Bull. Aust. Math. Soc. **109** (2024), 125–137
doi:10.1017/S000497272300028X

THE GROWTH OF SOLUTIONS OF MONGE–AMPÈRE EQUATIONS IN HALF SPACES AND ITS APPLICATION

SHANSHAN MA and XIAOBIAO JIA ✉

(Received 10 February 2023; accepted 24 February 2023; first published online 31 March 2023)

Abstract

We consider the growth of the convex viscosity solution of the Monge–Ampère equation $\det D^2 u = 1$ outside a bounded domain of the upper half space. We show that if u is a convex quadratic polynomial on the boundary $\{x_n = 0\}$ and there exists some $\varepsilon > 0$ such that $u = O(|x|^{3-\varepsilon})$ at infinity, then $u = O(|x|^2)$ at infinity. As an application, we improve the asymptotic result at infinity for viscosity solutions of Monge–Ampère equations in half spaces of Jia, Li and Li ['Asymptotic behavior at infinity of solutions of Monge–Ampère equations in half spaces', *J. Differential Equations* **269**(1) (2020), 326–348].

2020 *Mathematics subject classification*: primary 35J96; secondary 35B40.

Keywords and phrases: Monge–Ampère equation, growth, asymptotic behaviour.

1. Introduction

The existence and asymptotic behaviour at infinity of the convex viscosity solution of the Monge–Ampère equation:

$$\begin{cases} \det D^2 u(x) = f(x) & \text{in the half space } \mathbb{R}^n_+, \\ u(x', x_n) = \frac{1}{2}|x'|^2 & \text{on the boundary } \{x_n = 0\}, \end{cases} \tag{1.1}$$

is investigated in [5], when the space dimension $n \geq 2$ and $f \in C^0(\overline{\mathbb{R}^n_+})$ satisfies

$$\operatorname{supp}\{f - 1\} := \Omega_0 \quad \text{is bounded,} \tag{1.2}$$

and

$$0 < \lambda \leq \inf_{\mathbb{R}^n_+} f \leq \sup_{\mathbb{R}^n_+} f \leq \lambda^{-1} < \infty, \tag{1.3}$$

for some λ with $0 < \lambda < 1$. The main result in [5] states that any viscosity solution of (1.1) must tend to a quadratic polynomial at infinity if u satisfies the quadratic growth

The first author was supported by Natural Science Foundation of Henan Province (Grant No. 222300420321); the second author was supported by Natural Science Foundation of Henan Province (Grant No. 222300420232).

condition at infinity and f satisfies (1.2) and (1.3). We say a function u satisfies the *quadratic growth condition at infinity*, if

$$\tau|x|^2 \leq u(x) \leq \tau^{-1}|x|^2$$

at infinity for some constant $\tau \in (0, \frac{1}{2}]$.

For the whole space case, the Liouville type theorem and the asymptotic behaviour of solutions for Monge–Ampère equations over exterior domains have been widely studied (see, for example, [1, 11] and the references therein). For the half space case, the Liouville type theorem has been obtained by Mooney [7]. He showed that if u is a convex viscosity solution of (1.1) and satisfies the quadratic growth condition at infinity, then u is a quadratic polynomial. The reason for assuming the quadratic growth condition is that the convex function

$$u(x_1, \ldots, x_n) = \frac{x_1^2}{2(x_n+1)} + \frac{1}{2}(x_2^2 + \cdots + x_{n-1}^2) + \frac{1}{6}(x_n^3 + 3x_n^2)$$

solves the Monge–Ampère equation (1.1) with $f \equiv 1$, but it is obviously not a quadratic polynomial. Recently, for the Liouville type theorem, the quadratic growth condition at infinity has been weakened. Savin [9] showed that if there exists $\epsilon > 0$ such that $u = O(|x|^{3-\varepsilon})$ at infinity, then u is a quadratic polynomial. In [9], the important tools are the rescaling method and a key lemma (Lemma 8.3), which holds if the right-hand term f in (1.1) is very close to some constant near the origin. Obviously, the example mentioned above does not satisfy $u = O(|x|^{3-\varepsilon})$ at infinity.

This raises an interesting question for the general asymptotic result for the solutions of Monge–Ampère equations. Can the quadratic growth condition in [5] be weakened to $u = O(|x|^{3-\varepsilon})$ at infinity? The main difficulty comes from the fact that f may not be close to some constant near the origin and thus [9, Lemma 8.3] does not work directly.

We consider the convex viscosity solution of the Monge–Ampère equation:

$$\begin{cases} \det D^2 u(x) = f(x) & \text{in } \mathbb{R}^n_+, \\ u(x', x_n) = \varphi(x') & \text{on } \{x_n = 0\}, \end{cases} \tag{1.4}$$

where $n \geq 2$, $f \in C^0(\overline{\mathbb{R}^n_+})$ satisfies (1.3) and

$$|f - 1| = O(|x|^{-s}) \qquad \text{as } |x| \to \infty \tag{1.5}$$

for some $s > 0$, and $\varphi \in C^2(\mathbb{R}^{n-1})$ satisfies

$$\mu I_{n-1} \leq D^2\varphi \leq \mu^{-1} I_{n-1} \quad \text{on } \{x_n = 0\}, \quad I_{n-1} = \mathrm{diag}(\underbrace{1, 1, \cdots, 1}_{n-1}), \tag{1.6}$$

for some $\mu \in (0, 1]$. We also assume that for some $\varepsilon > 0$,

$$u = O(|x|^{3-\varepsilon}) \qquad \text{as } |x| \to \infty. \tag{1.7}$$

https://doi.org/10.1017/S000497272300028X Published online by Cambridge University Press

Our main result gives a positive answer to the question posed above. Throughout the paper, we use the following standard notation:

- for any $x \in \mathbb{R}^n$, $x = (x_1, x_2, \ldots, x_n) = (x', x_n)$, with $x' \in \mathbb{R}^{n-1}$;
- $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n > 0\}$; $\overline{\mathbb{R}^n_+} = \{x \in \mathbb{R}^n : x_n \geq 0\}$;
- for any $x \in \mathbb{R}^n$ and any $r > 0$, we write $B_r(x) = \{y \in \mathbb{R}^n : |y - x| < r\}$ and $B^+_r(0) = B_r(0) \cap \{x_n > 0\}$; for simplicity, we set $B_r = B_r(0)$ and $B^+_r = B^+_r(0)$.

