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Tracking a joint path for the walk of an underactuated biped

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SUMMARY

This paper presents a control law for the tracking of a cyclic reference path by an under-actuated biped robot. The robot studied is a five-link planar biped. The degree of under-actuation is one during the single support phase. The control law is defined in such a way that only the geometric evolution of the biped configuration is controlled, but not the temporal evolution. To achieve this objective, we consider a parametrized control. When a joint path is given, a five degree of freedom biped in single support becomes similar to a one degree of freedom inverted pendulum. The temporal evolution during the geometric tracking is completely defined and can be analyzed through the study of a model with one degree of freedom. Simple analytical conditions, which guarantee the existence of a cyclic motion and the convergence towards this motion, are deduced. These conditions are defined on the reference trajectory path. The analytical considerations are illustrated with some simulation results.

KEYWORDS: Biped robot; Walking; Under-actuated system; Reference joint path; Control; Limit cycle; Stability.

1. INTRODUCTION

Human walk is composed of disequilibrium phases caused by the gravity effect, and these phases produce the dynamics of the motion. For a biped, in fact, static equilibrium at each time instant is not necessary. To study this point specifically, a biped with unactuated ankles is interesting because no statically stable gait can be obtained. The reduction of the number of actuators is also a step towards simpler and cheaper robots. As a consequence, we choose to study a planar biped with only four actuators: two on the haunch and two on the knees. During the single support phase, the configuration of the biped is defined by five independent variables, but there are only four actuators. Hence, the biped is an under-actuated system. This simplification in terms of mechanics makes the design of the control law difficult.

Various studies exist about the control of an under-actuated biped. One method is based on the definition of the reference trajectory for m outputs (where m is the number of actuators), not as a function of time, but as a function of a configuration variable independent of the m outputs. With such a control, the configuration of the biped at impact time is the desired configuration, but its velocities can differ from the desired velocities. The convergence of the motion towards a cyclic motion is studied numerically using the Poincaré criterion in references [1, 2]. Another approach involves parameterized reference trajectories. In this case, one derivative of the parameter can be used as a "supplementary input", as it was shown in references [3–6]. In reference [4], the parameter is used to satisfy some constraints on the ground reaction applied to the supporting foot. In reference [7], a parameter involved in the zero dynamics is used as a supplementary input.

In the present paper, like in references [1, 2], only the geometric evolution of the robot is controlled, not the temporal evolution. To achieve this objective the reference trajectory is parameterized by a scalar path parameter: the arc length s. A time scaling control is defined as in references [8] and [9]. The second derivative of s is considered as a "supplementary control input". Thus, we deal with a model, for which the numbers of inputs and of independent configuration variables are equal. For a given reference joint path, the model of the five-dof-biped is reduced to a one-dof-model described by the variable s. This model is similar to one-link-pendulum model. Through the study of this dynamic model, the evolution of parameter s can be analyzed. In fact, the evolution of the second derivative $\ddot{s}$ is defined by the choice of the evolution of the configuration variables or, in other words, by the choice of the reference path. A condition on the reference path is defined to ensure the existence of a cyclic motion of the robot. When a cyclic motion exists, a condition to ensure the convergence towards the cyclic motion is deduced from this analysis. This condition is also related to the reference path. Moreover, the ground reaction force applied to the supporting leg is unilateral, the limits on the motion induced by this constraint are also taken into account.

In Section 2, the model of the biped is presented. In Section 3 the reference path and the control law are defined. The admissible reference motions are defined in Section 4, the condition of existence and uniqueness of a cyclic reference motion are presented in this section as well. A condition for the convergence towards the cyclic motion is deduced in Section 5. Section 6 presents some simulation results. The conditions for a cyclic motion to exist and for the convergence towards the cyclic motion are showed to be inequalities. These characteristics induce some robustness properties of the proposed control which are also illustrated in Section 6. Section 7 concludes the paper.

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2. THE BIPED MODELLING

2.1. The biped

The biped studied walks in a vertical sagittal xz plane. It is composed of a torso and two identical legs. Each leg is composed of two links articulated by a knee. The knees and the hips are one-degree-of-freedom rotational ideal (without friction) joints. The walk is composed of single support phases separated by impact phases (instantaneous double support phases). Vector $q_c = [q_1, q_2, q_3, q_4]^T$ of "internal" variables (Figure 1a) describes the shape of the biped. To define completely the biped position in the vertical plane with respect to a fixed frame, we add three coordinates q_5, x_g, z_g, where q_5 is the absolute orientation of the trunk, x_g and z_g are the abscissa and the ordinate of the robot mass centre, respectively. The vector of all coordinates of the robot is $x = [q_1, q_2, q_3, q_4, q_5, x_g, z_g]^T$ and the vector of the angular coordinates is $q = [q_1, q_2, q_3, q_4, q_5]^T$.

All links are assumed massive and rigid. In the simulation, we use the following biped parameters. The lengths of the thighs and of the shins are 0.4 *m*. However, their masses are different: 6.8 *kg* for each thigh and 3.2 *kg* for each shin. The length of the torso is 0.625 *m* and its mass is 20 *kg*. The total mass of the biped is $m = 40$ *kg*. A prototype with these characteristics is under construction.[10] The inertia moments of the links are also taken into account. Γ is the 4×1 vector of the torques applied in the hip and in the knee joints (Figure 1b).

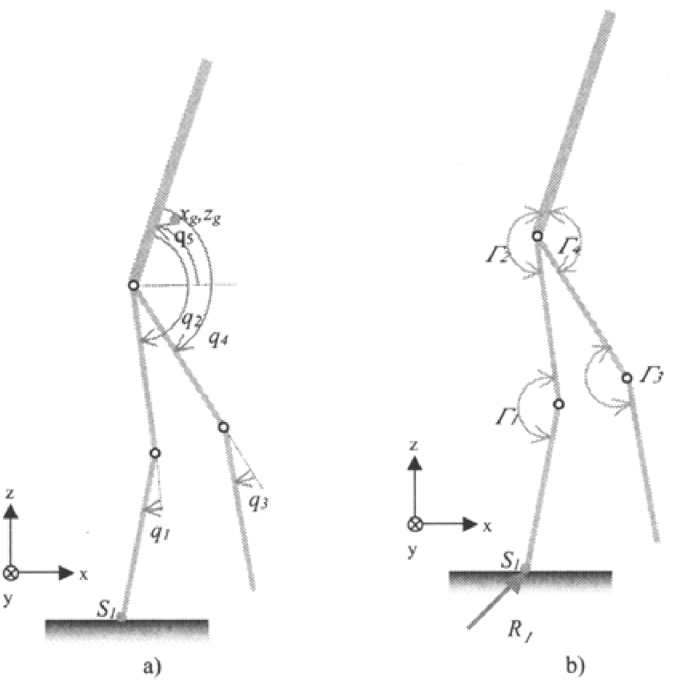

Fig. 1. The studied biped: a) generalized coordinates, b) applied torques and ground reaction.

2.2. Dynamic modelling

(i) **The complete model:** In the literature, different dynamic models of the biped are developed. In this paper, we present the dynamic model using the variable x that involves the biped mass centre coordinates, the trunk orientation coordinate and four relative joint variables. This particular choice of the coordinates is useful to highlight the role of the angular momentum and to derive easily the linear momentum theorem.

The l^{th} line of the dynamic model can be written using the Lagrange's formalism, for $l = 1, \ldots, 7$ (x_l is the l^{th} element of vector x):

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{x}_l}\right) - \frac{\partial K}{\partial x_l} + \frac{\partial P}{\partial x_l} = Q_l \tag{1}$$

where K is the kinetic energy, P is the potential energy. The virtual work δW of the external torques and forces, given by expression $\delta W = \sum Q_l\, \delta x_l = Q^T\, \delta x$, defines the vector Q of the generalized forces. When leg i is on the ground, a reaction force $R_i = [R_{ix}\;\; R_{iz}]^T$ is applied to the leg tip S_i by the ground (Figure 1b).When leg i is not on the ground, $R_i = 0_{2,1}$, where $0_{k,l}$ is the $k \times l$ zero-matrix.

The position of the mass centre of the biped can be expressed as function of the position of the leg tip S_i and on the angular coordinates vector q:

$$\begin{bmatrix} x_g \\ z_g \end{bmatrix} = \begin{bmatrix} x_{S_i} \\ z_{S_i} \end{bmatrix} + \begin{bmatrix} f_{ix}(q) \\ f_{iz}(q) \end{bmatrix} \tag{2}$$

The vector-function $f_i(q) = [f_{ix}(q)\;\; f_{iz}(q)]^T$ depends on vector q and on the biped parameters (lengths of the links, masses, positions of the mass centres).

Using Equation (2), we can deduce that the virtual displacement of the leg tip S_i is:

$$\begin{bmatrix} \delta x_{S_i} \\ \delta z_{S_i} \end{bmatrix} = \begin{bmatrix} -\dfrac{\partial f_i(q)}{\partial q} & I_2 \end{bmatrix} \delta x \tag{3}$$

where $\dfrac{\partial f_i(q)}{\partial q}$ is a 2×5 matrix, I_n is $n \times n$ identity matrix.

