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ARTICLES

THE ELASTICITY OF SUBSTITUTION AS AN ENGINE OF GROWTH

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This paper characterizes the elasticity of factor substitution in one-sector convex growth models with a general production function. It shows that an elasticity of substitution that is asymptotically greater than unity is a sufficient (but not a necessary) condition for the existence of a lower bound on the marginal product of capital, which in turn can lead to unbounded endogenous growth. Hence, an elasticity of substitution that eventually becomes greater than unity can counteract the role of diminishing returns to capital. This renders factor substitution a powerful engine of growth.

Keywords: Elasticity of Substitution, Endogenous Growth, Convex Models

1. INTRODUCTION

In a well-known paper, Jones and Manuelli (1990) have shown that if the market interest rate, which in a competitive economy equals the marginal product of capital, is bounded away from zero, then there can be endogenous growth even in the case where there exist diminishing returns with respect to factors that can be accumulated (see also Jones and Manuelli 1997).

On the other hand, starting with Solow (1956), a seemingly unrelated literature has shown that a *constant*-elasticity-of-substitution (CES) production function with an elasticity of substitution between capital and labor greater than unity makes the emergence of endogenous growth possible [see, for example, La Grandville (1989), Duffy and Papageorgiou (2000), Klump and La Grandville (2000), and Klump and Preissler (2000)].[1] Moreover, within the CES context, a higher elasticity of substitution can lead to higher long-term growth rates. Thus, "when two countries start from common initial conditions, the one with the higher elasticity of substitution will always experience, other things being equal, a higher income per head" [Klump and La Grandville (2000, p. 283)].[2]

This paper investigates more closely the connection between these two strands of the growth literature and establishes a relation between the existence of a lower bound on the marginal product of capital and the asymptotic value of the elasticity

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of substitution in one-sector convex growth models. In particular, it shows that if the asymptotic value of the elasticity of substitution is greater than unity, then the marginal product of capital is bounded strictly away from zero. Hence, the aforementioned result regarding the CES is generalized to any production function; namely, regardless of the functional form of the underlying production function, an elasticity of substitution between capital and labor that eventually becomes greater than unity can counteract the role of diminishing returns to capital and thus can potentially generate unbounded growth even in the presence of nonreproducible factors and the absence of technical progress. This renders factor substitution a powerful engine of growth.

We view this generalization as an important finding, because in a CES world capital and labor are unnecessarily restricted to be equally substitutable at all stages of development. Indeed, Duffy and Papageorgiou (2000), estimating a CES production function, find that the elasticity of substitution is greater than one in a subsample of 21 high-capital countries, whereas it is less than one in a subsample of 23 low-capital countries. Other authors have attempted to estimate more general production functions, in which σ is an endogenous variable right from the start. Karagiannis, Palivos, and Papageorgiou (2005), henceforth KPP, survey several empirical studies that have estimated a variable-elasticity-of-substitution (VES) production function. Most of these studies have rejected the Cobb–Douglas and/or the CES in favor of a VES production function, using either cross-section data sets consisting of different sectors or time-series for an entire country. KPP also estimate a VES production function proposed by Revankar (1971), using the same panel data set as in Duffy and Papageorgiou (2000). In doing so, they, as do Duffy and Papageorgiou, use raw labor data and data adjusted for human capital and allow for constant and nonconstant returns to scale. In all four cases, they reject the Cobb–Douglas specification in favor of a VES with a value of σ that is in general greater than unity.

Besides country estimates, there have also been estimates of the elasticity of substitution for specific sectors of the economy. For example, Ferguson (1965), May and Denny (1979), and Yuhn (1991) applied either translog or CES cost functions to the U.S. manufacturing industry and found the Allen–Uzawa elasticity of substitution between capital and labor to be less than unity. However, Feng and Serletis (2008) recently applied a more advanced factor-augmenting asymptotically ideal model to the U.S. manufacturing industry and found the Morishima elasticity of substitution between capital and labor to be greater than unity.[3]

In sum, the existing empirical evidence suggests that the substitutability between capital and labor may depend on the stage of economic development and more precisely on the extent of capital accumulation per worker. Moreover, in some cases, the elasticity of substitution may exceed unity, which, in light of the point put forward in this paper, can be used to explain why some countries have managed to exhibit a high rate of per capita growth without exhibiting much technical progress.

