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A geometric description of the extreme Khovanov cohomology

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We prove that the potential extreme Khovanov cohomology of a link is the cohomology of the independence simplicial complex of its Lando graph. We also provide a family of knots having as many non-trivial extreme Khovanov cohomology modules as desired, that is, examples of H-thick knots that are as far from being H-thin as desired.

Keywords: knots and links; Khovanov cohomology; independence complex; H-thick knots; Lando graph

2010 *Mathematics subject classification:* Primary 57M27; 57M25
Secondary 57M15

1. Introduction

The Khovanov cohomology of knots and links was introduced by Mikhail Khovanov at the end of last century (see [9]) and nicely explained by Bar-Natan in [2]. In [14] Viro interpreted it in terms of enhanced states of diagrams (these states are the well-known Kauffman states [8] enhanced with a sign assignment). Using Viro's point of view, in this paper we will prove that the hypothetical extreme Khovanov cohomology of a link coincides with the cohomology of the independence simplicial complex of its Lando graph.

The Lando graph [5] of a link diagram was studied by Morton and Bae in [1], where they proved that the hypothetical extreme coefficient of the Jones polynomial coincides with a certain numerical invariant of the graph, named in this paper as

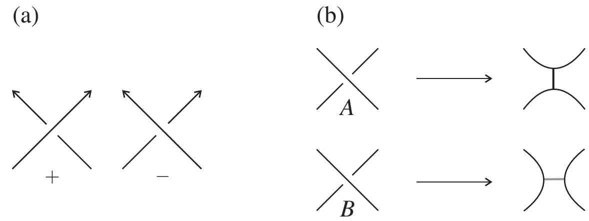

Figure 1. (a) Our convention of signs. (b) The smoothing of a crossing according to its A or B-label. A-chords (B-chords) are represented by dark (light) segments.

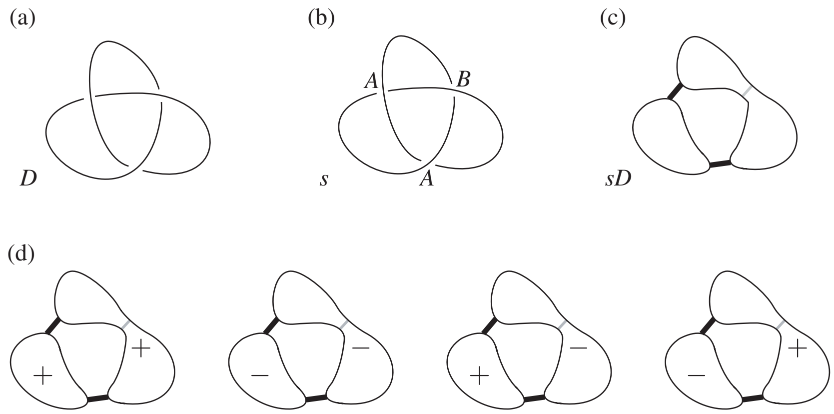

Figure 2. The figure shows (a) a diagram D representing 3_1, (b) a state s, and (c) sD. Here $|s| = 2$. The four related enhanced states are shown in (d). From left to right the values of τ are 2, −2, 0, 0.

its independence number. Now, on one hand the Jones polynomial can be seen as the bigraded Euler characteristic of the Khovanov cohomology. On the other hand, the formula for the independence number certainly suggests the formula of an Euler characteristic. Both ideas together have led us to a way of understanding the extreme Khovanov cohomology in terms of this graph. This is what we develop in this paper and reflect in theorem 4.4.

In [12] it was proved that the independence number can take any value, and hence there are links (in fact knots) with arbitrary extreme coefficients. This idea is also extended here to Khovanov cohomology by proving that there are links (in fact knots) with an arbitrary number of non-trivial extreme Khovanov cohomology modules (see theorem 6.2), which are of course examples of H-thick knots (see [10]). The basis of these examples is a link with exactly two non-trivial extreme Khovanov cohomology modules, constructed in theorem 5.7. However, we think that the construction of the example is more interesting than the example itself. The example is constructed using theorem 4.4, the Alexander duality and a construction by Jonsson [7].

There are many other interesting constructions of graphs starting with a link diagram (see, for example, [6]), not to be confused with the Lando graph. In addition, there are other very interesting ways of trying to understand the Khovanov cohomology as the cohomology of a chain complex. For example, in [11] Lipshitz and Sarkar construct, in an explicit and combinatorial way, a chain bicomplex that produces the Khovanov cohomology.

The paper is organized as follows. In § 2 we review the definition of the Khovanov cohomology of links, by using the notion of an enhanced state of a diagram, and explain what we understand by the potential extreme Khovanov complex and cohomology modules. In § 3 we review the notion of the Lando graph of a link diagram, and define the independence simplicial complex of this graph. We will refer to the corresponding cochain complex as the Lando cochain complex, and to the corresponding cohomology modules as the Lando cohomology. Indeed, we will note that the independence number of the Lando graph is the Euler characteristic associated with the Lando cohomology. In § 4 we prove that the Lando cochain complex is a copy of the potential extreme Khovanov complex, with the degrees shifted by a constant (theorem 4.4). In this section there is also an example showing how to apply this theorem in order to compute Khovanov extreme cohomology in a practical way. The rest of the paper deals with applications of theorem 4.4. In particular, § 5 shows how to relate Lando cohomology (and extreme Khovanov cohomology) to the homology of a simplicial complex, which allows us to give an example of a link diagram having exactly two non-trivial extreme Khovanov cohomology modules. Finally, in § 6 we use the previous example to give families of H-thick knots.

Note added in proof. After the first version of this paper was submitted for publication, we heard from Sergei Chmutov that he presented a talk, as yet unpublished, at the conference Knots in Washington XXI, held in Washington, DC, in 2005, with a similar statement to that appearing in theorem 4.4.