THEOREM 1.1. *Let $n \geq 2$, $u \in C^0(\overline{\mathbb{R}^n_+})$ be a convex viscosity solution of (1.4), where φ satisfies (1.6) and f satisfies (1.3) and (1.5). If $|u| \leq |x|^{3-\varepsilon}$ in $\mathbb{R}^n_+ \backslash B^+_1$ for some $\varepsilon > 0$, then*

$$\tau|x|^2 \leq u(x) \leq \tau^{-1}|x|^2 \quad \text{as } |x| \to \infty,$$

where $\tau \in (0, \frac{1}{2}]$ is a constant depending only on μ, ε, λ and n.

Theorem 1.1 yields the following theorem. To obtain it, we will apply the extension theorem [10, Theorem 3.2], which holds for all C^2 uniformly convex functions over nonconvex domains.

THEOREM 1.2. *Let $n \geq 2$, $u \in C^2(\overline{\mathbb{R}^n_+ \backslash B^+_R})$ be a convex viscosity solution of*

$$\begin{cases} \det D^2 u(x) = f(x) & \text{in } \mathbb{R}^n_+ \backslash \overline{B^+_R}, \\ u(x', x_n) = \varphi(x') & \text{on } \{x_n = 0 : |x'| \geq R\}, \end{cases} \tag{1.8}$$

where $R > 0$ is a constant, $\varphi \in C^2$ satisfies (1.6) on $\{x_n = 0 : |x'| \geq R\}$, and $f \in C^0(\overline{\mathbb{R}^n_+ \backslash B^+_R})$ satisfies (1.3) and (1.5). If (1.7) holds, then u satisfies the quadratic growth condition at infinity.

REMARK 1.3. In Theorem 1.2, the special domain B^+_R can be replaced by any bounded domain in $\mathbb{R}^n_+$. Indeed, for any bounded domain $\Omega \subset \mathbb{R}^n_+$, there exists some $R > 0$ such that $\Omega \subset B^+_R$.

This paper is organised as follows. In Section 2, we prove a basic and crucial lemma on the measure of cross sections of solutions and introduce some important lemmas which were obtained in [8, 9]. In Section 3, we prove Theorems 1.1 and 1.2 and then we use Theorem 1.1 in Section 4 to obtain an asymptotic result for Monge–Ampère equations in half spaces.

2. Preliminaries

From now on, for any $u \in C^0(\overline{\mathbb{R}^n_+})$ in Theorem 1.1, we assume

$$u(0) = 0, \quad \nabla u(0) = 0, \tag{2.1}$$

which means that $x_{n+1} = 0$ is a support plane of u at the origin, but for any $\epsilon > 0$, $x_{n+1} = \epsilon x_n$ is not a support plane. We can reduce the general case to (2.1) by subtracting a linear function. Then we can denote by

$$S_h(u) = \{x \in \mathbb{R}^n_+ : \ u(x) < h\}$$

https://doi.org/10.1017/S000497272300028X Published online by Cambridge University Press

the cross section of u at zero with height h (see [2, 4]). We write S_h for convenience if there is no confusion. Equation (2.1) together with (1.6) shows that

$$\frac{1}{2}\mu|x'|^2 \leq \varphi(x') \leq \frac{1}{2}\mu^{-1}|x'|^2. \tag{2.2}$$

2.1. Estimate of cross sections. In this subsection, we first estimate the Lebesgue measure of cross sections as in [8].

LEMMA 2.1. *Assume that the convex function $u \in C^0(\overline{\mathbb{R}^n_+})$ solves*

$$\begin{cases} \det D^2u(x) = f(x) & \text{in } \mathbb{R}^n_+, \\ u(x', x_n) = \varphi(x') & \text{on } \{x_n = 0\}, \end{cases} \tag{2.3}$$

where f satisfies (1.3), φ satisfies (1.6) and u satisfies (2.1). Then for any $h > 0$,

$$C^{-1}h^{n/2} \leq |S_h| \leq Ch^{n/2}, \tag{2.4}$$

where $C > 0$ depends only on λ, $\varphi(x')$ and n.