The remainder of the paper is organized as follows. Section 2 briefly reviews the standard convex optimal growth model. Section 3 shows that an elasticity of substitution between capital and labor that becomes asymptotically greater than unity is a sufficient condition for a marginal product of capital bounded away from zero and has the potential for generating growth endogenously. The paper explores also the converse relation between the elasticity of substitution and the marginal product of capital. It shows that endogenous unbounded growth can occur even if the elasticity of substitution becomes asymptotically equal to unity. In this case, however, it will first have to take values that are strictly greater than unity. Section 4 concludes the paper.

2. A CONVEX ONE-SECTOR GROWTH MODEL

We begin with a brief review of the standard convex one-sector optimal growth model [for details see Jones and Manuelli (1990, 1997)]. Consider a competitive economy in which agents maximize their lifetime utility

$$U = \int_0^\infty \frac{c_t^{1-\eta} - 1}{1-\eta} \exp(-\theta t) dt, \qquad \theta, \eta > 0, \tag{1}$$

where c_t denotes consumption at time t and θ is the rate of time preference.

Let K and $L \in \mathbf{R}_+$ denote capital and labor, respectively. The technology in this economy is described by the production function $F(K, L)$, which is assumed to exhibit constant returns to scale with respect to K *and* L; hence, we can define the production function in intensive form as $f(k) \equiv F(K/L, 1)$, where $k \equiv K/L$ is the capital–labor ratio. The following assumptions about f are maintained throughout:

(A1) $f(0) \geq 0$; furthermore, $\forall k \in \mathbf{R}_+$
(A2) f is at least twice continuously differentiable;
(A3) $f' > 0$ (strictly increasing);
(A4) $f'' < 0$ (strictly concave).

The budget constraint faced by each agent in every period is

$$\dot{k} = f(k) - c = rk + w - c, \tag{2}$$

where r $[= f'(k)]$ and w $[= f(k) - kf'(k)]$ denote the competitive interest rate and wage, respectively. For simplicity we assume that there is no population growth and no depreciation.

It is known (see, for example, Jones and Manuelli 1997) that maximizing (1) subject to (2) yields the long-run growth rate

$$\lim_{t\to\infty} \frac{\dot{c}}{c} = \lim_{t\to\infty} \frac{\dot{k}}{k} = \frac{\lim_{k\to\infty} f'(k) - \theta}{\eta}. \tag{3}$$

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Thus, if $\lim_{k\to\infty} f'(k) > \theta$, then there will be unbounded growth, despite the presence of nonreproducible factors (namely labor) and the absence of technical progress. In the remainder of the paper we seek conditions that ensure that $\lim_{k\to\infty} f'(k) > 0$ and assume, as is the case in most of the literature, that if this limit is positive then it is also higher than θ. In other words, if $\lim_{k\to\infty} f'(k) > 0$, then the model is capable of generating growth endogenously for certain parameter values.[4]

3. ENDOGENOUS GROWTH AND THE ELASTICITY OF SUBSTITUTION

In this section we provide a link between the limiting value of the elasticity of substitution and a marginal product of capital that is bounded away from zero. In particular, we prove that in an economy with a convex technology and nonreproducible factors, if the elasticity of factor substitution in the limit becomes greater than unity, then the marginal product of capital remains positive as the economy grows (i.e., $\lim_{k\to\infty} f'(k) > 0$). In view of (3), this establishes a causal relation between the elasticity of factor substitution and endogenous growth.

