\phi\,\mathrm{d}s + \int_{\Omega} f(u)\phi\,\mathrm{d}x\right] = \int_{\Omega} f_{uu}(u)|\nabla u|^2\phi\,\mathrm{d}x + \int_{\partial\Omega} [f_u(u)g(u) - f(u)g_u(u)]\phi\,\mathrm{d}x, \quad (4.1)$$

where σ_1 is the principal eigenvalue of (1.14), (1.15) with corresponding eigenfunction ϕ.

Proof of lemma 4.1. Suppose that u is a positive solution of (1.1), (1.2) and σ_1 is the principal eigenvalue of (1.14), (1.15) with corresponding eigenfunction ϕ chosen to be positive on $\bar{\Omega}$. Calculating $(1.1)f_u(u)\phi - (1.14)f(u)$ and integrating over Ω we have

$$\int_{\Omega} [(-\Delta u)f_u(u)\phi + (\Delta\phi)f(u)]\,\mathrm{d}x = -\sigma_1\int_{\Omega} f(u)\phi\,\mathrm{d}x. \quad (4.2)$$

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But, by Green's first identity,

$$\begin{aligned}\int_\Omega [(-\Delta u) f_u(u)\phi]\,\mathrm{d}x &= \int_\Omega \nabla u\cdot\nabla(f_u(u)\phi)\,\mathrm{d}x - \int_{\partial\Omega}\frac{\partial u}{\partial\eta} f_u(u)\phi\,\mathrm{d}s\\ &= \int_\Omega f_{uu}(u)|\nabla u|^2\phi\,\mathrm{d}x + \int_\Omega f_u(u)\nabla\phi\cdot\nabla u\,\mathrm{d}x\\ &\quad + \int_{\partial\Omega} f_u(u)g(u)\phi\,\mathrm{d}s \end{aligned}\tag{4.3}$$

and

$$\begin{aligned}\int_\Omega [(\Delta\phi) f(u)]\,\mathrm{d}x &= -\int_\Omega \nabla\phi\cdot\nabla(f(u))\,\mathrm{d}x + \int_{\partial\Omega}\frac{\partial\phi}{\partial\eta} f(u)\,\mathrm{d}s\\ &= -\int_\Omega f_u(u)\nabla\phi\cdot\nabla u\,\mathrm{d}x + \int_{\partial\Omega}[\sigma_1 - g_u(u)]f(u)\phi\,\mathrm{d}s. \end{aligned}\tag{4.4}$$

Combining (4.2)–(4.4) gives

$$\begin{aligned}-\sigma_1\left[\int_{\partial\Omega} f(u)\phi\,\mathrm{d}s + \int_\Omega f(u)\phi\,\mathrm{d}x\right] &= \int_\Omega f_{uu}(u)|\nabla u|^2\phi\,\mathrm{d}x\\ &\quad + \int_{\partial\Omega}[f_u(u)g(u) - f(u)g_u(u)]\phi\,\mathrm{d}s \end{aligned}\tag{4.5}$$

as desired. □

We now prove theorems 1.6–1.9.

Proof of theorem 1.6. Suppose u is a positive solution of (1.1), (1.2), and σ_1 is the principal eigenvalue of (1.14), (1.15) with corresponding eigenfunction ϕ chosen to be positive on $\bar{\Omega}$. If (F1) and (FG1) both hold, then by lemma 4.1 we have that

$$\begin{aligned}-\sigma_1&\left[\int_{\partial\Omega} f(u)\phi\,\mathrm{d}s + \int_\Omega f(u)\phi\,\mathrm{d}x\right]\\ &= \int_\Omega f_{uu}(u)|\nabla u|^2\phi\,\mathrm{d}x + \int_{\partial\Omega}[f_u(u)g(u) - f(u)g_u(u)]\phi\,\mathrm{d}s\\ &< 0, \end{aligned}\tag{4.6}$$

since $f_{uu}(u) < 0$ for $u > 0$, $\phi > 0$ in $\bar{\Omega}$ and $f_u(u)g(u) - f(u)g_u(u) < 0$ for $u > 0$ from (FG1). Thus, $\sigma_1 > 0$, since $f(u) > 0$ for $u > 0$, and the result follows from theorem 1.1. On the other hand, if (F2) and (FG2) both hold, then lemma 4.1 again gives that

$$\begin{aligned}-\sigma_1&\left[\int_{\partial\Omega} f(u)\phi\,\mathrm{d}s + \int_\Omega f(u)\phi\,\mathrm{d}x\right]\\ &= \int_\Omega f_{uu}(u)|\nabla u|^2\phi\,\mathrm{d}x + \int_{\partial\Omega}[f_u(u)g(u) - f(u)g_u(u)]\phi\,\mathrm{d}s\\ &> 0, \end{aligned}\tag{4.7}$$

since $f_{uu}(u) > 0$ for $u > 0$, $\phi > 0$ in $\bar{\Omega}$, and $f_u(u)g(u) - f(u)g_u(u) > 0$ for $u > 0$ from (FG2). Thus, $\sigma_1 < 0$ since $f(u) > 0$ for $u > 0$, and the result follows from theorem 1.1. □

Before proving theorem 1.7, we first establish the following lemma, which is crucial to the proof of this theorem.

LEMMA 4.2. *If (F2A) holds and u is a positive solution of (1.1), (1.2), then*

$$\begin{aligned} -\sigma_1\left[\int_{\partial\Omega}\bar{f}(u)\phi\,\mathrm{d}s+\int_{\Omega}\bar{f}(u)\phi\,\mathrm{d}x\right] &= \int_{\Omega}\bar{f}_{uu}(u)|\nabla u|^2\phi\,\mathrm{d}x-\lambda f(0)\int_{\Omega}\bar{f}_u(u)\phi\,\mathrm{d}x \\ &\quad +\lambda|\bar{f}_u(0)|\int_{\Omega}[u\bar{f}_u(u)-\bar{f}(u)]\phi\,\mathrm{d}x+\int_{\partial\Omega}[\bar{f}_u(u)g(u)-\bar{f}(u)g_u(u)]\phi\,\mathrm{d}s, \end{aligned} \tag{4.8}$$

where σ_1 is the principal eigenvalue of (1.14), (1.15) with corresponding eigenfunction ϕ and $\bar{f}(u) := f(u) - f(0) + |f_u(0)|u$.

