

DIVISORIAL MODELS OF NORMAL VARIETIES

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Abstract We prove that the canonical ring of a canonical variety in the sense of de Fernex and Hacon is finitely generated. We prove that canonical varieties are Kawamata log terminal (klt) if and only if $\mathcal{R}_X(-K_X)$ is finitely generated. We introduce a notion of nefness for non- \mathbb{Q} -Gorenstein varieties and study some of its properties. We then focus on these properties for non- \mathbb{Q} -Gorenstein toric varieties.

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1. Introduction

In this paper we continue the investigation of singularities of non- \mathbb{Q} -Gorenstein varieties that was initiated in [6] and [12]. In particular, we focus on the study of canonical singularities and non- \mathbb{Q} -Gorenstein toric varieties.

In the context of the minimal model programme, there are some natural classes of singularities to be considered, such as canonical and log terminal (see [2]). Also, while running the programme, we naturally encounter non- \mathbb{Q} -Gorenstein varieties, for which the canonical divisor is not \mathbb{Q} -Cartier, as the target of small contractions when performing a flip. In 2009, de Fernex and Hacon [6] extended the notion of canonical and log terminal singularities to the case of non- \mathbb{Q} -Gorenstein varieties. It is natural to wonder which of the properties connected to these types of singularity are preserved when extending to this more general context.

In § 2 we will briefly recall the fundamental definitions and properties introduced in [6].

In § 3 we show that if X is canonical in the sense of [6], then the relative canonical ring $\mathcal{R}_X(K_X)$ is a finitely generated \mathcal{O}_X -algebra (Theorem 3.4). Thus, if X is canonical, there exists a small proper birational morphism $\pi: X' \rightarrow X$ such that $K_{X'}$ is \mathbb{Q} -Cartier and π -ample. As a corollary we obtain that the canonical ring of any normal variety with canonical singularities (in the sense of [6]) is finitely generated.

We next turn our attention to log terminal singularities. Recall that in [12] we gave an example of canonical singularities that are not log terminal. In this paper we show that, if X is canonical, then finite generation of the relative anti-canonical ring $\mathcal{R}_X(-K_X)$ is equivalent to X being log terminal (Proposition 3.7).

In §4 we introduce a notion of nefness for Weil divisors (on non- \mathbb{Q} -factorial varieties). We call such divisors quasi-nef (q-nef) and we study their basic properties. We prove that if X is a normal variety with canonical singularities such that K_X is q-nef, then $X' = \text{Proj}_X(\mathcal{R}_X(K_X))$ is a minimal model.

In §§5 and 6 we focus our attention on toric varieties. We give a complete description of quasi-nef divisors on toric varieties and we note that they correspond to divisors whose divisorial sheaf is globally generated.

And finally, in §7, we give a new natural definition of minimal log discrepancies in the new setting and prove that even in the toric case they do not satisfy the ACC (ascending chain condition) conjecture.

2. Background

We work over the complex numbers.

Recall the following definition of de Fernex and Hacon [6].

Definition 2.1. Let $f: Y \rightarrow X$ a proper projective birational morphism of normal varieties. Given any Weil divisor D on X , we define the pullback of D on Y as

$$f^*(D) = \sum_{P \text{ prime on } Y} \lim_{k \rightarrow \infty} \frac{v_P(\mathcal{O}_Y \cdot \mathcal{O}_X(-kD))}{k} P.$$

Note that if D is \mathbb{Q} -Cartier, then $f^*(D)$ coincides with the usual notion of pullback.

Using this definition, de Fernex and Hacon define canonical and log terminal singularities for non- \mathbb{Q} -Gorenstein varieties. As usual, this is done in terms of the relative canonical divisor $K_{Y/X}$, for $f: Y \rightarrow X$ a proper morphism. Note, however, that there are two different choices for the relative canonical divisor (which coincide in the \mathbb{Q} -Gorenstein setting):

$$K_{Y/X}^- := K_Y - f^*(K_X) \quad \text{and} \quad K_{Y/X}^+ := K_Y + f^*(-K_X).$$

We will not use $K_{Y/X}^-$ in this paper but recall that it is the natural choice that is used to define log terminal singularities and multiplier ideal sheaves.

Definition 2.2. X is said to be *canonical* if

$$\text{ord}_F(K_{Y/X}^+) \geq 0$$

for every exceptional prime divisor F over X .

