

RIGIDIFIED TORSOR COCYCLES, HYPERCOVERINGS AND BUNDLE GERBES

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Abstract

We give a geometric interpretation of sheaf cohomology for higher degrees $n \geq 1$ in terms of torsors on the member of degree $d = n - 1$ in hypercoverings of type $r = n - 2$, endowed with an additional datum, the so-called rigidification. This generalizes the fact that cohomology in degree one is the group of isomorphism classes of torsors, where the rigidification becomes vacuous, and that cohomology in degree two can be expressed in terms of bundle gerbes, where the rigidification becomes an associativity constraint.

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1. Introduction

Let X be a topological space and \mathcal{F} be an abelian sheaf. It is a classical fact that the first sheaf cohomology $H^1(X, \mathcal{F})$, which is defined via injective resolutions, has a *combinatorial description* in terms of Čech cohomology $\check{H}^1(X, \mathcal{F})$, and also a *geometric interpretation* as the group $\pi_0(\mathcal{F}\text{-Tors})$ of isomorphism classes for the Picard category of *torsors*. This was discussed by Grothendieck in [20, Section 5.1] and treated in utmost generality in Giraud’s monograph [17, Ch. III]. In degree $n = 2$, one may interpret cohomology via *gerbes*, which are certain fibered categories.

It is natural to ask whether higher cohomology groups $H^n(X, \mathcal{F})$ also admit a geometric interpretation. The goal of this paper is to show that this indeed holds in all degrees $n \geq 1$. The idea is to use torsors \mathcal{T} living on pieces U_{n-1} of hypercoverings U_\bullet of type $r = n - 2$, endowed with some additional datum called *rigidification*. See Definition 6.1 for details. This generalizes Murray’s notion of *bundle gerbes* [29], which describe the situation in degree $n = 2$ and are used in the context of differential geometry and theoretical physics, mainly for the case $\mathcal{F} = \mathbb{C}^\times$. See [30] for an introduction, [31] and [32] for further development, and also the work of Caray, Murray and Wang on *higher bundle gerbes* [12].

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Our starting point is Lazard’s observation that higher cohomology groups already have a combinatorial description in terms of *hyper-Čech cohomology*

$$H^n(X, \mathcal{F}) = \check{H}^n(X, \mathcal{F}) = \varinjlim H^n\Gamma(U_\bullet, \mathcal{F}),$$

where the direct limit runs over all hypercoverings U_\bullet of type $r = n - 1$. Roughly speaking, *hypercoverings* are certain semi-simplicial coverings U_\bullet , which are more general than Čech coverings, in the sense that one allows in finitely many degrees the passage to open coverings. The *type of a hypercovering* indicates that from a certain degree $r \geq 0$, one does not pass to open coverings anymore, such that $U_\bullet = \text{cosk}_r(U_{\leq r})$ becomes a coskeleton. This beautiful idea was developed in utmost generality in [3, Exposé V, Section 7].

In contrast to common Čech coverings, where $U_n = U^{n+1} = U \times_X \cdots \times_X U$ are fiber products coming from a single open covering $U = \bigcup_{\lambda \in L} U_\lambda$, the hypercoverings never gained widespread attention. Of course, this relies on the fact that for paracompact spaces, sheaf cohomology coincides with Čech cohomology [19, Ch. II, Theorem 5.10.1]. Such a result also holds for étale cohomology on quasicompact schemes admitting an ample sheaf by Artin’s result [1], and for Nisnevich cohomology on quasicompact separated schemes by my findings in [33, 35]. In the realm of homotopy theory, Čech coverings also suffice, because the canonical map $|U_\bullet| \rightarrow X$ from the geometric realization is a weak equivalence. This is due to Segal [36] for countable coverings, and also holds in general as observed by Dugger and Isaksen [14].

However, hypercoverings play a crucial role for the étale homotopy type, as introduced by Artin and Mazur [5] and further studied in Friedlander’s monograph [16]. More generally, they appear in homotopical algebra, that is, abstract homotopy theory in the context of Quillen’s model categories, as explained in Jardine’s book [24]. Hypercoverings of type $r = 1$ are essential to understand gerbes, as exposed by Giraud [17], and the subsequent differential geometry of gerbes, which was developed by Brylinski [11], Hitchin [22] and Breen and Messing [10]; see also the survey [9]. Despite all these advances, hypercoverings should become more popular, in my opinion. Recently, I have used them to reduce Serre’s vanishing for general affine schemes to the noetherian situation [34].

To obtain a geometric description for higher cohomology groups $H^n(X, \mathcal{F})$, we introduce the notion of *rigidified torsor cocycles* $(U_\bullet, \mathcal{T}, \varphi)$ for the abelian sheaf \mathcal{F} in degree $n \geq 1$. Here U_\bullet is a hypercovering of type $r = n - 2$, and \mathcal{T} is a torsor for the abelian sheaf $\mathcal{F}|_{U_{n-1}}$, and the *rigidification* φ is a section over U_n of the *alternating preimage*

$$p_{\text{alt}}^*(\mathcal{T}) = p_0^*(\mathcal{T}) \wedge p_1^*(\mathcal{T}^{-1}) \wedge \cdots \wedge p_n^*(\mathcal{T}^{(-1)^n}),$$

subject to the cocycle condition $q_{\text{alt}}^*(\varphi) = 0$. See Definition 6.1 for details. The $p_i : U_n \rightarrow U_{n-1}$ and $q_j : U_{n+1} \rightarrow U_n$ denote face operators in the hypercovering. The rigidified torsor cocycles form a *fibred Picard category* $\mathfrak{R}_{\mathcal{F}}^n$, a notion going back to MacLane [26] and studied in depth by Deligne [4, Exposé XVIII, Section 4.1]. The category $\mathfrak{R}_{\mathcal{F}}^n$ contains ‘coboundary objects’ coming from alternating preimages

of $\mathcal{F}|U_{n-2}$ -torsors. Forming the resulting residue class groups fiberwise and passing to a filtered direct limit along homotopy classes of refinements, we obtain the abelian group $\text{RTC}^n(\mathcal{F})$ of *equivalence classes of rigidified torsor cocycles*. Our main result is that this indeed gives a geometric description for sheaf cohomology.

THEOREM 1.1 (See **Theorem 7.2**). *For each integer $n \geq 1$, there is a natural identification of abelian groups $\text{RTC}^n(\mathcal{F}) = H^n(X, \mathcal{F})$.*

To define the *comparison map* $\text{RTC}^n(\mathcal{F}) \rightarrow H^n(X, \mathcal{F})$, we analyze the spectral sequences attached to the double complex $\Gamma(U_\bullet, \mathcal{I}^\bullet)$, arising from the injective resolution \mathcal{I}^\bullet used to define sheaf cohomology and the hypercovering U_\bullet . This leads to the so-called *three-term complex* $C^{n-1} \xrightarrow{\Psi} C^n \xrightarrow{\Phi} C^{n+1}$, which captures a small but essential part of the double complex. The comparison map takes the form

$$\text{RTC}^n(\mathcal{F}) \mapsto \varinjlim \text{Ker}(\Phi) / \text{Im}(\Psi) = H^n(X, \mathcal{F}).$$

In degree $n = 1$, everything boils down to the classical identification

$$H^1(X, \mathcal{F}) = \check{H}^1(X, \mathcal{F}) = \pi_0(\mathcal{F}\text{-Tors});$$

the hypercovering is constant, and the rigidification is vacuous. In degree $n = 2$, we recover Murray's notion of *bundle gerbes*; the hypercovering reduces to a Čech covering, and the rigidification translates into an associativity constraint. In some sense, our construction gives a higher-dimensional generalization of bundle gerbes. The approach works for arbitrary sites \mathcal{C} satisfying mild technical assumptions, introduced mainly for the sake of exposition. In this setting, $X \in \mathcal{C}$ denotes a final object. One recurrent technical challenge is to understand the effect of simplicial homotopies between refinements of hypercoverings.

After the completion of the paper, I learned from M. Murray, D. Stevenson and D. Roberts that similar results appear in the work of Duskin [15], Glenn [18], Beke [6] and Jardine [23], albeit in different forms and in much more categorical settings. One advantage of the present approach is that rigidified torsor cocycles yield a concrete way to represent cohomology classes, and that we have an explicit comparison map to sheaf cohomology. This could make the theory more accessible in fields outside category theory.

The paper is organized as follows: In **Section 2** we discuss the notion of fibered Picard categories $\mathfrak{R} \rightarrow \mathcal{C}$ and certain resulting direct limits. **Section 3** contains basic facts on hypercoverings U_\bullet . We use them to express sheaf cohomology with the cohomology of the three-term complex $C^{n-1} \xrightarrow{\Psi} C^n \xrightarrow{\Phi} C^{n+1}$ in **Section 4**, by using spectral sequences. In **Section 5**, we introduce the notion of iterated alternating preimages $p_{\text{alt}}^*(\mathcal{F})$. **Section 6** contains the central definition of rigidified torsor cocycles $(U_\bullet, \mathcal{F}, \varphi)$. The main result appears in **Section 7**, where we define the comparison map $\text{RTC}^n(\mathcal{F}) \rightarrow H^n(X, \mathcal{F})$ and establish its bijectivity. In the closing **Section 8** we discuss how Murray's bundle gerbes can be seen as rigidified torsor cocycles.

2. Fibered Picard categories

In this section, we recall some notions from category theory that I found rather useful to phrase results in later sections, and discuss certain direct limits attached to fibered Picard categories.

Let \mathfrak{R} be a *symmetric monoidal category*. The monoidal structure is given by a functor $\mathfrak{R} \times \mathfrak{R} \rightarrow \mathfrak{R}$, which we usually write as $(A, B) \mapsto A \wedge B$. Moreover, we have natural isomorphisms

$$(A \wedge B) \wedge C \longrightarrow A \wedge (B \wedge C) \quad \text{and} \quad A \wedge B \xrightarrow{\sim} B \wedge A$$

satisfying MacLane’s axioms (see [26] and [4, Exposé XVIII, Section 4.1]). These natural isomorphisms are the *associativity and commutativity constraints*. If there is an object $E \in \mathfrak{R}$ together with a natural isomorphism $A \wedge E \xrightarrow{\sim} A$, then the set $\pi_0(\mathfrak{R})$ of isomorphism classes $[A]$, where $A \in \mathfrak{R}$ runs over all objects, acquires the structure of an abelian monoid, with addition $[A] + [B] = [A \wedge B]$ and zero element $0 = [E]$.

A symmetric monoidal category \mathfrak{R} is called *strict* if the commutativity constraint for $A = B$ becomes the identity transformation $A \wedge A \rightarrow A \wedge A$. Note that this does not hold for the category $\mathfrak{R} = (R\text{-Mod})$ of modules over a ring, where the monoidal structure is given by tensor products $M \otimes_R N$ and the commutativity constraint is $a \otimes b \mapsto b \otimes a$. But strictness actually holds on the full subcategory $\mathfrak{R}' = (R\text{-Inv})$ of invertible modules.

A *groupoid* is a category in which all morphisms are isomorphisms. A *Picard category* is a strictly symmetric monoidal groupoid \mathfrak{R} where the translation functors $X \mapsto A \wedge X$ are self-equivalences of categories, for all objects $A \in \mathfrak{R}$. See [7] for a nice discussion. The following is an important example. Let X be a topological space, \mathcal{F} be an abelian sheaf, and $\mathfrak{R} = (\mathcal{F}\text{-Tors})$ be the Picard category of \mathcal{F} -torsors \mathcal{T} . The monoidal structure is given by $\mathcal{T}' \wedge \mathcal{T}'' = \mathcal{F} \backslash (\mathcal{T}' \times \mathcal{T}'')$, which is the quotient of the product sheaf by the diagonal action.

For this category $\mathfrak{R} = (\mathcal{F}\text{-Tors})$, we have a slight notational problem. It is customary to write group actions and torsor structures in a multiplicative way. On the other hand, the group law for abelian groups and abelian sheaves is preferably written additively, in particular when it comes to Čech cohomology. So the axiom for a multiplicative action $\mathcal{F} \times \mathcal{T} \rightarrow \mathcal{T}$, $(f, \varphi) \mapsto f \cdot \varphi$ of an additive group takes the form $(f + g) \cdot \varphi = f \cdot (g \cdot \varphi)$ and $0 \cdot s = s$. For $\mathcal{T} = \mathcal{F}$, we usually revert back to purely additive notation.

Next, we recall the notion of fibered categories. For details, see [21, Exposé VI]. Given a functor $F : \mathfrak{R} \rightarrow \mathfrak{C}$ between arbitrary categories and an object $U \in \mathfrak{C}$, we write $\mathfrak{R}(U)$ for the *fiber category*, consisting of objects and morphisms in \mathfrak{R} sent to the object U and the morphism id_U , respectively. A morphism $f : A \rightarrow B$ in \mathfrak{R} over $\theta : U \rightarrow V$ in \mathfrak{C} is called *cartesian* if the map

$$\text{Hom}_U(A', A) \longrightarrow \text{Hom}_\theta(A', B), \quad h \mapsto f \circ h$$

is bijective, for all $A' \in \mathfrak{R}(U)$. One says that \mathfrak{R} is an *fibered category* if for each morphism $\theta : U \rightarrow V$ in \mathfrak{C} and each object $B \in \mathfrak{R}(V)$ there is a *cartesian morphism*

$f : A \rightarrow B$ in \mathfrak{R} inducing θ , and the composition of cartesian morphisms is cartesian. Furthermore, if the fiber categories are groupoids, one says that \mathfrak{R} is a *category fibered in groupoids*. Then all morphism in \mathfrak{R} are cartesian. The choice of a cartesian morphism $f : A \rightarrow B$, for each morphism $\theta : U \rightarrow V$ and each object $B \in \mathfrak{R}(V)$, is called a *cleavage*. One then regards A as a ‘fiber product’ $B \times_U V$ or ‘base-change’ $\theta^*(B)$ of the object B with respect to the morphism $\theta : U \rightarrow V$.