With our choice of coordinates, we have:

$$K = \dot{x}^T \begin{bmatrix} A(q_c) & 0_{5,2} \\ 0_{2,5} & mI_2 \end{bmatrix} \dot{x}, \qquad P = mgz_g,$$

$$Q = \begin{bmatrix} I_4 \\ 0_{3,4} \end{bmatrix} \Gamma + \sum_{i=1,2} \begin{bmatrix} -\dfrac{\partial f_i(q)^T}{\partial q} \\ I_2 \end{bmatrix} R_i \tag{4}$$

where m is the mass of the biped, g is the gravity acceleration, $A(q_c)$ is a 5×5 matrix. The presented model is convenient for all phases of planar bipedal locomotion. For double support phases, the both ground reactions are not zero. For single support phases, only one reaction force is not zero. For flight phases, both reaction forces are zero.

Remarks:

- The kinetic energy K is independent of the coordinate frame chosen. Since coordinates q_5, x_g, z_g define only the position and orientation of the biped as a rigid body,

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the inertia matrix is independent of these three variables, it depends only on vector q_c of "internal" variables.

- The fifth equation of system (1) describes the change of the angular momentum of the biped around its mass centre, corresponding to the angular momentum theorem.
- The last two equations of system (1) correspond to the linear momentum theorem for the biped.

(ii) **The single support phase model:** The ground and the robot links are assumed rigid. During the single support phase, supporting leg tip i is on the ground, thus x_{S_i}, z_{S_i} are constant in Equation (2) (no sliding), and Equation (3) gives:

$$\mathrm{x}=\begin{bmatrix} \mathrm{I}_5 \\ \dfrac{\partial f_i(q)}{\partial q} \end{bmatrix} q, \quad \dot{x}=\begin{bmatrix} I_5 \\ \dfrac{\partial f_i(q)}{\partial q} \end{bmatrix} \dot{q} \tag{5}$$

The various terms of the corresponding dynamic model can be expressed only as functions of q, we obtain:

$$K=\dot{q}^T\left[A(q_c)+m\,\frac{\partial f_i(q)^T}{\partial q}\,\frac{\partial f_i(q)}{\partial q}\right]\dot{q},$$

$$P=mgf_{iz}(q), \quad Q=\begin{bmatrix} I_4 \\ 0_{1,4} \end{bmatrix}\Gamma \tag{6}$$

As the supporting leg tip is motionless, the virtual work of the reaction force is zero.

The first four lines of this dynamic model can be grouped into different matrices and vectors to write:

$$M(q)\ddot{q}+h(q,\dot{q})=\Gamma \tag{7}$$

where $M(q)$ is a (4×5) matrix and vector $h(q,\dot{q})$ contains the centrifugal, Coriolis and gravity forces.

We pay more attention to the fifth line of the dynamic model, which characterizes the under-actuation of the biped. As mentioned previously, the inertia matrix is independent of the coordinate frame chosen. For the single support also, angle q_5 describes the orientation of the biped relative to the coordinate frame and not the shape of the robot. Thus K in Equation (6) is independent of angle q_5, and the fifth equation of system becomes:

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_5}\right)+\frac{\partial P}{\partial q_5}=0 \tag{8}$$

For our planar biped and our choice of the coordinates in the single support, the term $\dfrac{\partial K}{\partial \dot{q}_5}$ is the biped angular momentum around the stance leg tip S_i. We denote this term by σ. Thus we have:

$$\frac{\partial K}{\partial \dot{q}_5}=\sigma=N(q_c)\dot{q} \tag{9}$$

where $N(q_c)$ is the fifth line of matrix

$$\left(A(q_c)+m\frac{\partial f_i(q)^T}{\partial q}\,\frac{\partial f_i(q)}{\partial q}\right).$$

The expression $\dfrac{\partial P}{\partial q_5}$ is equal to $mg(x_{S_i}-x_g)$. Thus the fifth equation of the dynamic model of the biped in the single support can be written in the following simple form:

$$\dot{\sigma}=N(q_c)\ddot{q}+\dot{q}^T\,\frac{\partial N(q_c)}{\partial q}\,\dot{q}=mg(x_g-x_{S_i}) \tag{10}$$

(iii) **The reaction force during the single support phase:** When the leg i is on the ground, reaction force R_i exists (see Figure (1b). The last two lines of general model (1) make it possible to calculate this force:

$$m\begin{bmatrix} \ddot{x}_g \\ \ddot{z}_g \end{bmatrix}+mg\begin{bmatrix} 0 \\ 1 \end{bmatrix}=R_i \tag{11}$$

In the single support phase, Equation (11) can also be written:

$$m\,\frac{\partial f_{ix}(q)}{\partial q}\,\ddot{q}+m\dot{q}^T\,\frac{\partial^2 f_{ix}(q)}{\partial q^2}\,\dot{q}=R_{ix}$$

$$m\,\frac{\partial f_{iz}(q)}{\partial q}\,\ddot{q}+m\dot{q}^T\,\frac{\partial^2 f_{iz}(q)}{\partial q^2}\,\dot{q}+mg=R_{iz} \tag{12}$$

where $\dfrac{\partial^2 f_{ix}(q)}{\partial q^2}$ and $\dfrac{\partial^2 f_{iz}(q)}{\partial q^2}$ are (5×5) matrices.

The reaction force exerted by the ground can be directed upward only, and to avoid the sliding of the biped, the reaction force must be inside the friction cone. These conditions can be written at each time by:

$$R_{ix}+\mu R_{iz}>0$$
$$-R_{ix}+\mu R_{iz}>0$$

where μ is the friction coefficient (positive). It follows from these two inequalities that $R_{iz}>0$. These two scalar inequalities can be expressed by the following matrix inequality:

$$CR_i>0 \tag{13}$$

with $C=\begin{bmatrix} 1 & \mu \\ -1 & \mu \end{bmatrix}$. For the single support phase, these constraints can be written using Equation (12):

$$U(q)\ddot{q}+V(q,\dot{q})+W>0 \tag{14}$$

With

$$U(q)=C\,\frac{\partial f_i(q)}{\partial q},\; V(q,\dot{q})=C\begin{bmatrix} \dot{q}^T\,\dfrac{\partial^2 f_{ix}(q)}{\partial q^2}\,\dot{q} \\ \dot{q}^T\,\dfrac{\partial^2 f_{iz}(q)}{\partial q^2}\,\dot{q} \end{bmatrix},\; W=gC\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

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(iv) **The impact model:** When the swing leg j touches the ground at the end of single support, an impact takes place. We assume that the ground reaction at the instant of impact is described by a Dirac delta-function with intensity I_{R_j}. This impact is assumed inelastic. This means that the velocity of the foot j becomes zero just after the impact. Two kinds of impact can occur depending on whether the stance leg takes off or not. We study the gait with instantaneous double support phases. Thus during an impact the stance leg i takes off and $I_{R_i}=0$ at the instant of impact. The robot configuration q is assumed to be constant at the instant of double support, and there are jumps in the velocities. The velocity vectors just before and just after impact, are denoted $\dot{x}^-$, $\dot{q}^-$ and $\dot{x}^+$, $\dot{q}^+$ respectively. The torques Γ_l, $l=1,\ldots,4$ are limited, thus they do not influence the instantaneous double support. Using general model (1) and expressions (4) the impact model can be written:[11,12]

$$\begin{bmatrix} A(q_c) & 0_{5,2} \\ 0_{2,5} & mI_2 \end{bmatrix} \{\dot{x}^+ - \dot{x}^-\} = \begin{bmatrix} -\dfrac{\partial f_j(q)^T}{\partial q} \\ I_2 \end{bmatrix} I_{R_j} \tag{15}$$

Vector I_{R_j} of the ground reaction intensities can be expressed using the last two lines of matrix Equation (15). Substituting this expression into the first five lines (15) we obtain:

$$A(q_c)(\dot{q}^+ - \dot{q}^-\}) = -m\frac{\partial f_j(q)^T}{\partial q}\left(\begin{bmatrix}\dot{x}_g^+ \\ \dot{z}_g^+\end{bmatrix} - \begin{bmatrix}\dot{x}_g^- \\ \dot{z}_g^-\end{bmatrix}\right) \tag{16}$$

Before impact, leg i is in contact with the ground, and after impact, leg j is in contact with the ground. Thus the linear velocity of the mass centre, before and after the impact, can be expressed as function of the angular velocities (see the last two lines of (5)) and instead of (16) we obtain:

$$A(q_c)(\dot{q}^+ - \dot{q}^-) = -m\frac{\partial f_j(q)^T}{\partial q}\left(\frac{\partial f_j(q)}{\partial q}\dot{q}^+ - \frac{\partial f_i(q)}{\partial q}\dot{q}^-\right) \tag{17}$$

Thus the biped angular velocity vectors before and after impact are related by a linear equation:

$$\dot{q}^+ = \left(A(q_c) + m\frac{\partial f_j(q)^T}{\partial q}\frac{\partial f_j(q)}{\partial q}\right)^{-1}\left(A(q_c) + m\frac{\partial f_j(q)^T}{\partial q}\frac{\partial f_i(q)}{\partial q}\right)\dot{q}^- \tag{18}$$

This equation will be simply noted:

$$\dot{q}^+ = I(q)\dot{q}^- \tag{19}$$

Intensity I_{R_j} of the impulsive reaction force exerted by the ground can be calculated using the last two lines of matrix Equation (15) and Equation (5):

$$I_{R_j} = m\left(\frac{\partial f_j(q)}{\partial q}I(q) - \frac{\partial f_i(q)}{\partial q}\right)\dot{q}^-$$

This ground reaction force must be directed upward and be inside the friction cone. Thus the velocity $\dot{q}^-$ must satisfy the following matrix inequality:

$$C\left(\frac{\partial f_j(q)}{\partial q}I(q) - \frac{\partial f_i(q)}{\partial q}\right)\dot{q}^-$$

To ensure a take-off of leg i, the vertical velocity component of leg tip S_i must be positive. Using the mass centre vertical velocity as intermediate expression and due to the definition of the functions $f_{iz}(q)$, $f_{jz}(q)$, this condition can be written:

$$\left(\frac{\partial f_{jz}(q)}{\partial q} - \frac{\partial f_{iz}(q)}{\partial q}\right)I(q)\dot{q}^- > 0$$

These two types of constraint can be grouped into:

$$D(q)\dot{q}^- > 0 \tag{20}$$

where $D(q)$ is a 3×5 matrix.