First, we discuss briefly the general concept of the elasticity of substitution for a multifactor technology. Let $C(p, q)$ denote the cost function, where p is the input price vector and q is output. The Allen–Uzawa elasticity of substitution between two inputs i and j is given by [see, for example, Feng and Serletis (2008) and the references cited therein]

$$\sigma_{ij}^{A} = \frac{C_{ij}}{C_i C_j},$$

where subscripts denote partial derivatives. This concept of the elasticity of substitution, however, among others (i) is not a measure of the "ease" of substitution and (ii) provides information about the effect of price changes on *absolute* input shares [see Blackorby and Russel (1989)]. For these reasons, the more appropriate concept in the present context is the Morishima elasticity of substitution, which is given by

$$\sigma_{ij}^{M} = \frac{p_j C_{ij}}{C_i} - \frac{p_j C_{jj}}{C_j}$$

and (i) is a measure of the "ease" of substitution and (ii) provides information about the effect of price changes on relative input shares [see again Blackorby and Russel (1989)]. If there are only two inputs, K and L, and $q = F(K, L)$, then it follows immediately using the linear homogeneity of the cost function with respect to factor prices that these two concepts of the elasticity of substitution are equal to each other. Furthermore, it follows from the cost minimization problem that they are equal to

$$\sigma_{ij}^{M} = \sigma_{ij}^{A} = \frac{\Phi_{KL}}{|\Phi|}\frac{q}{KL} = \frac{F_K F_L}{F F_{KL}} \equiv \sigma,$$

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where Φ_{KL} and $|\Phi|$ denote respectively the cofactor of the term F_{KL} and the determinant of the bordered Hessian of the production surface,

$$\begin{bmatrix} 0 & F_K & F_L \\ F_K & F_{KK} & F_{KL} \\ F_L & F_{LK} & F_{LL} \end{bmatrix}.$$

Using the properties of constant-returns-to-scale production functions, we can write

$$\sigma(k) = -\frac{f^{'}(k)}{kf(k)}\frac{f(k) - kf^{'}(k)}{f^{''}(k)} > 0.$$

We make the following assumption regarding the limiting behavior of σ (nevertheless, see footnote 5):

(A5) $\lim_{k\to\infty} \sigma(k) = \overline{\sigma}$ exists.

Next, we denote the ratio of the marginal products of labor and capital, which is equal to the absolute value of the slope of an isoquant, as $m(k) \equiv \frac{f(k)-kf^{'}(k)}{f^{'}(k)} > 0$. The function $m(k)$ is increasing and hence we can write $k = h(m)$, where

$$h'(m) = \frac{1}{m'(k)} = -\frac{[f'(k)]^2}{f(k)f''(k)} > 0.$$

In this notation we can write

$$\sigma(k) = \frac{h'(m)m}{h(m)} > 0. \quad \mathbf{(4)}$$

We now establish the first of our two main results.

THEOREM 1.

$$\lim_{k\to\infty} \sigma(k) > 1 \Rightarrow \lim_{k\to\infty} f^{'}(k) > 0.$$

Proof. Let $k_1 > 0$ be such that $\sigma(k) > \gamma \ \forall\ k > k_1$, where $\gamma \in (1, \overline{\sigma})$. Then for every $k > k_1$,

$$\int_{m_1}^{m} \frac{\sigma(x)}{x} dx > \int_{m_1}^{m} \gamma \frac{1}{x} dx,$$

where $k_1 = h(m_1)$. Using (4), we have

$$\int_{m_1}^{m} \frac{h'(x)}{h(x)} dx > \gamma \ln\left[\frac{m}{m_1}\right]$$

or

$$\frac{k}{k_1} > \left(\frac{m}{m_1}\right)^{\gamma},$$

and finally

$$m < \Gamma k^{1/\gamma}, \tag{5}$$

where $\Gamma \equiv m_1 k_1^{-1/\gamma} > 0$. Substituting the definition of m into (5), we have

$$\frac{f(k) - f'(k)k}{f'(k)} < \Gamma k^{1/\gamma}$$

or

$$\frac{f'(k)}{f(k)} < \frac{1}{k + \Gamma k^{1/\gamma}}. \tag{6}$$

Integrating (6), we have

$$\int_{k_1}^{k} \frac{f'(x)}{f(x)} dx > \frac{1}{\Gamma} \int_{k_1}^{k} \frac{1}{x\left\{\left[\left(\frac{1}{\Gamma}\right)^{\gamma/(1-\gamma)}\right]^{(1-\gamma)/\gamma} + x^{(1-\gamma)/\gamma}\right\}} dx.$$