Furthermore, given a functor $\mathfrak{R} \times_{\mathcal{C}} \mathfrak{R} \rightarrow \mathfrak{R}$, together with associativity and commutativity constraints that turn all fibers $\mathfrak{R}(U)$ into Picard categories, we say that \mathfrak{R} is a *fibered Picard category*. This notion was introduced by Deligne in [4, Exposé XVIII, Section 1.4] where \mathcal{C} is furthermore endowed with a Grothendieck topology and \mathfrak{R} is regarded as a *Picard stack*. In any case, we get a contravariant functor

$$\mathcal{C} \longrightarrow (\text{Ab}), \quad U \longmapsto \pi_0(\mathfrak{R}(U)) \tag{2-1}$$

into the category of abelian groups. The effect of a morphism $\theta : U \rightarrow V$ is to send an isomorphism class $[B]$ for the fiber category $\mathfrak{R}(V)$ to the isomorphism class of its base-change $A = \theta^*(B) = B \times_U V$. Let us call $[B] \mapsto [A]$ the *transition map* for the above functor.

In the category of sets, one may now form the *direct limit*

$$\mathfrak{R}_\infty = \varinjlim \pi_0(\mathfrak{R}(U))$$

for (2-1), regarded as a covariant functor on the opposite category \mathcal{C}^{op} . Note that the group structures on the $\pi_0(\mathfrak{R}(U))$ do not necessarily induce a group structure on the direct limit, because direct limits do not necessarily commute with finite inverse limits.

For simplicity, suppose now that \mathcal{C} admits a *final object* $X \in \mathcal{C}$, and that the opposite category \mathcal{C}^{op} is *filtered*. For the category \mathcal{C} , this means that each pair of morphisms $V' \rightarrow V \leftarrow V''$ sits in some commutative diagram

$$\begin{array}{ccc} U & \longrightarrow & V'' \\ \downarrow & & \downarrow \\ V' & \longrightarrow & V \end{array}$$

and for each pair of morphisms $V \rightrightarrows V'$ there is a morphism $U \rightarrow V$ so that the two compositions $U \rightrightarrows V'$ coincide; cf. [2, Exposé I, Definition 2.7]. The first condition is automatic if fiber products exist, and usually poses no problem in practice, but the latter often becomes tricky for categories that do not come from ordered sets. If both conditions hold, the direct limit \mathfrak{R}_∞ indeed inherits a group structure. We call it the *stalk group* for the fibered Picard category \mathfrak{R} over the cofiltered category \mathcal{C} .

Often, the category \mathcal{C} is not cofiltered, but becomes cofiltered after passing to a *quotient category* \mathcal{C}/\sim , whose morphisms are equivalence classes of morphisms in \mathcal{C} , and whose objects are the same. In order to get a group structure on \mathfrak{R}_∞ , it then suffices to check that the base-change map $\pi_0(\mathfrak{R}(V)) \rightarrow \pi_0(\mathfrak{R}(U))$ depends only on the equivalence class of the morphism $\theta : U \rightarrow V$. In the application we have in mind, this independence actually takes place after passing to certain natural residue class groups $\pi_0(\mathfrak{R}(U))/\sim$.

3. Hypercoverings

In this section we recall some generalities on semi-simplicial coverings, review the notion of hypercoverings and discuss the ensuing hyper-Čech cohomology. All results are basically contained in [3, Exposé V, Sections 2 and 7], although in somewhat terse form. For generalities on simplicial objects, we refer to the monographs of May [28] and Weibel [38].

As is customary, we write Δ for the category comprising the sets $[n] = \{0, 1, \dots, n\}$ as objects and the monotonic maps $[m] \rightarrow [n]$ as morphisms. One may call this the *category of simplex types*. We will only be interested in the nonfull subcategory Δ_{semi} , which has the same object and where the morphisms are injective. The *face map* $\partial_i : [n - 1] \rightarrow [n]$, $0 \leq i \leq n$, is the unique injective monotonic map that omits the value $i \in [n]$. These face maps satisfy the simplicial identities $\partial_j \partial_i = \partial_i \partial_{j-1}$ for $i < j$. A *semi-simplicial object* in some category \mathcal{C} is a contravariant functor $U_\bullet : \Delta_{\text{semi}} \rightarrow \mathcal{C}$. This amounts to a sequence of objects $U_n \in \mathcal{C}$, $n \geq 0$, together with *face operators* $f_i = (\partial_i)^* : U_n \rightarrow U_{n-1}$, $0 \leq i \leq n$, satisfying the identities $f_i f_j = f_{j-1} f_i$ for $i < j$. Indeed, such data already specify the entire semi-simplicial object in a unique way. The semi-simplicial objects U_\bullet form a category $\text{Semi}(\mathcal{C})$, with natural transformations of functors as morphisms.

A *homotopy* between two morphisms $\theta_\bullet, \zeta_\bullet : U_\bullet \rightarrow V_\bullet$ is a collection of morphisms $h_0, \dots, h_n : U_n \rightarrow V_{n+1}$, given in each degree $n \geq 0$, satisfying $f_0 h_0 = \theta_n$ and $f_{n+1} h_n = \zeta_n$, together with the relations

$$f_i h_j = \begin{cases} h_{j-1} f_i & \text{if } 0 \leq i < j \leq n + 1, \\ f_i h_{i-1} & \text{if } 1 \leq i = j \leq n, \\ f_i h_i & \text{if } 1 \leq i = j + 1 \leq n, \\ h_j f_{i-1} & \text{if } 1 \leq j + 1 < i \leq n + 1. \end{cases} \tag{3-1}$$

We call them the *simplicial homotopy identities*. One says that the two morphisms θ_\bullet and ζ_\bullet are *homotopic* if they are equivalent under the equivalence relation generated by the homotopy relation. We then write $\text{Semi}(\mathcal{C})/\sim$ for the ensuing *quotient category*, where the objects are still the semi-simplicial objects in \mathcal{C} , but the morphisms are homotopy classes of natural transformations. Note that the equivalence relations generated by the homotopy relations on hom sets indeed from a *congruence* in the sense of category theory [27, Ch. II, Section 8].

Now suppose that \mathcal{C} is a *site*, that is, a category endowed with a *Grothendieck topology* (cf. [2, Exposé II]). For simplicity, we assume that there is a *final object* $X \in \mathcal{C}$, and that the topology comes from a *pretopology*. The latter means that for each object $V \in \mathcal{C}$, one has specified a collection $\text{Cov}(V)$ of *covering families* $(U_\lambda \rightarrow V)_{\lambda \in L}$ satisfying certain axioms. For the sake of exposition, we also assume that for each covering family the disjoint union $U = \bigcup_{\lambda \in L} U_\lambda$ exists. This ensures that one may refine each covering family to a one-member covering family, which one may call a *covering singleton*.

One should have the following example in mind. Given a topological space X , let \mathfrak{C} be the category of X -spaces (V, f) whose structure map $f : V \rightarrow X$ is a local homeomorphism, and $\text{Cov}(V)$ comprises those families where $\bigcup_{\lambda \in L} U_\lambda \rightarrow V$ is surjective. According to the comparison lemma [2, Exposé V, Theorem 4.1], the restriction functor $\text{Sh}(\mathfrak{C}) \rightarrow \text{Sh}(X)$ is an equivalence of categories. In turn, sheaves and cohomology for the space X are the same for the site \mathfrak{C} .

Let \mathcal{G} be an abelian presheaf on the site \mathfrak{C} . For each semi-simplicial object U_\bullet in \mathfrak{C} , we get the cochain complex $\Gamma(U_\bullet, \mathcal{G})$. The coboundary operators

$$\partial : \Gamma(U_n, \mathcal{G}) \longrightarrow \Gamma(U_{n+1}, \mathcal{G}), \quad g \longmapsto \sum_{j=0}^{n+1} (-1)^j q_j^*(g) \tag{3-2}$$

are induced from the face operators $q_j : U_{n+1} \rightarrow U_n, 0 \leq j \leq n + 1$ (see, for example, [25, Proposition 11.4.2]). This construction is functorial with respect to the presheaf \mathcal{G} and the hypercovering U_\bullet . Given two morphisms $\theta_\bullet, \zeta_\bullet : U_\bullet \rightarrow V_\bullet$ and a homotopy $h_0, \dots, h_n : U_n \rightarrow V_{n+1}$ between them, we can form the homomorphisms

$$s : \Gamma(V_{n+1}, \mathcal{G}) \longrightarrow \Gamma(U_n, \mathcal{G}), \quad g \longmapsto \sum_{i=0}^n (-1)^i h_i^*(g).$$

The following fact appears, in one form or another, over and over in homological algebra. The following form appears in [37, Tag 019S]; see also the proof of Proposition 6.3 below.

LEMMA 3.1. *In the above situation, we have $\theta_n^* - \zeta_n^* = s\partial - \partial s$ as homomorphisms $\Gamma(V_n, \mathcal{G}) \rightarrow \Gamma(U_n, \mathcal{G})$, for all degrees $n \geq 0$.*

In other words, simplicial homotopies yield cochain homotopies. In particular, the homomorphisms θ_n and ζ_n induce the same map on cohomology. More generally, the contravariant functor

$$\text{Semi}(\mathfrak{C}) \longrightarrow \text{Ch}(\text{Ab}), \quad U_\bullet \longmapsto \Gamma(U_\bullet, \mathcal{G})$$

into the abelian category $\text{Ch}(\text{Ab})$ of cochain complexes of abelian groups induces a functor

$$\text{Semi}(\mathfrak{C})/\sim \longrightarrow K(\text{Ab})$$

into the quotient category $K(\text{Ab}) = \text{Ch}(\text{Ab})/\sim$, where the morphisms are cochain maps modulo cochain homotopy. Note that this is still an additive category, in fact a triangulated category, but no longer an abelian category.

A semi-simplicial object U_\bullet in the site \mathfrak{C} is called a *semi-simplicial covering* if all face operators $p_i : U_n \rightarrow U_{n-1}$ and the augmentation $U_0 \rightarrow X$ are coverings. Morphisms between semi-simplicial coverings are called *refinements*. Using the covering families belonging to $\text{Cov}(U_n), n \geq 0$, we get categories of sheaves $\text{Sh}(U_n)$. The face operators thus yield contravariant functors

$$p_i^* : \text{Sh}(U_{n-1}) \longrightarrow \text{Sh}(U_n), \quad \mathcal{G} \longmapsto p_i^*(\mathcal{G}).$$

Here we write $p_i^*(\mathcal{G}) = p_i^{-1}(\mathcal{G})$ for the preimage sheaf, because there is no danger of confusion. In the presence of points and stalks, this means $p_i^*(\mathcal{G})_a = \mathcal{G}_{p_i(a)}$. These preimage functors are related by natural isomorphisms $q_j^* \circ p_i^* \simeq p_i^* \circ p_{j-1}^*$ for $i < j$, which we regard as identifications. Here

$$p_i : U_n \rightarrow U_{n-1} \quad \text{and} \quad q_j : U_{n+1} \rightarrow U_n$$

are the face operators defined in degree n and degree $n + 1$, respectively. In this context, we usually write $r_k : U_{n-1} \rightarrow U_{n-2}$ for face operators in degree $n - 1$. This convention will be used many times throughout.

Given a covering $U \rightarrow X$, we may form the fiber products

$$U_n = U^{n+1} = U \times_X \cdots \times_X U, \quad n \geq 0,$$

and use as face operators the projections $p_i : U_n \rightarrow U_{n-1}$ that omit the i th entry. Such semi-simplicial coverings occur in the definition of Čech cohomology, and we refer to them as Čech coverings. Note that in degree one, the face operators become the projections $p_0 = \text{pr}_2$ and $p_1 = \text{pr}_1$.

Čech coverings are special cases of the more flexible hypercoverings, which are certain semi-simplicial coverings U_\bullet constructed by the following recursive procedure. One formally starts with $U_{-1} = X$. If for some degree $n \geq 0$ the objects $U_{-1}, U_0, \dots, U_{n-1}$ are already defined, one extends this truncated semi-simplicial covering $U_{\leq n-1}$ to a full semi-simplicial covering L_\bullet by taking fiber products in the universal way, such that

$$\text{Hom}(T_{\leq n-1}, U_{\leq n-1}) = \text{Hom}(T_\bullet, L_\bullet)$$

for all other semi-simplicial coverings T_\bullet . Now one is allowed to choose a covering family $(W_\lambda \rightarrow L_n)_{\lambda \in L}$, and defines $U_n = \bigcup_{\lambda \in L} W_\lambda$ as the corresponding covering singleton. This concludes the recursive construction. However, one demands that for some degree $r = n - 1$, there is no passage to a covering family, such that $U_\bullet = L_\bullet$. In turn, the hypercovering U_\bullet is entirely determined by the truncated covering $U_{\leq r}$. Equivalently, the morphism $U_\bullet \rightarrow \text{cosk}_r(U_{\leq r})$ into the coskeleton is an isomorphism. One then says that U_\bullet is a hypercovering of type $r \geq 0$. Note that the hypercoverings of type $r = 0$ are precisely the Čech coverings. We refer to [3, Exposé V, Section 7] for details.

Now let $\mathfrak{H}_{X,r}$ be the category of hypercoverings U_\bullet of type $r \geq 0$. Recall that the morphisms are called refinements. We get a functor

$$\mathfrak{H}_{X,r}^{\text{op}} \longrightarrow \text{Ch}(\text{Ab}), \quad U \longmapsto \Gamma(U_\bullet, \mathcal{G})$$

into the abelian category $\text{Ch}(\text{Ab})$ of cochain complexes of abelian groups. This satisfies Grothendieck’s axiom (AB3), hence admits all direct sums and direct limits. One may form the direct limit of the above functor, but loses control over the resulting cochain groups, because the index category $\mathfrak{H}_{X,r}$ is in general not filtered.

However, we may pass to the quotient category $\overline{\mathfrak{H}}_{X,r} = \mathfrak{H}_{X,r}/\sim$, where the morphisms are homotopy classes of refinements. Let us call it the quotient category of hypercoverings. Thus we formally get a functor

$$\overline{\mathfrak{H}}_{X,r}^{\text{op}} \longrightarrow K(\text{Ab}), \quad U \longmapsto \Gamma(U_\bullet, \mathcal{G}).$$

As explained in [3, Exposé V, Theorem 7.3.2], the opposite quotient category $\overline{\mathfrak{H}}_{X,r}^{\text{op}}$ becomes filtered. In other words, the above may be regarded as an *ind-object* in the sense of [21, Exposé 1, Section 8.2]. Passing to cohomology, we obtain direct limits

$$\check{H}^p(X, \mathcal{G}) = \varinjlim H^p \Gamma(U_\bullet, \mathcal{G}) \in (\text{Ab}).$$

One may form the direct limits as sets or abelian groups, and use as index category either $\mathfrak{H}_{X,r}^{\text{op}}$ or the quotient category $\overline{\mathfrak{H}}_{X,r}^{\text{op}}$, and always gets the same group. These groups are called *hyper-Čech cohomology groups of type $r \geq 0$* , for each abelian presheaf \mathcal{G} . For type $r = 0$, this gives the usual Čech cohomology $\check{H}^p(X, \mathcal{G})$.