Definition 2.3. X is said to be *log terminal* if and only if there is an effective \mathbb{Q} -divisor Δ such that

- Δ is a boundary ($K_X + \Delta$ is \mathbb{Q} -Cartier) and
- (X, Δ) is Kawamata log terminal (klt).

Let us also recall some standard definitions from the minimal model programme (see [2, 10]).

Definition 2.4. Let X be a normal projective variety and let D be a \mathbb{Q} -divisor on X . We define

$$R_X(D) := \bigoplus_{m \geq 1} H^0(\mathcal{O}_X(\lfloor mD \rfloor))$$

and

$$\mathcal{R}_X(D) := \bigoplus_{m \geq 1} \mathcal{O}_X(\lfloor mD \rfloor),$$

where the first will always be considered as a \mathbb{C} -algebra and the second as an \mathcal{O}_X -algebra.

One important property that we will often need in the background is the following result [10, Lemma 6.2].

Lemma 2.5. *Let X be a normal algebraic variety and let B be a Weil divisor on it. The following are equivalent:*

- (1) $\mathcal{R}_X(B)$ is a finitely generated \mathcal{O}_X -algebra;
- (2) there exists a small projective birational morphism $f: Y \rightarrow X$ such that Y is normal, and $\bar{B} := f_*^{-1}B$ is \mathbb{Q} -Cartier and f -ample.

Moreover, $f: Y \rightarrow X$ is the unique morphism with the following properties: $Y \cong \text{Proj}_X(\mathcal{R}_X(B))$ and, for all $m \geq 0$, $f_*\mathcal{O}_Y(m\bar{B}) = \mathcal{O}_X(mB)$.

3. Canonical singularities

In this section we will show that if X has canonical singularities, then its canonical ring is finitely generated.

De Fernex and Hacon gave the following characterization of canonical singularities.

Proposition 3.1 (de Fernex and Hacon [6, Proposition 8.2]). *Let X be a normal variety. Then X is canonical if and only if for all sufficiently divisible $m \geq 1$, and for every resolution $f: Y \rightarrow X$, there is an inclusion*

$$\mathcal{O}_X(mK_X) \cdot \mathcal{O}_Y \subseteq \mathcal{O}_Y(mK_Y)$$

as sub- \mathcal{O}_Y -modules of \mathcal{K}_Y .

Lemma 3.2 (Urbinati [12, Lemma 2.14]). *Let $f: Y \rightarrow X$ be a proper birational morphism such that Y is \mathbb{Q} -Gorenstein with canonical singularities. If $\mathcal{O}_Y \cdot \mathcal{O}_X(mK_X) \subseteq \mathcal{O}_Y(mK_Y)$ for any sufficiently divisible $m \geq 1$, then X is canonical.*

The following immediate corollary of Lemma 3.2 is very useful.

Corollary 3.3. *Let $f: Y \rightarrow X$ be a proper birational morphism such that Y is \mathbb{Q} -Gorenstein and canonical. If $\text{ord}_F(K_{Y/X}^+) \geq 0$ for all divisors F on Y , then X is canonical.*

Proof. Set $f^\sharp(D) := \text{div}(\mathcal{O}_X(-D) \cdot \mathcal{O}_Y)$.

For all sufficiently divisible $m \geq 1$, $\text{ord}_F(K_{m,Y/X}^+) \geq 0$ (i.e. $mK_Y \geq -f^\sharp(-mK_X)$), so that

$$\mathcal{O}_Y \cdot \mathcal{O}_X(mK_X) \hookrightarrow (\mathcal{O}_Y \cdot \mathcal{O}_X(mK_X))^{\vee\vee} = \mathcal{O}_Y(-f^\sharp(-mK_X)) \hookrightarrow \mathcal{O}_Y(mK_Y).$$

□

The first result that we will prove is that if X is canonical, then $\mathcal{R}_X(K_X)$ is finitely generated over X . Note that this result is trivial for \mathbb{Q} -Gorenstein varieties.