2. Khovanov cohomology

Let D be an oriented diagram of a link L with c crossings and writhe $w = p - n$, with p and n being the number of positive and negative crossings in D, according to the sign convention shown in figure 1(a). A state s assigns a label, A or B, to each crossing of D. Let $\mathcal{S}$ be the collection of 2^c possible states of D. For $s \in \mathcal{S}$, assigning $a(s)$ A-labels and $b(s)$ B-labels, write $\sigma = \sigma(s) = a(s) - b(s)$. The result of smoothing each crossing of D according to its label, following figure 1(b), is sD, a collection of disjoint circles embedded in the plane together with some A- and B-chords (segments joining the circles at the site where there was a crossing). We represent A-chords as dark segments and B-chords as light ones (see figure 2). Enhance the state s with a map e that associates a sign $\varepsilon_i = \pm 1$ with each of the $|s|$ circles in sD. Unless otherwise stated, we will keep the letter s for the enhanced state (s, e) to avoid cumbersome notation. Write

$$\tau = \tau(s) = \sum_{i=1}^{|s|} \varepsilon_i,$$

and define, for the enhanced state s, the integers

$$i = i(s) = \frac{w - \sigma}{2}, \qquad j = j(s) = w + i + \tau.$$

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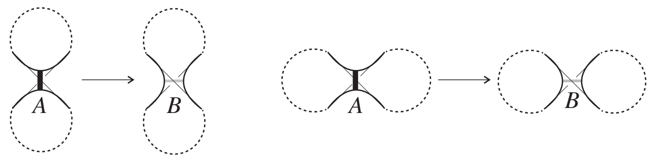

Figure 3. All possible enhancements when melting two circles are $(++ \to +)$, $(+- \to -)$, $(-+ \to -)$. The possibilities for the splitting are $(- \to --)$, $(+ \to +-)$ or $(+ \to -+)$.

Definition 2.1. Let s and t be enhanced states of an oriented link diagram D. We say that t is adjacent to s if the following conditions are satisfied:

(1) $i(t) = i(s) + 1$ and $j(t) = j(s)$;

(2) the labels assigned by t are identical to those assigned by s except at one (change) crossing $x = x(s,t)$, where s assigns an A-label and t a B-label;

(3) the signs assigned to the common circles in sD and tD are equal.

Note that the circles that are not common to sD and tD are those touching the crossing x. In fact, passing from sD to tD can be realized by melting two circles into one, or splitting one circle into two. The different possibilities according to the previous points are shown in figure 3.

Let R be a commutative ring with unit. Let $C^{i,j}(D)$ be the free module over R generated by the set of enhanced states s of D with $i(s) = i$ and $j(s) = j$. Order the crossings in D. Now fix an integer j and consider the ascendant complex

$$\cdots \to C^{i,j}(D) \xrightarrow{d_i} C^{i+1,j}(D) \to \cdots$$

with differential $d_i(s) = \sum (s : t)t$, where $(s : t) = 0$ if t is not adjacent to s and otherwise $(s : t) = (-1)^k$, with k being the number of B-labelled crossings coming after the change crossing x. It turns out that $d_{i+1} \circ d_i = 0$ and the corresponding cohomology modules over R

$$H^{i,j}(D) = \frac{\ker(d_i)}{\mathrm{im}(d_{i-1})}$$

are independent of the diagram D representing the link L and the ordering of its crossings, that is, these modules are link invariants. They are the Khovanov cohomology modules $H^{i,j}(L)$ of L (see [2, 9]), as presented by Viro in [14] in terms of enhanced states.

Let

$$j_{\min} = j_{\min}(D) = \min\{j(s)/s \text{ is an enhanced state of } D\}.$$

We will refer to the complex $\{C^{i,j_{\min}}(D), d_i\}$ as the extreme Khovanov complex, and to the corresponding modules $H^{i,j_{\min}}(D)$ as the (potential) extreme Khovanov modules. Indeed, there are analogous definitions for a certain $j_{\max}$.

We remark that the integers $j_{\min}$ and $j_{\max}$ depend on the diagram D, and may differ for two different diagrams representing the same link.

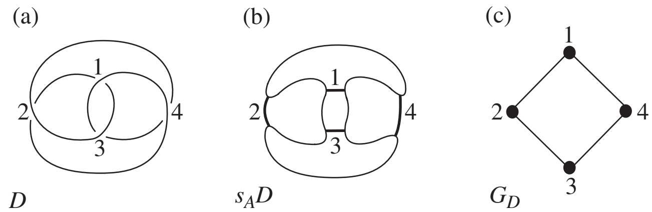

Figure 4. A diagram D representing the trefoil knot, $s_A D$ and the corresponding Lando graph G_D are shown in (a), (b) and (c), respectively.

3. The Lando graph and its cohomology

Let G be a graph. A set σ of vertices of G is said to be independent if no vertices in σ are adjacent. The independence number of G is defined to be

$$I(G) = \sum_{\sigma} (-1)^{|\sigma|},$$

where the sum is taken over all the independent sets of vertices of G. The empty set is considered as an independent set of vertices with $|\emptyset| = 0$. A point has independence number 0, a hexagon 2.

Starting from a link diagram D, we recall the construction of its Lando graph (see [1, 12]). An A- or B-chord of sD is admissible if it connects the same circle of sD to itself.

Definition 3.1. Let D be a link diagram and let s_A be the state assigning A-labels to all the crossings of D. The Lando graph G_D associated with D is constructed from $s_A D$ by considering a vertex for every admissible A-chord, and an edge joining two vertices if the ends of the corresponding A-chords alternate in the same circle.

Figure 4 exhibits a diagram D, the corresponding $s_A D$ and its Lando graph G_D.

Theorem 3.2 (Bae and Morton [1]). *Let L be a link represented by a diagram D. Then the coefficient of the potential lowest degree monomial of its Jones polynomial, $V(L)$ is, up to sign, the independence number $I(G_D)$ of the Lando graph G_D of D.*

Theorem 3.3 (Manchón [12]). *For any integer k there exists a link diagram D such that $I(G_D) = k$. Therefore, there are links with arbitrary extreme coefficients in their Jones polynomials.*

On one hand, the Jones polynomial can be seen as the bigraded Euler characteristic of the Khovanov cohomology. On the other hand, the formula for the independence number suggests that each extreme coefficient of the Jones polynomial is the Euler characteristic of a certain cohomology given in terms of independent sets of vertices of the Lando graph. Both ideas together have led us to a way of understanding the extreme Khovanov cohomology in terms of this graph; in other words, we have obtained a geometric realization of the potential extreme Khovanov cohomology of a link.