Proof of lemma 4.2. Suppose that (F2A) and (FG2) both hold, u is a positive solution of (1.1), (1.2) and σ_1 is the principal eigenvalue of (1.14), (1.15) with corresponding eigenfunction ϕ chosen to be positive on $\bar{\Omega}$. Define $\bar{f}(u) := f(u) - f(0) + |f_u(0)|u$. Thus, $\bar{f}(0) = 0$, $\bar{f}_u(u) = f_u(u) + |f_u(0)|$ and $\bar{f}_{uu}(u) = f_{uu}(u) > 0$. This implies that $\bar{f}_u(u) > 0$ for $u > 0$, and thus $\bar{f}(u) > 0$ for $u > 0$.

Now, u is a positive solution of (1.1), (1.2) rewritten as

$$-\Delta u = \lambda[\bar{f}(u) + f(0) - |f_u(0)|u] \quad \text{in } \Omega, \tag{4.9}$$

$$\frac{\partial u}{\partial \eta} = -g(u) \quad \text{on } \partial\Omega, \tag{4.10}$$

and σ_1 is the principal eigenvalue of (1.14), (1.15), also rewritten as

$$-\Delta\phi - \lambda[\bar{f}_u(u) - |f_u(0)|]\phi = \sigma\phi \quad \text{in } \Omega, \tag{4.11}$$

$$\frac{\partial\phi}{\partial\eta} + g_u(u)\phi = \sigma\phi \quad \text{on } \partial\Omega. \tag{4.12}$$

Integration of $(4.9)\bar{f}_u(u)\phi - (4.11)\bar{f}(u)$ over Ω gives

$$\begin{aligned} \int_{\Omega}(-\Delta u)\bar{f}_u(u)\phi + (\Delta\phi)\bar{f}(u) - \lambda f(0)\bar{f}_u(u)\phi & \\ + \lambda|\bar{f}_u(0)|u\bar{f}_u(u)\phi - \lambda|\bar{f}_u(0)|\bar{f}(u)\phi\,\mathrm{d}x & \\ = -\sigma_1\int_{\Omega}\bar{f}(u)\phi\,\mathrm{d}x. & \end{aligned} \tag{4.13}$$

But, by Green's first identity,

$$\begin{aligned} \int_{\Omega}[(-\Delta u)\bar{f}_u(u)\phi]\,\mathrm{d}x &= \int_{\Omega}\nabla u\cdot\nabla(\bar{f}_u(u)\phi)\,\mathrm{d}x - \int_{\partial\Omega}\frac{\partial u}{\partial\eta}\bar{f}_u(u)\phi\,\mathrm{d}s \\ &= \int_{\Omega}\bar{f}_{uu}(u)|\nabla u|^2\phi\,\mathrm{d}x + \int_{\Omega}\bar{f}_u(u)\nabla\phi\cdot\nabla u\,\mathrm{d}x \\ &\quad + \int_{\partial\Omega}\bar{f}_u(u)g(u)\phi\,\mathrm{d}s \end{aligned} \tag{4.14}$$

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and

$$\int_{\Omega}[(\Delta\phi)\bar{f}(u)]\,\mathrm{d}x = -\int_{\Omega}\nabla\phi\cdot\nabla\bar{f}(u)\,\mathrm{d}x + \int_{\partial\Omega}\frac{\partial\phi}{\partial\eta}\bar{f}(u)\,\mathrm{d}s$$
$$= -\int_{\Omega}\bar{f}_u(u)\nabla\phi\nabla u\,\mathrm{d}x + \int_{\partial\Omega}[\sigma_1 - g_u(u)]\bar{f}(u)\phi\,\mathrm{d}s. \quad (4.15)$$

Combining (4.13)–(4.15) gives

$$-\sigma_1\left[\int_{\partial\Omega}\bar{f}(u)\phi\,\mathrm{d}s + \int_{\Omega}\bar{f}(u)\phi\,\mathrm{d}x\right] = \int_{\Omega}\bar{f}_{uu}(u)|\nabla u|^2\phi\,\mathrm{d}x - \lambda f(0)\int_{\Omega}\bar{f}_u(u)\phi\,\mathrm{d}x$$
$$+ \lambda|\bar{f}_u(0)|\int_{\Omega}[u\bar{f}_u(u) - \bar{f}(u)]\phi\,\mathrm{d}x$$
$$+ \int_{\partial\Omega}[\bar{f}_u(u)g(u) - \bar{f}(u)g_u(u)]\phi\,\mathrm{d}s. \quad (4.16)$$

□

Proof of theorem 1.7. Suppose that (F2A) and (FG2) both hold, u is a positive solution of (1.1), (1.2), $\bar{f}(u)$ is defined as in lemma 4.2 and σ_1 is the principal eigenvalue of (1.14), (1.15) with corresponding eigenfunction ϕ chosen to be positive on $\bar{\Omega}$. We note that since $\bar{f}_{uu}(u) > 0$ for $u > 0$ we have

$$\int_{\Omega}\bar{f}_{uu}(u)|\nabla u|^2\phi\,\mathrm{d}x > 0. \quad (4.17)$$

In addition, note that since $\bar{f}(0) = 0$ and $\bar{f}_{uu}(u) > 0$ for $u > 0$ we must have that $u\bar{f}_u(u) - \bar{f}(u) > 0$ and $\bar{f}_u(u) > 0$ for $u > 0$. This fact combined with $\lambda > 0$ and $f(0) < 0$ yields

$$-\lambda f(0)\int_{\Omega}\bar{f}_u(u)\phi\,\mathrm{d}x > 0 \quad (4.18)$$

and

$$\lambda|\bar{f}_u(0)|\int_{\Omega}[u\bar{f}_u(u) - \bar{f}(u)]\phi\,\mathrm{d}x > 0. \quad (4.19)$$