PROOF. For all $h > 0$, let $x_h^* = ((x_h^*)', x_h^* \cdot e_n)$ be the centre of mass of S_h. We define

$$A_hx = x - \nu x_n, \quad \nu = \left(\frac{(x_h^*)'}{x_h^* \cdot e_n}, 0\right), \quad \widetilde{u}(A_hx) = u(x).$$

Then the centre of mass of $\widetilde{S}_h := \{x \in \mathbb{R}^n_+ : \widetilde{u}(x) < h\}$ satisfies

$$\widetilde{S}_h = A_hS_h \quad \text{and} \quad \widetilde{x}_h^* = A_hx_h^*.$$

In view of the definition of A_h:

- $\det |A_h| = 1$ and then

$$\det D^2\widetilde{u}(x) = \det((A_h^{-1})^T D^2uA_h^{-1})(A_h^{-1}x) = \det D^2u(A_h^{-1}x) = f(A_h^{-1}x);$$

- the sliding transformation preserves the volume of the level set, that is,

$$|\widetilde{S}_h| = \int_{\widetilde{S}_h} dy = \int_{S_h} dA_hx = \int_{S_h} |A_h|\, dx = |S_h|;$$

- the centre of mass $\widetilde{x}_h^*$ of $\widetilde{S}_h = A_hS_h$ lies on the x_n-axis, since for $i = 1, 2, \ldots, n-1$,

$$\widetilde{x}_h^* \cdot e_i = \frac{1}{|\widetilde{S}_h|}\int_{\widetilde{S}_h}\left(x_i - \frac{x_h^* \cdot e_i}{x_h^* \cdot e_n}x_n\right) \cdot \det A_h\, dx = \frac{1}{|S_h|}\int_{S_h} x_i\, dx - x_h^* \cdot e_i = 0.$$

By using John's lemma (relabelling the x' coordinates if necessary),

$$D_hB_1 \subset \widetilde{S}_h - \widetilde{x}_h^* \subset C(n)D_hB_1, \tag{2.5}$$

where $D_h = \operatorname{diag}(d_1, d_2 \ldots, d_n)$. We first claim that for all $h > 0$,

$$\prod_{i=1}^n d_i \geq c_0h^{n/2}, \tag{2.6}$$

https://doi.org/10.1017/S000497272300028X Published online by Cambridge University Press

where $c_0 > 0$ depends only on λ, $\varphi(x')$ and n. It is clear that (2.6) immediately implies $|\widetilde{S}_h| \geq ch^{n/2}$ for some constant c depending only on λ, $\varphi(x')$ and n. Now we show (2.6). Let

$$w = \varepsilon x_n + \sum_{i=1}^{n} ch\Big(\frac{x_i}{d_i}\Big)^2,$$

where $\varepsilon, c > 0$ are small constants to be determined. The boundary condition implies that $c_1|x'|^2 \leq \widetilde{u}(x', 0) \leq c_1^{-1}|x'|^2$ for some c_1 depending only on $\varphi(x')$. Then for all $h > 0$,

$$\begin{aligned}\widetilde{S}_h = \{x \in \overline{\mathbb{R}^n_+} : \widetilde{u}(x) < h\} &\supset \{(x', 0) \in \mathbb{R}^{n-1} \times \{x_n = 0\} : c_1^{-1}|x'|^2 < h\}\\ &= B_{c_1^{1/2}h^{1/2}} \cap \{x_n = 0\}.\end{aligned}$$

Consequently, since $\widetilde{x}_h^*$ lands on the x_n-axis and $C(n)D_hB_1$ is an ellipsoid,

$$Cd_i \geq c_1^{1/2}h^{1/2}, \quad \text{that is, } d_i \geq C^{-1}c_1^{1/2}h^{1/2}, \quad \text{for } i = 1, 2, \ldots, n-1,$$

where $C = C(n) > 0$. Using (1.6) and (2.1), we can choose $\varepsilon > 0$ depending only on h, and $c > 0$ small depending only on λ, $\varphi(x')$ and n, such that for all $h > 0$,

$$w \leq h \quad \text{on } \partial\widetilde{S}_h \cap \{x_n > 0\}, \quad \text{and} \quad w \leq \frac{c}{c_1}|x'|^2 \leq \varphi(x') \leq \widetilde{u} \quad \text{on } \{x_n = 0\}.$$

If (2.6) is false, then

$$\det D^2 w = (2ch)^n\Big(\prod_{i=1}^{n} d_i\Big)^{-2} > \Lambda.$$

By the comparison principle, we have $w \leq \widetilde{u}$ in $\widetilde{S}_h$. By the definition of $\widetilde{u}$, $x_{n+1} = 0$ is also the tangent plane of $\widetilde{u}$ at the origin. This gives a contradiction since $\widetilde{u} \geq w \geq \varepsilon x_n$. Thus, (2.6) is proved.

Next, we only need to show that for all $h > 0$,

$$|\widetilde{S}_h| \leq Ch^{n/2}, \tag{2.7}$$

where C is a large constant depending only on λ and n. In fact, for all $h > 0$, there exists $v \in C^0(\widetilde{S}_h)$ satisfying

$$\begin{cases}\det D^2 v = \lambda & \text{in } \widetilde{S}_h,\\ v = h & \text{on } \partial\widetilde{S}_h.\end{cases} \tag{2.8}$$

By the comparison principle, $v \geq u \geq 0$ in $\widetilde{S}_h$. There exists some constant $c > 0$ small depending only on λ and n such that

$$h \geq h - \min_{\widetilde{S}_h} v \geq c|\widetilde{S}_h|^{2/n},$$

(see [6, Lemma 2.2] and [4]). This establishes (2.7). □

https://doi.org/10.1017/S000497272300028X Published online by Cambridge University Press

2.2. Two kinds of function spaces. First, we define the classes $\mathbb{D}^{\mu}_{\sigma}(a_1, \ldots, a_{n-1})$ and $\mathbb{D}^{\mu}_{0}(a_1, \ldots, a_{n-1})$ (see [8, 9]).