Let X_D be the independence simplicial complex of the graph G_D. By definition, the simplices σ of X_D are the independent subsets of vertices of G_D. Recall that the dimension of a simplex σ is the number of its vertices minus 1.

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Definition 3.4. The Lando ascendant complex of a link diagram D is the cochain complex $\{C^i(X_D), \delta_i\}$, where $C^i(X_D)$ is the free module over R generated by the simplices of dimension i, and δ_i is the (standard) differential

$$\cdots \to C^i(X_D) \xrightarrow{\delta_i} C^{i+1}(X_D) \to \cdots$$

given by $\delta_i(\sigma) = \sum_v (-1)^k \sigma \cup \{v\}$, where v runs over the set of vertices of G_D that are not adjacent to any vertex of σ, and $k = k(\sigma, v)$ is the number of vertices of σ coming after v in the predetermined order of the vertices of G_D.

The Lando cohomology modules of D are the reduced cohomology modules

$$H^i(X_D) = \frac{\ker(\delta_i)}{\operatorname{im}(\delta_{i-1})}.$$

It is a straightforward algebraic exercise to check that if R is the field of the rational numbers, then $I(G_D)$ is the Euler characteristic associated with the cohomology of X_D.

4. Lando and extreme Khovanov cohomologies

Let D be an oriented link diagram and let s_A^- be the enhanced state assigning an A-label to each crossing and the sign -1 to each circle of $s_A D$.

Proposition 4.1. *Let $j_{\min} = j_{\min}(D)$. Then $j_{\min} = j(s_A^-)$ and $j(s) = j_{\min}$ if and only if $s \in S_{\min}$, where*

$$S_{\min} = \{\text{enhanced states } s \text{ such that } |s| = |s_A| + b(s),\ \tau(s) = -|s|\}.$$

Proof. Recall that $j(s) = (3w - \sigma)/2 + \tau$ with $\sigma = a(s) - b(s)$ and $\tau(s) = \sum_{i=1}^{|s|} \varepsilon_i$, where ε_i is the sign ($+1$ or -1) associated with the circle c_i in sD.

Given a diagram D, let s be an enhanced state associating a positive sign with at least one of the circles in sD. Then $j(s) \neq j_{\min}$, as the state given by associating negative signs with every circle in sD has a smaller value of j. Hence, all states s realizing $j_{\min}$ assign -1 to their circles, or equivalently $\tau(s) = -|s|$ (the second condition in the definition of $S_{\min}$).

Now identify a state with the set of crossings of D where the state assigns a B-label. Let $s = \{y_1, \dots, y_b\}$ be an enhanced state assigning $b = b(s)$ B-labels (at the crossings $y_1, \dots, y_b$) and negative signs to all of its circles. Consider the sequence of enhanced states

$$s_0 = s_A^-,\ s_1,\ \dots,\ s_b = s,$$

where $s_k = \{y_1, \dots, y_k\}$ for $k = 1, \dots, b$, and all circles in $s_k D$ have sign -1.

Since w is invariant and $\sigma(s_k) = \sigma(s_{k-1}) - 2$, there are two possibilities: if $|s_k| = |s_{k-1}| + 1$, then $\tau(s_k) = \tau(s_{k-1}) - 1$ and $j(s_k) = j(s_{k-1})$; if $|s_k| = |s_{k-1}| - 1$, then $\tau(s_k) = \tau(s_{k-1}) + 1$ and $j(s_k) = j(s_{k-1}) + 2$. Note that this is independent of the ordering of the crossings.

A first consequence is that $j(s_A^-) \leqslant j(s)$, where s was taken to be any state assigning -1 to all circles in sD, so $j(s_A^-) = j_{\min}$. A second consequence is that $j(s) = j_{\min}$ if and only if $|s_k| = |s_{k-1}| + 1$ for each $k \in \{1, \dots, b\}$, that is, if and only if $s \in S_{\min}$. □

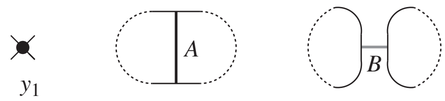

Figure 5. The vertex y_1 corresponds to a splitting from $s_AD = s_0D$ to s_1D.

There are analogous s_B^+, $j_{\max}$ and $S_{\max}$, with $j(s_B^+) = j_{\max}$ and $s \in S_{\max}$ if and only if $j(s) = j_{\max}$.

Corollary 4.2. *Fix an oriented link diagram D with c crossings, n negative and p positive. Then $j_{\min} = c - 3n - |s_A|$ and $j_{\max} = -c + 3p + |s_B|$.*

Proof. Since $w = p - n = c - 2n$ and $\sigma(s) = c - 2b(s)$ we deduce that $i(s) = b(s) - n$. In particular $i(s_A) = -n$. It follows that

$$\begin{aligned} j_{\min} &= j(s_A^-) \\ &= w + i(s_A^-) + \tau(s_A^-) \\ &= (c - 2n) - n - |s_A| \\ &= c - 3n - |s_A|. \end{aligned}$$

A similar argument works for $j_{\max}$ using s_B^+ instead of s_A^-. □

Recall that the vertices in the Lando graph of D, G_D, are associated with the admissible A-chords in s_AD (the ones having both ends in the same circle of s_AD). Let V_s be the set of vertices of G_D corresponding to the crossings of D with which s associates a B-label. Note that V_s can have less than $b(s)$ vertices, or even be empty.

Proposition 4.3. *The map that assigns V_s to each enhanced state s defines a bijection between $S_{\min}$ and the set of independent sets of vertices of G_D. Moreover, if $s \in S_{\min}$, then the cardinality of V_s is exactly $b(s)$.*

Proof. Let $s = \{y_1, \ldots, y_b\}$ be an enhanced state in $S_{\min}$ with $b = b(s)$ B-labels (at the crossings $y_1, \ldots, y_b$). Consider the sequence of enhanced states

$$s_0 = s_A^-, \ s_1, \ \ldots, \ s_b = s,$$

where $s_k = \{y_1, \ldots, y_k\}$ for $k = 1, \ldots, b$, and all circles in s_kD have sign -1. As $s \in S_{\min}$, according to the proof of proposition 4.1, $|s_k| = |s_{k-1}| + 1$ for each $k \in \{1, \ldots, b\}$ or, equivalently, one passes from $s_{k-1}D$ to s_kD by splitting one circle into two circles.

