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Decomposition of topological Azumaya algebras

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Abstract. Let $\mathscr{A}$ be a topological Azumaya algebra of degree mn over a CW complex X. We give conditions for the positive integers m and n, and the space X so that $\mathscr{A}$ can be decomposed as the tensor product of topological Azumaya algebras of degrees m and n. Then we prove that if $m < n$ and the dimension of X is higher than $2m + 1$, $\mathscr{A}$ may not have such decomposition.

1 Introduction

The classical theory of central simple algebras over a field was generalized by Azumaya [5] and Auslander–Goldman [4] by introducing the concept of an Azumaya algebra over a local commutative ring and over an arbitrary commutative ring, respectively. This concept was generalized by Grothendieck [7, 1.1] to the notion of a topological Azumaya algebra.

Grothendieck [7, Section 2] defined the notion of an Azumaya algebra over any locally-ringed topos $(X_{\text{ét}}, \mathscr{O}_X)$ where $X_{\text{ét}}$ is an étale topos of a scheme X, and the local ring $\mathscr{O}_X$ is the structure sheaf of X.

Definition 1.1 A *topological Azumaya algebra of degree n* over a topological space X is a bundle of associative and unital complex algebras over X that is locally isomorphic to the matrix algebra $\mathrm{M}_{n\times n}(\mathbb{C})$ where $\mathbb{C}$ has its ordinary topology, [7, 1.1].

Topological Azumaya algebras are classified by pointed homotopy classes of maps to $\mathrm{B}\,\mathrm{PGL}_n(\mathbb{C})$, as there is a bijective correspondence

$$\left\{\begin{matrix}\text{Isomorphism classes of topological}\\ \text{Azumaya algebras of degree } n \text{ over } X\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{Isomorphism classes of}\\ \text{principal } G\text{-bundles over } X\end{matrix}\right\},$$

where G is the topological group of automorphisms of $\mathrm{M}_{n\times n}(\mathbb{C})$ as an algebra, [12, 8.2]. The Skolem–Noether theorem asserts that this is $\mathrm{PGL}_n(\mathbb{C})$; i.e., matrices acting by conjugation.

For brevity of notation, we work with U_n instead of $\mathrm{GL}_n(\mathbb{C})$. Our choice of notation does not affect our results because U_n included in $\mathrm{GL}_n(\mathbb{C})$ as the maximal compact Lie subgroup is a deformation retract, in particular the inclusion is a weak

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equivalence. Hence, the homotopy type of U_n is that of $\mathrm{GL}_n(\mathbb{C})$. The homotopy equivalence is more than an equivalence of spaces, it upgrades to one of topological groups, hence of classifiying spaces.

Let a and m be positive integers. Let $\mu_m \subset \mathrm{U}_{am}$ be the cyclic subgroup of order m consisting of scalar matrices ζI_{am} for ζ an mth root of unity. If we have a principal U_{am}/μ_m-bundle on a topological space X, then the quotient map $q : \mathrm{U}_{am}/\mu_m \to \mathrm{PU}_{am}$ gives rise by an extension of structure group to a principal PU_{am}-bundle and therefore a topological Azumaya algebra of degree am.

The tensor product of complex algebras can be extended to topological Azumaya algebras by performing the operation fiberwise. The Brauer group of a topological space X classifies topological Azumaya algebras on X up to Brauer equivalence: $\mathscr{A}$ and $\mathscr{A}'$ are Brauer equivalent if there exist complex vector bundles $\mathscr{V}$ and $\mathscr{V}'$, and an isomorphism $\mathscr{A} \otimes \mathrm{End}(\mathscr{V}) \cong \mathscr{A}' \otimes \mathrm{End}(\mathscr{V}')$ of bundles of $\mathbb{C}$-algebras. If X is a finite dimensional CW complex, then $\mathrm{Br}(X) \cong \mathrm{H}^3(X;\mathbb{Z})_{\mathrm{tors}}$ the torsion part of the cohomology group $\mathrm{H}^3(X;\mathbb{Z})$ [7]. The order of a class $\alpha \in \mathrm{Br}(X)$ is called the *period* of α, and it is denoted by $\mathrm{per}(\alpha)$.

$$\begin{tikzcd}
 & \mathrm{B}\,\mathrm{U}_{am}/\mu_m \arrow[d] & \\
X \arrow[ur, bend left, "\mathscr{A}"] \arrow[r, "\chi_m"] & \mathrm{K}(\mathbb{Z}/m,2) \arrow[r, "\widetilde{\beta}_m"] & \mathrm{K}(\mathbb{Z},3).
\end{tikzcd}$$