Applying the formula $\int \frac{dx}{x(x^n + a^n)} = \frac{1}{na^n} \ln \frac{x^n}{x^n + a^n}$ and rearranging, we get

$$f(k) > A\left(\Gamma + k^{(\gamma-1)/\gamma}\right)^{\gamma/(\gamma-1)}, \tag{7}$$

where $A \equiv f(k_1)(\Gamma + k_1^{(\gamma-1)/\gamma})^{\gamma/(1-\gamma)} > 0$. Next we divide both sides of (7) by k to get

$$\frac{f(k)}{k} > A\left(\frac{\Gamma}{k^{(\gamma-1)/\gamma}} + 1\right)^{\gamma/(\gamma-1)}.$$

Applying L' Hôpital's rule and noting that $\gamma > 1$, we have $\lim_{k\to\infty} f'(k) > A > 0$.[5] ■

The idea behind the proof of Theorem 1 is that an arbitrary production function f with an elasticity of substitution $\sigma(k)$ can be bounded from below by a CES production function with elasticity of substitution γ that is eventually lower than $\sigma(k)$ [see inequality (7)]. Next note that in a diagram where K is measured on the vertical axis and L on the horizontal, $\Gamma k^{1/\gamma} > 0$ is the absolute value of the slope of an isoquant of the CES production function with elasticity γ. Inequality (5) says that, for a high enough value of k, the isoquant of the CES is steeper than the isoquant of f. Furthermore, as is easily shown, if $\gamma > 1$, then a CES isoquant intersects the K axis. Hence, the same is true for an isoquant of f. Without loss of generality, consider a unit-isoquant $\{(L, K) \in R_+^2 \mid L = 1/f(k)\}$. Then $\lim_{L\to 0} K = \lim_{L\to 0} kL = \lim_{k\to\infty} k/f(k) = \lim_{k\to\infty} 1/f'(k)$, where the last equality follows from L' Hôpital's rule. Because there is a vertical intercept, $\lim_{L\to 0} K$ is finite and positive, which implies that $\lim_{k\to\infty} f'(k) > 0$.

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Intuitively, unbounded growth occurs because a technology with an elasticity of substitution, $\sigma \equiv \frac{d\ln(K/L)}{d\ln(F_L/F_K)}$, greater than unity can accommodate large changes in the capital–labor ratio with small changes in the wage–rental ratio. In the limit, labor is not needed for production, that is, is not an essential factor, and the model becomes an AK one.[6]

EXAMPLES. Some well-known production functions for which the elasticity of substitution is greater than unity in the limit and hence can lead to unbounded growth, even in the absence of technical progress, are the following:

(a) CES. Consider the CES production function

$$f(k) = A\left[\alpha + \beta k^{-\rho}\right]^{-1/\rho}, \qquad A, \alpha, \beta > 0, \rho > -1,$$

where $\sigma = \frac{1}{1+\rho}$. If $\rho < 0$, then

$$\sigma(k) = \frac{1}{1+\rho} \in (1, \infty) \quad \text{and} \quad \lim_{k\to\infty} f'(k) = A\beta^{-1/\rho} > 0.$$

It is well known, since Solow's seminal contribution [Solow (1956)], that a CES production function with $\sigma > 1$ can lead to unbounded growth even in the absence of technical change. One can also see this from equation (3).

(b) Revankar. Consider the following VES production function [see Revankar (1971)]:

$$f(k) = Ak^{\alpha}\left[1 + \beta\alpha k\right]^{1-\alpha}, \qquad A > 0, \alpha \in [0, 1], \beta > -1,$$

where $\sigma = 1 + \beta k$. If $\beta > 0$, then

$$\lim_{k\to\infty} \sigma(k) = \infty \quad \text{and} \quad \lim_{k\to\infty} f'(k) = A(\beta\alpha)^{1-\alpha} > 0.$$

It follows that if $\beta > 0$ and the appropriate condition on θ holds, then there will be unbounded growth.