Theorem 3.4. *If X is canonical, then $\mathcal{R}_X(K_X)$ is finitely generated over X .*

Proof. We may assume that X is affine. Let $\tilde{X} \rightarrow X$ be a resolution. By [2], $\mathcal{R}_X(K_{\tilde{X}})$ is finitely generated. Let $X^c = \text{Proj}_X(\mathcal{R}_X(K_{\tilde{X}}))$ and let $f: X^c \rightarrow X$ be the induced morphism, where X^c is canonical. Since X is canonical, for any $m > 0$ there is an inclusion $\mathcal{O}_{X^c} \cdot \mathcal{O}_X(mK_X) \rightarrow \mathcal{O}_{X^c}(mK_{X^c})$. Pushing this forward we obtain inclusions

$$f_*(\mathcal{O}_{X^c} \cdot \mathcal{O}_X(mK_X)) \subset f_*\mathcal{O}_{X^c}(mK_{X^c}) \subset \mathcal{O}_X(mK_X).$$

Since the left- and right-hand sides have isomorphic global sections,

$$H^0(f_*\mathcal{O}_{X^c}(mK_{X^c})) \cong H^0(\mathcal{O}_X(mK_X)).$$

Since X is affine, $\mathcal{O}_X(mK_X)$ is globally generated and hence $f_*\mathcal{O}_{X^c}(mK_{X^c}) = \mathcal{O}_X(mK_X)$. But then

$$\mathcal{R}_X(K_X) \cong \mathcal{R}_X(K_{X^c}) := \bigoplus f_*\mathcal{O}_{X^c}(mK_{X^c})$$

is finitely generated. □

Remark 3.5. Note that we have seen that

$$\mathcal{R}_X(K_X) \cong \mathcal{R}_X(K_{X^c}) \cong \mathcal{R}_X(K_{\tilde{X}}),$$

hence

$$X^c = \text{Proj}_X(\mathcal{R}_X(K_X)),$$

and so $X^c \rightarrow X$ is a small morphism.

Corollary 3.6. *If X is canonical, then the canonical ring $\mathcal{R}_X(K_X)$ is finitely generated.*

Proof. Since $f: X^c \rightarrow X$ is small, it follows that $\mathcal{R}_X(K_X) \cong \mathcal{R}_X(K_{X^c})$. Since X^c is canonical and \mathbb{Q} -Gorenstein, it follows that $\mathcal{R}_X(K_{X^c})$ is finitely generated (see [2]). □

The next proposition strictly relates log terminal singularities to the finite generation of the canonical ring even in the non- \mathbb{Q} -Gorenstein case.

Proposition 3.7. *Let X be a normal variety with at most canonical singularities. $\mathcal{R}_X(-K_X)$ is a finitely generated \mathcal{O}_X -algebra if and only if X is log terminal.*

Proof. If X is log terminal, then $\mathcal{R}_X(-K_X)$ is a finitely generated \mathcal{O}_X -algebra by [9, Theorem 92].

We now want to show that, if $\mathcal{R}_X(-K_X)$ is finitely generated and X has at most canonical singularities, then it is also log terminal.

We will start constructing a \mathbb{Q} -Gorenstein model of X . Since $\mathcal{R}_X(-K_X)$ is finitely generated, by [10, Proposition 6.2], there exists a small map $\pi: X^- \rightarrow X$, such that $-K_{X^-} = \pi_*^{-1}(-K_X)$ is \mathbb{Q} -Cartier and π -ample. We will first show that X having at most canonical singularities implies for the model X^- to have at most canonical singularities in the usual sense.

For any m that is sufficiently divisible, consider the natural map $\mathcal{O}_{X^-} \cdot \mathcal{O}_X(-mK_X) \rightarrow \mathcal{O}_{X^-}(-mK_{X^-})$. Since $-K_{X^-}$ is π -ample, we can choose $A \subseteq X$ to be an ample divisor so that $\mathcal{O}_X(-mK_X + A)$ and $\mathcal{O}_X(-mK_{X^-} + \pi^*A)$ are both globally generated. The small map induces an isomorphism at the level of global sections,

$$H^0(X^-, \mathcal{O}_{X^-} \cdot \mathcal{O}_X(-mK_X + A)) \rightarrow H^0(X^-, \mathcal{O}_{X^-}(-mK_{X^-} + \pi^*A)),$$

and since the two sheaves are globally generated, this induces an isomorphism of sheaves

$$\mathcal{O}_{X^-} \cdot \mathcal{O}_X(-mK_X) \xrightarrow{\sim} \mathcal{O}_{X^-}(-mK_{X^-}).$$

Thus, considering $f: Y \rightarrow X$ and $g: Y \rightarrow X^-$, a common log resolution of both X and X^- , we have

$$K_Y + \frac{1}{m}g^*(-mK_{X^-}) = K_Y + \frac{1}{m}g^*(\pi^{\natural}(-mK_X)) = K_Y + \frac{1}{m}f^{\natural}(-mK_X) \geq 0;$$

the last inequality holds because X has at most canonical singularities. Thus X^- has at most canonical singularities. Furthermore, since K_{X^-} is \mathbb{Q} -Cartier and canonical, X^- is log terminal.