The *Brauer class* of a map $\mathscr{A} : X \longrightarrow \mathrm{B}\,\mathrm{U}_{am}/\mu_m$ is an element in $\mathrm{Br}(X)$ which will be denoted by $\mathrm{cl}(\mathscr{A})$. It is defined as follows. Let χ_m denote the composite of the projection of $\mathrm{B}\,\mathrm{U}_{am}/\mu_m$ on the the first nontrivial stage of its Postnikov tower, $\mathrm{B}\,\mathrm{U}_{am}/\mu_m \to \mathrm{K}(\mathbb{Z}/m,2)$, and the unreduced Bockstein map, $\widetilde{\beta}_m : \mathrm{K}(\mathbb{Z}/m,2) \to \mathrm{K}(\mathbb{Z},3)$, as illustrated in the diagram above. Then $\mathrm{cl}(\mathscr{A})$ is equal to the composite $\widetilde{\beta}_m \circ \chi_m$.

Remark 1.1 For a deeper discussion on topological Azumaya algebras and the Brauer group of a topological space, we refer the reader to [3].

Saltman asked in [10, p. 35] whether there is prime decomposition for Azumaya algebras under the tensor product operation, as there is for central simple algebras over a field. Antieau–Williams answered this question for topological Azumaya algebras in [2, Corollary 1.3] by showing the following result:

Theorem 1.2 *For $n > 1$ an odd integer, there exist a six-dimensional CW complex X and a topological Azumaya algebra $\mathscr{A}$ on X of degree $2n$ and period 2 such that $\mathscr{A}$ has no decomposition $\mathscr{A} \cong \mathscr{A}_2 \otimes \mathscr{A}_n$ for topological Azumaya algebras of degrees 2 and n, respectively.*

The aim of this paper is to provide conditions on a positive integer n and a topological space X such that a topological Azumaya algebra of degree n on X has a tensor product decomposition. The main result of this paper is the following theorem:

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Theorem 1.3 *Let m and n be positive integers such that m and n are relatively prime and $m < n$. Let X be a CW complex such that $\dim(X) \leq 2m+1$.*

If $\mathscr{A}$ is a topological Azumaya algebra of degree mn over X, then there exist topological Azumaya algebras $\mathscr{A}_m$ and $\mathscr{A}_n$ of degrees m and n, respectively, such that $\mathscr{A} \cong \mathscr{A}_m \otimes \mathscr{A}_n$.

Theorem 1.3 is a corollary of a more general result. We prove in Theorem 3.3 that a map $X \to \mathrm{B}\,\mathrm{U}_{abmn}/\mu_{mn}$ can be lifted to $\mathrm{B}\,\mathrm{U}_{am}/\mu_m \times \mathrm{B}\,\mathrm{U}_{bn}/\mu_n$ when the dimension of X is less than $2am+2$, the positive integers a, b, m and n are such that am is relatively prime to bn, and $am < bn$. The proof of Theorem 3.3 relies significantly in the description of the homomorphisms induced on homotopy groups by the r-fold direct sum of matrices $\oplus^r : \mathrm{U}_n \longrightarrow \mathrm{U}_{rn}$ in the range $\{0, 1, \ldots, 2n+1\}$. We call this set "*the stable range*" for U_n.

This paper is organized as follows. The Section 2 presents preliminaries on the effect of direct sum and tensor product operations on homotopy groups of compact Lie groups related to the unitary groups U_n. The Section 3 is devoted to the proof of Theorem 3.3. We explain in Remark 3.7 why the decomposition in Theorem 1.3 is not unique up to isomorphism.

2 Stabilization of operations on U_n

Let $m, n \in \mathbb{N}$, we consider the following matrix operations:

(1) The *direct sum of matrices*, $\oplus : \mathrm{U}_m \times \mathrm{U}_n \longrightarrow \mathrm{U}_{m+n}$ defined by

$$A \oplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

(2) The *r-fold direct sum*, $\oplus^r : \mathrm{U}_n \longrightarrow \mathrm{U}_{rn}$ given by $A^{\oplus r} = \underbrace{A \oplus \cdots \oplus A}_{r\text{-times}}$.

(3) The *tensor product of matrices*, $\otimes : \mathrm{U}_m \times \mathrm{U}_n \longrightarrow \mathrm{U}_{mn}$ defined by

$$A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1m}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mm}B \end{pmatrix},$$

for $A = (a_{ij}) \in \mathrm{U}_m$.

(4) The *r-fold tensor product*, $\otimes^r : \mathrm{U}_n \longrightarrow \mathrm{U}_{n^r}$ given by $A^{\otimes r} = \underbrace{A \otimes \cdots \otimes A}_{r\text{-times}}$.

The homomorphisms of homotopy groups induced by the operations above will be denoted by $\oplus_*$, $\oplus_*^r$, $\otimes_*$, and $\otimes_*^r$, respectively.