Note that the A-chord of s_AD corresponding to the crossing y_1 of D is admissible, since otherwise $|s_1| = |s_0| - 1$ (see figure 5). As the construction in the previous sequence does not depend on the order of the crossings, it follows that any A-chord of s_AD corresponding to a crossing y_i is admissible in s_AD, so G_D contains its associated vertex.

Moreover, there is no pair of A-chords in s_AD corresponding to B-labels of s with their ends alternating in the same circle, since otherwise two B-smoothings in these two crossings would not increase the number of circles by two, as figure 6 shows schematically. This implies that the corresponding vertices in V_s are independent.

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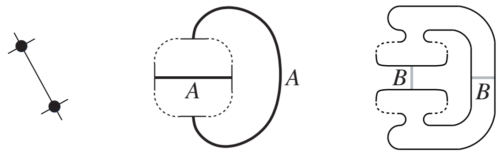

Figure 6. Adjacent vertices in G_D correspond to $|s_A| = |s|$ when smoothing.

Conversely, if C is an independent set of vertices of G_D, let s be the state that assigns B-labels to the crossings corresponding to the vertices of C. In particular, $b(s) = |C|$. Enhance this state by assigning -1 to each circle of sD. Since C is independent, $|s| = |s_A| + b(s)$, and hence $s \in S_{\min}$ as we wanted to show. □

The extreme Khovanov cohomology is constructed, according to proposition 4.1, in terms of the states in $S_{\min}$. For these states the notion of adjacency given in definition 2.1 is reduced to the second condition. Namely, if $s, t \in S_{\min}$, then t is adjacent to s if and only if t assigns the same labels as s except at one crossing x, where $s(x)$ and $t(x)$ are an A-label and a B-label, respectively. Now we are ready to establish the main result of this paper.

Theorem 4.4. *Let L be an oriented link represented by a diagram D having n negative crossings. Let G_D be the Lando graph of D and let $j = j_{\min}(D)$. Then the Lando ascendant complex $\{C^i(X_D), \delta_i\}$ is a copy of the extreme Khovanov complex $\{C^{i,j}(D), d_i\}$, shifted by $n-1$. Hence,*

$$H^{i,j}(D) \approx H^{i-1+n}(X_D).$$

Proof. According to proposition 4.1, the extreme Khovanov cohomology modules are generated by the states in $S_{\min}$. Suppose that $s \in S_{\min}$ and let V_s be the corresponding independent set of vertices of G_D. Since $i(s) = b(s) - n$ and $\dim(V_s) = |V_s| - 1 = b(s) - 1$, the bijection between $S_{\min}$ and the set of independent sets of vertices of G_D established in proposition 4.3 provides an isomorphism

$$C^{i,j}(D) \approx C^{i-1+n}(X_D).$$

Next, we need to show that the isomorphism described above respects both differentials (technically, that the assignment s to V_s defines a chain isomorphism). Recall that for two states $s, t \in S_{\min}$, t is adjacent to s if and only if t assigns the same labels as s except at one crossing x, where $s(x)$ and $t(x)$ are an A-label and a B-label, respectively. Moreover, in this case only a splitting is possible at the change crossing x when passing from sD to tD, since the degree $j_{\min}$ is preserved and the degree i is increased by one, $\tau(s) = -|s|$ and $\tau(t) = -|t|$. It follows that $V_t = V_s \cup \{v_x\}$, where v_x is the vertex in G_D corresponding to x.

In addition, if we order the vertices of G_D according to the order of the crossings in D (hence, the assignment 'vertex to crossing' is an increasing map), we get that the number of B-labelled crossings of D coming after the crossing x is exactly the number of vertices of V_s coming after the vertex v_x. □

Example 4.5. Consider the oriented link L represented by the oriented diagram D shown in figure 7(a). The Lando graph G_D is the hexagon shown in figure 7(d).

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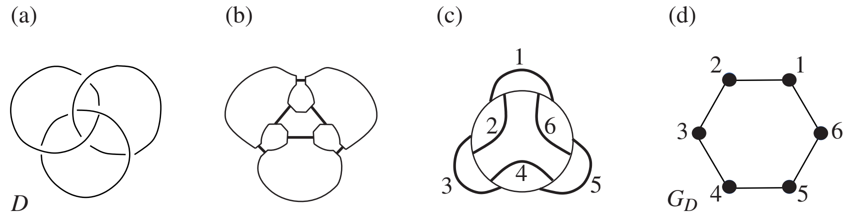

Figure 7. (a) A link diagram D; (b), (c) two versions of s_AD; and (d) its associated Lando graph G_D.

Number its vertices consecutively, from 1 to 6. Then $C^{-1}(X_D)$, $C^0(X_D)$, $C^1(X_D)$ and $C^2(X_D)$ have ranks 1, 6, 9 and 2, respectively, with bases

$$\{\emptyset\}, \qquad \{1,2,3,4,5,6\}, \qquad \{13,14,15,24,25,26,35,36,46\}, \qquad \{135,246\},$$

the other modules being trivial (note that we write, for example, 135 instead of $\{1,3,5\}$). The Lando ascendant complex is

$$0 \to C^{-1}(X_D) \xrightarrow{\delta_{-1}} C^0(X_D) \xrightarrow{\delta_0} C^1(X_D) \xrightarrow{\delta_1} C^2(X_D) \to 0,$$

with differentials δ_{-1}, δ_0 and δ_1 given by the matrices

$$\begin{pmatrix}1\\1\\1\\1\\1\\1\end{pmatrix}, \quad \begin{pmatrix}1&0&-1&0&0&0\\1&0&0&-1&0&0\\1&0&0&0&-1&0\\0&1&0&-1&0&0\\0&1&0&0&-1&0\\0&1&0&0&0&-1\\0&0&1&0&-1&0\\0&0&1&0&0&-1\\0&0&0&1&0&-1\end{pmatrix}, \quad \begin{pmatrix}1&0&-1&0&0&0&1&0&0\\0&0&0&1&0&-1&0&0&1\end{pmatrix}.$$

Let R be the field of rational numbers. The ranks of these matrices are 1, 5 and 2, respectively. Hence, $H^1(X_D)$ has dimension 2 as a rational vector space, the rest of the Lando cohomology vector spaces being trivial.