Also, (FG2) implies that $\bar{f}_u(u)g(u) - \bar{f}(u)g_u(u) > 0$ for $u > 0$. Combined with the fact that $\phi > 0$ in $\bar{\Omega}$, this gives

$$\int_{\partial\Omega}[\bar{f}_u(u)g(u) - \bar{f}(u)g_u(u)]\phi\,\mathrm{d}s > 0. \quad (4.20)$$

Combining lemma 4.2 with (4.17)–(4.20) gives

$$-\sigma_1\left[\int_{\partial\Omega}\bar{f}(u)\phi\,\mathrm{d}s + \int_{\Omega}\bar{f}(u)\phi\,\mathrm{d}x\right] = \int_{\Omega}\bar{f}_{uu}(u)|\nabla u|^2\phi\,\mathrm{d}x - \lambda f(0)\int_{\Omega}\bar{f}_u(u)\phi\,\mathrm{d}x$$
$$+ \lambda|\bar{f}_u(0)|\int_{\Omega}[u\bar{f}_u(u) - \bar{f}(u)]\phi\,\mathrm{d}x$$
$$+ \int_{\partial\Omega}[\bar{f}_u(u)g(u) - \bar{f}(u)g_u(u)]\phi\,\mathrm{d}s$$
$$> 0, \quad (4.21)$$

since $\bar{f}(u) > 0$ for $u > 0$. We then obtain $\sigma_1 < 0$, and the result now immediately follows from theorem 1.1. □

We close this section by presenting the proofs of theorems 1.8 and 1.9.

Proof of theorem 1.8. Suppose that (F3) holds, u is a positive solution of (1.1), (1.2) and σ_1 is the principal eigenvalue of (1.14), (1.15) with corresponding eigenfunction ϕ chosen to be positive on $\bar{\Omega}$.

Thus, lemma 4.1 gives

$$-\sigma_1\left[\int_{\partial\Omega} f(u)\phi\,\mathrm{d}s + \int_{\Omega} f(u)\phi\,\mathrm{d}x\right] = \int_{\partial\Omega}[f_u(u)g(u) - f(u)g_u(u)]\phi\,\mathrm{d}s. \tag{4.22}$$

Since $f(u) > 0$ for $u > 0$, and $\phi > 0$ in $\bar{\Omega}$, if (FG1) holds, then $f_u(u)g(u) - f(u)g_u(u) < 0$ for $u > 0$, and thus (4.22) implies that $\sigma_1 > 0$. On the other hand, if (FG2) holds, then $f_u(u)g(u) - f(u)g_u(u) > 0$ for $u > 0$, and thus (4.22) implies that $\sigma_1 > 0$. The result in these cases now follows from theorem 1.1. □

Proof of theorem 1.9. Suppose that (FG3) holds, u is a positive solution of (1.1), (1.2) and σ_1 is the principal eigenvalue of (1.14), (1.15) with corresponding eigenfunction ϕ chosen to be positive on $\bar{\Omega}$. Now, if (F1) holds, then with (FG3), lemma 4.1 yields

$$-\sigma_1\left[\int_{\partial\Omega} f(u)\phi\,\mathrm{d}s + \int_{\Omega} f(u)\phi\,\mathrm{d}x\right] = \int_{\Omega} f_{uu}(u)|\nabla u|^2\phi\,\mathrm{d}x < 0. \tag{4.23}$$

If (G1) holds, then from (FG3) and lemma 3.1 we have

$$-\sigma_1\left[\int_{\partial\Omega} g(u)\phi\,\mathrm{d}s + \int_{\Omega} g(u)\phi\,\mathrm{d}x\right] = \int_{\Omega} g_{uu}(u)|\nabla u|^2\phi\,\mathrm{d}x < 0. \tag{4.24}$$

In either case $\sigma_1 > 0$, and the result follows from theorem 1.1.

Now, note that if (F2) holds, then by lemma 4.1 with (FG3) we have that

$$-\sigma_1\left[\int_{\partial\Omega} f(u)\phi\,\mathrm{d}s + \int_{\Omega} f(u)\phi\,\mathrm{d}x\right] = \int_{\Omega} f_{uu}(u)|\nabla u|^2\phi\,\mathrm{d}x > 0. \tag{4.25}$$

Also, if (F2A) holds, then from lemma 4.2 and (FG3) we have that

$$\begin{aligned}-\sigma_1&\left[\int_{\partial\Omega} \bar{f}(u)\phi\,\mathrm{d}s + \int_{\Omega} \bar{f}(u)\phi\,\mathrm{d}x\right]\\ &= \int_{\Omega} \bar{f}_{uu}(u)|\nabla u|^2\phi\,\mathrm{d}x - \lambda f(0)\int_{\Omega}\bar{f}_u(u)\phi\,\mathrm{d}x\\ &\quad + \lambda|\bar{f}_u(0)|\int_{\Omega}[u\bar{f}_u(u) - \bar{f}(u)]\phi\,\mathrm{d}x\\ &> 0.\end{aligned} \tag{4.26}$$

Finally, if (G2) holds, then from (FG3) and lemma 3.1 we have

$$-\sigma_1\left[\int_{\partial\Omega} g(u)\phi\,\mathrm{d}s + \int_{\Omega} g(u)\phi\,\mathrm{d}x\right] = \int_{\Omega} g_{uu}(u)|\nabla u|^2\phi\,\mathrm{d}x > 0. \tag{4.27}$$