The class $\mathbb{D}^{\mu}_{\sigma}(a_1, \ldots, a_{n-1})$. Let μ and λ be positive small fixed constants. For an increasing sequence $\{a_i\}_{i=1}^{n-1}$ with $\mu \le a_1 \le a_2 \le \cdots \le a_{n-1}$, we say that the convex function $u : \overline{\Omega} \to \mathbb{R}$ belongs to the function space $\mathbb{D}^{\mu}_{\sigma}(a_1, \ldots, a_{n-1})$ if u and Ω satisfy

$$\lambda \le \det D^2 u \le \lambda^{-1} \quad \text{and} \quad 0 \le u \le 1 \quad \text{in } \Omega, \tag{2.9}$$

$$0 \in \partial\Omega \quad \text{and} \quad B_{\mu}(x_0) \subset \Omega \subset B^{+}_{1/\mu}(0) \quad \text{for some } x_0 \in \Omega, \tag{2.10}$$

$$\mu |h|^{n/2} \le |S_h| \le \mu^{-1} |h|^{n/2}, \tag{2.11}$$

the boundary $\partial\Omega$ has a closed subset $G \subset \{x_n \le \sigma\} \cap \partial\Omega$, which is a graph in the e_n direction with projection $\pi_n(G) \subset \mathbb{R}^{n-1}$ along e_n, that is,

$$\Big\{\mu^{-1} \sum_{i=1}^{n-1} a_i^2 x_i^2 \le 1\Big\} \subset \pi_n(G) \subset \Big\{\mu \sum_{i=1}^{n-1} a_i^2 x_i^2 \le 1\Big\},$$

and $u = \varphi$ on $\partial\Omega$ with

$$\varphi = 1 \quad \text{on } \partial\Omega \backslash G \quad \text{and} \quad \mu \sum_{i=1}^{n-1} a_i^2 x_i^2 \le \varphi \le \min\Big\{1, \mu^{-1} \sum_{i=1}^{n-1} a_i^2 x_i^2\Big\} \quad \text{on } G.$$

Note that in [8, 9], $\mathcal{D}^{\mu}_{\sigma}(a_1, \ldots, a_{n-1})$ denotes two different function spaces, both of which will be used in our paper.

The class $\mathbb{D}^{\mu}_{0}(a_1, \ldots, a_{n-1})$. We introduce the limiting solutions of $\mathbb{D}^{\mu}_{\sigma}(a_1, \ldots, a_{n-1})$ when $a_{k+1} \to \infty$ and $\sigma \to 0$. If $\mu \le a_1 \le \cdots \le a_k$, we denote by

$$\mathbb{D}^{\mu}_{0}(a_1, \ldots, a_k, \infty, \infty, \ldots, \infty), \quad 0 \le k \le n-2$$

the class of functions u satisfying (2.9), (2.10), (2.11) with $G \subset \{x_i = 0 : i > k\} \cap \partial\Omega$, and, if we restrict to the space generated by the first k coordinates, then

$$\Big\{\mu^{-1} \sum_{i=1}^{k} a_i^2 x_i^2 \le 1\Big\} \subset G \subset \Big\{\mu \sum_{i=1}^{k} a_i^2 x_i^2 \le 1\Big\},$$

and $u = \varphi$ on $\partial\Omega$ with

$$\varphi = 1 \quad \text{on } \partial\Omega \backslash G \quad \text{and} \quad \mu \sum_{i=1}^{k} a_i^2 x_i^2 \le \varphi \le \min\Big\{1, \mu^{-1} \sum_{i=1}^{k} a_i^2 x_i^2\Big\} \quad \text{on } G.$$

REMARK 2.2. By Lemma 2.1, there exists some μ, depending only on $\lambda, \varphi(x')$ and n, such that (2.10) holds. Consequently, the solution u in Lemma 2.1 belongs to some space $\mathbb{D}^{\mu}_{\sigma}(a_1, \ldots, a_{n-1})$.

LEMMA 2.3 [8, Theorem 2.7]. *Assume* $u_m \in \mathbb{D}^{\mu}_{\sigma_m}(a_1^m, \ldots, a_{n-1}^m)$ *is a sequence of functions with* $\sigma_m \to 0, a_{k+1} \to \infty$. *Then we can extract a subsequence converging to a function* u *with* $u \in \mathbb{D}^{\mu}_{0}(a_1, \ldots, a_l, \infty, \ldots, \infty)$ *for some* $l \in [0, k]$.

https://doi.org/10.1017/S000497272300028X Published online by Cambridge University Press

The class $\mathcal{D}^{\mu}_{\sigma}$. Let μ, σ be small positive fixed constants and let $\mu \le a_1 \le \cdots \le a_{n-1}$ be real numbers. We say that $u \in \mathcal{D}^{\mu}_{\sigma}(a_1, \ldots, a_{n-1})$ if u is a continuous convex function defined on a convex set $\bar{\Omega}$ such that:

(1) $0 \in \partial\Omega$, $B_{\mu}(x_0) \subset \Omega \subset B^{+}_{1/\mu}(0)$ for some x_0, $1 \ge u \ge 0$, $u(0) = 0$, $\nabla u(0) = 0$;
(2) in the interior of Ω, the function u satisfies $1 - \sigma \le \det D^2 u \le 1 + \sigma$;
(3) on $\partial\Omega$, the function u satisfies the following condition: there is a closed set $G \subset \partial\Omega$, which is a graph $(x', g(x'))$ with $g(x') \le \sigma |x'|^2$, such that $u = 1$ on $\partial\Omega \backslash G$ and $(1 - \sigma)\varphi_u(x') \le u \le (1 + \sigma)\varphi_u(x')$ on G for some function φ_u satisfying

$$\mu^{-1}\mathcal{N} \ge D^2_{x'}\varphi_u \ge \mu\mathcal{N}, \quad \text{with } \mathcal{N} = \operatorname{diag}(a_1^2, a_2^2, \ldots, a_{n-1}^2).$$