Now write $(\text{AbP}/\mathfrak{C})$ for the category of all abelian presheaves \mathcal{G} . Generalizing from Čech cohomology, one easily sees that for each fixed type $r \geq 0$, the

$$\check{H}^p : (\text{AbP}/\mathfrak{C}) \longrightarrow (\text{Ab}), \quad \mathcal{G} \longmapsto \check{H}^p(X, \mathcal{G})$$

form ∂ -functors, and the canonical inclusions $\mathfrak{H}_{X,r} \subset \mathfrak{H}_{X,r'}$ for $r \leq r'$ induce natural transformations between ∂ -functors. For type $r = 0$ we get the universal ∂ -functor and the universal natural transformations. Note further that the restriction to the full subcategory (Ab/\mathfrak{C}) of all abelian sheaves \mathcal{F} usually fails to be a ∂ -functor, because the inclusion functor is not exact and the subcategory contains more short exact sequences in general. However, we have the following vanishing result.

PROPOSITION 3.2. *Suppose $\mathcal{I} \in (\text{AbP}/\mathfrak{C})$ satisfies the sheaf axiom and becomes an injective object in the abelian category (Ab/\mathfrak{C}) . Then for each hypercovering U_\bullet , the cochain complex $\Gamma(U_\bullet, \mathcal{I})$ has no cohomology in degrees $p \geq 1$. In particular, \mathcal{I} is acyclic with respect to hyper-Čech cohomology for each type $r \geq 0$.*

PROOF. For each object $U \in \mathfrak{C}$, consider the corresponding representable presheaf $h_U : \mathfrak{C} \rightarrow (\text{Set})$ given by $h_U(V) = \text{Hom}_{\mathfrak{C}}(V, U)$. The Yoneda lemma yields $\text{Hom}_{(\text{AbP}/\mathfrak{C})}(h_U, \mathcal{G}) = \Gamma(U, \mathcal{G})$ for each set-valued presheaf \mathcal{G} . Write $\mathbb{Z}h_U : \mathfrak{C} \rightarrow (\text{Ab})$ for the resulting abelian presheaf, whose group of local sections over V is the free abelian group generated by the set $h_U(V)$. We thus get $\text{Hom}_{(\text{Ab}/\mathfrak{C})}(\mathbb{Z}h_U, \mathcal{G}) = \Gamma(U, \mathcal{G})$, for each abelian presheaf \mathcal{G} . Finally, write \mathbb{Z}_U for the sheafification of the abelian presheaf $\mathbb{Z}h_U$. By the universal property of sheafification, we obtain $\text{Hom}_{(\text{Ab}/\mathfrak{C})}(\mathbb{Z}_U, \mathcal{F}) = \Gamma(U, \mathcal{F})$ for each abelian sheaf \mathcal{F} .

Now let U_\bullet be a hypercovering of type $r \geq 0$, and consider the resulting semi-simplicial sheaf \mathbb{Z}_{U_\bullet} and the ensuing chain complex of abelian sheaves

$$\cdots \longrightarrow \mathbb{Z}_{U_2} \longrightarrow \mathbb{Z}_{U_1} \longrightarrow \mathbb{Z}_{U_0}$$

on \mathfrak{C} , where the boundary maps are alternating sums induced from the face operators $q_j : U_{n+1} \rightarrow U_n$. According to [3, Exposé V, Theorem 7.3.2], this chain complex is exact in degrees $p \geq 1$, and the cokernel for the map on the right is \mathbb{Z}_X . Since $\mathcal{I} \in (\text{Ab}/\mathfrak{C})$ is an injective object, the functor $\mathcal{F} \mapsto \text{Hom}_{(\text{Ab}/\mathfrak{C})}(\mathcal{F}, \mathcal{I})$ is exact. Applying

this exact hom functor to our exact sequence and using the preceding paragraph, we get an exact sequence

$$0 \longrightarrow \Gamma(X, \mathcal{F}) \longrightarrow \Gamma(U_0, \mathcal{F}) \longrightarrow \Gamma(U_1, \mathcal{F}) \longrightarrow \dots$$

Here the coboundary maps coincide with (3-2). In turn, the cohomology groups $H^p\Gamma(U_\bullet, \mathcal{F})$ vanish for $p \geq 1$. Passing to direct limits with respect to $U_\bullet \in \mathfrak{S}_{X,r}^{\text{op}}$, we get $\check{H}^p(X, \mathcal{F}) = 0$. □

4. The three-term complex

We keep the notation and assumptions from the preceding section, such that \mathfrak{C} is a site with final object $X \in \mathfrak{C}$. The goal now is to express sheaf cohomology in terms of certain three-term complexes, which arise from total complexes and spectral sequences involving hypercoverings. We use the standard conventions with regard to double complexes; cf. [13, Chs. IV and XV].

For each abelian sheaf \mathcal{F} , choose once and for all an injective resolution

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}^0 \xrightarrow{d} \mathcal{I}^1 \xrightarrow{d} \dots, \tag{4-1}$$

such that $H^p(X, \mathcal{F}) = H^p\Gamma(X, \mathcal{I}^\bullet)$. Furthermore, fix some hypercovering U_\bullet . This gives cochain complexes $\Gamma(U_\bullet, \mathcal{I}^q)$ with coboundary maps

$$\partial : \Gamma(U_n, \mathcal{I}^q) \longrightarrow \Gamma(U_{n+1}, \mathcal{I}^q), \quad f \longmapsto \sum_{j=0}^{n+1} (-1)^j q_j^*(f). \tag{4-2}$$

The two types of coboundary maps are compatible, such that the diagrams

$$\begin{array}{ccc} \Gamma(U_p, \mathcal{I}^{q+1}) & \xrightarrow{\partial_{p,q+1}} & \Gamma(U_{p+1}, \mathcal{I}^{q+1}) \\ d_{p,q} \uparrow & & \uparrow d_{p+1,q} \\ \Gamma(U_p, \mathcal{I}^q) & \xrightarrow{\partial_{p,q}} & \Gamma(U_{p+1}, \mathcal{I}^q) \end{array}$$

are commutative. To obtain a *double complex* $\Gamma(U_\bullet, \mathcal{I}^\bullet)$, we use the usual sign trick and replace the vertical differentials $d_{p,q}$ by $(-1)^p d_{p,q}$. Up to this sign change, the vertical differential comes from the injective resolution (4-1), whereas the horizontal differential comes from the hypercovering (4-2).

In turn, we obtain a *total complex* $\text{Tot } \Gamma(U_\bullet, \mathcal{I}^\bullet)$. First, we consider the *horizontal filtration*, which has

$$\text{Fil}^q = \bigoplus_{p \geq 0, b \geq q} \Gamma(U_p, \mathcal{I}^b).$$

The associated graded complex is $\text{gr}^q = \Gamma(U_\bullet, \mathcal{I}^q)$, with differential given by (4-2). This gives a spectral sequence with $E_1^{pq} = H^p\Gamma(U_\bullet, \mathcal{I}^q)$. Since the presheaf \mathcal{I}^q is an injective sheaf, we have $E_1^{pq} = 0$ for all $p > 0$, according to Proposition 3.2.

Furthermore, $E_1^{0,q} = H^0(X, \mathcal{I}^q)$. The differential $E_1^{0,q} \rightarrow E_1^{0,q+1}$ is induced by (4-1), and the definition of sheaf cohomology gives $E_2^{0,q} = H^q(X, \mathcal{F})$. Since there are no nontrivial differentials on the E_2 -page, the spectral sequence collapses, such that $E_2^{pq} = E_3^{pq} = \dots = E_\infty^{pq}$. In turn, the filtration $\text{Fil}^q H^{p+q} \text{Tot} \Gamma(U_\bullet, \mathcal{F})$ has just one nontrivial step. We thus have the following proposition.

PROPOSITION 4.1. *For each hypercovering U_\bullet and each degree $q \geq 0$, the edge map*

$$H^q \text{Tot} \Gamma(U_\bullet, \mathcal{I}^\bullet) \longrightarrow H^q(X, \mathcal{F})$$

for the above spectral sequence is bijective.

The total complex is functorial in the hypercovering U_\bullet , and the edge map is compatible with refinements. In turn, refinements induce identities on the cohomology of the total space.

Next, consider the vertical filtration $\text{Fil}^p = \bigoplus_{a \geq p, q \geq 0} \Gamma(U_a, \mathcal{I}^q)$ on the total complex, whose associated graded complex is $\text{gr}^p = \Gamma(U_p, \mathcal{I}^\bullet)$. This gives another spectral sequence, with pages

$$E_1^{pq} = H^q \Gamma(U_p, \mathcal{I}^\bullet) \quad \text{and} \quad E_2^{pq} = H^p \Gamma(U_\bullet, \underline{H}^q(\mathcal{F})).$$

Here $\underline{H}^q(\mathcal{F})$ denotes the presheaf defined by $\Gamma(V, \underline{H}^q(\mathcal{F})) = H^q(V, \mathcal{F})$. The spectral sequence is functorial in the hypercovering U_\bullet , and homotopic refinements induce identical maps. Passing to the direct limit over all $U_\bullet \in \overline{\mathfrak{H}}_{X,r}^{\text{op}}$ and using the preceding proposition, we obtain the spectral sequence

$$E_2^{pq} = \check{H}^p(X, \underline{H}^q(\mathcal{F})) \implies H^{p+q}(X, \mathcal{F}),$$

as given in [3, Exposé V, Section 7.4]. According to [3, Theorem 7.4.1] we have the following theorem.

THEOREM 4.2. *For each degree $n \leq r + 1$, the edge map*

$$\check{H}^n(X, \mathcal{F}) \longrightarrow H^n \text{Tot} \Gamma(U_\bullet, \mathcal{I}^\bullet) = H^n(X, \mathcal{F})$$

for the above spectral sequence is bijective.

This should be seen as a far-reaching generalization of the fact that cohomology in degree one coincides with Čech cohomology, which is the special case $r = 0$ and $n = 1$. Note, however, that in general the edge map is not bijective for $n = r + 2$.

We shall exploit this failure as follows. Fix some integer $n \geq 1$, and consider the three abelian groups

$$\begin{aligned} C^{n-1} &= \Gamma(U_{n-2}, \mathcal{I}^0/\mathcal{F}) \oplus \Gamma(U_{n-1}, \mathcal{I}^0), \\ C^n &= \Gamma(U_{n-1}, \mathcal{I}^1) \oplus \Gamma(U_n, \mathcal{I}^0), \\ C^{n+1} &= \Gamma(U_{n-1}, \mathcal{I}^2) \oplus \Gamma(U_n, \mathcal{I}^1) \oplus \Gamma(U_{n+1}, \mathcal{I}^0). \end{aligned}$$

Taking the horizontal and vertical differentials from (4-2) and (4-1), we get matrices

$$\Phi = \begin{pmatrix} \epsilon d & 0 \\ \partial & -\epsilon d \\ 0 & \partial \end{pmatrix} \quad \text{and} \quad \Psi = \begin{pmatrix} \partial & \epsilon d \\ 0 & \partial \end{pmatrix}$$

that define a *three-term cochain complex* $C^{n-1} \xrightarrow{\Psi} C^n \xrightarrow{\Phi} C^{n+1}$. Here $\epsilon = (-1)^{n-1}$ is the sign introduced for the double complex $\Gamma(U_\bullet, \mathcal{I}^\bullet)$. One should have in mind the following diagram, which arises from the double complex $\Gamma(U_\bullet, \mathcal{I}^\bullet)$:

$$\begin{array}{ccccc} & & \Gamma(U_{n-1}, \mathcal{I}^2) & & \\ & & \uparrow \epsilon d & & \\ \Gamma(U_{n-2}, \mathcal{I}^0/\mathcal{F}) & \xrightarrow{\partial} & \Gamma(U_{n-1}, \mathcal{I}^1) & \xrightarrow{\partial} & \Gamma(U_n, \mathcal{I}^1) \\ & & \uparrow \epsilon d & & \uparrow -\epsilon d \\ \Gamma(U_{n-1}, \mathcal{I}^0) & \xrightarrow{\partial} & \Gamma(U_n, \mathcal{I}^0) & \xrightarrow{\partial} & \Gamma(U_{n+1}, \mathcal{I}^0) \end{array}$$

Note that upper left entry of the matrix Ψ is given by the canonical inclusion $\mathcal{I}^0/\mathcal{F} \subset \mathcal{I}^1$ coming from the differential $\mathcal{I}^0 \rightarrow \mathcal{I}^1$, which ensures that the upper left entry of $\Phi \circ \Psi$ vanishes. The exact sequence $0 \rightarrow \mathcal{I}^0/\mathcal{F} \rightarrow \mathcal{I}^1 \rightarrow \mathcal{I}^2$ induces an exact sequence

$$0 \rightarrow \Gamma(U_{n-2}, \mathcal{I}^0/\mathcal{F}) \rightarrow \Gamma(U_{n-2}, \mathcal{I}^1) \xrightarrow{-\epsilon d} \Gamma(U_{n-2}, \mathcal{I}^2).$$

In turn, we have a homomorphism from the three-term cochain complex to the total cochain complex, which induces the map

$$\text{Ker}(\Phi)/\text{Im}(\Psi) \rightarrow H^n \text{Tot} \Gamma(U_\bullet, \mathcal{I}^\bullet)$$

on cohomology. Both sides depend functorially on the hypercovering: each refinement $U'_\bullet \rightarrow U_\bullet$ gives a restriction map, and we thus get a contravariant functor $\mathfrak{H}_{X,r} \rightarrow (\text{Ab})$ given by $U_\bullet \mapsto \text{Ker}(\Phi)/\text{Im}(\Psi)$.

Suppose that $\theta_\bullet, \zeta_\bullet : U'_\bullet \rightarrow U_\bullet$ are homotopic refinements. Each of them induces a cochain map between the three-term complexes formed with U_\bullet and U'_\bullet . Using that the operator s in Lemma 3.1 is natural in the sheaf \mathcal{G} , one easily infers that the induced maps $\text{Ker}(\Phi)/\text{Im}(\Psi) \rightarrow \text{Ker}(\Phi')/\text{Im}(\Psi')$ on cohomology coincide.