We begin by recalling low degree homotopy groups of the unitary groups and the special unitary groups. The first homotopy groups of U_n can be calculated by using Bott periodicity. Bott proves in [6] that

$$\pi_i(\mathrm{U}_n) \cong \begin{cases} 0 & \text{if } i < 2n \text{ is even,} \\ \mathbb{Z} & \text{if } i < 2n \text{ is odd,} \\ \mathbb{Z}/n! & \text{if } i = 2n. \end{cases}$$

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Since SU_n is the universal cover of U_n, and there is a fibration $\mathrm{SU}_n \hookrightarrow \mathrm{U}_n \xrightarrow{\det} S^1$, it follows that

$$\pi_i(\mathrm{SU}_n) \cong \begin{cases} 0 & \text{if } i = 1, \\ \pi_i(\mathrm{U}_n) & \text{otherwise.} \end{cases}$$

We now compute the low degree homotopy groups of U_{am}/μ_m and SU_{am}/μ_m. As SU_{am} is a simply connected m-cover of SU_{am}/μ_m we have

$$\pi_i(\mathrm{SU}_{am}/\mu_m) \cong \begin{cases} \mathbb{Z}/m & \text{if } i = 1, \\ \pi_i(\mathrm{SU}_{am}) & \text{otherwise.} \end{cases}$$

All columns as well as the two top rows of diagram (2.1) are short exact. The nine-lemma implies that the bottom row is also short exact.

(2.1)
$$\begin{tikzcd} \mu_m \arrow[r, equal] \arrow[d, hook] & \mu_m \arrow[r] \arrow[d, hook] & \{1\} \arrow[d] \\ \mathrm{SU}_{am} \arrow[r, hook] \arrow[d, two heads] & \mathrm{U}_{am} \arrow[r, two heads, "\det"] \arrow[d, two heads] & S^1 \arrow[d, equal] \\ \mathrm{SU}_{am}/\mu_m \arrow[r, "i"] & \mathrm{U}_{am}/\mu_m \arrow[r, "\det"] & S^1. \end{tikzcd}$$

Therefore, $\pi_i(\mathrm{U}_{am}/\mu_m) \cong \pi_i(\mathrm{SU}_{am}/\mu_m)$ for all $i > 1$. It remains to compute the fundamental group of U_{am}/μ_m.

By exactness of the bottom row of diagram (2.1), the induced sequence on fundamental groups is exact,

$$0 \longrightarrow \pi_1(\mathrm{SU}_{am}/\mu_m) \xrightarrow{i_*} \pi_1(\mathrm{U}_{am}/\mu_m) \xrightarrow{\det_*} \pi_1(S^1) \longrightarrow 0. \tag{2.2}$$

The map $\det : \mathrm{U}_{am} \to S^1$ has a section $t : S^1 \to \mathrm{U}_{am}$ defined by

$$t(\omega) = \begin{pmatrix} \omega & 0 \\ 0 & I_{am-1} \end{pmatrix}.$$

The section t is one of groups; in fact U_n is a semi-direct product of S^1 by SU_n. This section induces a section of $\det : \mathrm{U}_{am}/\mu_m \to S^1$, which we also denote by t,

(2.3)
$$\begin{tikzcd} 1 \arrow[r] & \mathrm{SU}_{am}/\mu_m \arrow[r, "i"] & \mathrm{U}_{am}/\mu_m \arrow[r, "\det"] & S^1 \arrow[r] \arrow[l, bend left, dotted, "t"] & 1. \end{tikzcd}$$

Since $\pi_1(S^1) \cong \mathbb{Z}$, sequence (2.2) splits. We describe $\pi_1(\mathrm{U}_{am}/\mu_m)$ in terms of $i_* : \pi_1(\mathrm{SU}_{am}/\mu_m) \to \pi_1(\mathrm{U}_{am}/\mu_m)$ and $t_* : \pi_1(S^1) \to \pi_1(\mathrm{U}_{am}/\mu_m)$ as $\pi_1(\mathrm{U}_{am}/\mu_m) = \operatorname{Im} i_* \oplus \operatorname{Im} t_* \cong \mathbb{Z}/m \oplus \mathbb{Z}$.

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2.1 Stabilization

Let $m, n \in \mathbb{N}$ and $m \leq n$. Define the map

$$\mathrm{s} : \mathrm{U}_m \longrightarrow \mathrm{U}_{m+n}$$
$$A \longmapsto A \oplus I_n.$$

The standard inclusion of unitary groups $\mathrm{U}_n \hookrightarrow \mathrm{U}_{n+1}$ is $2n$-connected. Since the map s is equal to the consecutive composite of standard inclusions, it follows that s is $2m$-connected. Hence, s induces a surjection in degree $2m$ and an isomorphism on homotopy groups in degrees less than $2m$.