Choosing a general ample \mathbb{Q} -divisor $H^- \sim_{\mathbb{Q}, X^-} -K_{X^-}$, let $m \gg 0$ and let $G^- \in |mH^-|$ be a general irreducible component. Picking $\Delta^- := G^-/m$, then $K_{X^-} + \Delta^- \sim_{\mathbb{Q}, X^-} 0$ is still log terminal and, because the sum is π -trivial, Δ^- will induce a boundary on X so that $(X, \Delta = \pi_*\Delta^-)$ is a log terminal pair on X . \square

4. Quasi-nef divisors

Given a divisor D on a variety X , it is useful to know if the divisor is nef. In particular, varieties such that the canonical divisor K_X is nef are minimal models.

Unfortunately, for arbitrary normal varieties there is no good notion of nefness (this is a numerical property that is well defined if the variety is \mathbb{Q} -factorial). A possible definition is given in [3] using the notion of b -divisors. A comparison between the approach given in this paper and the one in [3] is given in detail in [4]. In particular, whenever looking for a minimal model it is always necessary to either pass to a resolution of the singularities

or to perturb the canonical divisor adding a boundary (an auxiliary divisor Δ such that $K_X + \Delta$ is \mathbb{Q} -Cartier). However, both operations are not canonical and in either case different choices lead us to different minimal models. What we would like to do in this section is define a notion of a minimal model for an arbitrary normal variety.

We will start defining a notion of nefness for a divisor that is not \mathbb{Q} -Cartier.

Definition 4.1. Let X be a projective normal variety. A divisor $D \subseteq X$ is *quasi-nef* (*q-nef*) if for every ample \mathbb{Q} -divisor $A \subseteq X$, $\mathcal{O}_X(m(D + A))$ is generated by global sections for every $m > 0$ that is sufficiently divisible.

Remark 4.2. Let X be a normal \mathbb{Q} -factorial variety. A divisor $D \subseteq X$ is nef if and only if it is q-nef.

Proposition 4.3. Let D be a divisor on a normal variety X . If $g: Y \rightarrow X$ is a small projective birational map such that $\bar{D} := g_*^{-1}D$ is \mathbb{Q} -Cartier and g -ample, then D is q-nef if and only if \bar{D} is nef.

Proof. The existence of the map g is equivalent to the finite generation of $\mathcal{R}_X(D)$ as an \mathcal{O}_X -algebra [10, Lemma 6.2].

Let us first assume that D is q-nef. For every ample divisor $A \subseteq X$, by definition there exists a positive integer m such that $\mathcal{O}_X(m(D + A))$ is generated by global sections and $\mathcal{O}_Y(m\bar{D})$ is relatively globally generated. In particular, since g is small,

$$\varphi: \mathcal{O}_Y \cdot \mathcal{O}_X(m(D + A)) \rightarrow \mathcal{O}_Y(m(\bar{D} + g^*A))$$

induces an isomorphism at the level of global sections. Now, \bar{D} is g -ample, and there exists $k \gg 0$ such that $\mathcal{O}_Y(m(\bar{D} + g^*A)) \otimes \mathcal{O}_Y(kg^*A)$ is also generated by global sections, hence φ must be surjective and hence an isomorphism. Since $\mathcal{O}_Y \cdot \mathcal{O}_X(m(D + A))$ is generated by global sections, so is $\mathcal{O}_Y(m(\bar{D} + g^*A))$. This implies that $\bar{D} + g^*A$ is nef, and since nefness is a closed property, \bar{D} is nef.

Let us now suppose that \bar{D} is a nef divisor on Y . As in Proposition 3.7, since g is a small map we have an isomorphism of sheaves $g_*\mathcal{O}_Y(m\bar{D}) \cong \mathcal{O}_X(mD)$. Furthermore, since \bar{D} is g -ample, for any ample divisor A on X , $\varepsilon\bar{D} + g^*A$ is ample for all $0 < \varepsilon \ll 1$. If we assume that \bar{D} is nef, then

$$\bar{D} + g^*A = (1 - \varepsilon)\bar{D} + (\varepsilon\bar{D} + g^*A)$$

is ample, and $\mathcal{O}_Y(m(\bar{D} + g^*A))$ is thus globally generated for m sufficiently divisible. But then so is its direct image under g , which is isomorphic to $\mathcal{O}_X(m(D + A))$ by the projection formula since mA is Cartier. □

Let us recall the following conjecture from [12].