Passing to the direct limit and taking the identification in Proposition 4.1 into account, we get a map

$$\varinjlim \text{Ker}(\Phi)/\text{Im}(\Psi) \rightarrow \varinjlim H^n \text{Tot} \Gamma(U_\bullet, \mathcal{I}^\bullet) = H^n(X, \mathcal{F}),$$

where the direct limit runs over all hypercoverings $U_\bullet \in \mathfrak{H}_{X,r}^{\text{op}}$.

THEOREM 4.3. *For each integer $n \geq 2$ and each type $r \geq n - 2$, the above map is bijective, such that we get an identification $H^n(X, \mathcal{F}) = \varinjlim \text{Ker}(\Phi)/\text{Im}(\Psi)$.*

PROOF. First, we check that the map in question is surjective. Fix some cohomology class $[\alpha] \in H^n(X, \mathcal{F})$. Represent the class on some suitable hypercovering U_\bullet of type r by some tuple

$$\alpha = (\alpha_0, \alpha_1, \dots, \alpha_n),$$

where each entry is a local section $\alpha_i \in \Gamma(U_i, \mathcal{I}^{n-i})$ and the tuple is a cocycle in the total complex $\text{Tot} \Gamma(U_\bullet, \mathcal{I}^\bullet)$. We now show by induction on $0 \leq i \leq n - 2$ that after passing to finer hypercoverings, the tuple becomes cohomologous to a tuple of the form $\beta = (0, \dots, 0, \beta_i, \dots, \beta_n)$. In the final case $i = n - 2$, the remaining pair (β_{n-1}, β_n) gives a cocycle in the three-term complex inducing the class $[\alpha]$.

This assertion is trivial for $i = 0$. Now suppose that $1 \leq i \leq n - 3$, and that $\alpha_0 = \dots = \alpha_{i-1} = 0$. Then $\alpha_i \in \Gamma(U_i, \mathcal{I}^{n-i})$ vanishes in $\Gamma(U_i, \mathcal{I}^{n-i+1})$, thus defines a class $[\alpha_i] \in H^{n-i}(U_i, \mathcal{F})$. Choose some covering $W \rightarrow U_i$ on which the cohomology class vanishes, in other words, $\alpha_i|_W$ lies in the image of $\Gamma(W, \mathcal{I}^{n-i-1})$. Now we use $i < n - 2 \leq r$, so we may refine the hypercovering $U_\bullet \in \mathfrak{S}_{X,r}$ of type r and assume that α_i lies in the image of $\Gamma(U_i, \mathcal{I}^{n-i-1})$. Subtracting from α the ensuing coboundary, we obtain $\alpha_0 = \dots = \alpha_i = 0$. This shows that the canonical map is surjective.

It remains to check that our map is also injective. Suppose an n -cocycle of the form $\alpha = (0, \dots, 0, \alpha_{n-1}, \alpha_n)$ is a coboundary in the total complex $\text{Tot} \Gamma(U_\bullet, \mathcal{I}^\bullet)$. Then there is an $(n - 1)$ -cochain $\beta = (\beta_0, \dots, \beta_{n-1})$ mapping to α . Clearly, we may replace β by some cohomologous cochain. As in the preceding paragraph, we may thus assume that $\beta_0 = \dots = \beta_{n-3} = 0$. The entry $\beta_{n-2} \in \Gamma(U_{n-2}, \mathcal{I}^1)$ then vanishes in $\Gamma(U_{n-2}, \mathcal{I}^2)$, thus can be regarded as an local section of $\mathcal{I}^0/\mathcal{F}$. In turn, the pair $(\beta_{n-2}, \beta_{n-1})$ is an element from C^{n-1} mapping to $(\alpha_{n-1}, \alpha_n) \in C^n$. This means that the map in question is injective. □

5. Iterated alternating preimages

We keep the assumptions of the preceding section, such that \mathfrak{C} is a site with final object $X \in \mathfrak{C}$. Let \mathcal{F} be an abelian sheaf on \mathfrak{C} . We now collect some observations how torsors behave in simplicial coverings. Together with the identification $H^n(X, \mathcal{F}) = \varinjlim \text{Ker}(\Phi)/\text{Im}(\Psi)$ from the previous section, this will later lead to our geometric interpretation of sheaf cohomology.

First recall that one may view $H^1(X, \mathcal{F})$ as the group of isomorphism classes $\pi_0(\mathcal{F}\text{-Tors})$ of \mathcal{F} -torsors \mathcal{T} . We write the group law for \mathcal{F} additively and the group action on \mathcal{T} multiplicatively. Let \mathcal{T}^{-1} be the same set-valued sheaf \mathcal{T} , but endowed with the inverse \mathcal{F} -action, given on local sections by $(-f) \cdot s$. If $\mathcal{F} \rightarrow \mathcal{F}'$ is a homomorphism of abelian sheaves, we obtain an induced \mathcal{F}' -torsor $\mathcal{F}' \wedge^{\mathcal{F}} \mathcal{T}$ as the quotient

$$\mathcal{F}' \wedge^{\mathcal{F}} \mathcal{T} = \mathcal{F}' \backslash (\mathcal{F}' \times \mathcal{T})$$

by the diagonal \mathcal{F} -action. Given \mathcal{F} -torsors \mathcal{T}_i indexed by a finite set I , and a map $\epsilon : I \rightarrow \{\pm 1\}$, the product sheaf $\prod_{i \in I} \mathcal{T}_i^{\epsilon_i}$ becomes a torsor under the abelian product

sheaf $\mathcal{P} = \prod_{i \in I} \mathcal{F}_i$. We define the *contracted product* $\bigwedge_{i \in I} \mathcal{F}_i$ as the induced \mathcal{F} -torsor

$$\bigwedge_{i \in I} \mathcal{F}_i^{\epsilon_i} = \mathcal{F} \wedge^{\mathcal{P}} \left(\prod_{i \in I} \mathcal{F}_i^{\epsilon_i} \right)$$

with respect to the addition map $\mathcal{P} = \bigoplus_{i \in I} \mathcal{F} \rightarrow \mathcal{F}$. In the additive group $H^1(X, \mathcal{F}) = \pi_0(\mathcal{F}\text{-Tors})$ of isomorphism classes, we then have

$$\left[\bigwedge_{i \in I} \mathcal{F}_i^{\epsilon_i} \right] = \sum_{i \in I} \epsilon_i [\mathcal{F}_i].$$

The sum vanishes if and only if the contracted product admits a global section. Under suitable assumptions, the contracted product even acquires a *canonical section*. The precise meaning of this locution becomes apparent in the proof of the following assertion.

LEMMA 5.1. *Suppose there is a free involution $\sigma : I \rightarrow I$ on the index set such that $\epsilon_{\sigma(i)} = -\epsilon_i$ and $\mathcal{F}_i = \mathcal{F}_{\sigma(i)}$ for all $i \in I$. Then the contracted product $\bigwedge_{i \in I} \mathcal{F}_i^{\epsilon_i}$ admits a canonical section.*

PROOF. By taking contracted products of canonical sections, it suffices to treat the case where the index set I contains merely two elements, such that the contracted product takes the form $\mathcal{F} \wedge \mathcal{F}^{-1}$. This contracted product comes with a canonical map $\varphi : \mathcal{F} \wedge \mathcal{F}^{-1} \rightarrow \mathcal{F}$, which sends a pair of local sections (s, s') to the element f , defined via the condition $f \cdot s = s'$. Recall that $\mathcal{F} = \mathcal{F}^{-1}$ as set-valued sheaves. This map is well defined, and equivariant with respect to the canonical \mathcal{F} -actions. It must be an isomorphism, because the category (\mathcal{F} -tors) is a groupoid. The preimage of the zero-section $0 \in \Gamma(X, \mathcal{F})$ is the canonical global section of $\mathcal{F} \wedge \mathcal{F}^{-1}$. \square

Now let U_\bullet be any semi-simplicial covering. Fix some integer $n \geq 0$, and consider the face operators $p_i : U_n \rightarrow U_{n-1}$ for $0 \leq i \leq n$. Given an $\mathcal{F}|_{U_{n-1}}$ -torsor \mathcal{T} , we define the *alternating preimage* as

$$p_{\text{alt}}^*(\mathcal{T}) = p_0^*(\mathcal{T}) \wedge p_1^*(\mathcal{T}^{-1}) \wedge \cdots \wedge p_n^*(\mathcal{T})^{(-1)^n},$$

which is an $\mathcal{F}|_{U_n}$ -torsor. This torsor may or may not be trivial. However, we shall see that the *iterated alternating preimage* $q_{\text{alt}}^*(p_{\text{alt}}^*(\mathcal{T}))$ becomes trivial, and in fact acquires a canonical section. Here $q_j : U_{n+1} \rightarrow U_n$ for $0 \leq j \leq n+1$ denote face operators defined in degree $n+1$. The reason is as follows. Set

$$\mathcal{T}_{(j,i)} = q_j^*(p_i^*(\mathcal{T})) \quad \text{and} \quad \epsilon_{(j,i)} = (-1)^{i+j}.$$

By definition, the iterated alternating preimage is the contracted product for the $\mathcal{F}|_{U^{n+1}}$ -torsors $\mathcal{T}_{(j,i)}^{\epsilon_{(j,i)}}$, where the indices satisfy $0 \leq j \leq n+1$ and $0 \leq i \leq n$. The simplicial identities $\partial_j \partial_i = \partial_i \partial_{j-1}$ for $i < j$ yield canonical identifications $q_j^*(p_i^*(\mathcal{G})) = q_i^*(p_{j-1}^*(\mathcal{G}))$ that are natural in the sheaf \mathcal{G} . Consider the index set

$$L = [n] \times [n-1] = \{(j, i) \mid 0 \leq i \leq n \text{ and } 0 \leq j \leq n+1\},$$

which has cardinality $(n + 1)(n + 2)$, an even number. It comes with the partition

$$L' = \{(j, i) \mid i < j\} \quad \text{and} \quad L'' = \{(j, i) \mid i \geq j\},$$

and both of these sets have cardinality $(n + 1)(n + 2)/2$. In fact, we have a canonical map

$$\psi : L' \longrightarrow L'', \quad (j, i) \longmapsto (i, j - 1)$$

which reflects the simplicial identities. This map is injective, hence bijective, because domain and range have the same cardinality. In turn, we get a canonical free involution $\sigma : L \rightarrow L$, with $\sigma|L' = \psi$ and $\sigma|L'' = \psi^{-1}$. By construction,

$$\mathcal{T}_{\sigma(j,i)} = \mathcal{T}_{(j,i)} \quad \text{and} \quad \epsilon_{\sigma(i,j)} = -\epsilon_{(i,j)}.$$

So we may apply Lemma 5.1 to the iterated alternating preimage $q_{\text{alt}}^*(p_{\text{alt}}^*(\mathcal{T}))$ and infer that it comes with a canonical section $\varphi_{\text{can}} \in \Gamma(U_{n+1}, q_{\text{alt}}^*(p_{\text{alt}}^*(\mathcal{T})))$, which yields a canonical identification

$$q_{\text{alt}}^*(p_{\text{alt}}^*(\mathcal{T})) = \mathcal{F}|U_{n+1}.$$

This simple observation has remarkable consequences. If \mathcal{T} is an $\mathcal{F}|U_{n-1}$ torsor and $\varphi \in \Gamma(U_n, p_{\text{alt}}^*(\mathcal{T}))$ is a section of the alternating preimage of the torsor, the alternating preimage of the section can be regarded as a section $q_{\text{alt}}^*(\varphi) \in \Gamma(U_{n+1}, \mathcal{F})$. In turn, we get a ‘cocycle condition’ $q_{\text{alt}}^*(\varphi) = 0$ for pairs (\mathcal{T}, φ) .

Furthermore, each $\mathcal{F}|U_{n-2}$ -torsor \mathcal{T}_{n-2} yields a ‘coboundary pair’ (\mathcal{T}, φ) , where the $\mathcal{F}|U_{n-1}$ -torsor $\mathcal{T} = r_{\text{alt}}^*(\mathcal{T}_{n-2})$ is the alternating preimage, formed with the face operators $r_k : U_{n-1} \rightarrow U_{n-2}$, and $\varphi = \varphi_{\text{can}}$ is the canonical section of the iterated alternating preimage $p_{\text{alt}}^*(\mathcal{T}) = p_{\text{alt}}^*(r_{\text{alt}}^*(\mathcal{T}_{n-2}))$. Indeed, coboundary pairs satisfy the following cocycle condition.

PROPOSITION 5.2. *With the preceding notation, we have $q_{\text{alt}}^*(\varphi_{\text{can}}) = 0$.*

The proof of this innocuous assertion requires a little preparation. Consider the triple index set

$$M = \{(j, i, k) \mid 0 \leq j \leq n + 1, 0 \leq i \leq n \text{ and } 0 \leq k \leq n - 1\}.$$

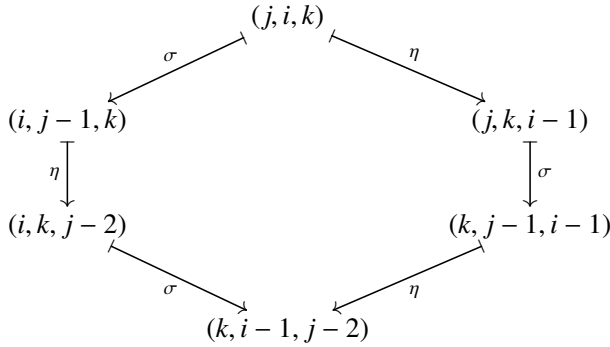
This set has cardinality $(n + 2)(n + 1)n$, which is divisible by six. In simplicial notation $M = [n + 1] \times [n] \times [n - 1]$. Furthermore, the simplicial identities yield two involutions $\sigma, \eta : M \rightarrow M$, determined by the condition

$$\sigma(j, i, k) = (i, j - 1, k) \quad \text{and} \quad \eta(j, i, k) = (j, k, i - 1)$$

for $i < j$ and $k < i$, respectively. Consider the dihedral permutation group $G = \langle \sigma, \eta \rangle$ inside the symmetric group S_M generated by the involutions, which satisfy the relations $\sigma^2 = \eta^2 = e$.