Notation 2.1 *Let* stab *denote* $\pi_i(\mathrm{s})$, *the homomorphism* s *induces on homotopy groups. Henceforth, the following isomorphism for* $i < 2m$ *will be needed throughout the paper*

(2.4) $$\mathrm{stab} : \pi_i(\mathrm{U}_m) \xrightarrow{\cong} \pi_i(\mathrm{U}_{m+n})$$

to identify $\pi_i(\mathrm{U}_m)$ *with* $\pi_i(\mathrm{U}_{m+n})$ *for all* $i < 2m$.

Lemma 2.2 *Let* $r : \mathrm{U}_m \to \mathrm{U}_m$ *be conjugation by* $P \in \mathrm{U}_m$. *There is a basepoint preserving homotopy* H *from* r *to* $\mathrm{id}_{\mathrm{U}_m}$ *such that for all* $t \in [0,1]$, $H(-,t)$ *is a homomorphism.*

Proof Since U_m is path-connected, there exists a path α from P to I_m in U_m. Define $H : \mathrm{U}_m \times [0,1] \to \mathrm{U}_m$ by $H(A,t) = \alpha(t)A\alpha(t)^{-1}$. Observe that $H(-,t) : \mathrm{U}_m \to \mathrm{U}_m$, $A \mapsto H(A,t)$ is a homomorphism. Moreover, H is such that

$$H(I_m,t) = I_m, \quad H(A,0) = r(A) \quad \text{and} \quad H(A,1) = A.$$

Therefore, the result follows. ∎

Lemma 2.3 *Let* $n, r \in \mathbb{N}$. *For all* $j = 1, \ldots, r$ *define* $\mathrm{s}_j : \mathrm{U}_n \longrightarrow \mathrm{U}_{rn}$ *by*

$$\mathrm{s}_j(A) = \mathrm{diag}(I_n, \ldots, I_n, A, I_n, \ldots, I_n),$$

where A *is in the jth position. The maps* s_j *and* s_{j+1} *are pointed homotopic for all* $j = 1, \ldots, r-1$.

Proof The block matrix

$$P_j = \begin{pmatrix} I_{(j-1)n} & & & \\ & 0 & I_n & \\ & I_n & 0 & \\ & & & I_{(r-j-1)n} \end{pmatrix}$$

is such that $P_jP_j = I_{rn}$ for $j = 1, \ldots, r-1$. Moreover, if $A, B \in \mathrm{U}_n$, then

$$P_j \,\mathrm{diag}(I_n, \ldots, I_n, A, B, I_n, \ldots, I_n) P_j = \mathrm{diag}(I_n, \ldots, I_n, B, A, I_n, \ldots, I_n),$$

where A and B are in positions (j,j), $(j+1,j+1)$, and $(j+1,j+1)$, (j,j), respectively.

From Lemma 2.2, s_j and s_{j+1} are pointed homotopic. ∎

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Notation 2.4 *We call the* s_j *maps* stabilization maps. *As* s_1 *is equal to* $\mathrm{s} : \mathrm{U}_n \to \mathrm{U}_{n+(r-1)n}$, *it follows that* s_j *is* $2n$*-connected for all* $j = 1, \dots, r$. *From Lemma 2.3 the homomorphisms induced on homotopy groups by the stabilization maps are equal, hence* stab *also denotes* $\pi_i(\mathrm{s}_1) = \cdots = \pi_i(\mathrm{s}_r)$. *Thus we identify* $\pi_i(\mathrm{U}_n)$ *with* $\pi_i(\mathrm{U}_{rn})$ *for* $i < 2n$ *through* stab. *The identification allows one to introduce a slight abuse of notation, namely to identify* x *and* $\mathrm{stab}(x)$ *for* $x \in \pi_i(\mathrm{U}_n)$ *and* $i < 2n$.

2.2 Operations

Proposition 2.5 *Let* $i \in \mathbb{N}$, *the homomorphism* $\oplus_* : \pi_i(\mathrm{U}_m) \times \pi_i(\mathrm{U}_n) \longrightarrow \pi_i(\mathrm{U}_{m+n})$ *is given by*

$$\oplus_*(x, y) = \mathrm{stab}(x) + \mathrm{stab}(y)$$

for $x \in \pi_i(\mathrm{U}_m)$ *and* $y \in \pi_i(\mathrm{U}_n)$.