In all of these cases $\sigma_1 < 0$, and the result follows from theorem 1.1. □

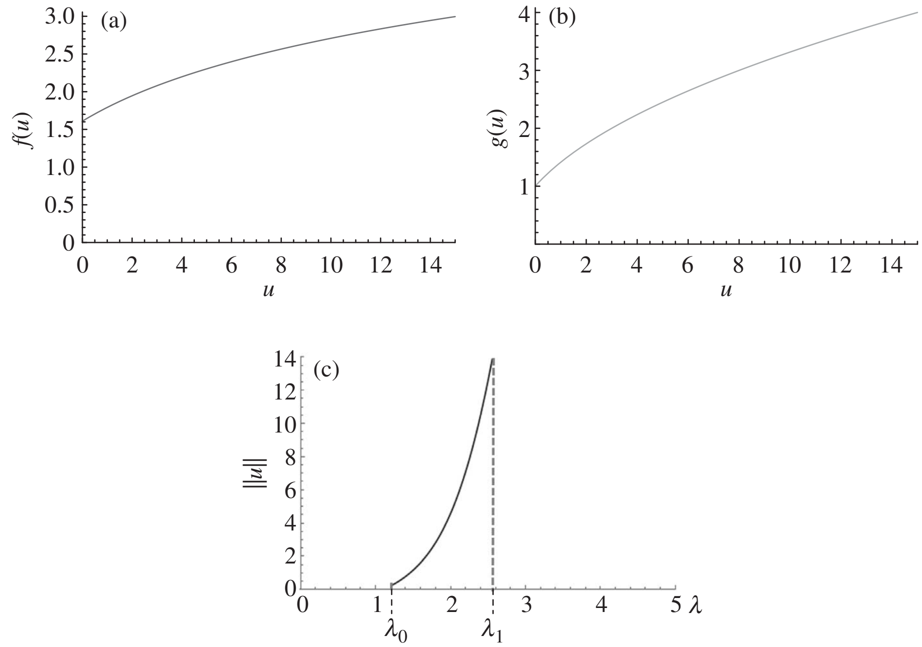

Figure 1. The bifurcation curve of positive solutions to (5.1)–(5.3) for (a) $f(u) = \ln(u+5)$ and (b) $g(u) = (u+1)^{1/2}$. (c) Bifurcation curve for (5.1)–(5.3) for the case (G1) and (FG1).

5. Bifurcation curves for the one-dimensional case

Here, we present complete bifurcation curves for the case when $n = 1$ and $\Omega = (0,1)$, namely

$$-u'' = \lambda f(u) \quad \text{in } (0,1), \tag{5.1}$$

$$u'(0) = g(u(0)), \tag{5.2}$$

$$u'(1) = -g(u(1)). \tag{5.3}$$

In [17], the quadrature method of Laetsch was adapted to study (5.1)–(5.3) in the $g(u) \equiv 1$ case. Modifying the quadrature method for the $g(u) \not\equiv 1$ case, the bifurcation curve of positive solutions of (5.1)–(5.3) is given by

$$G(\rho, q) := \frac{g(q)^2}{2(F(\rho) - F(q))} = \lambda, \tag{5.4}$$

where

$$F(u) := \int_0^u f(s)\,\mathrm{d}s,$$

$\rho = \|u\|_\infty$ and $q = u(0) = u(1)$ satisfies

$$\tilde{G}(\rho, q) := 2\int_0^\rho \frac{\mathrm{d}s}{\sqrt{F(\rho) - F(s)}} - \int_0^q \frac{\mathrm{d}s}{\sqrt{F(\rho) - F(s)}} - \frac{g(q)}{\sqrt{F(\rho) - F(q)}} = 0. \tag{5.5}$$

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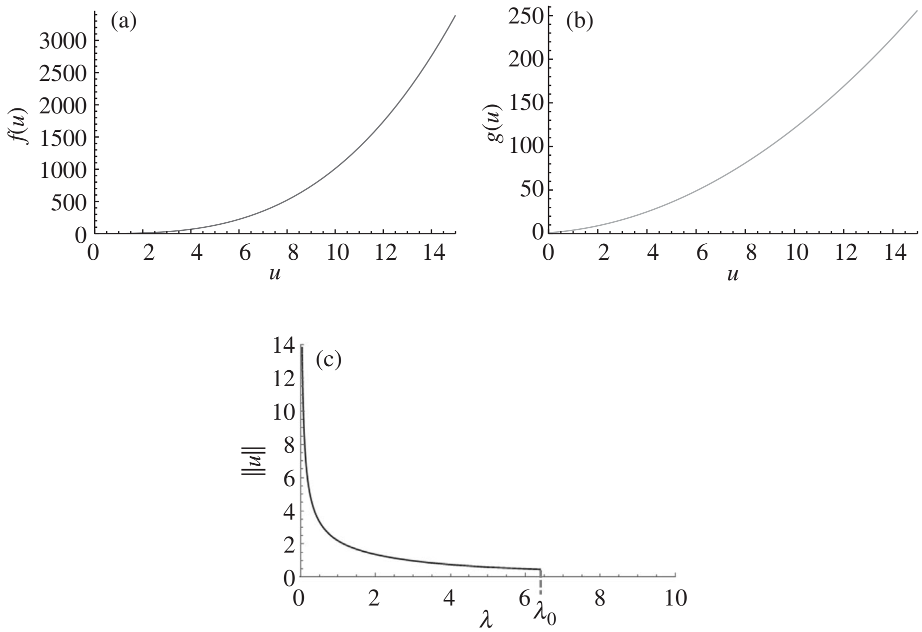

Figure 2. The bifurcation curve of positive solutions of (5.1)–(5.3) for (a) $f(u) = \ln(u+5)$ and (b) $g(u) = (u+1)^{1/2}$. (c) Bifurcation curve for (5.1)–(5.3) for the case (G2) and (FG2).

In particular, using the mathematics software package MATHEMATICA we obtain the bifurcation curves of positive solutions for (5.1)–(5.3) shown in the figures. The bifurcation curves are organized according to the assumptions on f, g and f/g.

5.1. Case (G1) and (FG1)

Setting $f(u) = \ln(u+5)$ and $g(u) = (u+1)^{1/2}$, the bifurcation curve of positive solutions for (5.1)–(5.3) is given in figure 1. The existence of a positive solution indicated by the bifurcation curve combined with theorem 1.3 shows that there exists an asymptotically stable positive solution for all $\lambda \in [\lambda_0, \lambda_1)$ (some $\lambda_0, \lambda_1 > 0$).