LEMMA 5.3. *The dihedral group G has order $\text{ord}(G) = 6$, and the G -action on the index set M is free.*

PROOF. Fix some element $(j, i, k) \in M$ with $k < i < j$. Its G -orbit consists of the following elements, arranged in a hexagonal pattern:



These six triples are pairwise different, because the entries j, i, k are pairwise different. Setting $g = \text{ord}(G)$, we obtain $6|g$.

By the simplicial identities, each element $(j', i', k') \in M$ lies in the G -orbit of some element (j, i, k) with $k < i < j$. Hence all G -orbits comprise six elements, and we have the relation $\sigma\eta\sigma = \eta\sigma\eta$ in G . Therefore, the generators $\sigma, \eta \in G$ yield a surjection from the dihedral group $D_3 = \langle \sigma, \eta \mid \sigma^2 = \eta^2 = (\sigma\eta)^3 \rangle$, thus $6|g$. The upshot is that $g = 6$, that the surjection $D_3 \rightarrow G$ is bijective, and that G acts freely on M . \square

PROOF OF PROPOSITION 5.2. By the very definition, the iterated alternating preimage $p_{\text{alt}}^*(\mathcal{S}) = p_{\text{alt}}^*(r_{\text{alt}}^*(\mathcal{T}_{U_{n-2}}))$ is a contracted product of the trivial $\mathcal{F}|U_n$ -torsors

$$\mathcal{P}_{ik} = p_i^*(r_k^*(\mathcal{T}_{U_{n-2}}))^{(-1)^{i+k}} \wedge p_k^*(r_{i-1}^*(\mathcal{T}_{U_{n-2}}))^{(-1)^{i+k-1}} \tag{5-1}$$

for $k < i$, and the canonical section φ_{can} is the contracted product of canonical sections $\varphi_{ik} \in \Gamma(U_n, \mathcal{P}_{ik})$. For each $0 \leq j \leq n + 1$ with $i < j$, define

$$\begin{aligned} \mathcal{P}_{j,i,k} &= q_j^*(\mathcal{P}_{ik})^{(-1)^j}, \\ \mathcal{P}_{k,j-1,i-1} &= q_k^*(\mathcal{P}_{j-1,i-1})^{(-1)^k}, \\ \mathcal{P}_{i,j-1,k} &= q_i^*(\mathcal{P}_{j-1,k})^{(-1)^i}. \end{aligned} \tag{5-2}$$

According to Lemma 5.3, the triple iterated alternating preimage $q_{\text{alt}}^*(p_{\text{alt}}^*(\mathcal{S})) = q_{\text{alt}}^*(p_{\text{alt}}^*(r_{\text{alt}}^*(\mathcal{T}_{U_{n-2}})))$ is the contracted product of the trivial $\mathcal{F}|U_{n+1}$ -torsors

$$\mathcal{P}_{j,i,k} \wedge \mathcal{P}_{k,j-1,i-1} \wedge \mathcal{P}_{i,j-1,k}, \tag{5-3}$$

where the contracted product runs over all triples (j, i, k) with $k < i < j$. Substituting (5-2) and (5-1), we see that (5-3) is the contracted product of six torsors, which are nontrivial in general. To simplify notation, set $\epsilon = (-1)^{i+j+k}$ and $\mathcal{S}_{b,a,c} =$

$q_b^*(p_a^*(r_c^*(\mathcal{T}_{U_{n-2}})))$, and arrange the six torsors in a hexagonal graph:

$$\begin{array}{ccccc}
 & & \mathcal{S}_{j,i,k}^\epsilon & & \\
 & \sigma \swarrow & & \searrow \eta & \\
 \mathcal{S}_{j,k,i-1}^{-\epsilon} & & & & \mathcal{S}_{j,k,i-1}^{-\epsilon} \\
 \eta \downarrow & & & & \downarrow \sigma \\
 \mathcal{S}_{i,k,j-2}^\epsilon & & & & \mathcal{S}_{k,j-1,i-1}^\epsilon \\
 & \sigma \swarrow & & \searrow \eta & \\
 & & \mathcal{S}_{k,i-1,j-2}^{-\epsilon} & &
 \end{array} \tag{5-4}$$

Here the edges reflect the action of the generators σ, η of the dihedral permutation group $G \subset S_M$ on the set of indices M . In light of the simplicial identities and the exponents $\pm\epsilon$, any two adjacent torsors are inverse to each other, and in particular have the same underlying set-valued sheaf. The contracted product of the six torsors is a trivial $\mathcal{F}|U_{n+1}$ -torsor. In fact, the contracted product has two *canonical sections*, coming from the two possible cyclic placement of parenthesis pairs. Pairing the torsors incident with the σ -edges gives

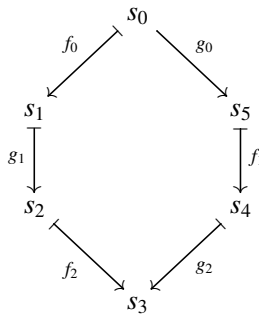
$$(\mathcal{S}_{j,i,k}^\epsilon \wedge \mathcal{S}_{j,k,i-1}^{-\epsilon}) \wedge (\mathcal{S}_{i,k,j-2}^\epsilon \wedge \mathcal{S}_{k,i-1,j-2}^{-\epsilon}) \wedge (\mathcal{S}_{k,j-1,i-1}^\epsilon \wedge \mathcal{S}_{j,k,i-1}^{-\epsilon}),$$

which yields the canonical section from the ‘outer iterated preimage’. Pairing the torsors incident with the η -edges gives

$$(\mathcal{S}_{j,i,k}^\epsilon \wedge \mathcal{S}_{j,k,i-1}^{-\epsilon}) \wedge (\mathcal{S}_{j,k,i-1}^{-\epsilon} \wedge \mathcal{S}_{k,i-1,j-2}^{-\epsilon}) \wedge (\mathcal{S}_{i,k,j-2}^\epsilon \wedge \mathcal{S}_{j,k,i-1}^{-\epsilon}),$$

which yields the canonical section from the ‘inner iterated preimage’.

Our task is to show that these two canonical sections coincide. To verify this, choose respective local sections



for the torsors in the hexagonal diagram (5-4). The equations $s_1 = f_0 \cdot s_0$ and $s_2 = g_1 \cdot s_1$ etc. define local sections f_0, f_1, f_2 and g_0, g_1, g_2 for the abelian sheaf \mathcal{F} , as indicated in the above diagram. Obviously, they satisfy $f_0 + g_1 + f_2 = g_0 + f_1 + g_2$, hence $f_0 - f_1 + f_2 = g_0 - g_1 + g_2$. The two sides of the latter equation correspond to the canonical sections of the sixfold contracted product, and our assertion follows. \square

6. Rigidified torsor cocycles

We keep the assumptions of the preceding section, such that \mathfrak{C} is a site with final object $X \in \mathfrak{C}$. For simplicity, we assume that the Grothendieck topology is given by a pretopology $\text{Cov}(V)$, $V \in \mathfrak{C}$, of covering families $(U_\lambda \rightarrow V)_{\lambda \in L}$. Furthermore, we suppose that for each covering family $(U_\lambda \rightarrow V)_{\lambda \in L}$, the disjoint union $U = \bigcup U_\lambda$ exists. We now introduce the central notion of this paper.

DEFINITION 6.1. Let \mathcal{F} be an abelian sheaf on the site \mathfrak{C} , and $n \geq 1$ be some integer. A *rigidified torsor cocycle* for \mathcal{F} in degree n is a triple $(U_\bullet, \mathcal{T}, \varphi)$ consisting of the following data:

- (1) U_\bullet is a hypercovering of type $r = n - 2$;
- (2) \mathcal{T} is a torsor for the abelian sheaf $\mathcal{F}|_{U_{n-1}}$ over U_{n-1} ;
- (3) φ is a section of the alternating preimage $p_{\text{alt}}^*(\mathcal{T})$ over U_n .

These data satisfy $q_{\text{alt}}^*(\varphi) = 0$ as section over U_{n+1} of the iterated alternating preimage under the canonical identification $q_{\text{alt}}^*(p_{\text{alt}}^*(\mathcal{T})) = \mathcal{F}|_{U_{n+1}}$.

For simplicity, we also say that $(U_\bullet, \mathcal{T}, \varphi)$ is a *rigidified \mathcal{F} -torsor n -cocycle*. Recall that $p_i : U_n \rightarrow U_{n-1}$ and $q_j : U_{n+1} \rightarrow U_n$ denote the face operators in the semi-simplicial covering U_\bullet defined in degree n and $n + 1$, respectively. One should have in mind the special case where the hypercovering U_\bullet has type $r = 0$, such that the $U_m = U^{m+1} = U \times_X \cdots \times_X U$ form a Čech covering. Also note that a hypercovering of type $r = -1$ is just the constant semi-simplicial covering with $U_m = X$ for all $m \geq 0$.

Some rigidified torsor cocycles arise as follows. Given a hypercovering U_\bullet of type $n - 2$, an $\mathcal{F}|_{U_{n-2}}$ -torsor $\mathcal{T}_{U_{n-2}}$ and a local section $s \in \Gamma(U_{n-1}, \mathcal{F})$, we may form the alternating preimage $\mathcal{T} = r_{\text{alt}}^*(\mathcal{T}_{U_{n-2}})$, where the $r_k : U_{n-1} \rightarrow U_n$, $0 \leq k \leq n - 1$, denote face operators defined in degree $n - 1$. Being an iterated alternating preimage, the $\mathcal{F}|_{U_n}$ -torsor $p_{\text{alt}}^*(\mathcal{T}) = p_{\text{alt}}^*(r_{\text{alt}}^*(\mathcal{T}_{U_{n-2}}))$ on U_n comes with a canonical section $\varphi = \varphi_{\text{can}}$, which we can multiply with $p_{\text{alt}}^*(s) \in \Gamma(U_{n-1}, \mathcal{F})$. It follows from Proposition 5.2 that the triple $(U_\bullet, r_{\text{alt}}^*(\mathcal{T}_{U_{n-2}}), p_{\text{alt}}^*(s) \cdot \varphi_{\text{can}})$ is a rigidified torsor cocycle. Let us call them *rigidified torsor coboundaries*.

The rigidified \mathcal{F} -torsor n -cocycles form a category $\mathfrak{R}_{\mathcal{F}}^n$, where the morphism

$$(\theta_\bullet, \tau) : (U_\bullet, \mathcal{T}, \varphi) \longrightarrow (U'_\bullet, \mathcal{T}', \varphi')$$

consists of a refinement $\theta_\bullet : U_\bullet \rightarrow U'_\bullet$, together with a morphism $\tau : \mathcal{T} \rightarrow \mathcal{T}'|_{U_{n-1}}$ of $\mathcal{F}|_{U_{n-1}}$ -torsors, such that $p_{\text{alt}}^*(\tau)(\varphi) = \varphi'|_{U_n}$. Recall that $\mathfrak{H}_{X,n-2}$ denotes the category of hypercoverings U_\bullet of type $r = n - 2$, and consider the forgetful functor

$$\mathfrak{R}_{\mathcal{F}}^n \longrightarrow \mathfrak{H}_{X,n-2}, \quad (U_\bullet, \mathcal{T}, \varphi) \longrightarrow U_\bullet$$

Clearly, all fiber categories $\mathfrak{R}_{\mathcal{F}}^n(U_\bullet)$ are groupoids, and all morphisms in $\mathfrak{R}_{\mathcal{F}}^n$ are cartesian. The fiber categories $\mathfrak{R}_{\mathcal{F}}^n(U_\bullet)$ are actually Picard categories. The monoidal structure is given by

$$(U_\bullet, \mathcal{T}, \varphi) = (U_\bullet, \mathcal{T}', \varphi') \wedge (U_\bullet, \mathcal{T}'', \varphi'')$$

where $\mathcal{T} = \mathcal{T}' \wedge \mathcal{T}''$ and $\varphi = \varphi' \wedge \varphi''$. The associativity and commutativity constraints come from the corresponding constraints for torsors. The neutral object is given by the triple $(U_\bullet, \mathcal{F}|_{U_{n-1}}, 0)$. The monoidal structure on fibers comes from the functor $\mathfrak{R}_{\mathcal{F}}^n \times_{\mathfrak{H}_{X,n-2}} \mathfrak{R}_{\mathcal{F}}^n \rightarrow \mathfrak{R}_{\mathcal{F}}^n$ between categories over $\mathfrak{H}_{X,n-2}$, where a triple consisting of $(U'_\bullet, \mathcal{T}', \varphi')$ and $(U''_\bullet, \mathcal{T}'', \varphi'')$ together with an isomorphism $\theta_\bullet : U'_\bullet \rightarrow U''_\bullet$ is mapped to the obvious wedge product taking θ_\bullet into account. Summing up, we have the following proposition.

PROPOSITION 6.2. *The forgetful functor $\mathfrak{R}_{\mathcal{F}}^n \rightarrow \mathfrak{H}_{X,n-2}$ is a fibered Picard category.*

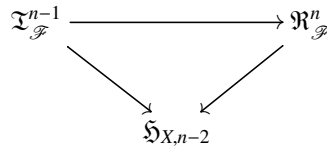
To proceed, let $\mathfrak{T}_{\mathcal{F}}^{n-1}$ be the fibered Picard category whose objects are triples $(U_\bullet, \mathcal{T}_{U_{n-2}}, s)$, where U_\bullet is a hypercovering of type $n - 2$, $\mathcal{T}_{U_{n-2}}$ is an $\mathcal{F}|_{U_{n-2}}$ -torsor, and $s \in \Gamma(U_{n-1}, \mathcal{F})$ is a local section. The morphism

$$(U_\bullet, \mathcal{T}_{U_{n-2}}, s) \longrightarrow (U'_\bullet, \mathcal{T}'_{U_{n-2}}, s')$$

consists of a refinement $\theta_\bullet : U_\bullet \rightarrow U'_\bullet$ such that $s = s'|_{U_{n-1}}$, together with a morphism $\tau : \mathcal{T}_{U_{n-2}} \rightarrow \mathcal{T}'_{U_{n-2}}|_{U_{n-2}}$ of $\mathcal{F}|_{U_{n-2}}$ -torsors.

Note that in the special case $n = 1$, our hypercovering U_\bullet of type $r = -1$ takes constant values $U_d = X, d \geq 0$. Hence $\mathcal{T}_{U_{-1}}$ is a torsor over X endowed with a global section $s \in \Gamma(X, \mathcal{F})$. In particular, all objects in the fiber categories $\mathfrak{T}_{\mathcal{F}}^{-1}(U_\bullet)$ are isomorphic.