Next, orient the three components of D in a counterclockwise sense. By corollary 4.2, $j_{\min} = c-3n-|s_A| = 6-3\cdot6-1 = -13$ and, by theorem 4.4, the complexes are shifted by $n-1=5$; hence, $H^{-4,-13}(L) \approx H^1(X_D)$ is two-dimensional and the other extreme Khovanov cohomology vector spaces are trivial.

This example illustrates that, in general, for different orientations of the components of a link, we obtain the same extreme Khovanov cohomology modules ($j_{\min}$ may change), with some shifting in the index i.

A complete bipartite graph $K_{r,s}$ is a graph whose vertices can be divided into two disjoint sets V and W, called parts, having r and s vertices, respectively, such that every edge connects a vertex in V to one in W, and every pair of vertices $v \in V$ and $w \in W$ are connected by an edge.

Corollary 4.6. *Let L be an oriented link represented by a diagram D having n negative crossings. Let G_D be the Lando graph of D and let $j = j_{\min}(D)$. If G_D is the complete bipartite graph $K_{r,s}$, then $H^{1-n,j}(L) \approx R$; otherwise, $H^{1-n,j}(L)$ is trivial.*

Proof. According to theorem 4.4, we just have to prove that $H^0(X_D) \approx R$ if G_D is $K_{r,s}$, and it is trivial otherwise. Let G_D^{c} be the complementary graph of G_D. A remarkable observation is that any Lando graph is always a bipartite graph, and $G_D = K_{r,s}$ if and only if G_D^{c} has exactly two connected components; otherwise, G_D^{c} is connected. The key observation is now that the connected components of G_D^{c} coincide exactly with the elements of a basis of $\ker(\delta_0)$. The fact that $\delta_{-1}(\emptyset) = 1 + 2 + \cdots + c \in C^0(X_D)$ completes the argument. □

5. Lando cohomology as homology of a simplicial complex

In this section we show how to construct a simplicial complex whose homology is equal to the cohomology of the Lando ascendant complex of a link diagram D up to a homological shift. This fact, together with theorem 4.4, implies that the homology of the simplicial complex determines the extreme Khovanov cohomology of the link represented by D.

A key point is the following result by Jonsson [7, theorem 3.1].

Theorem 5.1. *Let G be a bipartite graph with non-empty parts V and W. Then there exists a simplicial complex $X_{G,V}$ whose suspension is homotopy equivalent to the independence complex of G.*

In [7] Jonsson also gave the procedure for constructing the complex $X_{G,V}$. Starting with the bipartite graph G, a set $\sigma \subseteq V$ belongs to $X_{G,V}$ if and only if there is a vertex $w \in W$ such that $\sigma \cup \{w\}$ is an independent set in G. In other words, $\sigma \subseteq V$ is a face of $X_{G,V}$ if and only if σ is not adjacent to every $w \in W$.

Recall that the Alexander dual of a simplicial complex X with ground set V is a simplicial complex X^* whose faces are the complements of the non-faces of X. The combinatorial Alexander duality (see, for example, [4,13]) relates the homology and cohomology of a given simplicial complex and its Alexander dual.

Theorem 5.2. *Let X be a simplicial complex with a ground set of size n. Then for any $i \in \mathbb{Z}$ the reduced homology of X in degree i is equal to the reduced cohomology of the dual complex X^* in degree $n - i - 3$.*

As Lando graphs are bipartite, theorems 5.1 and 5.2, together with the fact that a simplicial complex X and its suspension $\mathrm{S}(X)$ have the same homology and cohomology up to a homological shift by 1, provide an algorithm for computing the cohomology of the independence simplicial complex associated with a Lando graph G_D (or, equivalently, the extreme Khovanov cohomology of the link represented by D) from the homology of a specific simplicial complex.

Theorem 5.3. *Let D be a diagram of an oriented link L with n negative crossings. Let $j = j_{\min}(D)$. Let $Y_D = (X_{G,V})^*$, where $G = G_D$ is the Lando graph of D, with parts V and W. Then*

$$H^{i,j}(L) \approx \tilde{H}_{|V|-i-1-n}(Y_D).$$

Proof. Let $Z = X_{G,V}$; hence $Y_D = Z^*$. Then

$$H^{i+1-n,j}(L) \approx \tilde{H}^i(X_D) \approx \tilde{H}^i(\mathrm{S}(Z)) \approx \tilde{H}^{i-1}(Z) \approx \tilde{H}_{|V|-i-2}(Y_D),$$

where we have applied theorem 4.4 (recall that $X_D = X_G$ is the independence complex of the Lando graph $G = G_D$), the homotopy equivalence $X_G \approx S(X_{G,V})$ given by theorem 5.1, the relation between the cohomology of a simplicial complex and its suspension, and finally the combinatorial Alexander duality theorem. □

Remark 5.4. One can also describe the complex Y_D in terms of $s_A D$, avoiding any reference to the Lando graph G_D. Start by colouring the regions of $s_A D$ in a chequerboard fashion. Call an A-chord white (black) if it is in a white (black) region. The ground set of Y_D is the set of admissible white arcs of $s_A D$, and a set of admissible white arcs σ is a simplex of Y_D if and only if for any admissible black arc there is at least an admissible white arc that is not in σ whose ends alternate with the ends of the black arc in the same circle of $s_A D$. Note that there are two different choices when colouring the regions; in order to get the simplest ground set of Y_D, choose colours in such a way that white regions contain a lower number of admissible A-chords than black regions.