5.2. Case (G2) and (FG2)

Defining $f(u) = u^3 + u$ and $g(u) = (u+1)^2$, the bifurcation curve of positive solutions for (5.1)–(5.3) is shown in figure 2. The existence of a positive solution indicated by the bifurcation curve combined with theorem 1.3 shows that there exists an unstable positive solution for $\lambda \in (0, \lambda_0]$ (some $\lambda_0 > 0$).

5.3. Case (G3) and (FG1)

Defining $f(u) = \mathrm{e}^{u/(1+u)}$ and $g(u) = 2u + 1$, the bifurcation curve of positive solutions for (5.1)–(5.3) is shown in figure 3. The existence of a positive solution indicated by the bifurcation curve combined with theorem 1.4 shows that there exists an asymptotically stable positive solution for $\lambda \geqslant \lambda_0$ (some $\lambda_0 > 0$).

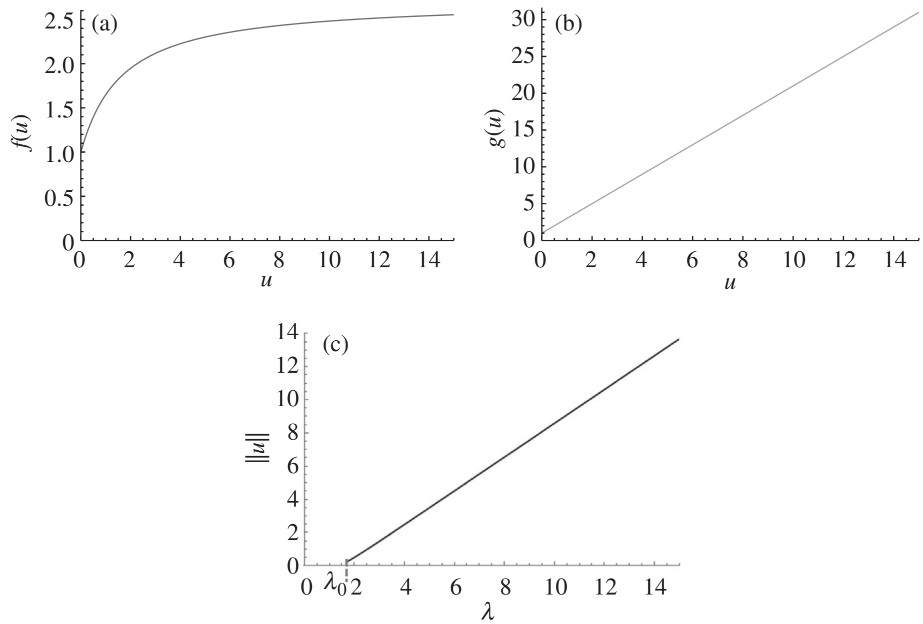

Figure 3. The bifurcation curve of positive solutions of (5.1)–(5.3) for (a) $f(u) = \mathrm{e}^{u/(1+u)}$ and (b) $g(u) = 2u + 1$. (c) Bifurcation curve for (5.1)–(5.3) for the case (G3) and (FG1).

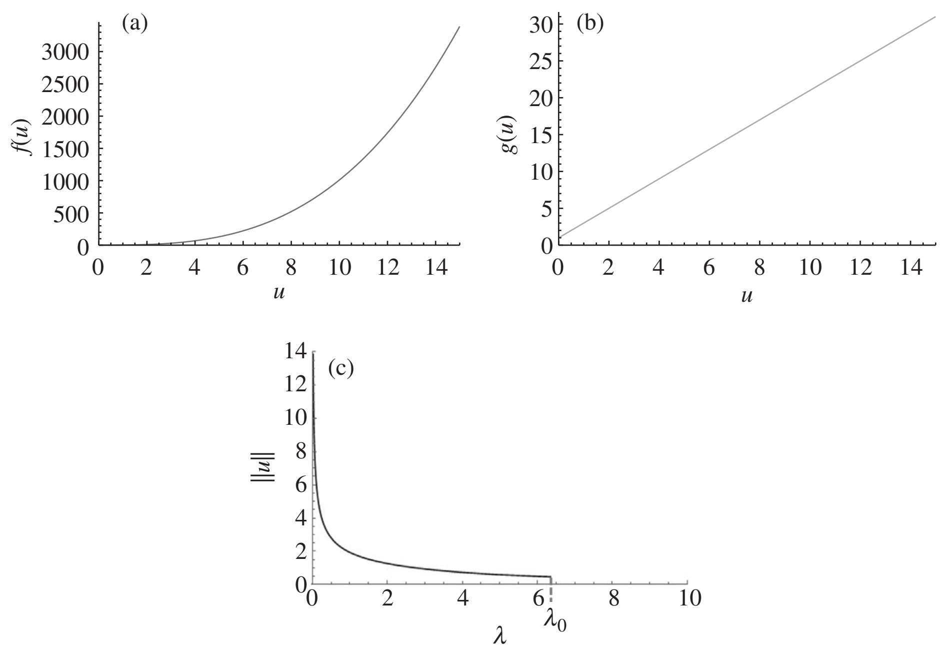

Figure 4. The bifurcation curve of positive solutions to (5.1)–(5.3) for (a) $f(u) = u^3 + u$ and (b) $g(u) = 2u + 1$. (c) Bifurcation curve for (5.1)–(5.3) for the case (G3) and (FG2).