The class $\mathcal{D}^{\sigma}_{0}$. Let μ be a small positive fixed constant and $\mu \le a_1 \le a_2 \le \cdots \le a_k$ be k real numbers with $0 \le k \le n - 1$. We say $u \in \mathcal{D}^{\mu}_{0}(a_1, \ldots, a_k, \infty, \ldots, \infty)$ if $u : \overline{\Omega} \to \mathbb{R}$ is a convex function defined on the convex domain $\overline{\Omega}$ such that

$$\det D^2 u = 1 \quad \text{in } \Omega; \quad u \ge 0, \quad u(0) = 0, \quad \nabla u(0) = 0; \tag{2.12}$$

$$0 \in \partial\Omega, \quad B_{\mu}(x_0) \subset \Omega \subset B^{+}_{1/\mu}(0) \quad \text{for some } x_0; \tag{2.13}$$

$$u = \begin{cases} \varphi_u & \text{on } G \subset \partial\Omega, \\ 1 & \text{on } \partial\Omega \backslash G, \end{cases} \tag{2.14}$$

where $\varphi_u(x_1, \ldots, x_k)$ is a nonnegative convex function of k variables satisfying

$$\mu\mathcal{N}_k \le D^2\varphi_u \le \mu^{-1}\mathcal{N}_k, \quad \text{with } \mathcal{N}_k := \operatorname{diag}(a_1^2, \ldots, a_k^2),$$

and G represents the k dimensional set of $\mathbb{R}^n$ where $\varphi_u \le 1$, that is,

$$G := \{x \in \mathbb{R}^n : \ \varphi_u(x_1, \ldots, x_k) \le 1, \ x_i = 0 \text{ if } i > k\}.$$

REMARK 2.4. The difference between the definitions of $\mathbb{D}^{\mu}_{\sigma}$ and $\mathcal{D}^{\mu}_{\sigma}$ is that the functions u solve different equations. The condition on f in $\mathcal{D}^{\mu}_{0}$ is stronger than that in $\mathbb{D}^{\mu}_{0}$.

Savin obtained the following crucial result (see [9, Lemma 8.3] with $\alpha = 0$).

LEMMA 2.5. *For any $\varepsilon' > 0$ small, there exists some C_* depending only on ε', μ, λ, Λ and n such that if $u \in \mathcal{D}^{\mu}_{0}(a_1, \ldots, a_{n-1})$ with $a_{n-1} \ge C_*$, then there exists some $t \in [C_*^{-1}, 1]$ such that*

$$\sup_{S_t(u)} x_n \ge \frac{2}{\mu} t^{1/(3-\varepsilon')}.$$

https://doi.org/10.1017/S000497272300028X Published online by Cambridge University Press

3. Proof of the main theorems

In this section, we prove Theorems 1.1 and 1.2.

PROOF OF THEOREM 1.1. For all $h > 0$, as in the proof of Lemma 2.1, we define

$$A_h x = x - \nu x_n, \quad \nu = \left(\frac{(x_h^*)'}{x_h^* \cdot e_n}, 0\right), \quad \widetilde{u}(A_h x) = u(x),$$

where $x_h^* = ((x_h^*)', x_h^* \cdot e_n)$ is the centre of mass of S_h. By using John's lemma (relabelling the x' coordinates if necessary),

$$D_h B_1 \subset \widetilde{S}_h - \widetilde{x}_h^* \subset C(n) D_h B_1,$$

where $D_h = \mathrm{diag}(d_1, \ldots, d_n)$. Then,

$$\begin{cases} \det D^2 \widetilde{u}(x) = \widetilde{f}(x) & \text{in } \widetilde{S}_h = A_h S_h, \\ \widetilde{u}(x) = \varphi(x') & \text{on } \partial \widetilde{S}_h \cap \{x_n = 0\}, \\ \widetilde{u}(x) = h & \text{on } \partial \widetilde{S}_h \cap \{x_n > 0\}, \end{cases}$$

where $\widetilde{f}(x) = f(A_h^{-1} x)$. For any $h > 0$, we denote

$$\varsigma_h = h^{-n/2} \prod_{i=1}^{n} d_i.$$

By Lemma 2.1, there exists $C > 0$ depending only on λ, $\varphi(x')$ and n such that

$$C^{-1} \le \varsigma_h \le C. \tag{3.1}$$

Note that ς_h has uniformly positive lower and upper bounds and is constant for any fixed $h > 0$. Now we divide this proof into two steps.

Step 1. We claim that

$$\widetilde{u}(0, x_n) \cdot x_n^{-2} \to 0 \quad \text{as } x_n \to \infty$$

will never happen. In fact, if it happens, then $\widetilde{u}(0, x_n) = o(x_n^2)$ as $x_n \to \infty$, which implies $d_n \cdot h^{-1/2} \to \infty$ as $h \to \infty$. By the boundary condition, the property of A_h and the convexity of $\widetilde{S}_h$,

$$d_i \ge C^{-1} \mu^{1/2} h^{1/2}, \quad i = 1, 2, \ldots, n - 1.$$

However, then $\varsigma_h \to \infty$ as $h \to \infty$, which contradicts (3.1). This completes Step 1.