We are now interested in reversing the process above, namely, starting with any simplicial complex X, we will construct a bipartite graph with an associated independence simplicial complex whose cohomology is equal to the homology of X shifted by some degree. Under certain conditions that will be clarified later, this allows us to construct a link diagram whose extreme Khovanov cohomology coincides with the homology of X. Again, the key point is the Alexander duality theorem together with the following result by Jonsson [7, theorem 3.2].

Theorem 5.5. *Let X be a simplicial complex. Then there is a bipartite graph G whose independence complex is homotopically equivalent to the suspension of X.*

The bipartite graph G can be constructed by taking as the set of vertices the disjoint union of the ground set V of the complex X, and the set M of maximal faces of X. The edges of G are all pairs $\{v, \mu\}$ such that $v \in V$, $\mu \in M$ and $v \notin \mu$.

We remark that, although theorem 5.5 holds for any simplicial complex, the graph obtained by the above procedure is not necessarily the Lando graph associated with a link diagram.

Definition 5.6. A graph G is said to be realizable if there is a link diagram D such that $G = G_D$ (in [12] these graphs were originally called convertible).

We are now ready to construct a link with exactly two non-trivial extreme Khovanov cohomology modules. In the following section this example will be an essential ingredient in the construction of families of H-thick knots.

Theorem 5.7. *There exist oriented link diagrams whose extreme Khovanov cohomology modules are non-trivial for two different values of i, that is, $H^{i,j_{\min}}(D)$ is non-trivial for two different values of i.*

Proof. Although our argument is equally valid for any commutative ring R with unit, just for convenience set $R = \mathbb{Z}$, the ring of integers.

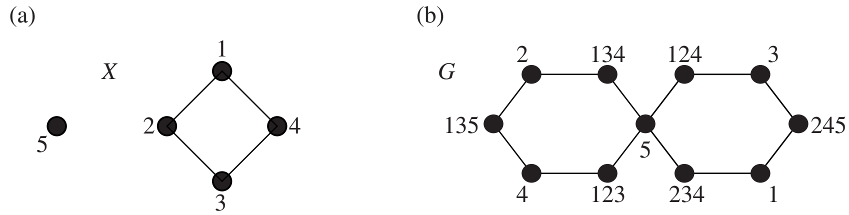

Figure 8. The topological realization of the simplicial complex X and the graph G are shown in (a) and (b), respectively. The independence complex of G is homotopy equivalent to the suspension of X^*.

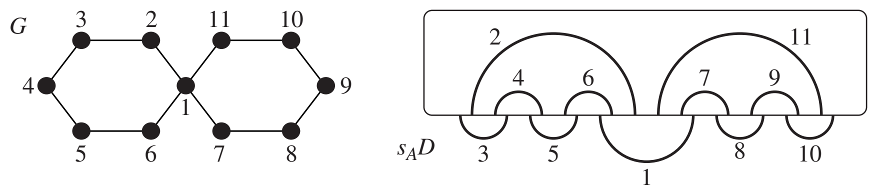

Figure 9. The graph G is realizable; the correspondence between its vertices and the A-chords in $s_A D$ is shown.

Let $X = \{\emptyset, 1, 2, 3, 4, 5, 12, 23, 34, 41\}$ be a simplicial complex with ground set $V = \{1, 2, 3, 4, 5\}$. Its topological realization is the disjoint union of a point and a square, as shown in figure 8(a), and hence its reduced homology is $\tilde{H}_0(X) \approx \tilde{H}_1(X) \approx \mathbb{Z}$ (the other homology groups being trivial).

Consider now the Alexander dual of X,

$$X^* = \{\emptyset, 1, 2, 3, 4, 5, 12, 13, 14, 15, 23, 24, 25, 34, 35, 45, 123, 124, 134, 135, 234, 245\}.$$

Note that $|X| + |X^*| = 32 = 2^{|V|}$. Applying the combinatorial Alexander duality leads to

$$\tilde{H}_i(X) \approx \tilde{H}^{|V|-i-3}(X^*) = \tilde{H}^{2-i}(X^*),$$

which implies that $\tilde{H}^2(X^*) \approx \tilde{H}^1(X^*) \approx \mathbb{Z}$, the other groups being trivial.

Applying theorem 5.5 (and the construction described right after) to the simplicial complex X^* leads to a graph G consisting in two hexagons sharing a common vertex, as shown in figure 8(b), whose independence complex X_G is homotopically equivalent to the suspension of X^*. In particular,

$$\tilde{H}^{i-1}(X^*) \approx \tilde{H}^i(S(X^*)) \approx \tilde{H}^i(X_G).$$

Hence, $\tilde{H}^3(X_G) \approx \tilde{H}^2(X_G) \approx \mathbb{Z}$ are the only non-trivial groups in the reduced cohomology of X_G. In fact, as the indices are different from zero, this is still true for the (non-reduced) cohomology, so $H^2(X_G) \approx H^3(X_G) \approx \mathbb{Z}$.

An important point now is the fact that the graph G is realizable. In fact, $G = G_D$ with D being the link diagram in figure 10. Indeed, figure 9 shows the correspondence between the vertices of G and the A-chords in $s_A D$.

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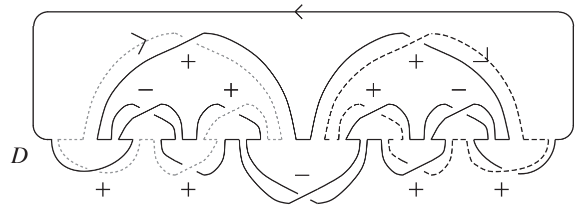

Figure 10. The oriented diagram D representing the link L.