Conjecture 4.4. Let X be a projective normal variety. Then, for any divisor $D \in \text{WDiv}_{\mathbb{Q}}(X)$, there exists a very ample divisor A such that $\mathcal{O}_X(mD) \otimes \mathcal{O}_X(A)^{\otimes m}$ is globally generated for every $m \geq 1$.

Definition 4.5. Let X be a normal projective variety, let D be any divisor on X , and let A be an ample divisor. If there exists a $t \in \mathbb{R}$ such that $D + tA$ is quasi-nef, we define the quasi-nef threshold with respect to A (qnt_A) as:

$$\text{qnt}_A(D) = \inf\{t \in \mathbb{R} \mid \mathcal{O}_X(m(D + tA)) \text{ is globally generated for all } m \text{ sufficiently divisible}\}.$$

Remark 4.6. Let X be a normal projective variety with at most log terminal singularities. For any divisor D on X and any ample divisor A , $\text{qnt}_A(D)$ exists and it is a rational number. This is a direct consequence of the fact that for any variety with at most klt singularities, every divisorial ring is finitely generated [9, Theorem 92].

5. Non- \mathbb{Q} -Gorenstein toric varieties

The aim of this section is to give an explicit description for toric varieties of the non- \mathbb{Q} -Gorenstein case. This is the more explicit example where it is possible to compute and understand the given definitions.

5.1. Quasi-nef divisors on toric varieties

For the notation and basic properties of toric varieties we refer the reader to [5]. See [3, § 2.4] for a b -divisorial interpretation of the same problem.

Consider a normal projective toric variety $X = X_\Sigma$ corresponding to a complete fan Σ in $N_{\mathbb{R}}$ (with no torus factor), with $\dim N_{\mathbb{R}} = n$. Recall that every T_N -invariant Weil divisor is represented by a sum

$$D = \sum_{\rho \in \Sigma(1)} d_\rho D_\rho,$$

where ρ is a one-dimensional sub-cone (a ray) and D_ρ is the associated T_N -invariant prime divisor. D is Cartier if for every maximal dimension sub-cone $\sigma \in \Sigma(n)$, $D|_{U_\sigma}$ is locally a divisor of a character $\text{div}(\chi^{m_\sigma})$, with $m_\sigma \in N^\vee = M$. If D is Cartier, we will say that $\{m_\sigma \mid \sigma \in \Sigma(n)\}$ is the Cartier data of D .

To every divisor we can associate a polyhedron:

$$P_D = \{m \in M_{\mathbb{R}} \mid \langle m, u_\rho \rangle \geq -d_\rho \text{ for every } \rho \in \Sigma(1)\}.$$

Even if the divisor is not Cartier, the polyhedron is still convex and rational but not necessarily integral.

For every divisor D and every cone $\sigma \in \Sigma(n)$, we can describe the local sections as

$$\mathcal{O}_X(D)(U_\sigma) = \mathbb{C}[W],$$

where $W = \{\chi^m \mid \langle m, u_\rho \rangle + d_\rho \geq 0 \text{ for all } \rho \in \sigma(1)\}$.

Let us recall the following proposition from [11].

Proposition 5.1. For a torus-invariant Weil divisor $D = \sum d_\rho D_\rho$, the following statements hold.

- (1) $\Gamma(X, D) = \bigoplus_{m \in P_D \cap M} \mathbb{C} \cdot \chi^m$.
- (2) Given that $\mathcal{O}_X(D)(U_\sigma) = \mathbb{C}[\sigma^\vee \cap M] \langle \chi^{m_{\sigma,1}}, \dots, \chi^{m_{\sigma,r_\sigma}} \rangle$ is a finitely generated $\mathbb{C}[\sigma^\vee \cap M]$ module for every $\sigma \in \Sigma(n)$ and a minimal set of generators is assumed to be chosen, $\mathcal{O}_X(D)$ is generated by its global sections if and only if $m_{\sigma,j} \in P_D$ for all σ and j .

We will also need the following result [7].