3. THE PROPOSED CONTROL LAW

The desired walking is essentially composed of single support phases. During these phases, the biped is an under-actuated system. The objective of the control law presented in this section is not to track a reference motion but only the associated path: only a geometrical tracking is desired and a time scaling control[8] is used. A reference joint path is assumed to be known. Thus the desired configuration q of the biped is not expressed as a function of time but as a function of the scalar path parameter, the arc length s: $q_r(s)$. The desired walking of the robot corresponds to an increasing function $s(t)$. In other words, function $s(t)$ defines the sequence of the biped configurations in time.

3.1. Reference joint path for the walking biped

Let us prescribe the desired configuration of the biped under the form:

$$q_d(t) = q_r(s(t)) \tag{21}$$

where $q_r(s)$ is a given vector-function of scalar parameter s.

Only cyclic walk of the biped is desired. The legs swap their roles from one step to the next one, so the reference path can be defined for one step only. For the first step, the scalar path parameter s varies from 0 to 1. The single support phase stands for $0<s<1$ and the impact occurs on the desired path for $s=1$. Vectors $q_r(0)$ and $q_r(1)$ describe the initial and final biped configurations of the single support, respectively. As the legs swap their roles from one step to the following one the desired configurations are such that $q_r(1)=Eq_r(0)$ where E is a permutation matrix describing the leg exchange. For the k^{th} step parameter s varies from $k-1$ to k. Here k is positive integer. We define a cyclic path, thus $q_r(s)$ has to satisfy the following condition of periodicity:

$$q_r(s+k) = E^k q_r(s)$$

where $0\le s\le 1$ and $E^2=I_5$.

For $k-1<s(t)<k$, the robot configuration $q_r(s)$ is such that the free leg tip is above the ground. The biped touches

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the ground at $s=k$ exactly. In consequence for any function $s(t)$, the configuration of the biped at the impact instant is the expected one.

The reference velocity of the robot $\dot{q}_d(t)=\frac{dq_r(s(t))}{ds}\dot{s}$ is proportional to $\dot{s}$. If parameter s increases strictly monotonically with respect to time, then this parameter can be chosen as independent variable. In this case, the reference velocity can be rewritten as: $\dot{q}_d(s)=\frac{dq_r(s)}{ds}\dot{s}(s)$. Derivative $\frac{dq_r(s)}{ds}$ is a discontinuous vector-function at points $s=0, 1, 2, \ldots$. The notation k^- (respectively k^+) means just before (respectively after) k^{th} impact. Just before and after the impact, the velocities are:

$$\dot{q}_d(k^-)=\frac{dq_r(k^-)}{ds}\dot{s}(k^-), \quad \dot{q}_d(k^+)=\frac{dq_r(k^+)}{ds}\dot{s}(k^+)$$

In order to obtain a cyclic path, the reference path $q_r(s)$ has to satisfy the impact Equation (19):

$$\frac{dq_r(k^+)}{ds}\dot{s}(k^+)=I(q_r(k))\frac{dq_r(k^-)}{ds}\dot{s}(k^-)$$

where $q_r(s)$ is a vector and $\dot{s}$ is a scalar. Thus $q_r(s)$ can not be arbitrary chosen. We choose this vector-function to have:

$$\frac{dq_r(k^+)}{ds}=I(q_r(k))\frac{dq_r(k^-)}{ds} \quad or \quad \frac{dq_r(0^+)}{ds}=EI(q_r(1))\frac{dq_r(1^-)}{ds} \tag{22}$$

With this choice we have the following equality: $\dot{s}(k^+)=\dot{s}(k^-)$. As a consequence time derivative $\dot{s}$ is a continuous function of parameter s.

During the impact, the ground reaction must be directed upwards and be inside the friction cone, the stance leg must take off, thus function $q_r(s)$ must be chosen such that (see inequality (20):

$$D(q_r(k))\frac{dq_r(k^-)}{ds}>0 \tag{23}$$

Thus the reference joint path $q_r(s)$ has to satisfy relations (22), (23) and the condition of periodicity.

3.2. *Definition of the control law*

It follows from (21) that the desired velocity and desired acceleration of the joint variables are:

$$\dot{q}_d(t)=\frac{dq_r(s(t))}{ds}\dot{s}$$

$$\ddot{q}_d(t)=\frac{dq_r(s(t))}{ds}\ddot{s}+\frac{d^2q_r(s(t))}{ds^2}\dot{s}^2 \tag{24}$$

So we assume that the reference path is a chosen periodical vector-function $q_r(s)$ that is twice differentiable except for the integer value of s.

The increasing function $s(t)$ defines the desired motion, but since the control objective is only to track a reference path, the evolution $s(t)$ is free and the second derivative $\ddot{s}$ will be treated as a "supplementary control input". Thus, the control law will be designed for a system with equal number of inputs and outputs. The control inputs are the four torques Γ_j, $j=1, \ldots, 4$, plus $\ddot{s}$. The chosen outputs are the five angular variables of vector q.

The control law is a non-linear control law classically used in robotics. But in order to obtain a finite-time stabilization around one of the desired trajectories, the feedback function proposed in references [2, 13] is used. The tracking errors are defined with respect to the trajectories satisfying (21):

$$e_q(t)=q_r(s(t))-q(t)$$
$$\dot{e}_q(t)=\frac{dq_r(s(t))}{ds}\dot{s}-\dot{q}(t) \tag{25}$$

The desired behaviour in a closed loop is:

$$\ddot{q}=\ddot{q}_d+\frac{1}{\epsilon^2}\psi \tag{26}$$

where ψ is a vector of five components ψ_l, $l=1, \ldots, 5$ with:

$$\psi_l=-sign(\epsilon\dot{e}_{q_l})|\epsilon\dot{e}_{q_l}|^{\nu}-sign(\phi_l)|\phi_l|^{\nu} \tag{27}$$

and $0<\nu<1$, $\epsilon>0$, $\phi_l=e_{q_l}+\frac{1}{2-\nu}sign(\epsilon\dot{e}_{q_l})|\epsilon\dot{e}_{q_l}|^{2-\nu}$, ν and ϵ are parameters to adjust the settling time of the controller. Taking into account expression (21) of the reference motion, Equation (26) can be rewritten as:

$$\ddot{q}=\frac{dq_r(s)}{ds}\ddot{s}+v(s,\dot{s},q,\dot{q}) \tag{28}$$

with

$$v(s,\dot{s},q,\dot{q})=\frac{d^2q_r(s)}{ds^2}\dot{s}^2+\frac{1}{\epsilon^2}\psi$$

The dynamic model of the robot is described by Equations (7) and (10), thus the control law must be such that:

$$M(q)\left(\frac{dq_r(s)}{ds}\ddot{s}+v\right)+h(q,\dot{q})=\Gamma$$

$$N(q_c)\left(\frac{dq_r(s)}{ds}\ddot{s}+v\right)+\dot{q}^T\frac{\partial N(q_c)}{\partial q}\dot{q}=mg(x_g-x_{S_i}) \tag{29}$$

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We can deduce that, in order to obtain the desired closed loop behaviour, it is necessary and sufficient to choose:

$$\ddot{s}=\frac{-N(q_c)v-\dot{q}^T\dfrac{\partial N(q_c)}{\partial q}\dot{q}+mg(x_g-x_{S_i})}{N(q_c)\dfrac{dq_r(s)}{ds}}$$

$$\Gamma=M(q)\left(\frac{dq_r(s)}{ds}\ddot{s}+v\right)+h(q,\dot{q}) \tag{30}$$

If $N(q_c)\dfrac{dq_r(s)}{ds}\neq 0$, the control law (30) ensures that $q(t)$ converges to $q_r(s(t))$ in a finite time, which can be chosen as less than the duration of one step.[2, 13] Without initial errors, a perfect tracking of $q_r(s(t))$ is obtained.

The first Equation (30) defines $\ddot{s}$. The evolution s can be calculated from this equation (but not chosen), if $s(0)$ and $\dot{s}(0)$ are known. We choose $s(0)=0$ and we define $\dot{s}(0)$ to minimize the error on the joint velocity $\epsilon=|\dot{q}(0)-\dot{q}_r(0)|^2=|\dot{q}(0)-\dfrac{dq_r(s(0))}{ds}\dot{s}(0)|^2$. Thus, $\dot{s}(0)$ is such that $\dfrac{d\epsilon}{d\dot{s}(0)}=0$. We obtain:

$$\dot{s}(0)=\frac{\dot{q}(0)^t\dfrac{dq_r(0)}{ds}}{\dfrac{dq_r(0)^t}{ds}\dfrac{dq_r(0)}{ds}} \tag{31}$$

3.3. *The singularities for the proposed control law*

It follows from the first Equation (30) that for the proposed control law, a singularity occurs if $N(q_c)\dfrac{dq_r(s)}{ds}=0$. For the reference motion $q(s)=q_r(s)$, we define:

$$f(s)=N(q_r(s))\frac{dq_r(s)}{ds}$$

In fact, matrix N depends only on the first four components of vector $q_r(s)$ (see Equation (9)), but here the notation $N(q_r(s))$ is used for simplicity.

