Fibonacci-Lucas hyperbolas

HOWARD SPORN

Let us define a *Fibonacci-Lucas hyperbola* as a hyperbola passing through an infinite number of points of the form (F_m, L_n), where the F_m are distinct Fibonacci numbers (0, 1, 1, 2, 3, 5, 8, 13, 21, . . ., where $F_0 = 0$), and the L_n are distinct Lucas numbers (2, 1, 3, 4, 7, 11, 18, 29, . . ., where $L_0 = 2$). The simplest examples are $5x^2 - y^2 = 4$, which contains the points (F_k, L_k) with odd subscripts, e.g. (1, 1), (2, 4), (5, 11), and $5x^2 - y^2 = -4$, which contains the points with even subscripts, e.g. (0, 2), (1, 3), (3, 7); (see [1, 2]). These follow immediately from the identity

$$L_n^2 - 5F_n^2 = 4(-1)^n. \tag{1}$$

Our goal is to find more of these Fibonacci-Lucas hyperbolas.

We assume that such a hyperbola has the equation

$$ax^2 + bxy + cy^2 + dx + ey + f = 0, \tag{2}$$

where a, b, c, d, e, and f are constants to be determined. We consider a hyperbola containing points of the form (F_k, L_{k+1}), where k is even. Then it contains the points (0, 1), (1, 4), (3, 11), (8, 29), (21, 76) and (55, 199). Substituting these into (2) gives a system of linear equations, which can be solved. The result is the hyperbola $5x^2 - 5xy + y^2 = 1$.

By the same procedure, we find that points of the form (F_k, L_{k+2}), where k is odd, lie on the hyperbola $5x^2 + 5xy - y^2 = 9$, and points of the form (F_k, L_{k+3}), where k is even, lie on the hyperbola $5x^2 - 10xy + y^2 = 16$.

Based on this, we guess that points of the form (F_k, L_{k+n}), where n is a constant such that $k + n$ is an odd number, lie on the hyperbola $5x^2 + 5(-1)^n F_n xy + (-1)^{n+1} y^2 = L_n^2$. Now we have to prove it.

We will make use of *Fibonacci hyperbolas*, which were studied by Clark Kimberling [3]. A Fibonacci hyperbola is a hyperbola passing through an infinite number of points of the form (F_m, F_n). where the F are distinct Fibonacci numbers. Kimberling proved that all Fibonacci hyperbolas are of the form

$$x^2 + (-1)^{n+1} L_n xy + (-1)^n y^2 = -F_n^2, \text{ for } n = 1, 2, 3, \ldots \tag{3}$$

or of the form

$$x^2 + (-1)^{n+1} L_n xy + (-1)^n y^2 = F_n^2, \text{ for } n = 1, 2, 3, \ldots \tag{4}$$

(or a reflection of one of those in the y-axis). Hyperbolas satisfying (3) contain points of the form

$$(F_k, F_{k+n}), (F_{k+2}, F_{k+n+2}), (F_{k+4}, F_{k+n+4}), \ldots , \tag{5}$$

where k is an integer such that $k + n$ is odd.

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Hyperbolas satisfying (4) contain points of the form

$$\left(F_k, F_{k+n}\right), \left(F_{k+2}, F_{k+n+2}\right), \left(F_{k+4}, F_{k+n+4}\right), \ldots, \tag{6}$$

where k is an integer such that $k + n$ is even.

We begin by using Catalan's identity [4, p. 83],

$$F_i^2 - F_{i+j}F_{i-j} = (-1)^{i-j}F_j^2.$$

Replace i with $k + n$, replace j with n, and then multiply both sides by $2(-1)^{n+1}$, and we have

$$2(-1)^{n+1}F_{k+n}^2 - (-1)^{n+1}F_k \cdot 2F_{k+2n} = (-1)^{k+n+1}2F_n^2. \tag{7}$$

Now we use an identity of Blazej [4, p. 93],

$$2F_{k+2n} = L_nF_{k+n} + F_nL_{k+n}.$$

Substitute into (7) and we have

$$2(-1)^{n+1}F_{k+n}^2 - (-1)^{n+1}L_nF_kF_{k+n} - (-1)^{n+1}F_nF_kL_{k+n} = (-1)^{k+n+1}2F_n^2. \tag{8}$$