(c) Jones and Manuelli. Finally, consider the following VES production function, which can result in endogenous growth [see Jones and Manuelli (1990) and equation (3)]:

$$f(k) = Ak + Bk^{\alpha}, \qquad A, B > 0, \alpha \in [0, 1),$$

where $\sigma = \frac{Ak + \alpha Bk^{\alpha}}{\alpha(Ak + Bk^{\alpha})}$. For this production function

$$\lim_{k\to\infty} \sigma(k) = \frac{1}{\alpha} > 1 \quad \text{and} \quad \lim_{k\to\infty} f'(k) = A > 0.$$

The same result holds even in the more general case considered in Jones and Manuelli (1990, 1997), where $f(k) = Ak + g(k)$ and $\lim_{k\to\infty} g'(k) = 0$.

Next we clarify Theorem 1 further with the following two remarks. The first remark shows that for unbounded growth to potentially emerge the elasticity of substitution need not be everywhere greater than unity; it suffices to be greater than unity only in the limit. The second remark provides an example of a production function where the elasticity of substitution is greater than unity for every finite value of the capital stock and is equal to unity only in the limit; such a production function cannot generate unbounded growth.

Remark 1. Let $f(k) = Ak + 1 - e^{-\beta k}$, where $A, \beta > 0$. Then $\sigma(k) = \frac{A+\beta e^{-\beta k}}{Ak+1-e^{-\beta k}} \frac{1-e^{-\beta k}(1+\beta k)}{\beta^2 k e^{-\beta k}}$, $\lim_{k\to\infty} \sigma(k) = \infty$, $f'(k) = A + \beta e^{-\beta k}$, and $\lim_{k\to\infty} f'(k) = A > 0$. Hence, this production function can lead to unbounded growth, despite the fact that for appropriately selected values of A and β, $\sigma(k)$ is less than 1 for low values of k.

Remark 2. Let $f(k) = Ak^a - \gamma$, where $\alpha \in (0, 1)$, known as the constant marginal shares production function [see Bruno (1968)]. Then

$$\sigma(k) = 1 - \frac{\alpha\gamma}{1-\alpha} \frac{1}{f(k)} > 1 \text{ if } \gamma < 0 \quad \text{and} \lim_{k\to\infty} \sigma(k) = 1.$$

Substituting in the appropriate formulas yields $\lim_{k\to\infty} f^{'}(k) = 0$, and hence there is no unbounded growth.

Next, we examine the converse relation between $\lim_{k\to\infty} \sigma(k)$ and $\lim_{k\to\infty} f'(k)$. Consider first the following lemma.

LEMMA.

$$\lim_{k\to\infty} f^{'}(k) > 0 \Rightarrow \lim_{k\to\infty} \frac{f^{'}(k)k}{f(k)} = 1.$$

Proof: The function $f(k)/k$ is nonincreasing and positive; hence, it converges. By the mean value theorem, there exists $\widetilde{k} \in (k, 2k)$ such that

$$\frac{f(2k) - f(k)}{2k - k} = f^{'}(\widetilde{k}), \quad \text{or equivalently,} \quad \frac{2f(2k)}{2k} - \frac{f(k)}{k} = f^{'}(\widetilde{k}).$$

Taking limits yields

$$\lim_{k\to\infty} \frac{f(k)}{k} = \lim_{k\to\infty} f^{'}(k),$$

and because $\lim_{k\to\infty} f^{'}(k) > 0$, we have

$$\lim_{k\to\infty} \frac{f^{'}(k)k}{f(k)} = 1.$$ [7] ■

Intuitively, as k increases, the marginal ($f^{'}(k)$) and the average product ($f(k)/k$) of capital, which in the limit are equal to each other, approach a positive constant. Given that markets are competitive and thus capital is paid its marginal product, it follows that as k increases without bound all income goes to capital; that is, $\lim_{k\to\infty} f(k) = \lim_{k\to\infty}[f(k)/k]k = \lim_{k\to\infty} f^{'}(k)k$.[8]

THEOREM 2.