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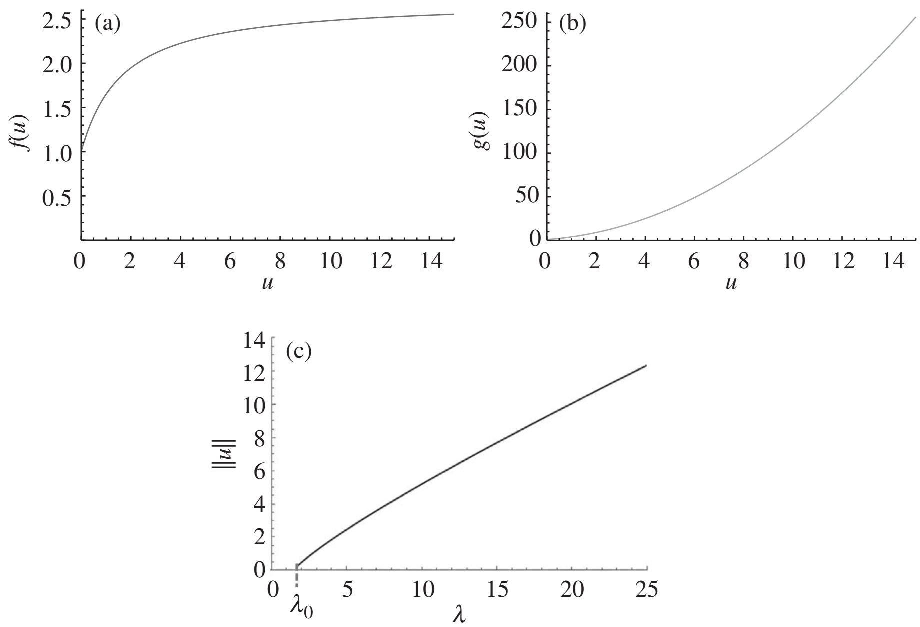

Figure 5. The bifurcation curve of positive solutions of (5.1)–(5.3) for (a) $f(u) = \mathrm{e}^{u/(1+u)}$ and (b) $g(u) = (u+1)^2$. (c) Bifurcation curve for (5.1)–(5.3) for the case (F1) and (FG1).

5.4. Case (G3) and (FG2)

Defining $f(u) = u^3 + u$ and $g(u) = 2u + 1$, the bifurcation curve of positive solutions for (5.1)–(5.3) is shown in figure 4. The existence of a positive solution indicated by the bifurcation curve combined with theorem 1.4 shows that there exists an unstable positive solution for $\lambda \in (0, \lambda_0]$ (some $\lambda_0 > 0$).

5.5. Case (F1) and (FG1)

Defining $f(u) = \mathrm{e}^{u/(1+u)}$ and $g(u) = (u+1)^2$, the bifurcation curve of positive solutions for (5.1)–(5.3) is shown in figure 5. The existence of a positive solution indicated by the bifurcation curve combined with theorem 1.6 shows that there exists an asymptotically stable positive solution for $\lambda \geqslant \lambda_0$ (some $\lambda_0 > 0$).

5.6. Case (F2A) and (FG2)

Defining $f(u) = u^3 + u - 0.1$ and $g(u) = (u+1)^2$, the bifurcation curve of positive solutions for (5.1)–(5.3) is shown in figure 6. The existence of a positive solution indicated by the bifurcation curve combined with theorem 1.7 shows that there exists an unstable positive solution for $\lambda \in (0, \lambda_0]$ (some $\lambda_0 > 0$).

5.7. Case (F2) and (FG2)

Defining $f(u) = \mathrm{e}^u$ and $g(u) = (u+1)^{1/2}$, the bifurcation curve of positive solutions for (5.1)–(5.3) is shown in figure 7. The existence of a positive solution

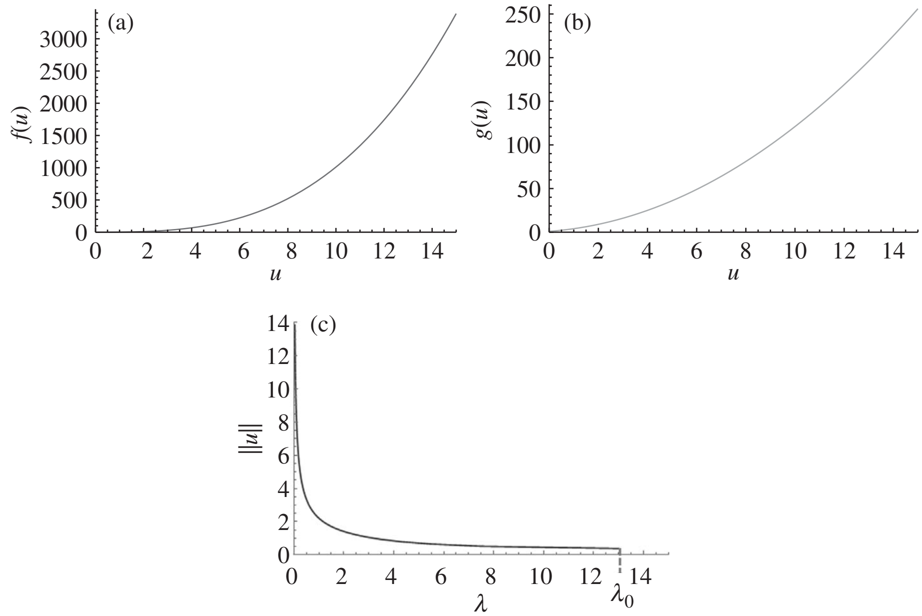

Figure 6. The bifurcation curve of positive solutions of (5.1)–(5.3) for (a) $f(u) = u^3 + u - 0.1$ and (b) $g(u) = (u+1)^2$. (c) Bifurcation curve for (5.1)–(5.3) for the case (F2A) and (FG2).