Proof It is enough to observe that the direct sum factors as

$$\begin{array}{ccccc} \mathrm{U}_m \times \mathrm{U}_n & \xrightarrow{\mathrm{s}_1 \times \mathrm{s}_2} & \mathrm{U}_{m+n} \times \mathrm{U}_{m+n} & \xrightarrow{\mathrm{mult}} & \mathrm{U}_{m+n} \\ (A, B) & \longmapsto & \left(\begin{pmatrix} A & 0 \\ 0 & I_n \end{pmatrix}, \begin{pmatrix} I_m & 0 \\ 0 & B \end{pmatrix}\right) & \longmapsto & \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}. \end{array}$$

Thus $\oplus_*(x, y) = \mathrm{mult}_* \circ (\mathrm{stab} \times \mathrm{stab})(x, y) = \mathrm{stab}(x) + \mathrm{stab}(y)$, where the last equality is true by the Eckmann–Hilton argument, [11, Theorem 1.6.8]. ∎

Corollary 2.6 *If* $m < n$ *and* $i < 2m$, *then* $\oplus_*(x, y) = x + y$ *for* $x \in \pi_i(\mathrm{U}_m)$ *and* $y \in \pi_i(\mathrm{U}_n)$.

Proof Since s_1 and s_2 are $2m$-connected, the homomorphisms $\mathrm{stab} : \pi_i(\mathrm{U}_m) \longrightarrow \pi_i(\mathrm{U}_{m+n})$ and $\mathrm{stab} : \pi_i(\mathrm{U}_n) \longrightarrow \pi_i(\mathrm{U}_{m+n})$ are isomorphisms $i < 2m$ and $i < 2n$, respectively. We use these isomorphisms to identify source and target.

From Proposition 2.5, $\oplus_*(x, y) = \mathrm{stab}(x) + \mathrm{stab}(y) = x + y$ for $i < 2m$. ∎

Proposition 2.7 *Let* $i \in \mathbb{N}$, *the homomorphism* $\oplus_*^r : \pi_i(\mathrm{U}_n) \longrightarrow \pi_i(\mathrm{U}_{rn})$ *is given by*

$$\oplus_*^r(x) = r\,\mathrm{stab}(x)$$

for $x \in \pi_i(\mathrm{U}_n)$.

Proof Let $\Delta : \mathrm{U}_n \to (\mathrm{U}_n)^{\times r}$ denote the diagonal map. The r-block summation factors as

$$\begin{array}{ccccccc} \mathrm{U}_n & \xrightarrow{\Delta} & (\mathrm{U}_n)^{\times r} & \xrightarrow{\mathrm{s}_1 \times \cdots \times \mathrm{s}_r} & (\mathrm{U}_{rn})^{\times r} & \xrightarrow{\mathrm{mult}} & \mathrm{U}_{rn} \\ A & \longmapsto & (A, \dots, A) & \longmapsto & \big(\mathrm{s}_1(A), \dots, \mathrm{s}_r(A)\big) & \longmapsto & \mathrm{s}_1(A) \cdots \mathrm{s}_r(A). \end{array}$$

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By the Eckmann–Hilton argument $\mathrm{mult}_* : \pi_i(\mathrm{U}_{rn})^r \longrightarrow \pi_i(\mathrm{U}_{rn})$ is given by

$$\mathrm{mult}_*(x_1, \ldots, x_r) = x_1 + \cdots + x_r$$

for $x_j \in \pi_i(\mathrm{U}_{rn})$ and $j = 1, \ldots, r$. From this $\oplus_*^r$ takes the form

$$\begin{array}{ccc} \pi_i(\mathrm{U}_n) & & x \\ \Big\downarrow{\scriptstyle \Delta} & & \Big\downarrow \\ \pi_i(\mathrm{U}_n)^{\times r} & & (x, \ldots, x) \\ \Big\downarrow{\scriptstyle \mathrm{stab}^{\times r}} & & \Big\downarrow \\ \pi_i(\mathrm{U}_{rn})^{\times r} & & \big(\mathrm{stab}(x), \ldots, \mathrm{stab}(x)\big) \\ \Big\downarrow{\scriptstyle \mathrm{mult}_*} & & \Big\downarrow \\ \pi_i(\mathrm{U}_{rn}) & & \underbrace{\mathrm{stab}(x) + \cdots + \mathrm{stab}(x)}_{r \text{ times}} \end{array}$$

This proves the statement. ∎

Corollary 2.8 *If $i < 2n$, then $\oplus_*^r(x) = rx$ for $x \in \pi_i(\mathrm{U}_n)$.*

Proof The homomorphism $\mathrm{stab}^{\times r} : \pi_i(\mathrm{U}_n)^{\times r} \longrightarrow \pi_i(\mathrm{U}_{rn})^{\times r}$ is an isomorphism for all $i < 2n$ because so is $\mathrm{stab} : \pi_i(\mathrm{U}_n) \longrightarrow \pi_i(\mathrm{U}_{rn})$. By Proposition 2.7, we conclude $\oplus_*^r(x) = r\,\mathrm{s}_*(x) = rx$ for $i < 2n$. ∎

Lemma 2.9 *Let $L, R : \mathrm{U}_m \to \mathrm{U}_{mn}$ be the maps $L(A) = A \otimes I_n$ and $R(A) = I_n \otimes A$. There is a basepoint preserving homotopy H from L to R such that for all $t \in [0,1]$, $H(-, t)$ is a homomorphism.*