Now we take the case for which $k + n$ is odd. In that case we can use (3) and (5) to obtain

$$F_k^2 + (-1)^{n+1}L_nF_kF_{k+n} + (-1)^nF_{k+n}^2 = -F_n^2.$$

We add this to (8) and use the fact that $k + n$ is odd. The result is

$$F_k^2 - (-1)^{n+1}F_nF_kL_{k+n} + (-1)^{n+1}F_{k+n}^2 = F_n^2. \tag{9}$$

Now, from (1) and the fact that $k + n$ is odd, we have

$$5F_{k+n}^2 = L_{k+n}^2 + 4. \tag{10}$$

We multiply both sides of (9) by 5, and substitute (10) in to obtain

$$5F_k^2 + 5(-1)^nF_nF_kL_{k+n} + (-1)^{n+1}L_{k+n}^2 - 4(-1)^n = 5F_n^2.$$

We move the $4(-1)^n$ term to the right and use (1) to get

$$5F_k^2 + 5(-1)^nF_nF_kL_{k+n} + (-1)^{n+1}L_{k+n}^2 = L_n^2. \tag{11}$$

We now let

$$x = F_k \text{ and } y = L_{k+n}.$$

Then (11) becomes

$$5x^2 + 5(-1)^nF_nxy + (-1)^{n+1}y^2 = L_n^2, \tag{12}$$

as desired.

We list some hyperbolas of the form of (12), along with some points that lie on the hyperbolas. See Table 1. Graphs of those hyperbolas are given in Figures 1 and 2.

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n	*Hyperbola*	*Representative Points*
0	$5x^2 - y^2 = 4$	(1, 1) (2, 4) (5, 11) (13, 29)
1	$5x^2 - 5xy + y^2 = 1$	(0, 1) (1, 4) (3, 11) (8, 29)
−1	$5x^2 - 5xy + y^2 = 1$	(1, 1) (3, 4) (8, 11) (21, 29)
2	$5x^2 + 5xy - y^2 = 9$	(1, 1) (1, 4) (2, 11) (5, 29)
−2	$5x^2 - 5xy - y^2 = 9$	(2, 1) (5, 4) (13, 11) (34, 29)

TABLE 1: Hyperbolas of the form (12), showing representative points

Note that we get the same hyperbola for $n = 1$ and $n = -1$. However, the points for $n = 1$ lie on one branch of the hyperbola, and the points for $n = -1$ lie on the other branch.

We had taken the case for which $k + n$ is odd. Now we will consider the case for which $k + n$ is even. In that case, we use (4) and (6), along with (8). The derivation is analogous to that for (11). The result is

$$5F_k^2 + 5(-1)^n F_n F_k L_{k+n} + (-1)^{n+1} L_{k+n}^2 = -L_n^2. \quad (13)$$

We now let

$$x = F_k \text{ and } y = L_{k+n}.$$

Then (13) becomes

$$5x^2 + 5(-1)^n F_n xy + (-1)^{n+1} y^2 = -L_n^2. \quad (14)$$

We list some hyperbolas of the form of (14), along with some points that lie on the hyperbolas. See Table 2. Graphs of those hyperbolas are given in Figures 3 and 4. Note that we get the same hyperbola for $n = 1$ and $n = -1$.