Step 2. We show that

$$\widetilde{u}(0, x_n) \cdot x_n^{-2} \text{ is bounded} \quad \text{as } x_n \to \infty. \tag{3.2}$$

Indeed, if not, then

$$d_n \cdot h^{-1/2} \to 0 \quad \text{as } h \to \infty. \tag{3.3}$$

By Lemma 2.1, there exists at least one of $\{d_i\}_{i=1}^{n-1}$ (we denote it by d_{n-1}) such that

$$d_{n-1} \cdot h^{-1/2} \to \infty \quad \text{as } h \to \infty. \tag{3.4}$$

Let

$$U_h(x) = \frac{\widetilde{u}(F_h x)}{h}, \quad F_h x = \Big(d_1 x_1, d_2 x_2, \dots, \frac{1}{\varsigma_h} d_n x_n\Big).$$

It is easy to show that $U_h(x)$ solves

$$\begin{cases} \det D^2 U_h = \widetilde{f}(F_h x) & \text{in } \mathcal{O}_h, \\ U_h(x) = \varphi((F_h x)') & \text{on } \partial\mathcal{O}_h \cap \{x_n = 0\}, \\ U_h(x) = 1 & \text{on } \partial\mathcal{O}_h \cap \{x_n > 0\}, \end{cases} \tag{3.5}$$

where $\mathcal{O}_h = F_h^{-1}\widetilde{S}_h$ satisfies

$$DB_1 \subset \mathcal{O}_h - F_h^{-1}(\widetilde{x}_h^*) \subset C(n)DB_1, \tag{3.6}$$

with $D = \operatorname{diag}(1, 1, \dots, \varsigma_h)$. By its definition, $U_h \in \mathbb{D}_0^{\mu}(a_1, a_2, \dots, a_{n-1})$ with

$$a_i = \tfrac{1}{2} d_i h^{-1/2}, \quad \text{for } i = 1, 2, \dots n-1, \quad a_n = \tfrac{1}{2} d_n h^{-1/2}.$$

By (3.4), $a_{n-1} \to \infty$ as $h \to \infty$. The equations

$$\begin{cases} \det D^2 w_h = 1 & \text{in } \mathcal{O}_h, \\ w_h = \varphi((F_h x)') & \text{on } \partial\mathcal{O}_h \cap \{x_n = 0\}, \\ w_h = 1 & \text{on } \partial\mathcal{O}_h \cap \{x_n > 0\}, \end{cases} \tag{3.7}$$

determine a unique convex function $w_h(x) \in C^0(\overline{\mathcal{O}_h})$ (see [5]). We now divide this step into three parts.

Step 2.1. We claim that for all $h > 0$ small, there exists a constant $C > 0$ depending only on f, λ, φ and n such that

$$|w_h - U_h| \le C h^{-s/2(1+s)} \quad \text{in } \mathcal{O}_h,$$

where s is the constant in (1.5). In fact, by the Alexandrov estimate (see, [3, Lemma 9.2]),

$$-\min_{\mathcal{O}_h}(w_h - U_h) \le C\Big\{\int_{\mathcal{O}_{h,1}} \det D^2(w_h - U_h)\Big\}^{1/n},$$

where $\mathcal{O}_{h,1} := \{x \in \mathcal{O}_h : D^2(w_h - U_h) > 0\}$ and $C > 0$ depends only on n and the diameter of the domain $\mathcal{O}_h$. By virtue of (3.6) and (3.1), the diameter can be controlled uniformly. By the convexity of $(\det \cdot)^{1/n}$,

$$\Big\{\det D^2\Big(\frac{w_h}{2}\Big)\Big\}^{1/n} \ge \frac{1}{2}\{\det D^2(w_h - U_h)\}^{1/n} + \frac{1}{2}\{\det D^2(U_h)\}^{1/n}.$$

https://doi.org/10.1017/S000497272300028X Published online by Cambridge University Press

This implies that

$$\{\det D^2(w_h - U_h)\}^{1/n} \leq 1 - \widetilde{f}^{1/n}(F_h x), \quad x \in O_{h,1}.$$

Together with (1.3) and (1.5), this yields

$$\begin{aligned} -\min_{O_h}(w_h - U_h) &\leq C\Big\{\int_{O_{h,1}} (1 - \widetilde{f}^{1/n}(F_h x))^n \, dx\Big\}^{1/n} \\ &\leq Ch^{-1/2}\Big\{\Big(\int_{S_h \setminus B^+_{h^{1/2(1+s)}}} + \int_{B^+_{h^{1/2(1+s)}}}\Big)(1 - \widetilde{f}^{1/n}(x))^n \, dx\Big\}^{1/n} \\ &\leq Ch^{-1/2}\{h^{n/2(1+s)} + h^{-sn/2(1+s)+n/2}\}^{1/n} \\ &\leq Ch^{-s/2(1+s)}. \end{aligned}$$

Similarly,

$$-\min_{O_h}(U_h - w_h) \leq C\Big\{\int_{O_{h,2}} (\widetilde{f}^{1/n}(F_h x) - 1)^n\Big\}^{1/n} \leq Ch^{-s/2(1+s)},$$

where $O_{h,2} := \{x \in O_h : D^2(U_h - w_h) > 0\}$. Thus, $|w_h - U_h| \leq Ch^{-s/2(1+s)}$ in O_h.

Step 2.2. We claim that

$$D_n w_h(0) \to 0 \quad \text{as } h \to \infty.$$

In fact, if not, then for any small $\varepsilon > 0$, there exists a subsequence w_{h_m} such that $D_n w_{w_{h_m}}(0) \nrightarrow 0$ as $m \to \infty$. Then, by Lemma 2.3, we can extract a convergent subsequence $U_{h_{m_k}}$ to a function $U_\infty \in \mathbb{D}_0^\mu(a_1, a_2, \ldots, a_{n-1})$ with $U_{h_{m_k}} \to U_\infty$ as $k \to \infty$. Combined with Step 2.1, we obtain $w_{h_{m_k}} \to U_\infty$ as $k \to \infty$.