If for the reference path, function $f(s)$ is sufficiently far from zero, and if the tracking error is sufficiently small, no singularity occurs.

4. EXISTENCE AND UNIQUENESS OF A CYCLIC MOTION

Our main goal is to design a control strategy, which ensures a stable periodic motion of the biped. The control law (30) ensures that the motion of the biped converges in a finite time towards a reference path described by (21). This time can be chosen to be less than the duration of the first step. With this choice, the biped with control law (30) follows perfectly the reference path, starting from the second step.

In this section, the five degree of freedom biped model is reduced to a one degree of freedom model with respect to variable s using the given reference path. This model is similar to the model of an inverted pendulum. Then we study the properties of this simpler model. Like a stable cyclic motion of the biped is desired, we study the conditions of existence and uniqueness of cyclic admissible reference motions.

4.1. *Properties of the admissible reference motion*

During the single support phase, the biped is an under-actuated system, thus it cannot follow any desired motion $q_d(t)$. We denote "admissible reference motion", the motion $q_r(s(t))$ satisfying the dynamic model (7), (10).

Analyzing the angular momentum σ is sufficient to study the evolution of parameter s. The motion of the robot can in turn be deduced from the evolution of parameter s. The angular momentum σ is linear with respect to vector $\dot{q}$ (see Equation (9)) and for the reference motion the velocity of the robot is proportional to $\dot{s}$ (see first Equation (24)). Thus, the angular momentum can be expressed by:

$$\sigma=f(s)\dot{s} \tag{32}$$

Scalar function $f(s)$ depends on vector $q_r(s)$ and on the biped parameters. Let us assume that function $q_r(s)$ and the biped parameters are such that $f(s)\neq 0$ for $0\leq s\leq 1$. If $f(s)\neq 0$ in the interval $0\leq s\leq 1$, then $f(s)<0$ or $f(s)>0$ in this interval. The sign of $f(s)$ changes with the sense of the axis y. In the following we assume that $q_r(s)$ is such that $f(s)>0$. Some examples of function $f(s)$ are given in Section 6. If $f(s)\neq 0$, we obtain from (32):

$$\dot{s}=\frac{\sigma}{f(s)} \tag{33}$$

If vector-function $q_r(s)$ is given, then the abscissa x_g of the mass centre is known as function of parameter s: $x_g=x_g(s)$. In this case, Equation (10) can be rewritten as:

$$\dot{\sigma}=mg(x_g(s)-x_{S_i}) \tag{34}$$

Under a given joint path, model (33), (34) is equivalent to the dynamic model (7), (10). Thus, both Equations (33), (34) define the admissible reference motion. Functions $\sigma(t)$ and $s(t)$ can be calculated from system (33), (34), when their initial values are known.

The system of second order (33), (34) is similar to the system describing the motion of usual physical pendulum with one degree of freedom.[14] Thus it has an integral similar to the energy integral of the pendulum motion:

$$\sigma^2-\Phi(s)=C=const \tag{35}$$

where,

$$\Phi(s)=2mg\int_{k^+}^{s}(x_g(\xi)-x_{S_i})f(\xi)\,d\xi \tag{36}$$

Using Equation (32), we can rewrite relation (35) in the form:

$$f^2(s)\dot{s}^2(s)-\Phi(s)=C=const$$

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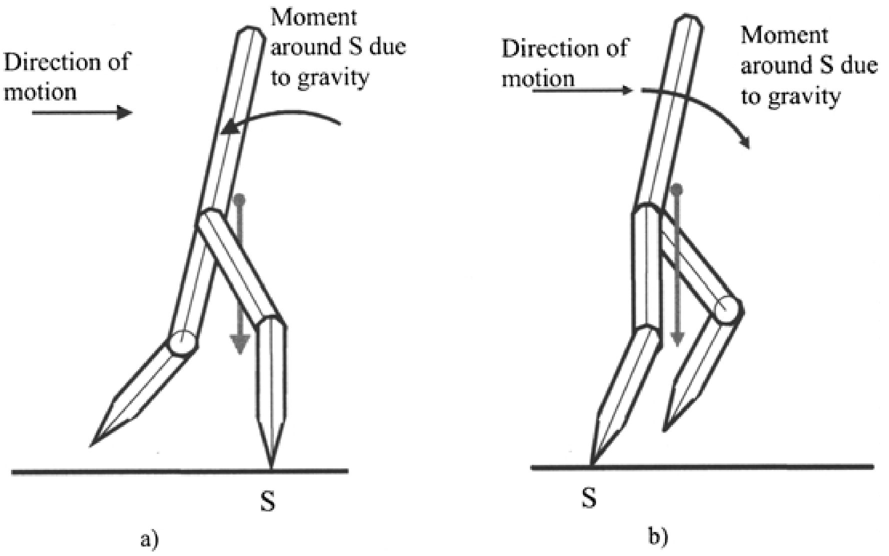

Fig. 2. The effect of gravity for one step: a) the gravity slows down the motion, b) the gravity accelerates the motion.

or

$$f^2(s)\dot{s}^2(s) - f^2(k^+)\dot{s}^2(k^+) = \Phi(s)$$

The functions $f(s)$ and $\Phi(s)$ can be calculated when $q_r(s)$ is known. These functions are periodic, with period equal to 1. Thus the characteristics of the robot behaviour can be studied only for one step, $0 \le s \le 1$.

For human gait, abscissa x_g of the mass centre increases during walking. In order to be close to the human gait, we choose function $q_r(s)$ such that abscissa x_g increases when parameter s increases from 0 to 1. The single support begins with $x_g < x_{S_i}$ and finishes with $x_g > x_{S_i}$. Figure 2 illustrates the action of the gravity during the single support and the behaviour of the angular momentum according to Equation (34).

Functions $\Phi(s)$ are shown in Figure 3 for some biped parameters[10] and some vector-functions $q_r(s)$:

- Function $\Phi(s)$ initially decreases (see Equation (36)) strictly monotonically, starting from zero.
- The negative minimal value Φ_m:

$$\Phi_m = \min_{0<s<1} \Phi(s) = \Phi(s_g) \tag{38}$$

  is reached at $s = s_g$, such that $x_g(s_g) = x_{S_i}$
- After, $x_g > x_{S_i}$ and function $\Phi(s)$ increases strictly monotonically.

In fact, under the described above properties of function $x_g(s)$, the shape of function $\Phi(s)$ is always the same as in Figure 3.

At the end of the single support phase, the angular momentum is greater than at the beginning if $\Phi(1) > 0$, and smaller if $\Phi(1) < 0$ (see integral (35)).

4.2. Minimal angular momentum to achieve a step

We have shown in previous section that at the beginning of the single support, the angular momentum decreases due to the gravity effect. Now we will show that the initial angular momentum $\sigma(0)$, or the initial velocity $\dot{s}(0)$, must be high enough to reach the configuration such that $x_g > x_{S_i}$. When $x_g > x_{S_i}$, the angular momentum increases due to gravity.

Using integral (35) and above mentioned properties of function $\Phi(s)$, it is easy to define the trajectories of system (33), (34). These trajectories are drawn for $0 \le s \le 1$ in the phase plane (s, σ) in Figure 4. The arrows indicate the direction in which the point representing the motion moves as time increases.

Equations (33), (34), and the phase portrait in Figure 4 show that the behaviour of the biped with the given joint

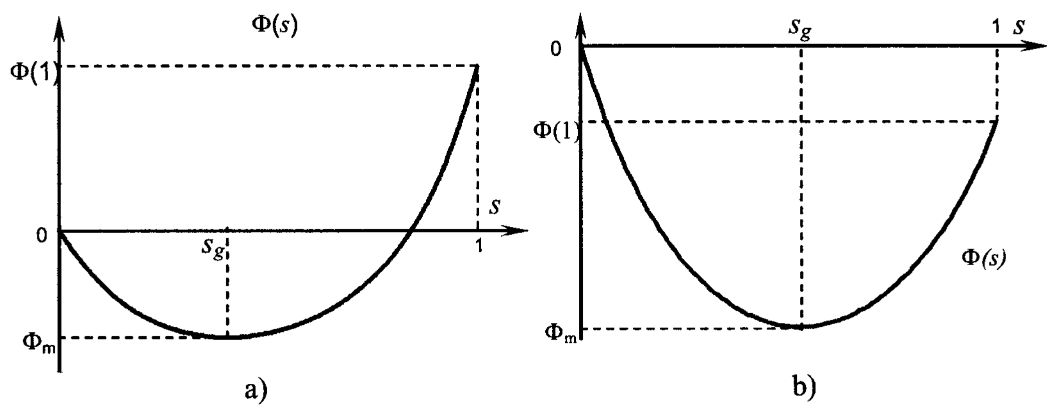

Fig. 3. Two typical behaviours of $\Phi(s)$ for one step: in case a) $\Phi(1) > 0$, in case b) $\Phi(1) < 0$.