Theorem 5.2. Let X be a complete toric variety and let D be a Cartier divisor on X . Then the ring

$$\mathcal{R}_D := \bigoplus_{n \geq 0} H^0(X, \mathcal{O}_X(nD))$$

is a finitely generated \mathbb{C} -algebra.

Corollary 5.3. Since every toric variety admits a \mathbb{Q} -factorialization, a small morphism from a \mathbb{Q} -factorial variety [8, Corollary 3.6], the above result holds for Weil divisors as well.

We then easily get the following result.

Proposition 5.4. Conjecture 4.4 holds for $X = X_\Sigma$, a complete toric variety.

We can now focus our attention on q-nef divisors.

Remark 5.5. A small birational map of toric varieties $f: Y \rightarrow X$ is given by adding faces of dimension greater than or equal to 2 to the fan. This operation increases the number of sub-cones. In particular, a sub-cone in the fan corresponding to Y may be strictly contained in one of the original sub-cones.

Proposition 5.6. Let us consider a Weil divisor $D \subseteq X$ on a normal toric variety. If $f: Y \rightarrow X$ is a small birational map, then $P_D = P_{f_*^{-1}D}$.

Proof. This is clear, since the definition of the polyhedron only depends on the rays generating the fan and not on the structure of the sub-cones. \square

We assume that the polyhedron P_D is of maximal dimension and that zero is inside the polyhedron.

To have a better description of the relation between a small morphism and the local sections of a Weil divisor, we will introduce a new polyhedron associated to the divisor, the dual of P_D .

Definition 5.7. Let $D = \sum d_\rho D_\rho \subseteq X = X_\Sigma$ be a Weil divisor on a normal toric variety. We define $Q_D \subseteq N_{\mathbb{R}}$ to be the convex hull generated by $(1/d_\rho)u_\rho$, where $\rho \in \Sigma(1)$. In particular,

$$Q_D = P_D^* = \{u \in N_{\mathbb{R}} \mid \langle m, u \rangle \geq -1 \text{ for all } m \in P_D\}.$$

Recall that a divisor D is Cartier if and only if for each $\sigma \in \Sigma$, there is $m_\sigma \in M$ with $\langle m_\sigma, u_\rho \rangle = -d_\rho$ for all $\rho \in \sigma(1)$, with $D|_{U_\sigma} = \text{div}(\chi^{-m_\sigma})$ [5, Theorem 4.2.8].

We will define Σ' to be the fan generated by Σ and the faces of Q_D . In particular, any face of Q_D is contained in a hyperplane corresponding to m_σ for some $\sigma \in \Sigma(n)$. Note that the vertices of Q_D are all contained in the one-dimensional faces of Σ , hence $Y := X_{\Sigma'} \rightarrow X$ is a small birational map.

For every cone $\sigma \in \Sigma(n)$, if $\mathcal{O}_X(D)(U_\sigma)$ is locally generated by a single equation (is locally Cartier), nothing changes. Otherwise we substitute the cone σ with the set of sub-cones generated by the faces of Q_D contained in σ .

Lemma 5.8. *With the notation above, suppose that $\bar{\sigma} \subseteq \Sigma'(n)$ corresponds to a face of Q_D . Let $\bar{m} \in M_{\mathbb{R}}$ be the element corresponding to the hyperplane containing the face, hence $\mathcal{O}_X(D)(U_{\bar{\sigma}}) = \mathbb{C}[\bar{\sigma}^\vee \cap M][\chi^{\bar{m}}]$. If $\mathcal{O}_X(D)$ is globally generated, then $\bar{m} \in P_D$.*

Proof. Since $\chi^{\bar{m}}$ is a generator of $\mathcal{O}_X(D)(U_{\bar{\sigma}})$, we have that $\langle \bar{m}, u_\rho \rangle = -d_\rho$ for every $\rho \in \bar{\sigma}(1)$. Also, since Q_D is convex and $\langle \bar{m}, 0 \rangle = 0 > -1$, we have that $\langle \bar{m}, u_\rho \rangle \geq -d_\rho$ for every $\rho \in \Sigma(1)$, hence $\bar{m} \in P_D$. \square

Proposition 5.9. *Let X be a normal toric variety and let D be a Weil divisor whose corresponding reflexive sheaf is generated by global sections. Then there exists a small map $f: Y \rightarrow X$ of toric varieties such that $\bar{D} := f_*^{-1}D$ is \mathbb{Q} -Cartier and f -ample, and the vertices of P_D are given by the Cartier data $\{m_\sigma \mid \sigma \in \Sigma'(n)\}$ of \bar{D} , where Σ' is the fan associated to Y .*

Proof. We will consider the toric variety associated with the fan Σ' generated by Σ and the convex polytope Q_D . It follows from the construction that the divisor \bar{D} is \mathbb{Q} -Cartier. By Lemma 5.8, since Q_D is convex, we have that the reflexive sheaf corresponding to \bar{D} is still generated by global sections.