We have a commutative diagram



where the horizontal functor sends the triple $(U_\bullet, \mathcal{T}_{U_{n-2}}, s)$ to the rigidified torsor coboundary $(U_\bullet, r_{\text{alt}}^*(\mathcal{T}_{U_{n-2}}), p_{\text{alt}}^*(s) \cdot \varphi_{\text{can}})$, and the two diagonal arrows are the forgetful functors. The horizontal functor respects the monoidal structure. The cokernels

$$\text{Coker}(\pi_0(\mathfrak{T}_{\mathcal{F}}^{n-1}(U_\bullet)) \longrightarrow \pi_0(\mathfrak{R}_{\mathcal{F}}^n(U_\bullet)))$$

for the resulting homomorphisms between fiberwise groups of isomorphism classes depend functorially on the hypercovering U_\bullet . The transition maps arise from base-changes. We write the ensuing contravariant functor as

$$\mathfrak{H}_{X,n-2} \longrightarrow (\text{Ab}), \quad U_\bullet \longmapsto \pi_0(\mathfrak{R}_{\mathcal{F}}^n(U_\bullet)) / \pi_0(\mathfrak{T}_{\mathcal{F}}^{n-1}(U_\bullet)).$$

The ensuing direct limit

$$\text{RTC}^n(\mathcal{F}) = \varinjlim \pi_0(\mathfrak{R}_{\mathcal{F}}^n(U_\bullet)) / \pi_0(\mathfrak{T}_{\mathcal{F}}^{n-1}(U_\bullet))$$

in the category of sets is called the set of *equivalence classes of rigidified torsor cocycles* for the abelian sheaf \mathcal{F} in degree $n \geq 1$. Here the direct limit runs over all objects $U_\bullet \in \mathfrak{H}_{X,n-2}^{\text{op}}$.

It is not *a priori* clear that the set $\text{RTC}^n(\mathcal{F})$ inherits a group structure, because the category $\mathfrak{S}_{X,n-2}^{\text{op}}$ is usually not filtered. We thus have to check that the transition maps depend only on homotopy classes of refinements, that is, the functor factors over the quotient category $\overline{\mathfrak{S}}_{X,n-2}^{\text{op}}$. To see this, suppose we have a homotopy $h_0, \dots, h_n : U_n \rightarrow V_{n+1}$ between two refinements $\theta_\bullet, \zeta_\bullet : U_\bullet \rightarrow V_\bullet$, as discussed in Section 3. Write $h_{\text{alt}}^*(\mathcal{P}) = h_0^*(\mathcal{P}) \wedge h_1^*(\mathcal{P}^{-1}) \wedge \dots \wedge h_n^*(\mathcal{P}^{(-1)^n})$ for each $\mathcal{F}|V_{n+1}$ -torsor \mathcal{P} .

LEMMA 6.3. *Let V_\bullet be a semi-simplicial covering and \mathcal{T} be an $\mathcal{F}|V_{n-1}$ -torsor. With the above notation, we have a canonical identification*

$$\theta_{n-1}^*(\mathcal{T}) \wedge r_{\text{alt}}^*(h_{\text{alt}}^*(\mathcal{T}^{-1})) = \zeta_{n-1}^*(\mathcal{T}) \wedge h_{\text{alt}}^*(p_{\text{alt}}^*(\mathcal{T})) \tag{6-1}$$

of $\mathcal{F}|U_{n-1}$ -torsors, which is natural in \mathcal{T} . Here p_i and r_k denote face operators defined in degree n and $n - 1$, respectively.

PROOF. The torsor $h_{\text{alt}}^*(p_{\text{alt}}^*(\mathcal{T}))$ on the right-hand side is a wedge product of torsors $h_j^*(p_i^*(\mathcal{T}))^\epsilon$, with exponents $\epsilon = (-1)^{i+j}$ and indices $0 \leq i \leq n$ and $0 \leq j \leq n - 1$. Now recall from Section 3 the simplicial homotopy identities (3-1). In the two boundary cases $i = j = 0$ and $i = j + 1 = n$ we get $\theta_{n-1}^*(\mathcal{T})$ and $\zeta_{n-1}^*(\mathcal{T})^{-1}$, respectively. The two diagonal strips $1 \leq i = j \leq n$ and $1 \leq i = j + 1 \leq n + 1$ give torsors $h_i^*(p_i^*(\mathcal{T}))$ and $h_i^*(p_{i-1}^*(\mathcal{T}))^{-1}$ that are inverse to each other. The remaining $n(n + 1) - 2 - 2(n - 1) = n(n - 1)$ torsors take the form

$$h_j^*(p_i^*(\mathcal{T}))^\epsilon = \begin{cases} r_i^*(h_{j-1}^*(\mathcal{T}))^\epsilon & \text{if } 0 \leq i < j \leq n + 1, \\ r_{i-1}^*(h_j^*(\mathcal{T}))^\epsilon & \text{if } n + 1 \geq i > j + 1 > 0. \end{cases}$$

These comprise the factors in the torsor $r_{\text{alt}}^*(h_{\text{alt}}^*(\mathcal{T}^{-1}))$ occurring on the left hand side of the natural identification. Summing up, the simplicial homotopy identities, together with $\mathcal{P} \wedge \mathcal{P}^{-1} = \mathcal{F}|U_{n-1}$, give the desired identification, which is therefore natural in \mathcal{T} . □

This ensures the desired homotopy invariance.

PROPOSITION 6.4. *Let $(V_\bullet, \mathcal{T}, \varphi)$ be a rigidified torsor cocycle in degree $n \geq 1$, and $\theta_\bullet, \zeta_\bullet : U_\bullet \rightarrow V_\bullet$ be two homotopic refinements. Then the resulting transition maps*

$$\pi_0(\mathfrak{R}_{\mathcal{F}}^n(V_\bullet))/\pi_0(\mathfrak{T}_{\mathcal{F}}^{n-1}(V_\bullet)) \longrightarrow \pi_0(\mathfrak{R}_{\mathcal{F}}^n(U_\bullet))/\pi_0(\mathfrak{T}_{\mathcal{F}}^{n-1}(U_\bullet))$$

induced by θ_\bullet and ζ_\bullet coincide.

PROOF. It suffices to treat the case where there is a homotopy $h_0, \dots, h_d : U_d \rightarrow V_{d+1}$ between the refinements θ_\bullet and ζ_\bullet . The idea is to extend the natural identification of $\mathcal{F}|U_{n-1}$ -torsors in Lemma 6.3 to an isomorphism of rigidified torsor cocycles, where the additional factors

$$\mathcal{T}' = r_{\text{alt}}^*(h_{\text{alt}}^*(\mathcal{T})) \quad \text{and} \quad \mathcal{T}'' = h_{\text{alt}}^*(p_{\text{alt}}^*(\mathcal{T}))$$

are equipped with the structure of rigidified torsor coboundaries.

For \mathcal{T}' , we take the canonical section $\varphi' = \varphi_{\text{can}}$ for the iterated alternating preimage $q_{\text{alt}}^*(\mathcal{T}')$ of $h_{\text{alt}}^*(\mathcal{T})$. Note that in the special case $\mathcal{T} = \mathcal{F}|_{V_{n-1}}$, this becomes $\mathcal{T}' = \mathcal{F}|_{U_{n-1}}$ and $\varphi' = 0$.

For \mathcal{T}'' , we use the given $\varphi \in \Gamma(V_n, p_{\text{alt}}^*(\mathcal{T}))$ to define $\varphi'' = -p_{\text{alt}}^*(h_{\text{alt}}^*(\varphi))$. The ensuing triple $(U_\bullet, \mathcal{T}'', \varphi'')$ is indeed a rigidified coboundary: using the identification $\mathcal{T}'' = \mathcal{F}|_{U_{n-1}}$ stemming from $h_{\text{alt}}^*(\varphi)$, we get $\mathcal{T}'' = \mathcal{F}|_{U_{n-1}} = r_{\text{alt}}^*(\mathcal{F}|_{U_{n-2}})$ and $\varphi'' = 0$, and in particular $q^*(\varphi'') = 0$.

It remains to see that the natural identification (6-1) is compatible with the four rigidifications $\theta_n^*(\varphi), \varphi', \zeta_n^*(\varphi), \varphi''$. Applying p_{alt}^* and using that θ_\bullet and ζ_\bullet are natural transformations, we get

$$\theta_n^*(p_{\text{alt}}^*(\mathcal{T})) \wedge p_{\text{alt}}^*(\mathcal{T}') = \zeta_n^*(p_{\text{alt}}^*(\mathcal{T})) \wedge p_{\text{alt}}^*(\mathcal{T}'').$$

This is an identification of $\mathcal{F}|_{U_n}$ -torsors, which is natural in the $\mathcal{F}|_{V_{n-1}}$ -torsor \mathcal{T} . Our task is to verify

$$\theta_n^*(\varphi) \wedge \varphi' = \zeta_n^*(\varphi) \wedge \varphi'' \tag{6-2}$$

as sections in the above torsor. This problem is local in U_n ; to check it we may refine our hypercoverings, and even increase their type. Now choose some covering $\tilde{V}_{n-1} \rightarrow V_{n-1}$ on which \mathcal{T} acquires a section, and form the fiber product $\tilde{U}_{n-1} = \tilde{V}_{n-1} \times_{V_{n-1}} U_{n-1}$. Setting $\tilde{V}_d = V_d$ and $\tilde{U}_d = U_d$ for $d < n - 1$, we obtain truncated semi-simplicial coverings and hypercoverings

$$\tilde{U}_\bullet = \text{cosk}_{n-1}(\tilde{U}_{\leq n-1}) \quad \text{and} \quad \tilde{V}_\bullet = \text{cosk}_{n-1}(\tilde{V}_{\leq n-1})$$

of type $r = n - 1$. In order to check (6-2), we may restrict to the new hypercoverings. Replacing the old hypercoverings by the new ones and using the naturalness of our construction, we have reduced the problem to the special case $\mathcal{T} = \mathcal{F}|_{V_{n-1}}$ and thus $\varphi' = 0 = h_{\text{alt}}^*(q_{\text{alt}}^*(\varphi))$.

Consider the cochain complexes $\Gamma(V_\bullet, \mathcal{F})$ and $\Gamma(U_\bullet, \mathcal{F})$ of abelian groups, where the coboundary operators ∂ are alternating sums as in (3-2). The refinements induce cochain maps $\theta_\bullet^*, \zeta_\bullet^* : \Gamma(V_\bullet, \mathcal{F}) \rightarrow \Gamma(U_\bullet, \mathcal{F})$. With the notation from Lemma 3.1, we have $\theta_d^* - \zeta_d^* = s\partial - \partial s$ in all degrees $d \geq 0$. For $d = n$, we find

$$\theta_n^*(\varphi) + \varphi' = \theta_n^*(\varphi) - h_{\text{alt}}^*(q_{\text{alt}}^*(\varphi)) = \zeta_n^*(\varphi) - p_{\text{alt}}^*(h_{\text{alt}}^*(\varphi)) = \zeta_n^*(\varphi) + \varphi''.$$

It follows that equality (6-2) holds. □

Let us unravel our setup in degree $n = 1$. Recall that a hypercovering U_\bullet of type $r = -1$ is given by $U_m = X$ for all $m \geq 0$, and every face operator is the identity on X . Hence the alternating preimage for the face operators $p_0, p_1 : U_1 \rightarrow U_0$ is given by $p_{\text{alt}}^*(\mathcal{T}) = \mathcal{T} \wedge \mathcal{T}^{-1} = \mathcal{F}$. Likewise, the iterated alternating preimage is

$$q_{\text{alt}}^*(p_{\text{alt}}^*(\mathcal{T})) = q_{\text{alt}}^*(\mathcal{F}) = \mathcal{F} \wedge \mathcal{F}^{-1} \wedge \mathcal{F} = \mathcal{F},$$

such that $q_{\text{alt}}^*(\varphi) = \varphi\varphi^{-1}\varphi = \varphi$. In turn, a rigidified torsor cocycle $(U_\bullet, \mathcal{T}, \varphi)$ in degree $n = 1$ is entirely determined by the \mathcal{F} -torsor \mathcal{T} over the final object $U_0 = X$, and φ is the zero-section of $\mathcal{T} \wedge \mathcal{T}^{-1} = \mathcal{F}|_{U_n}$. Using the description of $H^1(X, \mathcal{F}) = \pi_0(\mathcal{F}\text{-Tors})$, we immediately get the following result.

PROPOSITION 6.5. *There is a functorial identification $\text{RTC}^1(\mathcal{F}) = H^1(X, \mathcal{F})$.*

The situation is more challenging in degree $n = 2$. Suppose we have an \mathcal{F} -gerbe $\mathcal{G} \rightarrow \mathcal{C}$. Choose a covering $U \rightarrow X$ over which there is an object $T \in \mathcal{G}_U$. The fiber products $U_m = U^{m+1}$ form a Čech covering U_\bullet , that is, a hypercovering of type $r = 0$. Write

$$U_3 \rightrightarrows^{q_j} U_2 \rightrightarrows^{p_i} U_1 \rightrightarrows^{r_k} U_0 \longrightarrow X$$

for the face operators defined in degrees up to three. We get two objects $T_0 = r_0^*(T)$ and $T_1 = r_1^*(T)$ in the fiber category \mathcal{G}_{U_1} , and thus an $\mathcal{F}|U_1$ -torsor $\mathcal{T} = \underline{\text{Hom}}(T_1, T_0)$. To compute the alternating preimage $p_{\text{alt}}^*(\mathcal{T})$, we use the three simplicial identities $r_0 p_1 = p_0 r_1$, $r_0 p_2 = r_1 p_0$ and $r_1 p_2 = r_1 p_1$ to get

$$\begin{aligned} p_0^*(\mathcal{T})^{-1} &= \underline{\text{Hom}}(p_0^* r_0^* T, p_0^* r_1^* T), \\ p_1^*(\mathcal{T})^{-1} &= \underline{\text{Hom}}(p_1^* r_0^* T, p_1^* r_1^* T) = \underline{\text{Hom}}(p_0^* r_0^* T, p_1^* r_1^* T), \\ p_2^*(\mathcal{T})^{-1} &= \underline{\text{Hom}}(p_2^* r_0^* T, p_2^* r_1^* T) = \underline{\text{Hom}}(p_0^* r_1^* T, p_1^* r_1^* T). \end{aligned}$$

Composition for hom sets yields a canonical isomorphism of $\mathcal{F}|U^2$ -torsors

$$\varphi : p_2^*(\mathcal{T})^{-1} \wedge p_0^*(\mathcal{T})^{-1} \longrightarrow p_1^*(\mathcal{T})^{-1}, \quad (h_2, h_0) \longmapsto h_2 \circ h_0 \tag{6-3}$$

which can be regarded as a section $\varphi \in \Gamma(U_2, p_{\text{alt}}^*(\mathcal{T}))$. This gives us a triple $(U_\bullet, \mathcal{T}, \varphi)$.