(a)

$$\lim_{k\to\infty} f^{'}(k) > 0 \Rightarrow \lim_{k\to\infty} \sigma(k) \geq 1.$$

(b) *If* $\lim_{k\to\infty} f^{'}(k) > 0$ *then there exists at least one interval* (k', k'') *such that* $\sigma(k) > 1$ *for every* $k \in (k', k'')$.

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Proof. (a) Recall that $m(k) \equiv \frac{f(k)-kf^{'}(k)}{f^{'}(k)} > 0$. Let $\phi(k) \equiv \ln m(k)$. Then

$$\sigma(k) = \frac{1}{\phi'(k)k} > 0. \qquad (\mathbf{4'})$$

Next notice that

$$\lim_{k\to\infty} \frac{e^{\phi(k)}}{k} = \lim_{k\to\infty} \frac{m(k)}{k} = \lim_{k\to\infty} \frac{f(k)}{f'(k)k} - 1 = 0,$$

where the last equality follows from the lemma. Because $\lim_{k\to\infty} e^{\phi(k)}/k = 0$, there exists $k_2' > 0$ such that $e^{\phi(k)}/k < 1\ \forall\ k > k_2'$, or $\phi(k) < \ln k\ \forall k > k_2'$. Also, given $\varepsilon > 0$ there exists $k_2'' > 0$ such that $\overline{\sigma} - \varepsilon < \sigma(k) < \overline{\sigma} + \varepsilon, \forall k > k_2''$. Let $k_2 \equiv \max\{k_2', k_2''\}$. Then because $\phi(k) < \ln k\ \forall k > k_2$, it follows that

$$\int_{k_2}^{k} \phi'(x)dx < \int_{k_2}^{k} \frac{1}{x}dx + \tilde{c}, \qquad (\mathbf{8})$$

where $\tilde{c} \equiv \ln k_2 - \phi(k_2)$ results from the combination of the two constants of integration. Note that $\tilde{c} > 0$ (from the definition of k_2).

Using (4′), equation (8) can be written as

$$\int_{k_2}^{k} \frac{1}{x\sigma(x)}dx < \int_{k_2}^{k} \frac{1}{x}dx + \tilde{c},$$

or, because $\overline{\sigma} - \varepsilon < \sigma(k) < \overline{\sigma} + \varepsilon\ \forall k > k_2$,

$$\left[\frac{1}{\overline{\sigma} + \varepsilon} - 1\right] \int_{k_2}^{k} \frac{1}{x}dx < \tilde{c}. \qquad (\mathbf{9})$$

The integral on the left-hand side of (9) is increasing in k and tends to infinity, whereas the right-hand side of (9) is a positive constant. Thus, for this inequality to hold, it must be the case that $\overline{\sigma} \equiv \lim_{k\to\infty} \sigma(k) \geq 1$.

(b) Using similar steps very to those followed in the proof of Theorem 1, one can show that

$$\text{there exists } k_3 \text{ such that } \sigma(k) \leq 1 \forall k > k_3 \Rightarrow \lim_{k\to\infty} f^{'}(k) = 0.$$

The contrapositive of this statement is

$$\lim_{k\to\infty} f^{'}(k) > 0 \Rightarrow \forall k_3 \text{ there exists } k > k_3 \text{ such that } \sigma(k) > 1,$$

and by continuity the result follows. ■

The meaning of Theorem 2 is that a technology with a bounded marginal product of capital will exhibit an elasticity of factor substitution whose limit is greater than or equal to unity. Even if the limit is equal to unity, however, the elasticity cannot

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approach unity from below, but rather it will first take some values that are greater than one. The following example illustrates this case.

EXAMPLE. We seek an example of a production function such that the elasticity of substitution is equal to unity in the limit, but the marginal product of capital is bounded away from zero. Recall that $\sigma \equiv \frac{dk/k}{dm/m}$, where $m \equiv f(k)/f'(k) - k$. Integrating, we have

$$m = \alpha \exp\left(\int \frac{d\ln k}{\sigma(k)}\right),$$

where α is a positive constant of integration. Substituting the definition of m and integrating again yields

$$f(k) = A \exp\left\{\int \frac{1}{k + \alpha \exp\left(\int \frac{d\ln k}{\sigma(k)}\right)} dk\right\}, \qquad \textbf{(10)}$$

where A is another positive constant of integration. Equation (10) suggests the following functional form: $\sigma(k) = 1 + b/\ln k$, $b > 0$, which approaches one from above.[9] After integration, we have

$$f(k) = A \exp\left(y - \alpha \int \frac{1}{y^b + \alpha} dy\right), \qquad \text{where} \quad y = \ln k + b. \qquad \textbf{(11)}$$