Furthermore, every curve C extracted via the map f will correspond to a face $\tau \subseteq \Sigma'$. Since \bar{D} is globally generated, we already know that $(\bar{D}.C) \geq 0$. In particular τ is given as the intersection of two maximal cones $\tau = \sigma \cap \sigma'$, and for each of the cones we have local generators of \bar{D} , m and m' . The intersection is computed as $(\bar{D}.C) = \langle m, u \rangle - \langle m', u \rangle$, where u is a ray in $\sigma \setminus \sigma'$, where this is zero if and only if $m = m'$, and this would not be one of the curves to be extracted by the map f by definition. \square

Because of Proposition 5.9 we get the following.

Theorem 5.10. *Let D be a Weil divisor on a normal toric variety X . Then D is q -nef if and only if $\mathcal{O}_X(mD)$ is globally generated for $m \gg 0$.*

Example 5.11. Let X be a normal projective toric variety and let $A = \sum a_\rho D_\rho \subseteq X$ be an ample divisor. Then $\text{qnt}_A(D)$ can be explicitly computed. In particular, let $D = \sum d_\rho D_\rho$ be any Weil divisor. If no multiple of D is globally generated, this implies that for every $b \in \mathbb{N}$, there exists $\sigma_b \in \Sigma(n)$ such that $u_{\rho_b} \notin \sigma_b$ and $\langle m_{\sigma_b}^{bD}, u_{\rho_b} \rangle < -bd_{\rho_b}$, where $m_{\sigma_b}^{bD}$ is one of the generators of $\mathcal{O}_X(bD)(U_{\sigma_b})$, i.e. there exists a positive rational number δ_{ρ_b} such that $\langle m_{\sigma_b}^{bD}, u_{\rho_b} \rangle = -bd_{\rho_b} - \delta_{\rho_b}$. Since A is ample, the support function

of A is strictly convex, and in particular $\langle m_{\sigma_a}^A, u_{\rho_b} \rangle > -a_{\rho_b}$, i.e. there exists a positive rational number ε_{ρ_b} such that $\langle m_{\sigma_b}^A, u_{\rho_b} \rangle = -a_{\rho_b} + \varepsilon_{\rho_b}$.

For every $\rho_b \notin \sigma_b$ and every σ_b we can find a rational number t_{b,σ_b,ρ_b} such that

$$\langle m_{\sigma_b}^D + t_{b,\sigma_b,\rho_b} m_{\sigma_b}^A, u_{\rho_b} \rangle = -bd_{\rho_b} - \delta_{\rho_b} - t_{b,\sigma_b,\rho_b} a_{\rho_b} + t_{b,\sigma_b,\rho_b} \varepsilon_{\rho_b} = -bd_{\rho_b} - t_{b,\sigma_b,\rho_b} a_{\rho_b}.$$

Then

$$\text{qnt}_A(D) = \inf_b \max_{\sigma_b, \rho_b \notin \sigma_b} -t_{b,\sigma_b,\rho_b}.$$

5.2. Minimal log discrepancies for terminal toric threefolds

In this final subsection we return to properties of log discrepancies in the setting of [6] in the context of toric varieties.

Depending on our choice of a relative canonical divisor, we have two possible definitions for the minimal log discrepancies (MLDs).

Definition 5.12. Let X be a normal variety over the complex numbers. We associate two numbers to the variety X :

$$\text{MLD}^-(X) = \inf_E \text{ord}_E(K_{Y/X}^-)$$

and

$$\text{MLD}^+(X) = \inf_E \text{ord}_E(K_{Y/X}^+),$$

where $E \subseteq Y$ is any prime divisor and $Y \rightarrow X$ is any proper birational morphism of normal varieties.

It is natural to wonder if these MLDs also satisfy the ACC conjecture. If X is assumed to be \mathbb{Q} -Gorenstein, then this is conjectured to hold by Shokurov. In view of [6, Theorem 5.4], the MLD^+ s correspond to MLDs of appropriate pairs (X, Δ) . However, the coefficients of Δ do not necessarily belong to a DCC (descending chain condition) set (the investigation for log-pairs given in the \mathbb{Q} -Gorenstein context has this extra assumption in [1]).