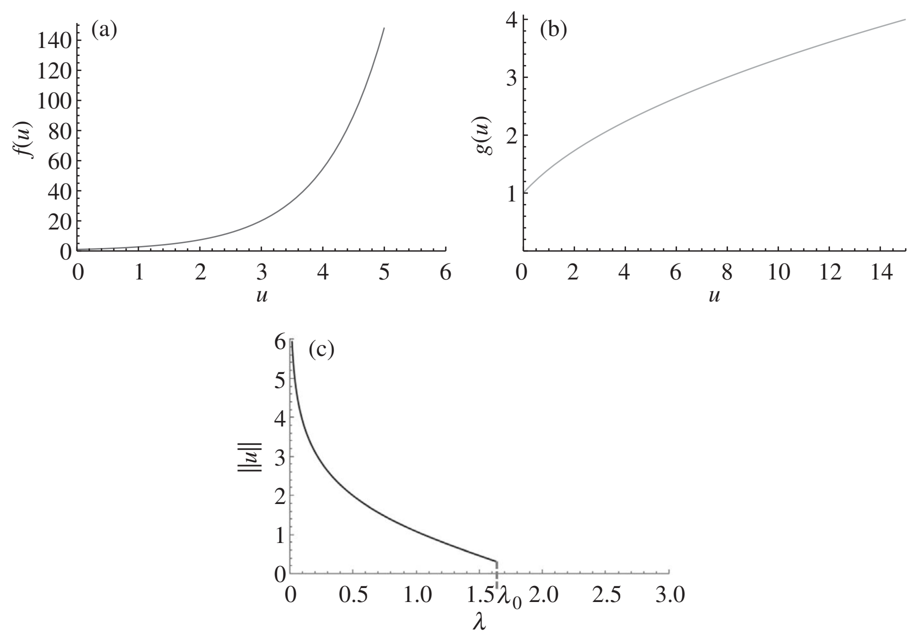

Figure 7. The bifurcation curve of positive solutions of (5.1)–(5.3) for (a) $f(u) = \mathrm{e}^u$ and (b) $g(u) = (u+1)^{1/2}$. (c) Bifurcation curve for (5.1)–(5.3) for the case (F2) and (FG2).

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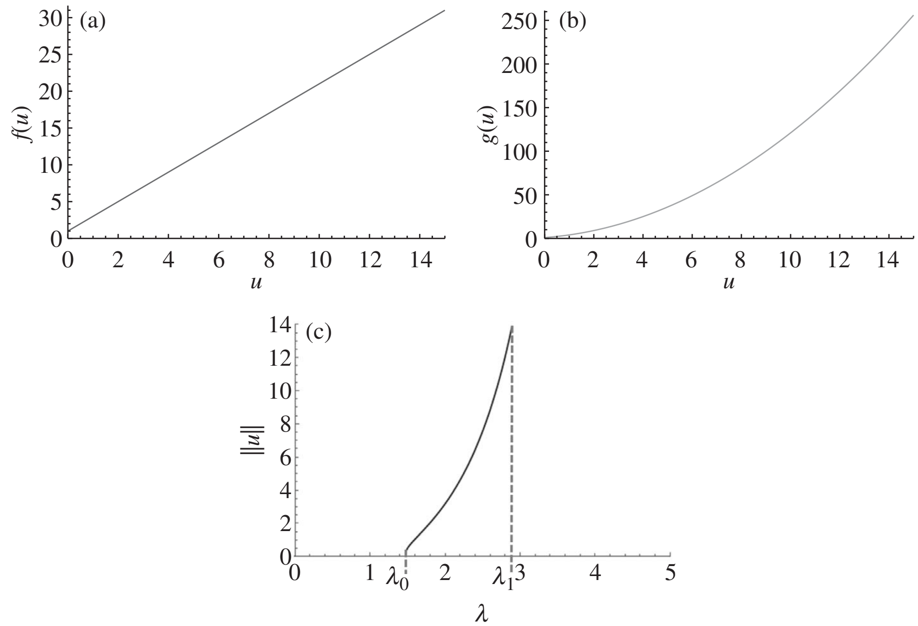

Figure 8. The bifurcation curve of positive solutions of (5.1)–(5.3) for (a) $f(u) = 2u + 1$ and (b) $g(u) = (u+1)^2$. (c) Bifurcation curve for (5.1)–(5.3) for the case (F3) and (FG1).

indicated by the bifurcation curve combined with theorem 1.7 shows that there exists an unstable positive solution for $\lambda \in (0, \lambda_0]$ (some $\lambda_0 > 0$).

5.8. Case (F3) and (FG1)

Defining $f(u) = 2u + 1$ and $g(u) = (u + 1)^2$, the bifurcation curve of positive solutions for (5.1)–(5.3) is shown in figure 8. The existence of a positive solution indicated by the bifurcation curve combined with theorem 1.8 shows that there exists an asymptotically stable positive solution for $\lambda \in [\lambda_0, \lambda_1)$ (some $\lambda_0, \lambda_1 > 0$).

5.9. Case (F3) and (FG2)

Defining $f(u) = 2u + 1$ and $g(u) = (u + 1)^{1/10}$, the bifurcation curve of positive solutions for (5.1)–(5.3) is shown in figure 9. The existence of a positive solution indicated by the bifurcation curve combined with theorem 1.8 shows that there exists an unstable positive solution for $\lambda \in (\lambda_0, \lambda_1]$ (some $\lambda_0, \lambda_1 > 0$).

5.10. Case (F1) or (G1) and (FG3)

Defining $f(u) = (u+1)^{1/2}$ and $g(u) = 2(u+1)^{1/2}$, the bifurcation curve of positive solutions for (5.1)–(5.3) is shown in figure 10. The existence of a positive solution indicated by the bifurcation curve combined with theorem 1.9 shows that there exists an asymptotically stable positive solution for $\lambda \in [\lambda_0, \lambda_1)$ (some $\lambda_0, \lambda_1 > 0$).