PROPOSITION 6.6. *The triple $(U_\bullet, \mathcal{T}, \varphi)$ is a rigidified torsor cocycle.*

PROOF. We have to check that $q_{\text{alt}}^*(\varphi) = 0$ as section of the iterated alternating preimages $q_{\text{alt}}^*(p_{\text{alt}}^*(\mathcal{T})) = \mathcal{F}|U^3$. Choose a covering $V \rightarrow U_1$ over which the $\mathcal{F}|U_2$ -torsor \mathcal{T} acquires a section. This section defines an isomorphism $r_1^*(T)|V \rightarrow r_0^*(T)|V$ of objects in the fiber category \mathcal{G}_V , and we write $h : r_0^*(T)|V \rightarrow r_1^*(T)|V$ for its inverse. Consider the truncated semi-simplicial covering $V_{\leq 2}$ with $V_2 = V$, $V_1 = U_1$ and $V_0 = U_0$, and the ensuing hypercovering $V_\bullet = \text{cosk}_2(V_{\leq 2})$ of type two. The composition (6-3) of hom sets defines via the equation

$$p_2^*(h) \circ p_0^*(h) = f \cdot p_1^*(h)$$

a local section $f \in \Gamma(V_2, \mathcal{F})$. Using the six simplicial identities $p_i q_j = p_{j-1} q_i$ for $i < j$, the above equality gives four equations:

$$\begin{aligned} q_0^* p_2^*(h) \circ q_0^* p_0^*(h) &= q_0^*(f) \cdot q_0^* p_1^*(h), \\ q_1^* p_2^*(h) \circ q_0^* p_0^*(h) &= q_1^*(f) \cdot q_1^* p_1^*(h), \\ q_2^* p_2^*(h) \circ q_0^* p_1^*(h) &= q_2^*(f) \cdot q_1^* p_1^*(h), \\ q_2^* p_2^*(h) \circ q_0^* p_2^*(h) &= q_3^*(f) \cdot q_1^* p_2^*(h). \end{aligned}$$

Here $q_j^*(f) \in \Gamma(V_3, \mathcal{F})$, and the equations hold as morphisms $q_j^* p_i^* r_0^*(T) \rightarrow q_j^* p_i^* r_1^*(T)$ in the fiber category \mathcal{G}_{V_3} . In the above four equations, each of the terms $q_j^* p_i^*(h)$

appears twice, and the pair members come with ‘opposite signs’ in the alternating preimage. This ensures $q_0^*(f) + q_2^*(f) = q_1^*(f) + q_3^*(f)$. By definition of the iterated alternating preimage, one has

$$q_{\text{alt}}^*(\varphi)|V_3 = q_0^*(f) - q_1^*(f) + q_2^*(f) - q_3^*(f).$$

In turn, we have $q_{\text{alt}}^*(\varphi)|V_3 = 0$. Since the induced morphism $V_3 \rightarrow U_3$ is a covering, the sheaf axiom ensures that $q_{\text{alt}}^*(\varphi) = 0$. \square

It is easy to see that the class of $(U_\bullet, \mathcal{T}, \varphi)$ in the group $\text{RTC}^2(\mathcal{F})$ is independent of the choice of the equivalence class of the \mathcal{F} -gerbe $\mathfrak{G} \rightarrow \mathfrak{C}$, the covering $U \rightarrow X$, and the object $T \in \mathfrak{G}_U$. One thus gets a well-defined map

$$H^2(X, \mathcal{F}) \longrightarrow \text{RTC}^2(\mathcal{F}), \quad \mathfrak{G} \longmapsto (U_\bullet, \mathcal{T}, \varphi).$$

In the next section, we shall see that this map is bijective.

7. The comparison map

We keep the assumptions of the preceding section, such that \mathfrak{C} is a site having a final object $X \in \mathfrak{C}$, and that the Grothendieck topology is given by a pretopology $\text{Cov}(V)$, $V \in \mathfrak{C}$, of covering families $(U_\lambda \rightarrow V)_{\lambda \in L}$. Furthermore, we suppose that for each covering family $(U_\lambda \rightarrow V)_{\lambda \in L}$, the disjoint union $U = \bigcup U_\lambda$ exists, such that each covering family can be refined to a covering singleton. For each abelian sheaf \mathcal{F} , fix an injective resolution $0 \rightarrow \mathcal{F} \rightarrow \mathcal{I}^0 \rightarrow \dots$, such that sheaf cohomology becomes $H^n(X, \mathcal{F}) = H^n\Gamma(X, \mathcal{I}^\bullet)$.

The goal of this section is to identify the group $\text{RTC}^n(\mathcal{F})$ of equivalence classes of rigidified torsor cocycles with the cohomology group $H^n(X, \mathcal{F})$, for each degree $n \geq 1$. The crucial ingredient is the three-term complex $C^{n-1} \xrightarrow{\Psi} C^n \xrightarrow{\Phi} C^{n+1}$ constructed in Section 4. The main task is to define the *comparison map*

$$\text{RTC}^n(\mathcal{F}) \longrightarrow \varinjlim \text{Ker}(\Phi) / \text{Im}(\Psi) = H^n(X, \mathcal{F}),$$

where the identification on the right comes from Theorem 4.3. We start by defining the comparison map on objects $(U_\bullet, \mathcal{T}, \varphi)$ from a fixed fiber category $\mathfrak{R}_{\mathcal{F}}^n(U_\bullet)$. Recall that $\mathfrak{R}_{\mathcal{F}}^n \rightarrow \mathfrak{S}_{X, n-2}$ is the fibered Picard category of rigidified torsor cocycles.

Consider the induced $\mathcal{I}^0|U_{n-1}$ -torsor $\mathcal{T}^0 = (\mathcal{I}^0|U_{n-1}) \wedge^{\mathcal{F}|U_{n-1}} \mathcal{T}$, obtained by extending the structure sheaf with respect to the inclusion $\mathcal{F} \subset \mathcal{I}^0$. This torsor admits a section $s \in \Gamma(U_{n-1}, \mathcal{T}^0)$, because \mathcal{I}^0 is an injective and hence acyclic sheaf. The resulting bijection $\mathcal{T}^0 \rightarrow \mathcal{I}^0|U_{n-1}$ yields an injection $\mathcal{T} \subset \mathcal{I}^0|U_{n-1}$, whose image under the vertical differential $\epsilon d : \mathcal{I}^0 \rightarrow \mathcal{I}^1$ can be regarded as a section $\alpha_{n-1} \in \Gamma(U_{n-1}, \mathcal{I}^1)$ mapping to zero in $\Gamma(U_{n-1}, \mathcal{I}^2)$. Here $\epsilon = (-1)^{n-1}$ is the sign introduced for the double complex.

Next, consider the section $\varphi \in \Gamma(U_n, p_{\text{alt}}^*(\mathcal{T}))$ for the alternating preimage. Recall that $p_i : U_n \rightarrow U_{n-1}$, $0 \leq i \leq n$, denote the face operators in the hypercovering U_\bullet . The inclusion $\mathcal{T} \subset \mathcal{T}^0 = \mathcal{I}^0|U_{n-1}$ induces an inclusion of alternating preimages

$$p_{\text{alt}}^*(\mathcal{T}) \subset p_{\text{alt}}^*(\mathcal{T}^0) = p_{\text{alt}}^*(\mathcal{I}^0|U_{n-1}) = \mathcal{I}^0|U_n,$$

so our section φ becomes an element $\alpha_n \in \Gamma(U_n, \mathcal{I}^0)$. In turn, we obtain a cochain $(\alpha_{n-1}, \alpha_n) \in C^n$ in the three-term complex $C^{n-1} \xrightarrow{\Psi} C^n \xrightarrow{\Phi} C^{n+1}$, and the comparison map will be given by the assignment

$$(U_\bullet, \mathcal{T}, \varphi) \mapsto (\alpha_{n-1}, \alpha_n). \tag{7-1}$$

This cochain is a cocycle. We have already remarked that $d(\alpha_{n-1}) = 0$. By construction, α_n is the section φ of the subsheaf $p_{\text{alt}}^*(\mathcal{T}) \subset \mathcal{I}^0|_{U_n}$, hence $-\epsilon d(\alpha_n) = \delta(\alpha_{n-1})$. Furthermore, we have $q_{\text{alt}}^*(\varphi) = 0$ as section of the iterated alternating preimage $q_{\text{alt}}^*(p_{\text{alt}}^*(\mathcal{T}))$, which ensures $\partial(\alpha_n) = 0$. Summing up, the pair (α_{n-1}, α_n) lies in the kernel of the differential Φ , hence is a cocycle.

This attaches to each object $(U_\bullet, \mathcal{T}, \varphi) \in \mathfrak{R}_{\mathcal{F}}^n(U_\bullet)$ an element $(\alpha_{n-1}, \alpha_n) \in \text{Ker}(\Phi)$. Note that the assignment depends on the choice of sections $s \in \Gamma(U_{n-1}, \mathcal{I}^0)$. Passing to cohomology gives a map

$$\mathfrak{R}_{\mathcal{F}}^n(U_\bullet) \longrightarrow \text{Ker}(\Phi)/\text{Im}(\Psi) \longrightarrow H^n(X, \mathcal{F}), \quad (U_\bullet, \mathcal{T}, \varphi) \mapsto (\alpha_{n-1}, \alpha_n). \tag{7-2}$$

This map does not depend anymore on the choice of sections: any other section is of the form $s' = s + \beta_{n-1}$ for some unique $\beta_{n-1} \in \Gamma(U_{n-1}, \mathcal{I}^0)$. One easily checks that the resulting cocycle $(\alpha'_{n-1}, \alpha'_n)$ differs by the coboundary $\Psi(0, \beta_{n-1})$. The following proposition is also immediate.

PROPOSITION 7.1. *The above map (7-2) sends isomorphic objects to the same cohomology class, and turns wedge products in the Picard category $\mathfrak{R}_{\mathcal{F}}^n(U_\bullet)$ into addition of classes. Furthermore, for each $\mathcal{F}|_{U_{n-2}}$ -torsor \mathcal{T}_{n-2} and each local section $s \in \Gamma(U_{n-1}, \mathcal{F})$, the rigidified torsor coboundary $(U_\bullet, \mathcal{T}, \varphi)$ given by $\mathcal{T} = r_{\text{alt}}^*(\mathcal{T}_{n-2})$ and $\varphi = p_{\text{alt}}^*(s) \cdot \varphi_{\text{can}}$ is sent to the zero class.*

Clearly, our map is functorial in U_\bullet . Recall that the group $\text{RTC}^n(\mathcal{F})$ is the direct limit for the functor

$$\overline{\mathfrak{H}}_{X, n-2}^{\text{op}} \longrightarrow (\text{Ab}), \quad U_\bullet \mapsto \pi_0(\mathfrak{R}_{\mathcal{F}}^n(U_\bullet))/\pi_0(\mathfrak{T}_{\mathcal{F}}^{n-1}(U_\bullet))$$

which is defined on the quotient category of hypercoverings of type $r = n - 2$. Passing to direct limits, the maps in (7-2) give the desired comparison map

$$\text{RTC}^n(\mathcal{F}) \longrightarrow \varinjlim \text{Ker}(\Phi)/\text{Im}(\Psi) = H^n(X, \mathcal{F}).$$

We now come to our main result, which gives the desired geometric interpretation of higher cohomology.

THEOREM 7.2. *For each abelian sheaf \mathcal{F} and each degree $n \geq 1$, the comparison map is bijective, such that we have an identification $\text{RTC}^n(\mathcal{F}) = H^n(X, \mathcal{F})$.*

PROOF. To see surjectivity, we represent a given cohomology class $[\alpha] \in H^n(X, \mathcal{F})$ by a cocycle (α_{n-1}, α_n) in the three-term complex $C^{n-1} \xrightarrow{\Psi} C^n \xrightarrow{\Phi} C^{n+1}$, with respect to some

hypercovering U_\bullet of type $r = n - 2$. In particular, the entry $\alpha_{n-1} \in \Gamma(U_{n-1}, \mathcal{F}^1)$ is a local section that vanishes in $\Gamma(U_{n-1}, \mathcal{F}^2)$. The cartesian square of set-valued sheaves

$$\begin{array}{ccc} \mathcal{T} & \longrightarrow & h_{U_{n-1}} \\ \downarrow & & \downarrow \alpha_{n-1} \\ \mathcal{F}^0|_{U_{n-1}} & \xrightarrow{\epsilon d} & \mathcal{F}^1|_{U_{n-1}} \end{array}$$

defines an $\mathcal{F}|_{U_{n-1}}$ -torsor \mathcal{T} . Here $\epsilon = (-1)^{n-1}$ is the sign introduced for the double complex. One easily checks that its alternating preimage sits in the cartesian square

$$\begin{array}{ccc} p_{\text{alt}}^*(\mathcal{T}) & \xrightarrow{\quad \varphi \quad} & h_{U_n} \\ \downarrow & \swarrow -\alpha_n & \downarrow \partial(\alpha_{n-1}) \\ \mathcal{F}^0|_{U_n} & \xrightarrow{\epsilon d} & \mathcal{F}^1|_{U_n} \end{array}$$

The dotted diagonal arrow arises from the entry $\alpha_n \in \Gamma(U_n, \mathcal{F}^0)$, in light of the cocycle condition $\partial(\alpha_{n-1}) = -\epsilon d(\alpha_n)$. This dotted diagonal arrow corresponds to a section $\varphi \in \Gamma(U_n, p_{\text{alt}}^*(\mathcal{T}))$. In turn, we obtain a triple $(U_\bullet, \mathcal{T}, \varphi)$. The condition $\partial(\alpha_n) = 0$ ensures that $q_{\text{alt}}^*(\varphi) = 0$, hence $(U_\bullet, \mathcal{T}, \varphi)$ is a rigidified torsor cocycle.