Since $\nabla U_{h_{m_k}}(0) = 0$, we have $D_n U_\infty(0) = 0$ (which means that the support plane of U_∞ at the origin is $x_{n+1} = 0$, that is, for any small $\varepsilon > 0$, the plane $x_{n+1} = \varepsilon x_n$ is not a support plane of U_∞ at the origin). However, this contradicts the previous convergence result. Therefore, $D_n w_h(0) \to 0$ as $h \to \infty$.

Step 2.3. Now we show (3.2). For h large enough, by Step 2.2, we can subtract a linear function $l_h = D_n w_h(0) x_n$, which is very close to the plane $x_{n+1} = 0$, such that $w_h - l_h$ belongs to $\mathcal{D}_0^\mu(a_1, a_2, \ldots, a_{n-1})$. This together with Lemma 2.5 implies that for any $\varepsilon' > 0$ small, there exists some $t \in [C_*^{-1}, 1]$ such that

$$\sup_{S_t(w_h)} x_n \geq \frac{7}{4\mu} t^{1/(3-\varepsilon')} \tag{3.8}$$

for large h, where $t < 1$ since $\sup_{S_1(w_h)} x_n \leq \mu^{-1}$ and a_{n-1} is large enough. Indeed, if $a_{n-1} \leq C_*^{-1}$, one can deduce that $d_n \cdot h^{-1/2} \to 0$ will never happen. This completes the proof of Step 2.3.

https://doi.org/10.1017/S000497272300028X Published online by Cambridge University Press

Equation (3.8) together with Step 2.1 yields

$$\sup_{S_{t+\varrho}(U_h)} x_n \geq \frac{7}{4\mu} t^{1/(3-\varepsilon')} \geq \frac{3}{2\mu}(t+\varrho)^{1/(3-\varepsilon')},$$

for all ϱ small enough and large h. Without loss of generality, we may assume that for some $t \in [C_*^{-1}, 1]$ and all $h \geq h_0$,

$$b(t) := \sup_{S_t(U_h)} x_n \geq \frac{3}{2\mu} t^{1/(3-\varepsilon')}.$$

Together with the definition of U_h, this means that for some $t \in [C_*^{-1}, 1]$,

$$\frac{b_{\widetilde{u}}(th)}{b_{\widetilde{u}}(h)} = \frac{b(t)}{b(1)} \geq \frac{3}{2\mu} t^{1/(3-\varepsilon')},$$

where

$$b(1) := \sup_{S_1(U_h)} x_n, \quad b_{\widetilde{u}}(h) = \sup_{S_h(\widetilde{u})} x_n, \quad b_{\widetilde{u}}(th) = \sup_{\{x \in S_{h(\widetilde{u})},\, \widetilde{u} < th\}} x_n.$$

Note that $b_{\widetilde{u}}(th)$ is different from $\sup_{S_{th}(\widetilde{u})} x_n$. Then for any h large enough,

$$q(h) \leq \frac{2\mu}{3} q(th),$$

where $q(h) = b_{\widetilde{u}}(h) h^{-1/(3-\varepsilon')}$. This implies that $q(h) \to 0$ as $h \to \infty$, contradicting $u = O(|x|^{3-\varepsilon})$ at infinity, when $\varepsilon' < \varepsilon$. Thus, Step 2 is finished.

Combining Steps 1 and 2, we have shown that there is a large constant C such that $C^{-1} x_n^2 \leq \widetilde{u}(0, x_n) \leq C x_n^2$ at infinity. This yields $C^{-1} h^{1/2} \leq d_n \leq C h^{1/2}$. Then by the boundary condition, the convexity of S_h and Lemma 2.1,

$$C^{-1} h^{1/2} \leq d_i \leq C h^{1/2}, \quad i = 1, 2, \ldots, n,$$

which completes the proof of Theorem 1.1. □

PROOF OF THEOREM 1.2. We deduce Theorem 1.2 from Theorem 1.1.

We claim that u is uniformly convex in $\overline{B_{10R}^+} \backslash B_{3R}^+$. Otherwise, $D_{ee} u(x) = 0$ at some interior point x for some unit vector e. This gives a contradiction because $u \in C^2$ and u solves (1.8).

Now we construct a solution of (1.4). Near $\{x_n = 0\}$, by the uniform convexity of u, we can extend u to $\widetilde{u}$ defined in $(B_{10R} \backslash \overline{B_{2R}}) \cap \{x_n > -\varepsilon\}$ for ε small such that $\widetilde{u}$ is uniformly convex and $\widetilde{u} = u$ in $\overline{B_{10R}^+} \backslash B_{2R}^+$. By the extension theorem for uniformly convex functions on nonconvex domains (see [10, Theorem 3.2]), we can obtain a uniformly convex $\widetilde{\widetilde{u}}$ which is equal to u in $\overline{B_{10R}^+} \backslash B_{3R}^+$. We extend $\widetilde{\widetilde{u}}$ to the half space and equal to u in $\overline{\mathbb{R}_+^n} \backslash B_{10R}^+$. Then, $\widetilde{\widetilde{u}} = O(|x|^{3-\varepsilon})$ at infinity for some $\varepsilon > 0$, and $\widetilde{\widetilde{u}}$ solves (1.4) with proper f and φ satisfying (1.3), (1.5) and (1.6), respectively, where μ and λ will be chosen smaller if necessary. Applying Theorem 1.1 to $\widetilde{\widetilde{u}}$ shows that $\widetilde{\widetilde{u}}$ enjoys the quadratic growth condition and so does u. □

https://doi.org/10.1017/S000497272300028X Published online by Cambridge University Press

4. An application

We first state a result on the asymptotic behaviour at infinity of convex viscosity solutions of Monge–Ampère equations in half spaces.