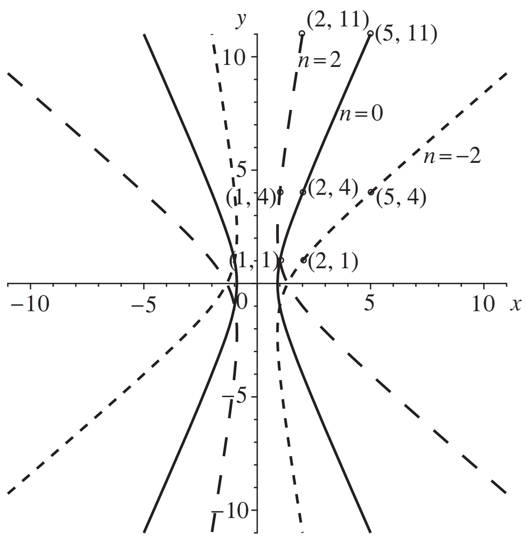

FIGURE 1: Hyperbolas of the form (12), for $n = 0$, 2 and −2

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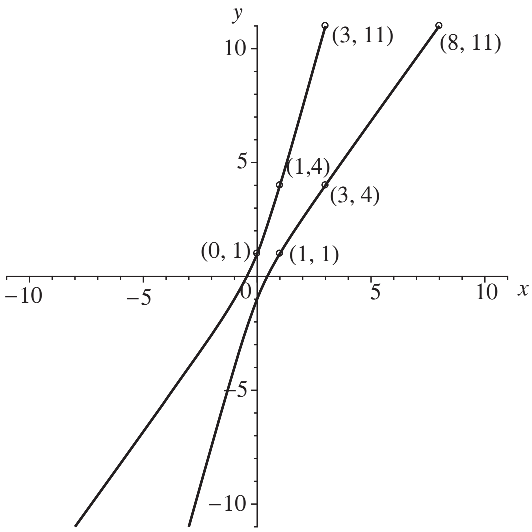

FIGURE 2: Hyperbola $5x^2 - 5xy + y^2 = 1$ of the form (12), for $n = \pm 1$

n	*Hyperbola*	*Representative Points*
0	$5x^2 - y^2 = -4$	(0, 2) (1, 3) (3, 7) (8, 18)
1	$5x^2 - 5xy + y^2 = -1$	(1, 2) (1, 3) (2, 7) (5, 18)
−1	$5x^2 - 5xy + y^2 = -1$	(1, 2) (2, 3) (5, 7) (13, 18)
2	$5x^2 + 5xy - y^2 = -9$	(0, 3) (1, 7) (3, 18) (8, 47)
−2	$5x^2 - 5xy - y^2 = -9$	(0, 3) (1, 2) (3, 3) (8, 7)

TABLE 2: Hyperbolas of the form (14), showing representative points

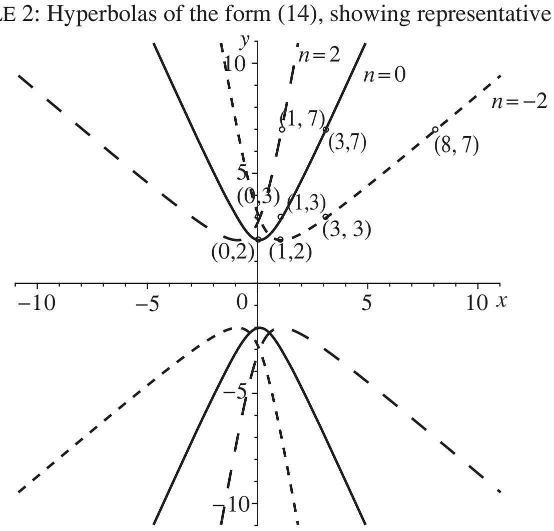

FIGURE 3: Hyperbolas of the form (14), for $n = 0,\ 2$ and -2

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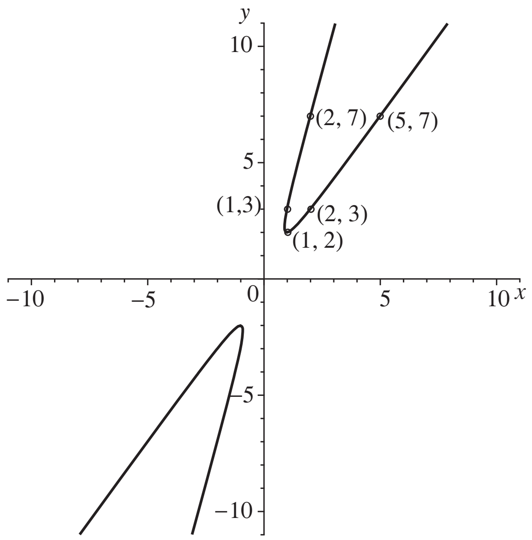

FIGURE 4: Hyperbola $5x^2 - 5xy + y^2 = -1$ of the form (14) for $n = \pm 1$

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HOWARD SPORN
Department of Mathematics and Computer Science,
Queensborough Community College,
Bayside, NY, USA 11364
e-mail: *hsporn@qcc.cuny.edu*