We now check that $(U_\bullet, \mathcal{T}, \varphi) \mapsto (\alpha_n, \alpha_{n-1})$ under the comparison map, as described on objects in (7-1). Indeed, the inclusion $\mathcal{T} \subset \mathcal{F}^0|_{U_{n-1}}$ yields a canonical isomorphism

$$\mathcal{T}^0 = (\mathcal{F}^0|_{U_{n-1}}) \wedge^{\mathcal{F}|_{U_{n-1}}} \mathcal{T} \longrightarrow \mathcal{F}^0|_{U_{n-1}},$$

and we may take for the section $s \in \Gamma(U_{n-1}, \mathcal{T}^0)$ the zero-section of $\mathcal{F}^0|_{U_{n-1}}$. Then the image of $\mathcal{T} \subset \mathcal{F}^0|_{U_{n-1}}$ under the differential ϵd is the entry $\alpha_{n-1} \in \Gamma(U_{n-1}, \mathcal{F}^1)$. Furthermore, the section φ corresponds to $-\alpha_n \in \Gamma(U_n, \mathcal{F}^0)$. Thus the cocycle (α_{n-1}, α_n) lies in the image.

It remains to check that the comparison map is injective. Let $(U_\bullet, \mathcal{T}, \varphi)$ be a rigidified torsor cocycle, and suppose that the comparison map (7-1) sends it to a cocycle (α_{n-1}, α_n) in the three-term complex $C^{n-1} \xrightarrow{\Psi} C^n \xrightarrow{\Phi} C^{n+1}$ that vanishes in $H^n(X, \mathcal{F})$. Refining the hypercovering U_\bullet , we may assume that (α_{n-1}, α_n) is already the coboundary of some $(\beta_{n-2}, \beta_{n-1})$; in other words,

$$\alpha_{n-1} = \epsilon d(\beta_{n-1}) + \sum_{k=0}^{n-1} (-1)^k r_k^*(\beta_{n-2}) \quad \text{and} \quad \alpha_n = \sum_{i=0}^n (-1)^i p_i^*(\beta_{n-1}).$$

The local section $\beta_{n-2} \in \Gamma(U_{n-2}, \mathcal{F}^0/\mathcal{F})$ yields an $\mathcal{F}|_{U_{n-2}}$ -torsor $\mathcal{T}_{U_{n-2}}$, by taking the preimage sheaf of this local section. Adding the rigidified torsor coboundary $(U_\bullet, r_{\text{alt}}^*(\mathcal{T}^{-1}), \varphi_{\text{can}})$, we reduce to the situation $\beta_{n-2} = 0$, hence $\alpha_{n-1} = \epsilon d(\beta_{n-1})$. By definition of the comparison map, $\mathcal{T} \subset \mathcal{F}^0|_{U_{n-1}}$ is the preimage sheaf of $\alpha_{n-1} \in$

$\Gamma(U_{n-1}, \mathcal{I}^1)$, so $\epsilon\beta_{n-1} \in \Gamma(U_{n-1}, \mathcal{I}^0)$ defines a section for \mathcal{I} . In turn, we may assume that $\mathcal{I} = \mathcal{F}|_{U_{n-1}}$ and $\beta_{n-1} = 0$. Consequently $\alpha_n = 0$, which means that the section $\varphi \in \Gamma(U_{n+1}, q_{\text{alt}}^*(\mathcal{I}))$ coincides with the zero-section. Summing up, our rigidified torsor cocycle is a rigidified torsor coboundary. \square

Let us record the following two consequences.

COROLLARY 7.3. *Two rigidified \mathcal{F} -torsor n -cocycles $A' = (U'_\bullet, \mathcal{I}', \varphi')$ and $A'' = (U''_\bullet, \mathcal{I}'', \varphi'')$ have the same cohomology class in $H^n(X, \mathcal{F})$ if and only if there exist a common refinement $U'_\bullet \leftarrow U_\bullet \rightarrow U''_\bullet$ and some rigidified torsor coboundary $B = (U_\bullet, r_{\text{alt}}^*(\mathcal{I}_{n-2}), p_{\text{alt}}^*(s) \cdot \varphi_{\text{can}})$ such that $A'|U_\bullet \simeq (A''|U_\bullet) \wedge B$.*

COROLLARY 7.4. *A rigidified \mathcal{F} -torsor n -cocycle $A' = (U'_\bullet, \mathcal{I}', \varphi')$ has trivial cohomology class in $H^n(X, \mathcal{F})$ if and only if there exist a refinement $U_\bullet \rightarrow U'_\bullet$ and some rigidified torsor coboundary $B = (U_\bullet, r_{\text{alt}}^*(\mathcal{I}_{n-2}), p_{\text{alt}}^*(s) \cdot \varphi_{\text{can}})$ such that $A'|U_\bullet \simeq B$.*

8. Bundle gerbes and Dixmier–Douady classes

Let us now connect our theory of rigidified torsor cocycles with Murray’s notion of bundle gerbes from [29, Section 3]; see also [32, Section 2]. Let M be a differential manifold, $\pi : Y \rightarrow M$ be a fibration, and $P \rightarrow Y^{[2]}$ be a \mathbb{C}^\times -principal bundle. In this context, Y is also a differentiable manifold and fibrations denote differentiable maps that are surjective on points and tangent vectors. The total space Y is allowed to be infinite-dimensional, and the fibration $Y \rightarrow M$ is assumed to admit local sections. Conforming with the notation in [32], we here write $Y^{[2]} = Y \times_M Y$ for the fiber product. Now define

$$Y^{[2]} \circ Y^{[2]} \subset Y^{[2]} \times Y^{[2]}$$

as the set of all pairs of the form $((a, b), (b, c))$, with $a, b, c \in Y$ all mapping to the same point in M . Write $\pi_i : Y^{[2]} \circ Y^{[2]} \rightarrow Y^{[2]}$ with $i = 1, 2$ for the two projections.

A bundle gerbe consists of the choice of a fibration $\pi : Y \rightarrow M$ and a principal \mathbb{C}^\times -bundle $P \rightarrow Y^{[2]}$, together with a map of \mathbb{C}^\times -bundles

$$\mu : \pi_1^{-1}(P) \otimes \pi_2^{-1}(P) \longrightarrow P \tag{8-1}$$

covering the map

$$\pi_3 : Y^{[2]} \circ Y^{[2]} \longrightarrow Y^{[2]}, \quad ((a, b), (b, c)) \longmapsto (a, c). \tag{8-2}$$

The product in (8-1) is assumed to be *associative* whenever triple products in (8-2) are defined. Bundle gerbes arising from $P = \pi_1^{-1}(Q^*) \otimes \pi_2^{-1}(Q)$, where $Q \rightarrow Y$ is a principal \mathbb{C}^\times -bundle, are called *trivial*.

Let us translate this into the setup and notation of the present paper. The tensor product in (8-1) is vector bundle notation for the contracted product of \mathbb{C}^\times -bundles. Furthermore, the morphism can be regarded as a map $\mu : \pi_1^*(P) \otimes \pi_2^*(P) \rightarrow \pi_3^*(P)$

of principal \mathbb{C}^\times -bundles over $Y^{[2]} \circ Y^{[2]}$. Using semi-simplicial notation, we write $Y_d = Y^{[d+1]}$. Clearly, the canonical map

$$Y^{[2]} \circ Y^{[2]} \longrightarrow Y^{[3]} = Y_2, \quad ((a, b), (b, c)) \mapsto (a, b, c)$$

into the threefold fiber product is a homeomorphism. With respect to this identification, we have

$$\pi_1 = p_2, \quad \pi_2 = p_0, \quad \pi_3 = p_1$$

as face operators $Y_2 \rightarrow Y_1$, in simplicial notation. So μ in (8-1) may be regarded as a section into the alternating preimage $p_{\text{alt}}^*(P^{-1}) = p_0^*(P^{-1}) \otimes p_1^*(P) \otimes p_2^*(P^{-1})$ for the inverse principal bundle P^{-1} . On the other hand, we may regard μ as the collection of fiberwise maps of principal \mathbb{C}^\times -sets

$$\mu_{abc} : P_{(a,b)} \otimes P_{(b,c)} \longrightarrow P_{(a,c)},$$

where $a, b, c \in Y$ map to the same point in M . Write $\text{id}_{ab} : P_{(a,b)} \rightarrow P_{(a,b)}$ for the identity map. The associativity condition for bundle gerbes becomes

$$\mu_{acd} \circ (\mu_{abc} \otimes \text{id}_{cd}) = \mu_{abd} \circ (\text{id}_{ab} \otimes \mu_{bcd}) \tag{8-3}$$

as maps $P_{(a,b)} \otimes P_{(b,c)} \otimes P_{(c,d)} \rightarrow P_{(a,d)}$ of principal \mathbb{C}^\times -sets, for all $a, b, c, d \in Y$ mapping to the same point in M . For the following observation, recall that the $p_i : Y_2 \rightarrow Y_1$ and $q_j : Y_3 \rightarrow Y_2$ denote face operators.

PROPOSITION 8.1. *The associativity condition (8-3) for bundle gerbes holds if and only if $q_{\text{alt}}^*(\mu) = 1_{Y_3}$ with respect to the identification $q_{\text{alt}}^*(p_{\text{alt}}^*(P^{-1})) = \mathbb{C}^\times \times Y_3$ of iterated alternating preimages.*

PROOF. Over each point $(a, b, c, d) \in Y^{[4]} = Y_3$, the fiber of the iterated alternating preimage $q_{\text{alt}}^*(p_{\text{alt}}^*(P^{-1}))$ is the tensor product of the following 12 principal \mathbb{C}^\times -sets:

$$\begin{matrix} P_{(c,d)}^{-1}, & P_{(b,d)}, & P_{(b,c)}^{-1}, \\ P_{(c,d)}, & P_{(a,d)}^{-1}, & P_{(a,c)}, \\ P_{(b,d)}^{-1}, & P_{(a,d)}, & P_{(a,b)}^{-1}, \\ P_{(b,c)}, & P_{(a,c)}^{-1}, & P_{(a,b)}. \end{matrix}$$

Now choose for each of the six occurring points $(c, d), \dots, (a, b) \in Y^{[2]}$ some elements $s_{cd} \in P_{(c,d)}, \dots, s_{ab} \in P_{(a,b)}$. Regarding μ as a pairing, the four equations

$$\begin{aligned} \mu_{bcd}(s_{bc} \otimes s_{cd}) &= f_a s_{bd}, \\ \mu_{acd}(s_{ac} \otimes s_{cd}) &= f_b s_{ad}, \\ \mu_{abd}(s_{ab} \otimes s_{bd}) &= f_c s_{ad}, \\ \mu_{abc}(s_{ab} \otimes s_{bc}) &= f_d s_{ac} \end{aligned}$$

define scalars $f_a, \dots, f_d \in \mathbb{C}^\times$. The condition $q_{\text{alt}}^*(\mu) = 1_{Y_3}$ on the alternating preimage translates into $f_a f_c = f_b f_d$. Note that here we use multiplicative rather than additive notation. Applying the two sides of (8-3) to the element

$$s_{ab} \otimes s_{bc} \otimes s_{cd} \in P_{(a,b)} \otimes P_{(b,c)} \otimes P_{(c,d)},$$

we see that the associativity conditions is equivalent to $f_a f_c = f_b f_d$ as well. □

It is now straightforward to verify that Murray’s bundle gerbes correspond to our rigidified torsor cocycles for $n = 2$. Let $\mathcal{F} = \mathcal{C}_M^\times$ be the abelian sheaf of invertible complex-valued functions. Each principal \mathbb{C}^\times -bundle $P \rightarrow M$ yields the \mathcal{F} -torsor \mathcal{T} of local sections, and each \mathcal{F} -torsor \mathcal{T} yields the principal \mathbb{C}^\times -bundle defined as the relative spectrum of the \mathcal{C}_M -algebra $\bigoplus_{d \in \mathbb{Z}} \mathcal{L}^{\otimes -d}$, where $\mathcal{L} = \mathcal{C}_M \wedge^{\mathcal{C}_M^\times} \mathcal{T}$ is the invertible sheaf attached to the \mathcal{C}_M^\times -torsor. In this way, one gets an equivalence of categories between principal \mathbb{C}^\times -bundles and \mathcal{C}_M^\times -torsors.

In turn, a bundle gerbe, which consists of a fibration $P \rightarrow M$, a principal bundle $P \rightarrow Y^{[2]}$ and a pairing $\mu : \pi_1^{-1}(P) \otimes \pi_2^{-1}(P) \rightarrow P$ satisfying the associativity condition, corresponds to a covering $U = Y$ with respect to some suitable site \mathbb{C} , an $\mathcal{F}|U_1$ -torsor \mathcal{T} and a section $\varphi \in \Gamma(U_2, p_{\text{alt}}^*(\mathcal{T}))$ satisfying $q_{\text{alt}}^*(\varphi) = 1$, in multiplicative notation.

The Dixmier–Douady class arises as follows. The exponential sequence of abelian sheaves $0 \rightarrow 2\pi i\mathbb{Z}_M \rightarrow \mathcal{C}_M \rightarrow \mathcal{C}_M^\times \rightarrow 1$ induces long exact sequences

$$H^n(M, \mathcal{C}_M) \longrightarrow H^n(X, \mathcal{C}_M^\times) \longrightarrow H^n(M, 2\pi i\mathbb{Z}) \longrightarrow H^{n+1}(M, \mathcal{C}_M).$$

The outer terms vanish for $n \geq 1$, because the sheaf \mathcal{C}_M is soft, whence acyclic. In turn, we get the identifications

$$\text{RTC}^n(\mathcal{C}_M^\times) = H^n(M, \mathcal{C}_M^\times) = H^{n+1}(M, 2\pi i\mathbb{Z}).$$

Here one may interpret the right-hand side both as sheaf cohomology and singular cohomology. The integral cohomology class attached to a bundle gerbe or a rigidified torsor cocycle in degree $n = 2$ is called the *Dixmier–Douady class*. We see that it is defined for all degrees $n \geq 1$.

Note also that the theory works well if the differentiable manifold M is merely a topological space that is paracompact and locally contractible, because then sheaf cohomology with locally constant \mathbb{Z} -coefficients coincides with singular cohomology, as explained in [8, Ch. III].

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