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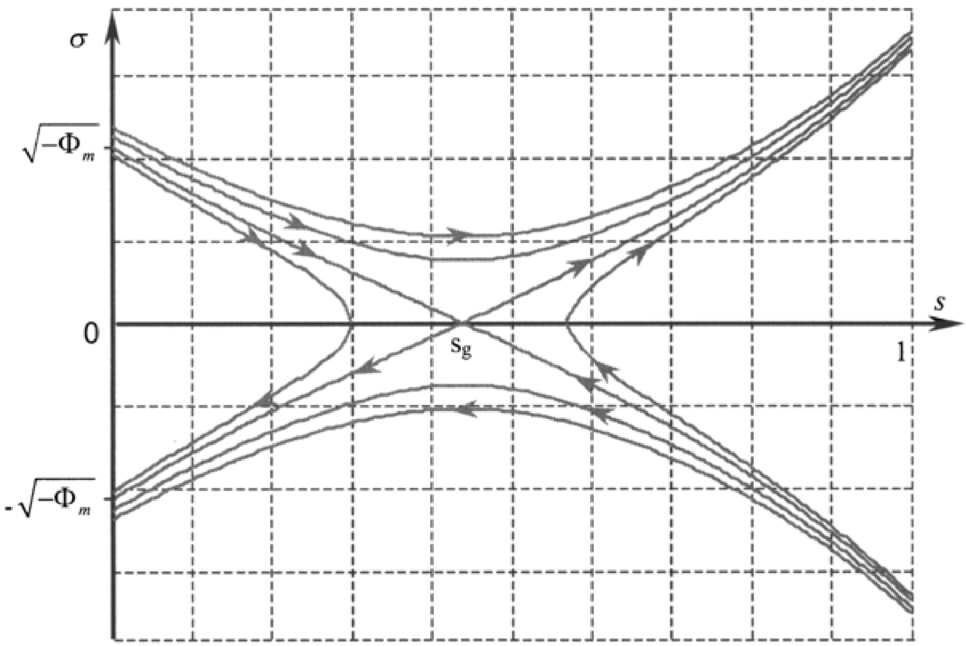

Fig. 4. The phase portrait of system (33), (34) in the plane (s, σ), corresponding to the function $\Phi(s)$ given in Figure 3a.

path is similar to the behaviour of an inverted pendulum. The phase portrait is symmetric with respect to axis s. The point $s=s_g$, $\sigma=0$ is a *saddle point* or *col*, it corresponds to an unstable equilibrium of the biped. Two separatrices intersect at the saddle point. The ordinates of these separatrices are $\sigma=\pm\sqrt{-\Phi_m}$ at $s=0$. Each separatrix is close to a straight line because the graph of function $\Phi(s)$ is close to a parabola. The phase portrait in Figure 4 shows that:

- If $\sigma(0)<\sqrt{-\Phi_m}$, then the curve $\sigma(s)$ defined by Equation (35) crosses axis $\sigma=0$ for $s<s_g$, after angular momentum becomes negative as velocity $\dot{s}$; the parameter s decreases and the biped according to the given path $q_r(s)$ falls backward.
- If $\sigma(0)=\sqrt{-\Phi_m}$, the motion converges asymptotically when $t\to\infty$ to the unstable equilibrium: $s=s_g$, $\sigma=0$.
- If $\sigma(0)>\sqrt{-\Phi_m}$, the step can be achieved because the angular momentum σ and the velocity $\dot{s}$ are positive during all the step $0\leq s\leq 1$.

The results are summarized in the following theorem.

Theorem 1: The path $q_r(s)$ *with* $k<s<k+1$ can be achieved by the biped, if and only if $\sigma(k^+)>\sqrt{-\Phi_m}$ or $\dot{s}(k^+)>\dfrac{\sqrt{-\Phi_m}}{f(0^+)}$.

4.3. Conditions of existence and uniqueness of cyclic motion

A cyclic admissible reference motion is defined by a cyclic evolution of angular momentum σ or equivalently of a cyclic velocity $\dot{s}$ denoted by $\dot{s}_c$. All the admissible reference motions are defined by Equations (35) or (37). Thus a cyclic admissible reference motion exists if and only if there exists an initial angular momentum $\sigma(k^+)$ such that: $\sigma(k+1^+)=\sigma(k^+)$, or in another words, if and only if there exists an initial velocity $\dot{s}(k)$ denoted by $\dot{s}_c(k)$, such that s increases when time increases and $\dot{s}(k+1)=\dot{s}(k)=\dot{s}_c(k)=\dot{s}_c(0)$ (note that $\dot{s}(s)$ is a continuous function at $s=k$). Under these conditions, the states of the biped are identical at the beginning of the steps k and $k+1$ (but the legs swap their role).

Since the functions $f(s)$ and $\Phi(s)$ are cyclic, writing Equation (37) for $s=k+1^-$ or for $s=1^-$ implies that the initial velocity $\dot{s}_c(0)$ is such that:

$$f^2(1^-)\dot{s}_c(0)^2-f^2(0^+)\dot{s}_c(0)^2=\Phi(1^-)$$

Analyzing Equation (39), we conclude:

- If $f(0^+)=f(1^-)$ and $\Phi(1^-)=0$, then any initial value $\dot{s}(k)>\dot{s}_m$ produces a cyclic reference motion.
- If $f(0^+)=f(1^-)$ and $\Phi(1^-)\neq 0$ or if values $\Phi(1^-)$ and $f^2(1^-)-f^2(0^+)$ have different signs, then Equation (39) has no solution, and consequently there is no cyclic reference motion.
- Equation (39) has a unique solution $\dot{s}_c(0)$:

$$\dot{s}_c(0)\}=\sqrt{\frac{\Phi(1^-)}{f^2(1^-)-f^2(0^+)}} \tag{40}$$

  if and only if values $\Phi(1^-)$ and $f^2(1^-)-f^2(0^+)$ have the same sign.

According to Theorem 1, solution Equation (40) is the initial velocity for the cyclic reference motion if and only if $\dot{s}_c(0)>\dfrac{\sqrt{-\Phi_m}}{f(0^+)}$. Using Equation (40), the following theorem can be formulated.

Theorem 2: A unique cyclic reference motion exists if and only if $\dfrac{\Phi(1^-)}{f^2(1^-)-f^2(0^+)}+\dfrac{\Phi_m}{f^2(0^+)}>0$. The initial cyclic velocity for one step is defined by Equation (40).

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Remark: If $f^2(1^-) - f^2(0^+) > 0$, then the angular momentum decreases during the impact (the supporting leg changes); in opposite case, it increases. If $\Phi(1^-) > 0$ the angular momentum increases during the single support phase; in opposite case, it decreases. A cycle is achieved only when an increase (decrease) of the angular momentum at the impact instant is compensated by a decrease (increase) during the single support motion.

5. CONVERGENCE TOWARDS THE CYCLIC REFERENCE MOTION

In this section, a condition of convergence of the admissible reference motion to the cyclic motion is obtained.

We assume that a unique cyclic reference motion exists and that the initial velocity $\dot{s}$ is high enough to have a monotonic evolution of parameter s. The relative difference between velocity $\dot{s}(s)$ and cyclic velocity $\dot{s}_c(s)$, which is referred to as "velocity difference", is defined by:

$$e(s) = \frac{\dot{s}(s) - \dot{s}_c(s)}{\dot{s}_c(s)} \quad (41)$$

The biped motion converges towards the cyclic one if and only if $\dot{s}(s)$ converges towards $\dot{s}_c(s)$ or equivalently if $e(s)$ converges to 0 when $s \to \infty$.

5.1. *Evolution of the "velocity difference" e(s)*

Under definition (41), the velocity $\dot{s}(s)$ can be expressed using the cyclic velocity as:

$$\dot{s}(s) = \dot{s}_c(s)(1 + e(s)) \quad (42)$$

The cyclic motion is an admissible reference motion. Thus Equation (37) can be written for the cyclic motion, in the following form ($k < s < k+1$):

$$f^2(s)\dot{s}_c^2(s) - f^2(k^+)\dot{s}_c(0)^2 = \Phi(s) \quad (43)$$

Taking Equation (42) into account in Equation (37), and using Equation (43), we have:

$$(f^2(0^+)\dot{s}_c(0)^2 + \Phi(s))(1 + e(s))^2 - f^2(0^+)\dot{s}_c(0)^2(1 + e(k))^2 = \Phi(s)$$

Then $e(s)$ can be expressed as function of $e(k)$ for $k < s < k+1$:

$$e(s) = \sqrt{1 + e(k)(e(k)+2)\,\frac{f^2(0^+)\dot{s}_c(0)^2}{f^2(0^+)\dot{s}_c(0)^2 + \Phi(s)}} - 1 \quad (44)$$

The function $e(s)$ includes a square root and is defined only for:

$$e(k) > \frac{\sqrt{-\Phi_m}}{f(0^+)\dot{s}_c(0)} - 1$$

This condition is equivalent to inequality $\dot{s}(k) > \dfrac{\sqrt{-\Phi_m}}{f(0^+)}$ (see Theorem 1).

The evolution of the velocity difference $e(s)$ for one step can be directly deduced from the evolution of $\Phi(s)$. For the evolution of $\Phi(s)$ given in Figure 3, the velocity difference evolutions are shown in Figure 5.