Consider now the link L represented by the diagram D oriented as shown in figure 10. Then applying corollary 4.2, we get that $j_{\min} = c - 3n - |s_A| = 11 - 3 \cdot 3 - 1 = 1$, and by theorem 4.4 one gets $H^{i,1}(L) \approx \mathbb{Z}$ for $i = 0, 1$, the other cohomology groups being trivial. This concludes the proof. □

For the example in the previous proof we have checked with computer assistance that the ranks of the chain groups $C^i = C^i(X_D)$ and differentials δ_i for the Lando ascendant complex

$$0 \to C^{-1} \xrightarrow{\delta_{-1}} C^0 \xrightarrow{\delta_0} C^1 \xrightarrow{\delta_1} C^2 \xrightarrow{\delta_2} C^3 \xrightarrow{\delta_3} C^4 \xrightarrow{\delta_4} C^5 \to 0$$

are

$$\begin{array}{llll} \mathrm{rk}(C^{-1}) = 1, & \mathrm{rk}(\delta_{-1}) = 1, & \mathrm{rk}(C^0) = 11, & \mathrm{rk}(\delta_0) = 10, \\ \mathrm{rk}(C^1) = 43, & \mathrm{rk}(\delta_1) = 33, & \mathrm{rk}(C^2) = 73, & \mathrm{rk}(\delta_2) = 39, \\ \mathrm{rk}(C^3) = 52, & \mathrm{rk}(\delta_3) = 12, & \mathrm{rk}(C^4) = 13, & \mathrm{rk}(\delta_4) = 1, \\ \mathrm{rk}(C^5) = 1. & & & \end{array}$$

Hence,

$$\mathrm{rk}(H^2(X_D)) = 73 - 39 - 33 = 1 \quad \text{and} \quad \mathrm{rk}(H^3(X_D)) = 52 - 12 - 39 = 1.$$

REMARK 5.8. The proof of theorem 5.7 does not work if, for example, we start with the simplicial complex whose topological realization is a point plus a triangle. Although one again gets a graph G such that X_G has two non-trivial cohomology groups, G consists of two hexagons with four common consecutive edges (a total of eight vertices), which is no longer a realizable graph.

6. Families of *H*-thick knots

Citing Khovanov [10], there are 249 prime unoriented knots with at most 10 crossings (not counting mirror images). It is known that all but 12 of these knots are H-thin, that is, their Khovanov cohomology is supported on two adjacent diagonals, in a matrix in which rows are indexed by j and columns by i. An H-thick knot is a knot that is not H-thin. For example, any non-split alternating link is H-thin, and any adequate non-alternating knot is H-thick (see [10, theorem 2.1 and proposition 5.1]).

Up to eleven crossings, there are no knots with more than one non-trivial cohomology group in the rows corresponding to the potential extreme $j_{\max}$ or $j_{\min}$ ob-

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tained from the associated diagrams in [3]. There are examples that seem to contradict this statement. For example, knot 10_{132}, whose Khovanov cohomology groups are trivial for $j > -1$ and which has two non-trivial groups for $j = -1$, but for the diagram of 10_{132} taken from [3], we have $j_{\max}(D) = -c + 3p + |s_B| = -10 + 3 \cdot 3 + 2 = 1$. We do not know if there exists a diagram D representing knot 10_{132} with $j_{\max}(D) = -1$. Related to this fact we pose the following question.

QUESTION. Does any oriented link L have a diagram D whose associated $j_{\min}(D)$ equals the minimum value of j such that $H^{i,j}(L)$ is non-trivial for at least one value of i?

In this section we show examples of H-thick knots having any arbitrary number of non-trivial cohomology groups in the non-trivial row of smallest possible index. More precisely, we will provide a diagram D whose row indexed by $j_{\min}(D)$ is non-trivial, and hence corresponds to the non-trivial row of smallest possible index. Moreover, this row has as many non-trivial cohomology groups as desired. The basis of our construction is the link given in the proof of theorem 5.7.

We remark that theorem 4.4 allows us to compute the extreme Khovanov cohomology of any link diagram D by considering independently each of the circles appearing in $s_A D$, since the non-admissible A-chords do not take part in the construction of the simplicial complex Y_D described in § 5. More precisely, let D be a link diagram and let $c_1, \dots, c_n$ be the circles of $s_A D$. Write C_i for the circle c_i together with the admissible A-chords having both ends in the circle c_i, and let D_i be the diagram reconstructed from C_i by reversing the corresponding smoothings. Then, from the construction in remark 5.4 it follows that $Y_D = Y_{D_1} * \cdots * Y_{D_n}$, with $*$ being the join of simplicial complexes. Recall that the join $X * Y$ of two simplicial complexes X and Y is defined as the simplicial complex whose simplices are the disjoint unions of simplices of X and Y.

The reduced homology of the join of two simplicial complexes can be computed directly from the reduced homology of each of the complexes, namely,

$$\tilde{H}_i(X * Y) = \sum_{r+s=i-1} \tilde{H}_r(X) \otimes \tilde{H}_s(Y) \oplus \sum_{r+s=i-2} \operatorname{Tor}(\tilde{H}_r(X), \tilde{H}_s(Y)).$$

Taking copies of the example in the proof of theorem 5.7, one obtains a link that, by theorem 5.3 and the above formula, has any number of non-trivial extreme Khovanov cohomology groups. Note that one could also use the general formula for the Khovanov cohomology of a split link (see [9, corollary 12]). At this point, our understanding of extreme Khovanov cohomology in terms of Lando cohomology allows us to slightly modify a link in such a way that one obtains a knot with the same extreme Khovanov cohomology. We explain this construction in detail in theorem 6.2 and remark 6.3. We first need the following result.

PROPOSITION 6.1. *Let* $*_n X$ *be the join of* n *copies of the simplicial complex* $X = \{\emptyset, 1, 2, 3, 4, 5, 12, 23, 34, 41\}$. *Then* $\tilde{H}_i(*_n X) \approx \mathbb{Z}^{\binom{n}{i-n+1}}$ *if* $n - 1 \leqslant i \leqslant 2n - 1$, *and it is trivial otherwise.*

Proof. We proceed by way of induction on n. In the proof of theorem 5.7 we saw that $\tilde{H}_0(X) \approx \tilde{H}_1(X) \approx \mathbb{Z}$, which is the $n = 1$ case. For $n > 1$ we apply the formula

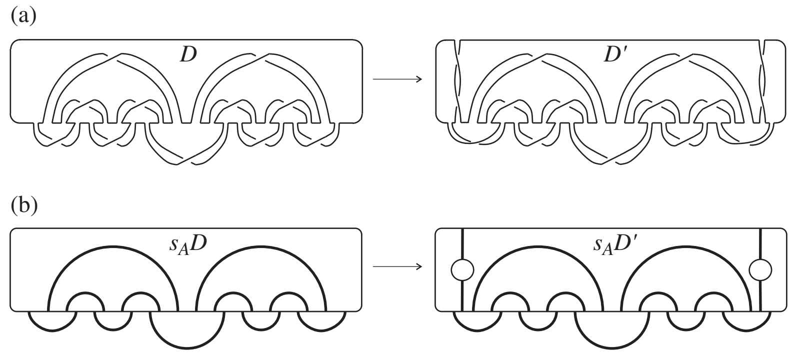

Figure 11. D and D' are shown in (a). The corresponding s_AD and s_AD' are shown in (b). Note that D' has one component.