Proof Let $A \in \mathrm{U}_m$.

$$L(A) = \begin{pmatrix} a_{11}I_n & \cdots & a_{1m}I_n \\ \vdots & \ddots & \vdots \\ a_{m1}I_n & \cdots & a_{mm}I_n \end{pmatrix} \quad \text{and} \quad R(A) = \begin{pmatrix} A & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A \end{pmatrix} = A^{\oplus n}.$$

Let $P_{m,n}$ be the permutation matrix

$$\begin{aligned} P_{m,n} = [&e_1, e_{n+1}, e_{2n+1}, \ldots, e_{(m-1)n+1}, e_2, e_{n+2}, e_{2n+2}, \ldots, e_{(m-1)n+2}, \\ &\ldots, \\ &e_{n-1}, e_{2n-1}, e_{3n-1}, \ldots, e_{mn-1}, e_n, e_{2n}, e_{3n}, \ldots, e_{(m-1)n}, e_{mn}], \end{aligned}$$

where, e_i is the ith standard basis vector of $\mathbb{C}^{mn}$ written as a column vector. Observe that $L(A) = P_{m,n}R(A)P_{m,n}^{-1}$. The result follows from Lemma 2.2. ∎

Proposition 2.10 *Let $i \in \mathbb{N}$, the homomorphism $\otimes_* : \pi_i(\mathrm{U}_m) \times \pi_i(\mathrm{U}_n) \longrightarrow \pi_i(\mathrm{U}_{mn})$ is given by*

$$\otimes_*(x, y) = n\,\mathrm{stab}(x) + m\,\mathrm{stab}(y)$$

for $x \in \pi_i(\mathrm{U}_m)$ and $y \in \pi_i(\mathrm{U}_n)$.

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Proof By the mixed-product property of the tensor product of matrices

$$A \otimes B = (A \otimes I_n)(I_m \otimes B) = L(A)R(B).$$

Lemma 2.9 gives $L_* = \oplus_*^n : \pi_i(\mathrm{U}_m) \to \pi_i(\mathrm{U}_{mn})$. Proposition 2.7 now yields $\otimes_*(x, y) = n\,\mathrm{stab}(x) + m\,\mathrm{stab}(y)$. ■

Corollary 2.11 *If $m < n$ and $i < 2m$, then $\otimes_*(x, y) = nx + my$ for $x \in \pi_i(\mathrm{U}_m)$ and $y \in \pi_i(\mathrm{U}_n)$.*

Proof The statement follows from Corollary 2.8 and Proposition 2.10. ■

Proposition 2.12 *Let $i \in \mathbb{N}$, the homomorphism $\otimes_*^r : \pi_i(\mathrm{U}_n) \longrightarrow \pi_i(\mathrm{U}_{n^r})$ is given by*

$$\otimes_*^r(x) = rn^{r-1}\,\mathrm{stab}(x)$$

for $x \in \pi_i(\mathrm{U}_n)$.

Corollary 2.13 *If $i < 2n$, then $\otimes_*^r(x) = rn^{r-1}x$ for $x \in \pi_i(\mathrm{U}_n)$.*

Proof Corollary 2.8 and Proposition 2.12 yield the result. ■

2.2.1 Tensor product on the quotient

Let a, b, m and n be positive integers so that $m < n$. The tensor product operation $\otimes : \mathrm{U}_{am} \times \mathrm{U}_{bn} \longrightarrow \mathrm{U}_{abmn}$ sends the group $\mu_m \times \mu_n$ to μ_{mn}. In consequence, the operation descends to the quotient

$$\otimes : \mathrm{U}_{am}/\mu_m \times \mathrm{U}_{bn}/\mu_n \longrightarrow \mathrm{U}_{abmn}/\mu_{mn}. \tag{2.5}$$

Proposition 2.14 *If $i > 1$, the homomorphism*

$$\otimes_* : \pi_i(\mathrm{U}_{am}/\mu_m) \times \pi_i(\mathrm{U}_{bn}/\mu_n) \longrightarrow \pi_i(\mathrm{U}_{abmn}/\mu_{mn})$$

is given by

$$\otimes_*(x, y) = bn\,\mathrm{stab}(x) + am\,\mathrm{stab}(y)$$

for $x \in \pi_i(\mathrm{U}_{am}/\mu_m)$ and $y \in \pi_i(\mathrm{U}_{bn}/\mu_n)$.