- For $k < s < k + s_g$, $|e(s)|$ increases because $\Phi(s)$ decreases (see Equation (44)).
- For $s = k + s_g$, $|e(s)|$ has a maximum,
- For $k + s_g < s < k+1$, $|e(s)|$ decreases because $\Phi(s)$ increases.

From the beginning of the step to its end, the error increases or decreases depending on the sign of $\Phi(1)$.

Function $\Phi(s)$ is cyclic but not continuous at $s = k$, thus formula (44) is convenient only for one step $k < s < k+1$. The velocity difference $e(s)$ (see (41)) is a continuous function at $s = k$ because $\dot{s}(s)$ and $\dot{s}_c(s)$ are continuous functions at $s = k$. Using Equation (39), we obtain the iterative formula from one step to the following one:

$$e(k+1) = \sqrt{1 + e(k)(e(k)+2)\left(\frac{f(0^+)}{f(1^-)}\right)^2} - 1$$

5.2. *Condition of convergence*

The following theorem can be proved.

Theorem 3: The admissible reference motion converges towards the cyclic admissible reference motion if and only if $\dot{s}(0) > \dfrac{\sqrt{-\Phi_m}}{f(0^+)}$ and $f(0^+) < f(1^-)$ (or equivalently $\Phi(1^-) > 0$).

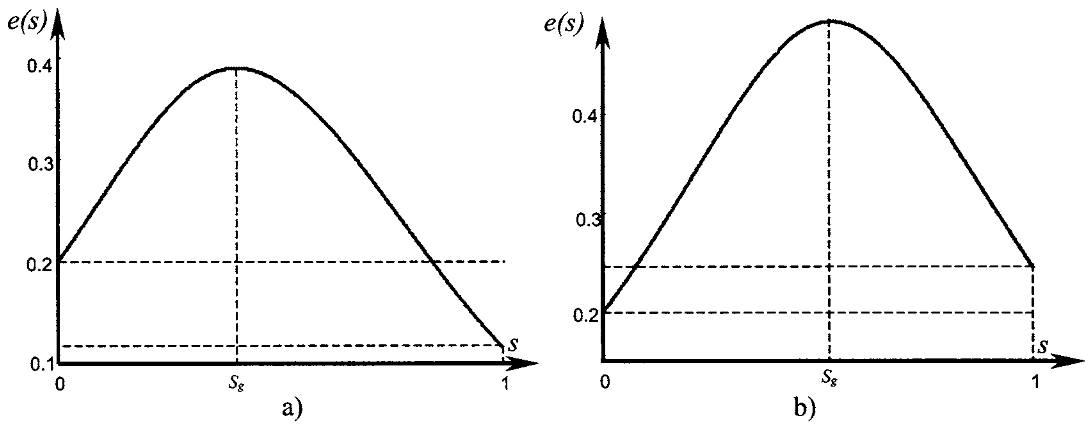

Fig. 5. Two typical evolutions $e(s)$ for one step: in case a) $e(1) < e(0)$, in case b) $e(1) > e(0)$.

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Proof: With $k<s<k+1$, if $k\to\infty$ then error $e(s)\to 0$ uniformly for any s, if and only if $e(k)\to 0$ when $k\to\infty$ because $e(s)$ is defined by Equation (44) and the function $\frac{f^2(0^+)\dot{s}_c(0)^2}{f^2(0^+)\dot{s}_c(0)^2+\Phi(s)}$ is cyclic and bounded. Thus, to prove that the biped motion converges to the cyclic admissible reference motion, it is necessary and sufficient to prove the convergence of $e(k)$ towards 0 when $k\to\infty$.

If $f(0^+)<f(1^-)$, then using Equation (46) and inequality $f(s)>0$, we can deduce that:

$$|e(k+1)|\leq\frac{f(0^+)}{f(1^-)}|e(k)| \tag{47}$$

And we can conclude that $e(k)\to 0$ when $k\to\infty$.

It follows from Equation (46) that if $f(0^+)=f(1^-)$, then $e(k+1)=e(k)$.

If $f(0^+)>f(1^-)$, then $|e(k+1)|\geq\frac{f(0^+)}{f(1^-)}|e(k)|$ and there is no convergence.

The condition $\dot{s}(0)>\frac{\sqrt{-\Phi_m}}{f(0^+)}$ ensures that $s(t)$ is an increasing function during the first step. If $f(0^+)<f(1^-)$, the condition $\dot{s}(k)>\frac{\sqrt{-\Phi_m}}{f(0^+)}$ will be satisfied for all k, and the function $s(t)$ increases for all steps. ■

Remark 1: The convergence of the admissible reference motion can also be shown using a section of the Poincaré map as in references [1, 2]. Equation (46) allows to draw easily $e(k+1)$ as a function of $e(k)$. The linearization of Equation (46) around point $e(k)=0$ defines the convergence ratio from one step to the next one around the cyclic motion.

After linearization, we have: $e(k+1)=\left(\frac{f(0^+)}{f(1^-)}\right)^2 e(k)$. This equation describes a geometrical progression with ratio $\left(\frac{f(0^+)}{f(1^-)}\right)^2$, which is less than the ratio in Equation (47). The lower the ratio $\left(\frac{f(0^+)}{f(1^-)}\right)^2$, the faster the convergence.

Remark 2: Theorem 3 concerns in fact *orbital* stability of the admissible reference cyclic motion, because in this theorem, we consider parameter s as independent variable but we do not consider time t.

Combining Theorems 1, 2 and 3, the following corollary can be deduced.

Corollary: *The admissible reference cyclic motion is orbitally asymptotically stable if and only if the reference joint path is such that:* $f(0^+)<f(1^-)$ *and* $\Phi(1^-)f^2(0^+)+\Phi_m(f^2(1^-)-f^2(0^+))>0$.

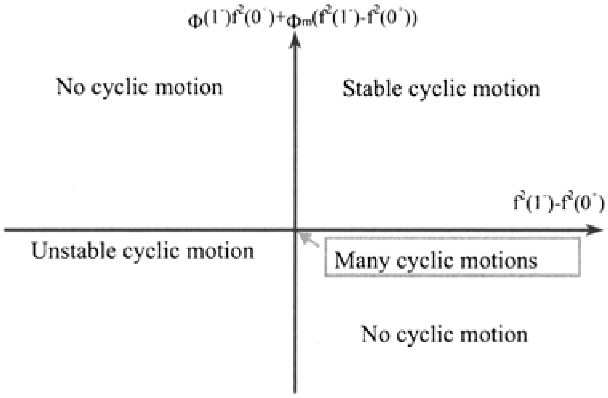

Fig. 6. Existence and stability of cyclic motion in different cases.

These conditions may be not satisfied for some reference joint paths. Figure 6 presents the different occurring cases.

- In Figure 6, the first quadrant corresponds to the cases satisfying the condition of the corollary.
- In the second and fourth quadrants $\frac{\Phi(1^-)}{f^2(1^-)-f^2(0^+)}+\frac{\Phi_m}{f^2(0^+)}<0$, the condition of Theorem 2 is not satisfied, thus there is no cyclic motion.
- In the third quadrant $\frac{\Phi(1^-)}{f^2(1^-)-f^2(0^+)}+\frac{\Phi_m}{f^2(0^+)}>0$, the condition of Theorem 2 is satisfied, but $f^2(1^-)-f^2(0^+)<0$, so the condition of Theorem 3 is not satisfied, thus there is an unstable cyclic motion.
- In the origin $f^2(1^-)-f^2(0^+)=0$ and $\Phi(1^-)=0$, thus any initial velocity $\dot{s}(k)$ provides a cyclic motion.

Only stable cyclic motions are interesting for the biped control design. These motions will be illustrated in Section 6 devoted to the simulation.

5.3. *Unilateral contact*

For an admissible reference motion to be followed, it must be such that the reaction force satisfies inequality Equation (14). For example, if the initial velocity is too large, the centrifugal forces are higher than the gravitational forces and a take-off of the biped occurs.

For an admissible reference motion the constraint Equation (14) becomes:

$$U(q_r)\left(\frac{dq_r}{ds}\ddot{s}+\frac{d^2q_r}{ds^2}\dot{s}^2\right)+V\left(q_r,\frac{dq_r}{ds}\right)\dot{s}^2+W>0$$

Using Equations (10) and (32), the acceleration $\ddot{s}$ can be calculated by:

$$\ddot{s}=-\frac{1}{f(s)}\frac{df(s)}{ds}\dot{s}^2+\frac{mg}{f(s)}(x_g(q_r(s))-x_{S_i})$$

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Thus, the constraint can be written:

$$\left(-U(q_r)\frac{dq_r}{ds}\frac{1}{f(s)}\frac{df(s)}{ds}+V(q_r\frac{dq_r}{ds})+U(q_r)\frac{d^2q_r}{ds^2}\right)\dot{s}^2$$

$$+U(q_r)\frac{dq_r}{ds}\frac{mg}{f(s)}(x_g(q_r(s))-x_{S_i})+W>0 \tag{48}$$

But the evolution of $\dot{s}$ is defined by the initial velocity for the step by Equation (37):

$$\dot{s}^2(s)=\frac{f^2(0^+)\dot{s}^2(k)+\Phi(s)}{f^2(s)} \tag{49}$$

Thus, combining Equations (48) and (49), the conditions for $\dot{s}^2(k)$ to satisfy the constraint on the reaction forces, have the form:

$$J(s)\dot{s}^2(k)+L(s)>0 \tag{50}$$

We recall that $J(s)$ and $L(s)$ are 2×1 vector-functions: $J(s)=[J_j(s)]$, $L(s)=[L_j(s)]$. Vectorial inequality (50) is equivalent to the following two scalar inequalities:

$$J_j(s)\dot{s}^2(k)+L_j(s)>0, \qquad j=1,2$$

Different cases exist depending on the signs of the functions $J_j(s)$, $L_j(s)$. Let us introduce the following three sets:

$$\mathrm{U_j}=\{\mathrm{s}\in[0,1]\text{: } \mathrm{J_j(s)}\le 0 \quad \text{and} \quad L_j(s)\le 0\}$$
$$V_j=\{s\in[0,1]\text{: } J_j(s)<0 \quad \text{and} \quad L_j(s)>0\}$$
$$W_j=\{s\in[0,1]\text{: } J_j(s)>0 \quad \text{and} \quad L_j(s)\le 0\}$$