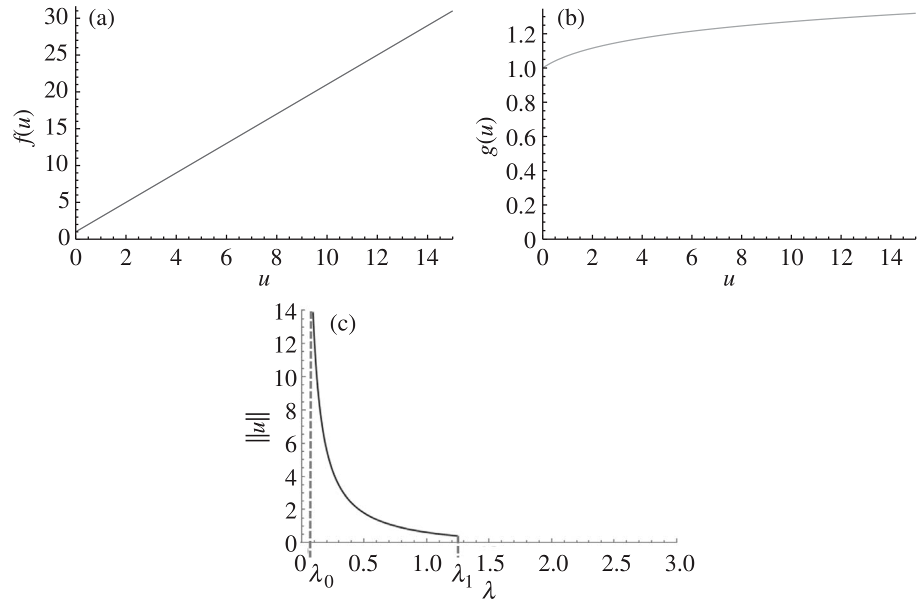

Figure 9. The bifurcation curve of positive solutions of (5.1)–(5.3) for (a) $f(u) = 2u + 1$ and (b) $g(u) = (u + 1)^{1/10}$. (c) Bifurcation curve for (5.1)–(5.3) for the case (F3) and (FG2).

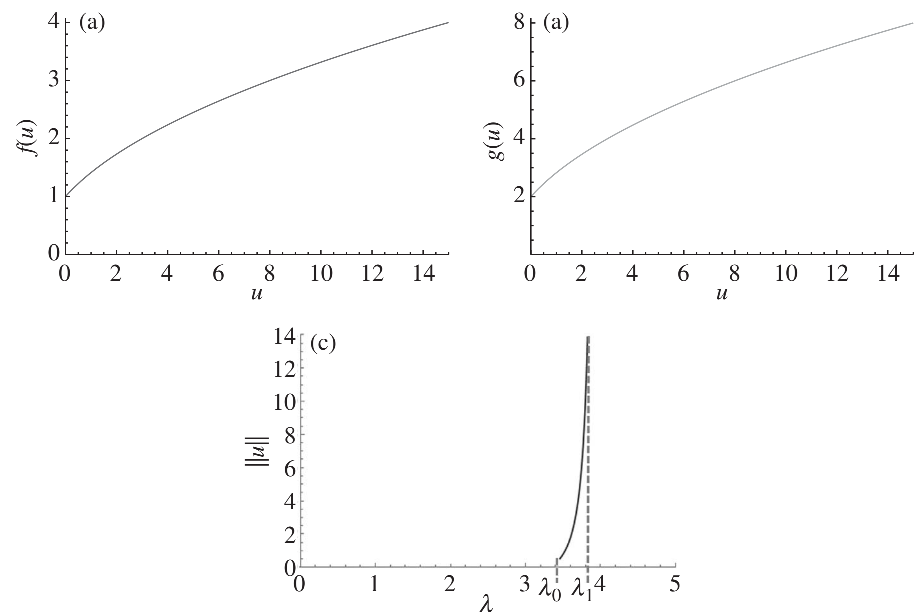

Figure 10. The bifurcation curve of positive solutions of (5.1)–(5.3) for (a) $f(u) = (u+1)^{1/2}$ and (b) $g(u) = 2(u + 1)^{1/2}$. (c) Bifurcation curve for (5.1)–(5.3) for the case (F1) or (G1) and (FG3).

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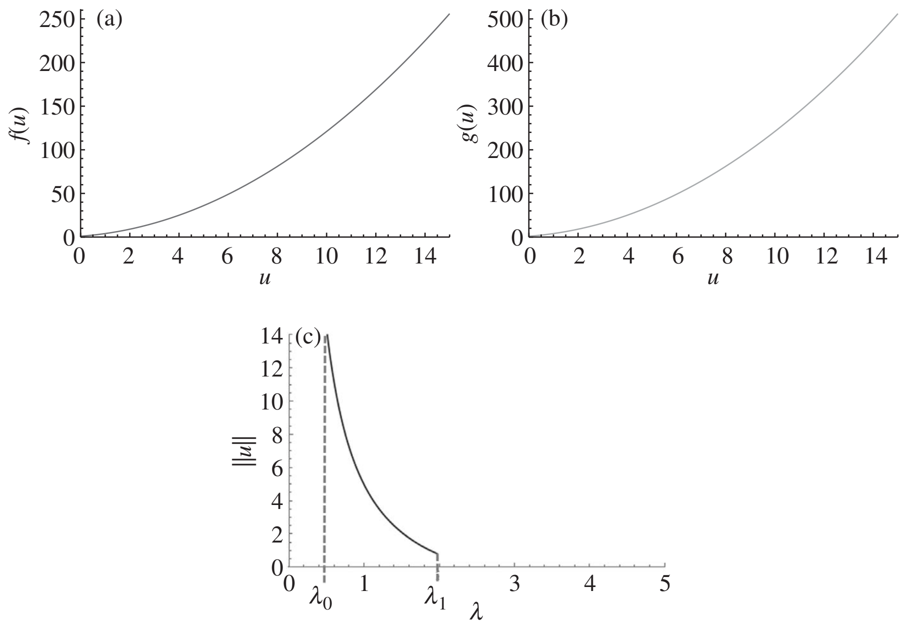

Figure 11. The bifurcation curve of positive solutions of (5.1)–(5.3) for (a) $f(u) = (u+1)^2$ and (b) $g(u) = 2(u+1)^2$. (c) Bifurcation curve for (5.1)–(5.3) for the case (F2A) or (G2) and (FG3).

5.11. Case (F2A) or (G2) and (FG3)

Defining $f(u) = (u+1)^2$ and $g(u) = 2(u+1)^2$, the bifurcation curve of positive solutions for (5.1)–(5.3) is shown in figure 11. The existence of a positive solution indicated by the bifurcation curve combined with theorem 1.9 shows that there exists an asymptotically stable positive solution for $\lambda \in (\lambda_0, \lambda_1]$ (some $\lambda_0, \lambda_1 > 0$).

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