Proof There is a map of fibrations

$$\begin{tikzcd}
\mu_m \times \mu_n \arrow[r, hook] \arrow[d, "\mathrm{mult}"] & \mathrm{U}_{am} \times \mathrm{U}_{bn} \arrow[r] \arrow[d, "\otimes"] & \mathrm{U}_{am}/\mu_m \times \mathrm{U}_{bn}/\mu_n \arrow[d, "\otimes"] \\
\mu_{mn} \arrow[r, hook] & \mathrm{U}_{abmn} \arrow[r] & \mathrm{U}_{abmn}/\mu_{mn}.
\end{tikzcd} \tag{2.6}$$

From the homomorphism of long exact sequences associated to the fibrations in diagram (2.6), we obtain a commutative square

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$$\begin{tikzcd} \pi_i(\mathrm{U}_{am}) \times \pi_i(\mathrm{U}_{bn}) \arrow[r, "\cong"] \arrow[d, "\otimes_*"] & \pi_i(\mathrm{U}_{am}/\mu_m) \times \pi_i(\mathrm{U}_{bn}/\mu_n) \arrow[d, "\otimes_*"] \\ \pi_i(\mathrm{U}_{abmn}) \arrow[r, "\cong"] & \pi_i(\mathrm{U}_{abmn}/\mu_{mn}). \end{tikzcd}$$

for $i > 1$. This diagram and Proposition 2.10 gives $\otimes_*(x, y) = \oplus_*^{bn}(x) + \oplus_*^{am}(y) = bn\,\mathrm{stab}(x) + am\,\mathrm{stab}(y)$ for all $i > 1$. ■

In the following proposition, we identify $\pi_1(\mathrm{U}_{am}/\mu_m)$ with $\mathrm{Im}\, i_* \oplus \mathrm{Im}\, t_* \cong \mathbb{Z}/m \oplus \mathbb{Z}$, where i and t are the maps in diagram (2.3). We also identify $\mathbb{Z}/m$ and $\mathbb{Z}/n$ with the subgroups $\{0, n, 2n, \ldots, (m-1)n\} \subset \mathbb{Z}/mn$ and $\{0, m, 2m, \ldots, (n-1)m\} \subset \mathbb{Z}/mn$, respectively.

Proposition 2.15 *The homomorphism*

$$\otimes_* : \pi_1(\mathrm{U}_{am}/\mu_m) \times \pi_1(\mathrm{U}_{bn}/\mu_n) \longrightarrow \pi_1(\mathrm{U}_{abmn}/\mu_{mn})$$

is given by

$$\otimes_*(\alpha + x, \beta + y) = (\alpha + \beta) + (bnx + amy)$$

for $\alpha \in \mathbb{Z}/m \subset \mathbb{Z}/mn$, $\beta \in \mathbb{Z}/n \subset \mathbb{Z}/mn$, and $x, y \in \mathbb{Z}$.

Proof Since the determinant of a tensor product is the product of powers of the determinants, we define $\phi : S^1 \times S^1 \to S^1$ by $\phi(v, \omega) = v^{bn}\omega^{am}$ so that the diagram below is a map of fibrations.

$$\begin{tikzcd} \mathrm{SU}_{am}/\mu_m \times \mathrm{SU}_{bn}/\mu_n \arrow[r, hook, "i \times i"] \arrow[d, "\otimes"] & \mathrm{U}_{am}/\mu_m \times \mathrm{U}_{bn}/\mu_n \arrow[r, "\det \times \det"] \arrow[d, "\otimes"] & S^1 \times S^1 \arrow[d, "\phi"] \arrow[l, bend right, dotted, "t \times t"'] \\ \mathrm{SU}_{abmn}/\mu_{mn} \arrow[r, hook, "i"] & \mathrm{U}_{abmn}/\mu_{mn} \arrow[r, "\det"] & S^1. \arrow[l, bend left, dotted, "t"] \end{tikzcd}$$

This map of fibrations induces a homomorphism of short exact sequences

$$\begin{tikzcd} \pi_1(\mathrm{SU}_{am}/\mu_m) \times \pi_1(\mathrm{SU}_{bn}/\mu_n) \arrow[r, "\otimes_*"] \arrow[d, hook, "i_* \times i_*"] & \pi_1(\mathrm{SU}_{abmn}/\mu_{mn}) \arrow[d, hook, "i_*"] \\ \pi_1(\mathrm{U}_{am}/\mu_m) \times \pi_1(\mathrm{U}_{bn}/\mu_n) \arrow[r, "\otimes_*"] \arrow[d, two heads] & \pi_1(\mathrm{U}_{abmn}/\mu_{mn}) \arrow[d, two heads] \\ \pi_1(S^1) \times \pi_1(S^1) \arrow[r, "\phi_*"] \arrow[u, bend left, dotted, "t_* \times t_*"] & \pi_1(S^1). \arrow[u, bend right, dotted, "t_*"'] \end{tikzcd} \tag{2.7}$$