- If there exists at least one index j such that $U_j\neq\emptyset$, then inequality (51) for this index j, and consequently inequality (50), cannot be satisfied for any value $\dot{s}(k)$.
- If for all j, $U_j=\emptyset$ and $W_j=\emptyset$, and if there exists at least one index j such that $V_j\neq\emptyset$, then the reaction forces satisfy the constraints if and only if $$\dot{s}(k)<\sqrt{\min_{j=1,2}\min_{s\in V_j}\left(\frac{-L_j(s)}{J_j(s)}\right)}.$$
- If for all j, $U_j=\emptyset$ and $V_j=\emptyset$, and if there exists at least one index j such that $W_j\neq\emptyset$, then the reaction forces satisfy the constraints if and only if $\dot{s}(k)>$ $$\sqrt{\max_{j=1,2}\max_{s\in W_j}\left(\frac{-L_j(s)}{J_j(s)}\right)}.$$
- If for all j, $U_j=\emptyset$ and if there exists at least one index j such that $V_j\neq\emptyset$ and one index l such that $W_l=\emptyset$, then the reaction forces satisfy the constraints if and only if $$\sqrt{\max_{l=1,2}\max_{s\in W_l}\left(\frac{-L_l(s)}{J_j(s)}\right)}<\dot{s}(k)<\sqrt{\min_{j=1,2}\min_{s\in V_j}\left(\frac{-L_j(s)}{J_j(s)}\right)}.$$
- If for all j, $U_j=\emptyset$, $W_j=\emptyset$ and $V_j=\emptyset$, then the reaction forces satisfy the constraints for any positive $\dot{s}(k)$.

The size of the attraction domain is important for practical applications. The larger this domain, the more robust the control law. This domain is also interesting to study possible changes of the velocity for the robot walking. The constraint on the torque limits can be taken into account in a similar way.

5.4. *Control law*

We have defined the conditions such that a joint path corresponds to a stable admissible cyclic motion. The attraction region of this cyclic motion has been found, this region is based on the value of the angular momentum. The constraints on the reaction force (no take-off, no sliding) give also some limits on the initial velocity (or angular momentum).

Control law (30) ensures that the motion of the biped converges in a finite time towards a reference path. Thus, the robot follows an admissible reference motion. And the following assertion is correct:

The control law (30) ensures an orbitally asymptotically stable motion of the robot if and only if the reference joint path is such that: $f(0^+)<f(1^-)$ *and* $\Phi(1^-)f^2(0^+)+\Phi_m(f^2(1^-)-f^2(0^+))>0$, *and the angular momentum at the beginning of the walking is within some limits.*

Remark: If the control law converges to the reference path during the first step, the limits on the velocity $\dot{s}$ (and consequently on the angular momentum) at the beginning of the second step are $\dot{s}(1)>\frac{\sqrt{-\Phi_m}}{f(0^+)}$ plus the limits defined in the previous subsection 5.3.

6. SIMULATION RESULTS

The proposed control law has been tested on the reference path presented under the stick-diagram form in Figure 7. The joint path $q_r(s)$ is defined by a polynomial evolution q with respect to s. We use a fourth order polynomial for each component of vector q.

The corresponding periodic functions $f(s)$ and $\Phi(s)$ are plotted in Figure 8.

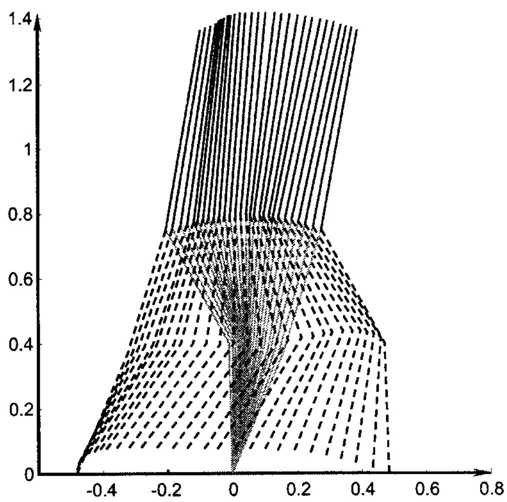

Fig. 7. The reference path.

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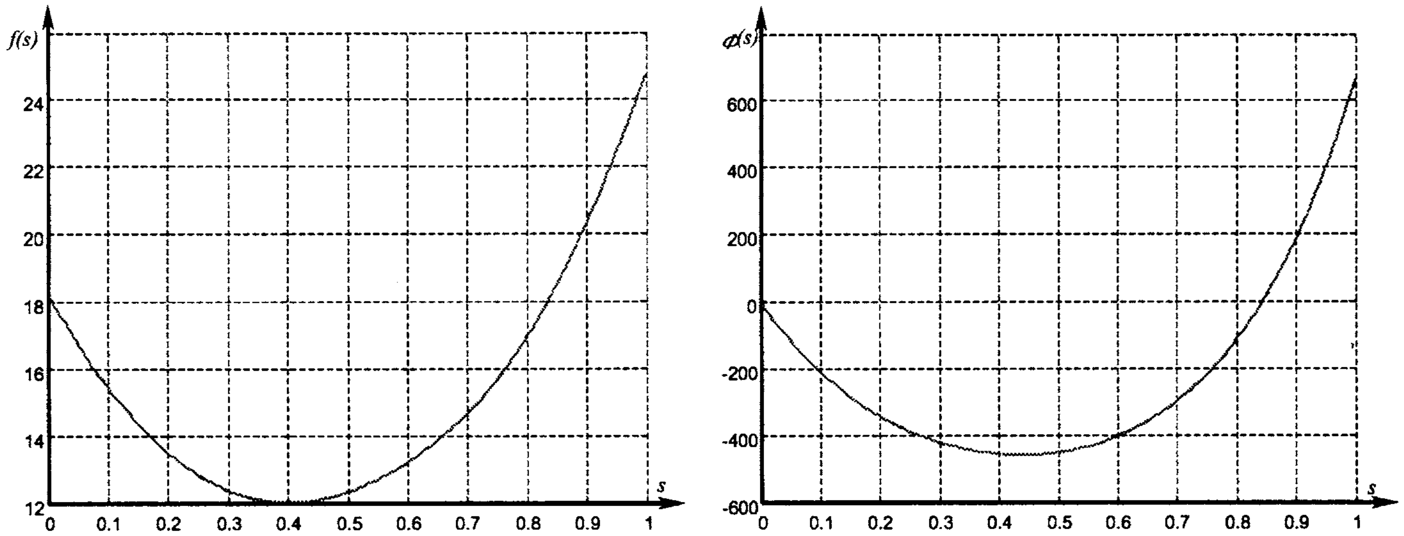

Fig. 8. Functions f(s) and $\Phi(s)$.

For the chosen reference path $f(0^+)=18.30$, $f(1^-)=24.76$ and $\Phi(1^-)f^2(0^+)+\Phi_m(f^2(1^-)-f^2(0^+))=94172$, thus $f(0^+)<f(1^-)$ and $\Phi(1^-)f^2(0^+)+\Phi_m(f^2(1^-)-f^2(0^+))>0$. In consequence, the biped motion converges to a cyclic motion. The minimal value $\dot{s}(k)$ to achieve a complete step is (see Theorem 1) $\dot{s}_m=1.17$. The constraints on the reaction force induce only a higher limit on the velocity $\dot{s}(k)$: this velocity must be less than 2.55 in order to avoid the sliding ($\mu=0.66$) of the supporting leg. For an initial velocity of the robot such that $1.17<\dot{s}(0)<2.55$, the motion of the biped converges to the cyclic motion defined by $\dot{s}_c(0)=1.54$ (see Equation (40)).

6.1. *Perfect modelling*

Figure 9a shows the behaviour obtained in simulation with control law (30) for a "large" initial velocity, $\dot{s}(0)=2.5$. The initial state of the biped belongs to the set of reference motions. Thus, the robot follows the parameterized reference path without tracking error and converges towards the cyclic motion. In Figure 9a, the velocity $\dot{s}(s)$ is shown. The function $\dot{s}_c(s)$ corresponding to the cyclic motion is also presented in order to point out the convergence of the robot motion to the cyclic one. But our control ensures only orbital stability, thus the velocity $\dot{s}(t)$ does not converge to $\dot{s}_c(t)$ as shown in Figure 9b. In Figure 10, the trunk orientation is drawn in its phase plane. This phase portrait allows us to illustrate the convergence to the cyclic motion and the effect of the impact with the ground (there is a jump in the velocity in the phase portrait).