Proposition 5.13. *The set of all MLD^+ s for terminal toric threefolds does not satisfy the ACC conjecture.*

Proof. We give an explicit example of a set of terminal toric threefolds whose associated MLD^+ s converge to a number from below. The problem is local, so we will consider a set of affine toric threefolds given by the following data.

Let X be the affine toric variety associated with the cone $\sigma = \langle u_1, u_2, u_3, u_4 \rangle$, $u_1 = (2, -1, 0)$, $u_2 = (2, 0, 1)$, $u_3 = (1, 1, 1)$, $u_4 = (a, 1, 0)$ with $a \in \mathbb{N}$. The associated toric variety is non- \mathbb{Q} -Gorenstein, i.e. the canonical divisor $K_X = \sum -D_i$ is not \mathbb{Q} -Cartier.

Let $\Delta = \sum d_i D_i$ be a \mathbb{Q} -divisor such that $0 \leq d_i \leq 1$ and $-K_X + \Delta$ is \mathbb{Q} -Cartier. This means that there exists $m = (x, y, z)$ such that $-K_X + \Delta = \sum (m, u_i) D_i$. Hence

$$\Delta = (2x - y - 1)D_1 + (2x + z - 1)D_2 + (x + y + z - 1)D_3 + (ax + y - 1)D_4.$$

The exceptional divisor E giving the smallest discrepancy is the one corresponding to the element $u_E = u_1 + u_2 + u_3$ (it is the exceptional divisor generated by the ray of smallest norm). In particular, we have

$$\text{ord}_E(K_{Y/X}^+) = \inf_{\Delta \text{ boundary}} 5x - 2z.$$

Increasing the value of the parameter a we see that the minimal valuation is given by $(4a+5)/(a+2)$, which accumulates from below at the value 4. Note that the key is solving a problem of minimality with constraints, corresponding to the fact that the coefficients of Δ need to be in the interval $[0, 1)$. \square

5.3. Towards a revised toric minimal model programme

Let $X = X_\Sigma$, as in the previous section. Using Theorem 5.4 we will try to define a minimal model programme for any normal projective toric variety without attaching any boundary to it.

Let us recall the following well-known fact [10, Theorem 3.25 (3)].

Proposition 5.14. *Let (Y, Δ) be a klt pair, Δ effective, and let $f: Y \rightarrow X$ be a projective morphism of normal projective varieties. Let $F \subset \overline{NE}(Y/X)$ be a $(K_Y + \Delta)$ -negative extremal face. Then there is a unique morphism $\text{cont}_F: Y/X \rightarrow Z/X$*

Using this proposition and the minimal model programme with scaling we will prove the following.

Proposition 5.15. *For any normal projective toric variety X there exists a small birational morphism $f: Z \rightarrow X$ such that K_Z is nef over X .*

Proof. Let H on X be an ample divisor such that $K_X + H$ is q-nef (we know that this is possible because of Theorem 5.2). We know that f is a small birational morphism and that $K_Y + f^*H$ is nef over X [10, Proposition 6.2]; note that Y is \mathbb{Q} -Gorenstein. At this point we run a minimal model programme with scaling of H over X to obtain another small morphism $\pi: Z \rightarrow X$. \square

Proposition 5.16. *Let $\pi: Z \rightarrow X$ be a small morphism of toric varieties such that K_Z is nef over X . Then, for any K_Z -negative curve C , there exists a boundary Δ on X such that $(K_X + \Delta). \pi_*C < 0$, i.e. every contraction on Z arising from the cone theorem, induces a contraction on X .*

Proof. Let $C \subset Z$ be such that $K_Z.C < 0$. Let D be an ample effective Cartier divisor on X ; in particular, $D.C > 0$. Also, writing $D = \sum d_\rho D_\rho$, we can assume that there exists an integer $k > 0$ such that $D' := D/k$ is a \mathbb{Q} -Cartier divisor such that $|d_\rho| < 1$ for every $\rho \in \Sigma(1)$. Let us choose $\Delta = -K_X - D'$, which by assumption is an effective divisor, and $K_X + \Delta = -D'$ is \mathbb{Q} -Cartier and $(K_X + \Delta).C < 0$. \square

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