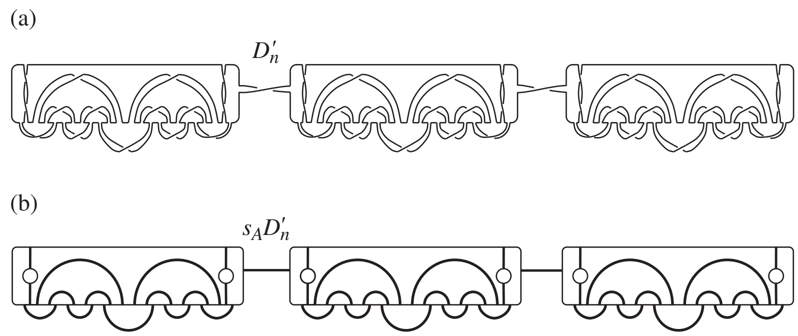

Figure 12. D'_n and $s_AD'_n$ for the $n = 3$ case are shown in (a) and (b), respectively.

for the homology of a join (torsion terms do not appear in any case):

$$\begin{aligned}\tilde{H}_i(*_nX) &\approx \bigoplus_{r+s=i-1} [\tilde{H}_r(*_{n-1}X) \otimes \tilde{H}_s(X)] \\ &\approx \tilde{H}_{i-1}(*_{n-1}X) \oplus \tilde{H}_{i-2}(*_{n-1}X) \\ &\approx \mathbb{Z}^{\binom{n-1}{(i-1)-(n-1)+1}} \oplus \mathbb{Z}^{\binom{n-1}{(i-2)-(n-1)+1}} \\ &\approx \mathbb{Z}^{\binom{n-1}{i-n+1}} \oplus \mathbb{Z}^{\binom{n-1}{i-n}} \\ &\approx \mathbb{Z}^{\binom{n}{i-n+1}}.\end{aligned}$$

□

Theorem 6.2. *For every $n > 0$ there exists an oriented knot diagram D with exactly $n + 1$ non-trivial extreme Khovanov integer cohomology groups $H^{i,j_{\min}}(D)$.*

Proof. Let L be the oriented link represented by the diagram D in figure 10. Considering as ground set the chords in the unbounded region of s_AD, the associated simplicial complex Y_D is the simplicial complex X appearing in the proof of theorem 5.7, whose topological realization is the disjoint union of a point and a square (figure 8(a)). Hence it has two non-trivial reduced homology groups, $\tilde{H}_0(Y_D) \approx \tilde{H}_1(Y_D) \approx \mathbb{Z}$.

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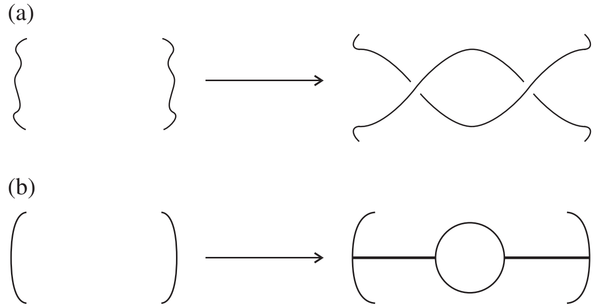

Figure 13. In part (a) two crossings are added in such a way that two different link components are merged. The effect of this transformation at the level of $s_A D$ is shown in part (b).

Now consider the link L_n consisting of the split union of n copies of L. It can be represented by D_n, the disjoint union of n copies of D, so $s_A D_n$ is the disjoint union of n copies of $s_A D$, shown in figure 9(b). Hence its associated simplicial complex is $Y_{D_n} = *_n Y_D = *_n X$, that is, the join of n copies of X. Applying proposition 6.1 to $*_n X$ one gets

$$\tilde{H}_i(Y_{D_n}) \approx \mathbb{Z}^{\binom{n}{i-n+1}}$$

for $n-1 \leqslant i \leqslant 2n-1$.

This fact together with theorem 5.3 shows that the extreme Khovanov cohomology of L_n has $n+1$ non-trivial groups.

Now we will give a knot having the same extreme Khovanov cohomology groups as L_n (the value of $j_{\min}$ changes in general). Starting from the diagram D in figure 10, add four crossings, as shown in figure 11(a), in such a way that the resulting diagram D' has one component. Note that $s_A D'$ is obtained from $s_A D$ by adding two circles with four A-chords (figure 11(b)). Consider now n copies of D' and join them as shown in figure 12(a). The resulting diagram D'_n is a knot diagram. Since $s_A D'_n$ just adds $5n-1$ non-admissible A-chords to $s_A D_n$, both diagrams D_n and D'_n share the same Lando graph. Hence D'_n represents a knot having $n+1$ non-trivial groups in its extreme Khovanov cohomology. □

The proof of theorem 6.2 provides a family of knots having as many non-trivial extreme Khovanov cohomology groups as desired. These are examples of H-thick knots with arbitrarily large thickness.

Remark 6.3. Every link L with μ components can be turned into a knot preserving its extreme Khovanov cohomology ($j_{\min}$ can change). One just needs to consider a diagram D of L and add two extra crossings merging two different components into one, as shown in figure 13(a). Since $s_A D'$ just adds two non-admissible A-chords to $s_A D$ (see figure 13(b)), both diagrams share the same Lando graph. After repeating this procedure $\mu - 1$ times, the link L is transformed into a knot.

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