6.2. *Presence of modelling error*

To illustrate a robustness property of the proposed approach, the following case is simulated:

- The mass errors are +10% for the thighs, +30% for the shins and +50% for the trunk. The error on the inertia

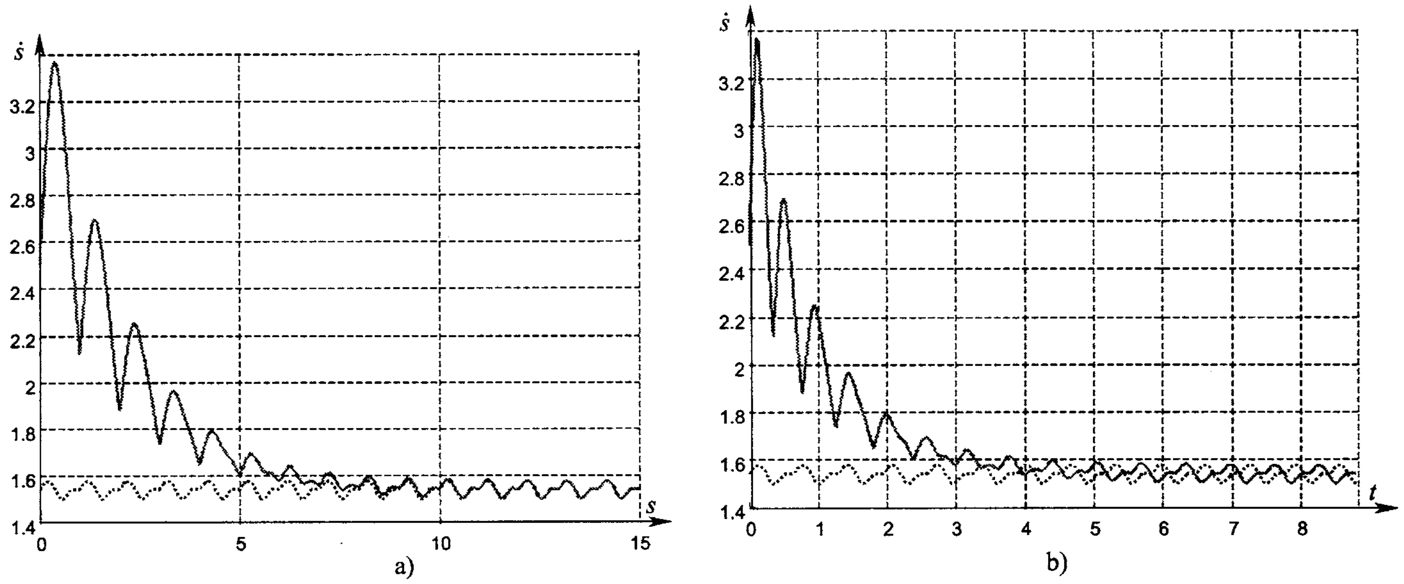

Fig. 9. The proposed control ensures orbital stability: a) evolution of velocity $\dot{s}(s)$ (solid line), and of cyclic velocity $\dot{s}_c(s)$ (dotted line) for 15 steps, b) evolution of velocity $\dot{s}(t)$ (solid line), and of cyclic velocity $\dot{s}_c(t)$ (dotted line) for 15 steps.

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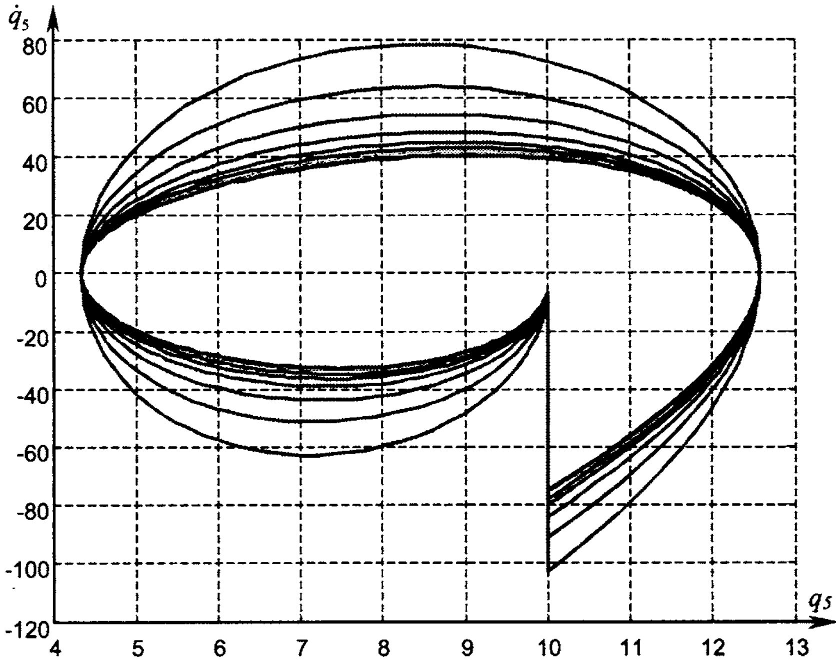

Fig. 10. Evolution in degrees of angle q_5 in its phase plane.

moments are +40% for the thighs, +10% for the shins and +30% for the trunk.

- The control law used is a classical computed torque control, thus the desired closed loop behaviour is

$$\ddot{q} = \ddot{q}_d + K_v \dot{e}_q + K_p e_q \quad (52)$$

instead of Equation (26).

- Since the reference path is designed with a false model of the robot, the velocity after the impact is not the expected one.

The behaviour obtained in simulation is presented in Figure 11. In this figure, velocity $\dot{s}(s)$ is shown. The function $\dot{s}_c(s)$ corresponding to the cyclic motion of the modelled robot is also presented. The velocity $\dot{s}$ does not converge to the "expected" motion because this motion is not compatible with the real dynamics of the biped, but a cyclic stable motion is obtained.

The simulation results show that the leg tip does not touch the ground during the single support (before $s = 1.006$), the ground reaction is directed upwards and is inside the friction cone. Some tracking errors exist particularly at the beginning of each step due to the effect of impact, thus the path followed is not exactly the expected one but the tracking errors in angular variables are cyclic and smaller than 0.005 rad.

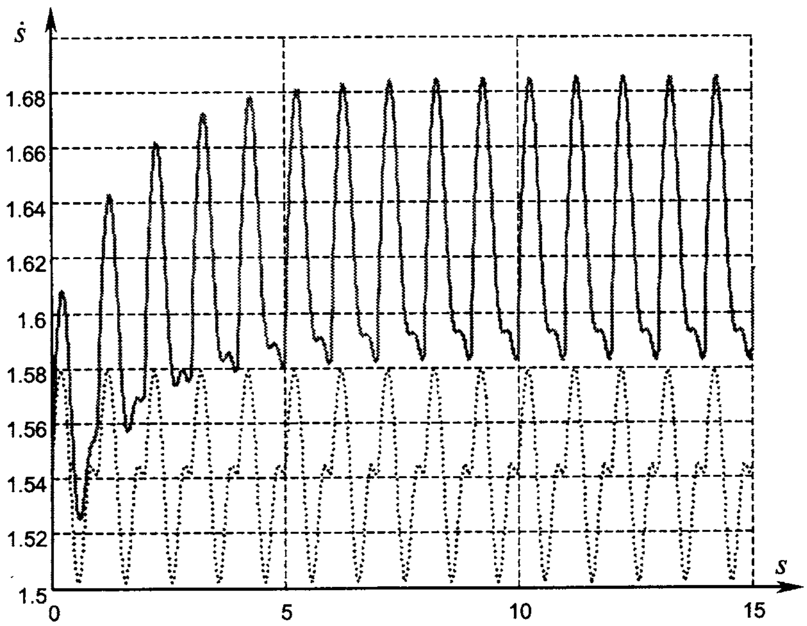

Fig. 11. Evolution of velocity $\dot{s}(s)$ (solid line), and of cyclic velocity $\dot{s}_c(s)$ (dotted line) for 15 steps in presence of modelling error.

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7. CONCLUSION

For a planar biped under-actuated during the single support phases, the proposed control strategy consists in tracking a reference path instead of a reference motion. The robot adapts its temporal evolution according to the effect of gravity. In this context a complete study has been presented. Some analytical conditions that can be easily tested have been proposed: conditions of existence and uniqueness of a cyclic motion, condition of convergence towards this cyclic motion. These conditions are defined on the reference path.

The conditions of a cyclic motion existence and convergence to it are inequalities. Thus some robustness is naturally contained in the proposed control strategy. In spite of tracking errors and/or modelling error, the behaviour of the robot converges to a cyclic motion, for a convenient reference path (i.e. satisfying the inequality with some margins). In the presence of modelling errors, the obtained cycle is slightly modified with respect to the predicted cycle, but stable walking is obtained as it has been observed in simulation.

Since a reference path must satisfy some conditions (inequalities) in order to produce stable cyclic walking, there exist some reference paths that cannot be used with the proposed strategy. But we want to point out that most of the tested paths are convenient for our control strategy. To correspond to a stable motion, the path must satisfy the two following conditions: The angular momentum must decrease during the impact phase (the contact point changes); during the single support phase, the sub-phase where the gravity speeds up the motion, must have a higher contribution to the change of the angular momentum than the sub-phase where the gravity slows down the motion (see Figure 2).

All the cyclic optimal reference trajectories defined in reference [15] for this biped produce a path that corresponds to a stable motion with the proposed control strategy.

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