Setting $b = 1$ in (11) yields[10]

$$f(k) = A \exp(1 + \ln k)(\alpha + 1 + \ln k)^{-\alpha}$$

and hence $\lim_{k\to\infty} f'(k) = 0$. Setting $b = 2$, however, results in

$$f(k) = A \exp(2 + \ln k) \exp\left(-\alpha^{1/2} \arctan \frac{2 + \ln k}{\alpha^{1/2}}\right),$$

which after differentiating and taking limits yields

$$\lim_{k\to\infty} f'(k) = A \exp\left(-\frac{\alpha^{1/2}}{2}\pi + 2\right) > 0.$$

4. CONCLUSIONS

This paper has investigated the role of the elasticity of factor substitution in one-sector convex growth models in which there are present nonreproducible factors, such as labor, and hence there exist diminishing returns with respect to the capital stock (the augmentable factor). It has established a connection between the existence of a lower bound on the marginal product of capital and the asymptotic value of the elasticity of substitution between capital and labor. Accordingly, it has documented that an elasticity of substitution that is greater than unity only in the limit, joined with an accommodating rate of time preference, is sufficient for unbounded endogenous growth. Furthermore, it has explored the converse relation and has shown that a production function with an elasticity of substitution that approaches unity asymptotically after first taking some values that are greater than one might be capable of generating unbounded endogenous growth.

Our finding has important implications regarding cross-country convergence and long-run growth. As mentioned in the Introduction, Klump and de La Grandville (2000), using a normalized CES production function, have examined the implications of differences in the elasticity of substitution, which in the CES case is a constant parameter, for the short-run and long-run levels of per capita income. Within the more general production function framework used in this paper, the implications of such differences are even more striking. Consider two countries that at some point in time are similar. If the lower bound of the elasticity of substitution for one of them is below unity and that for the other above unity, then any small differences in their per capita income will be magnified. In particular, the fist country will approach a finite level of per capita income, whereas the other will experience unbounded growth. As such an example, consider the findings in Duffy and Papageorgiou (2000): they estimate the elasticity of substitution for their low-middle-k subsample of 18 countries to be 1 (Cobb–Douglas) and for their high-k subsample of 21 countries to be 1.089. If this small difference persists, then it will have profound implications for growth performance. In the absence of technical progress, countries that belong to the first group will stagnate at a fixed level of per capita income, whereas countries that belong to the second group will grow without bound.

NOTES

1. See Nakamura and Nakamura (2008) on how a CES production function in which the elasticity of substitution exceeds unity can arise endogenously as the envelope of Cobb–Douglas production functions.

2. See also Turnovsky (2008) for an analysis of how the elasticity of factor substitution affects the speed of convergence, the distribution of income and wealth, and the relative merits of tied and untied foreign aid.

3. As we argue later in the paper, the Morishima elasticity of substitution is the more appropriate concept in the present context.

4. Our results hold in the Solow model as well, because in that model $\lim_{t\to\infty} \dot{k}/k = s \lim_{k\to\infty}[f(k)/k] - \delta$, where $\lim_{k\to\infty}[f(k)/k] = \lim_{k\to\infty} f'(k)$, s denotes the (exogenous) saving rate, and δ is the depreciation rate. If $\lim_{k\to\infty} f'(k) > \delta/s > 0$, then there will be endogenous growth. We also note that in a standard overlapping-generations model unbounded growth cannot exist in the presence of a convex technology, such as the one considered here [see Boldrin (1992) and Jones and Manuelli (1992)].

5. We note that the existence of $\lim_{k\to\infty} \sigma(k)$ is not actually required for the proof of Theorem 1. Following basically the same steps, one can show that if there exists k_1 such that for every $k > k_1$ $\sigma(k) > 1$, then $\lim_{k\to\infty} f'(k) > 0$.