THEOREM 4.1 [5]. *Let $n \geq 2$ and $u \in C^0(\overline{\mathbb{R}^n_+})$ be a convex viscosity solution of (1.1) satisfying*

$$\mu|x|^2 \leq u(x) \leq \mu^{-1}|x|^2 \quad \text{in } \overline{\mathbb{R}^n_+}\backslash B^+_{R_0} \tag{4.1}$$

for some constants $0 < \mu \leq \frac{1}{2}$, $R_0 > 0$, where $f(x)$ satisfies (1.2) and (1.3). Then there exist some symmetric positive definite matrix A with $\det A = 1$, vector $b \in \mathbb{R}^n$ and constant $c \in \mathbb{R}$ such that

$$\left|u(x) - \left(\frac{1}{2}x^T Ax + b \cdot x + c\right)\right| \leq C\frac{x_n}{|x|^n} \quad \text{in } \overline{\mathbb{R}^n_+}\backslash B^+_R,$$

where $x = (x', x_n)$, and C and $R \geq R_0$ depend only on n, μ, R_0 and the diameter of the domain Ω_0. Moreover, $u \in C^\infty(\overline{\mathbb{R}^n_+}\backslash\Omega_0)$ and for any $k \geq 1$,

$$|x|^{n-1+k}|D^k(u(x) - \tfrac{1}{2}x^T Ax - b \cdot x - c)| \leq C \quad \text{in } \overline{\mathbb{R}^n_+}\backslash B^+_R,$$

where C depends only on n, μ, k, R_0 and the diameter of the domain Ω_0.

By Theorem 1.2, the quadratic growth condition (4.1) in Theorem 4.1 can be deduced from $u = O(|x|^{3-\varepsilon})$ at infinity for some $\varepsilon > 0$. Therefore, we can obtain an improved theorem as follows.

THEOREM 4.2. *Let $n \geq 2$, $u \in C^0(\overline{\mathbb{R}^n_+})$ be a convex viscosity solution of (1.1) with f satisfying (1.2) and (1.3). If $u = O(|x|^{3-\varepsilon})$ at infinity for some $\varepsilon > 0$, then the conclusions of Theorem 4.1 hold.*

REMARK 4.3. Theorem 4.2 also holds over exterior domains in half spaces. In fact, the asymptotic behaviour at infinity of solutions of Monge–Ampère equations only depends on the datum at infinity, for example, the right-hand term of the equation, boundary values and the growth condition. Additionally, by Theorem 1.2, Theorem 4.2 also holds for $u \in C^2$ over exterior domains in half spaces if $u = O(|x|^{3-\varepsilon})$ at infinity with $\varepsilon > 0$.

As a corollary of Theorem 4.2, it is easy to deduce the following Liouville theorem, which improves the result in [7].

THEOREM 4.4. *Let $n \geq 2$ and $u \in C^0(\overline{\mathbb{R}^n_+})$ be a convex viscosity solution of (1.1), where $f(x) \equiv 1$. If $u = O(|x|^{3-\varepsilon})$ at infinity for some $\varepsilon > 0$, then u must be a quadratic polynomial.*

Acknowledgement

We are grateful to the anonymous referee for valuable comments and suggestions.

References

[1] L. A. Caffarelli and Y. Y. Li, 'An extension to a theorem of Jörgens, Calabi and Pogorelov', *Comm. Pure Appl. Math.* **56**(5) (2003), 549–583.

[2] A. Figalli, *The Monge–Ampère Equation and Its Applications*, Zurich Lectures in Advanced Mathematics (European Mathematical Society (EMS), Zürich, 2017).

[3] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Classics in Mathematics, 224 (Springer-Verlag, Berlin, 2001).

[4] C. E. Gutiérrez, *The Monge–Ampère Equation*, 2nd edn, Progress in Nonlinear Differential Equations and their Applications, 89 (Birkhäuser/Springer, Cham, 2016).

[5] X. B. Jia, D. S. Li and Z. S. Li, 'Asymptotic behavior at infinity of solutions of Monge–Ampère equations in half spaces', *J. Differential Equations* **269**(1) (2020), 326–348.

[6] C. Mooney, 'Partial regularity for singular solutions to the Monge–Ampère equation', *Comm. Pure Appl. Math.* **68**(6) (2015), 1066–1084.

[7] C. Mooney, *The Monge–Ampère equation*, available online at https://www.math.uci.edu/~mooneycr/MongeAmpere_Notes.pdf.

[8] O. Savin, 'Pointwise $C^{2,\alpha}$ estimates at the boundary for the Monge–Ampère equation', *J. Amer. Math. Soc.* **26**(1) (2013), 63–99.

[9] O. Savin, 'A localization theorem and boundary regularity for a class of degenerate Monge–Ampère equations', *J. Differential Equations* **256**(2) (2014), 327–388.

[10] M. Yan, 'Extension of convex function', *J. Convex Anal.* **21**(4) (2014), 965–987.

[11] Z. W. Zhou, S. Y. Gong and J. G. Bao, 'Ancient solutions of exterior problem of parabolic Monge–Ampère equations', *Ann. Mat. Pura Appl. (4)* **200**(4) (2021), 1605–1624.

SHANSHAN MA, School of Mathematics and Statistics,
Zhengzhou University, Zhengzhou 450001, PR China
e-mail: mass1210@zzu.edu.cn

XIAOBIAO JIA, School of Mathematics and Statistics,
North China University of Water Resources and Electric Power,
Zhengzhou 450046, PR China
e-mail: jiaxiaobiao@ncwu.edu.cn