6. Nakamura and Nakamura (2008) provide some micro foundations for this process.

7. Alternatively, because $\lim_{k\to\infty} f'(k) > 0$, it follows that $\lim_{k\to\infty} f(k) = \infty$. Hence one can apply L'Hôpital's rule to get $\lim_{k\to\infty} f(k)/k = \lim_{k\to\infty} f'(k)$.

8. For a discussion on possible extensions and reinterpretations of the model so as to avoid the undesirable property that labor's share in income converges to zero see Jones and Manuelli (1997, pp. 84–87).

9. Functions of the form $1 + b/k^n$, where n is a positive integer and $b > 0$, either yield $\lim_{k\to\infty} f^{'}(k) = 0$ or do not yield an explicit functional form for the production function.

10. The reader can easily verify that if $b = 0$, then (11) gives the Cobb–Douglas production function.

REFERENCES

Blackorby, Charles and Robert R. Russell (1989) Will the real elasticity of substitution please stand up? (A comparison of the Allen/Uzawa and Morishima elasticities). *American Economic Review* 79, 882–888.

Boldrin, Michele (1992) Dynamic externalities, multiple equilibria, and growth. *Journal of Economic Theory* 58, 198–218.

Bruno, Michael (1968) Estimation of factor contribution to growth under structural disequilibrium. *International Economic Review* 9, 49–62.

Duffy, John and Chris Papageorgiou (2000) A cross-country empirical investigation of the aggregate production function specification. *Journal of Economic Growth* 5, 87–120.

Feng, Guohua and Apostolos Serletis (2008) Productivity trends in U.S. manufacturing: Evidence from the NQ and the AIM cost functions. *Journal of Econometrics* 142, 281–311.

Ferguson, Charles E. (1965) Substitution, technical Progress, and returns to scale. *American Economic Review Papers and Proceedings* 55, 296–305.

Jones, Larry E. and Rodolfo E. Manuelli (1990) A convex model of equilibrium growth: Theory and policy implications. *Journal of Political Economy* 98, 1008–1038.

Jones, Larry E. and Rodolfo E. Manuelli (1992) Finite lifetimes and growth. *Journal of Economic Theory* 58, 171–197.

Jones, Larry E. and Rodolfo E. Manuelli (1997) Sources of growth. *Journal of Economic Dynamics and Control* 21, 75–114.

Karagiannis, Giannis, Theodore Palivos, and Chris Papageorgiou (2005) Variable elasticity of substitution and economic growth. In Diebolt Claude and Catherine Kyrtsou (eds.), *New Trends in Macroeconomics*, pp. 21–37. Berlin: Springer.

Klump, Rainer and Olivier de La Grandville (2000) Economic growth and the elasticity of substitution: Two theorems and some suggestions. *American Economic Review* 90, 282–291.

Klump, Rainer and Harald Preissler (2000) CES production functions and economic growth. *Scandinavian Journal of Economics* 102, 41–56.

La Grandville, Olivier de (1989) In quest of the Slutsky diamond. *American Economic Review* 79, 468–481.

May, Douglas J. and Michael Denny (1979) Factor-augmenting technical progress and productivity in U.S. manufacturing. *International Economic Review* 20, 759–774.

Nakamura, Hideki and Masakatsu Nakamura (2008) Constant-elasticity-of-substitution production function. *Macroeconomic Dynamics* 12, 694–701.

Revankar, Nagesh S. (1971) A class of variable elasticity of substitution production function. *Econometrica* 39, 61–71.

Solow, Robert M. (1956) A contribution to the theory of economic growth. *Quarterly Journal of Economics* 70, 65–94.

Turnovsky, Stephen J. (2008) The role of factor substitution in the theory of economic growth and income distribution: Two examples. *Journal of Macroeconomics* 30, 604–629.

Yuhn, Ky-Hyang (1991) Economic growth, technical change biases, and the elasticity of substitution: A test of the de La Grandville hypothesis. *Review of Economics and Statistics* 73, 340–346.