We want to determine the homomorphism $\otimes_*$ in the middle of diagram (2.7). In order to do this, we will determine $\otimes_* : \pi_1(\mathrm{SU}_{am}/\mu_m) \times \pi_1(\mathrm{SU}_{bn}/\mu_n) \longrightarrow$

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$\pi_1(\mathrm{SU}_{abmn}/\mu_{mn})$, and show that the short exact sequences in diagram (2.7) split compatibly so that $\otimes_* : \pi_1(\mathrm{U}_{am}/\mu_m) \times \pi_1(\mathrm{U}_{bn}/\mu_n) \longrightarrow \pi_1(\mathrm{U}_{abmn}/\mu_{mn})$ is equal to

$$\begin{array}{c}\Big(\pi_1(\mathrm{SU}_{am}/\mu_m) \times \pi_1(\mathrm{SU}_{bn}/\mu_n)\Big) \oplus \Big(\pi_1(S^1) \times \pi_1(S^1)\Big)\\ \Big\downarrow{\scriptstyle \otimes_* \oplus \phi_*}\\ \pi_1(\mathrm{SU}_{abmn}/\mu_{mn}) \oplus \pi_1(S^1).\end{array}$$

We begin by observing that there exists a similar map of fibrations to the one in diagram (2.6), but with the spaces SU_{am} and SU_{bn} instead of U_{am} and U_{bn}, respectively. In this case, we obtain the commutative square,

$$\begin{tikzcd} \pi_1(\mathrm{SU}_{am}/\mu_m) \times \pi_1(\mathrm{SU}_{bn}/\mu_n) \arrow[r, "\cong"] \arrow[d, "\otimes_*"] & \mathbb{Z}/m \times \mathbb{Z}/n \arrow[d, "\psi"] \\ \pi_1(\mathrm{SU}_{abmn}/\mu_{mn}) \arrow[r, "\cong"] & \mathbb{Z}/mn, \end{tikzcd}$$

where $\mathbb{Z}/m$ and $\mathbb{Z}/n$ are considered as subgroups of $\mathbb{Z}/mn$, and $\psi : \mathbb{Z}/m \times \mathbb{Z}/n \to \mathbb{Z}/mn$ is addition. From this $\otimes_* : \pi_1(\mathrm{SU}_{am}/\mu_m) \times \pi_1(\mathrm{SU}_{bn}/\mu_n) \to \pi_1(\mathrm{SU}_{abmn}/\mu_{mn})$ is equal to the addition.

In order to prove the compatibility, we observe that even though diagram (2.8) below does not commute, Claim 2.16 implies that it is commutative up to a pointed homotopy. Therefore, the induced diagram on homotopy groups does commute

$$\begin{tikzcd} \pi_1(S^1) \times \pi_1(S^1) \arrow[r, "t_* \times t_*"] \arrow[d, "\phi_*"] & \pi_1(\mathrm{U}_{am}/\mu_m) \times \pi_1(\mathrm{U}_{bn}/\mu_n) \arrow[d, "\otimes_*"] \\ \pi_1(S^1) \arrow[r, "t_*"] & \pi_1(\mathrm{U}_{abmn}/\mu_{mn}). \end{tikzcd}$$

Consequently, the diagram below commutes

$$\begin{tikzcd} \pi_1(\mathrm{U}_{am}/\mu_m) \times \pi_1(\mathrm{U}_{bn}/\mu_n) \arrow[r, "\cong"] \arrow[d, "\otimes_*"] & \mathrm{Im}(i_* \times i_*) \oplus \mathrm{Im}(u_* \times u_*) \arrow[d, "\psi \oplus \phi_*"] \\ \pi_1(\mathrm{SU}_{abmn}/\mu_{mn}) \arrow[r, "\cong"] & \mathrm{Im}\, i_* \oplus \mathrm{Im}\, u_*, \end{tikzcd}$$

this is, $\otimes_*(\alpha + x, \beta + y) = \psi(\alpha, \beta) + \phi_*(x, y)$ for $\alpha + x \in \mathbb{Z}/m \oplus \mathbb{Z}$ and $\beta + y \in \mathbb{Z}/n \oplus \mathbb{Z}$. By the Eckmann–Hilton argument and Corollary 2.11, $\otimes_*(\alpha + x, \beta + y) = \psi(\alpha, \beta) + (bnx + amy)$. ∎

Claim 2.16 Diagram (2.8) commutes up to a pointed homotopy,

$$\begin{tikzcd} S^1 \times S^1 \arrow[r, "t \times t"] \arrow[d, "\phi"] & \mathrm{U}_{am}/\mu_m \times \mathrm{U}_{bn}/\mu_n \arrow[d, "\otimes"] \\ S^1 \arrow[r, "t"] & \mathrm{U}_{abmn}/\mu_{mn}. \end{tikzcd} \tag{2.8}$$

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Proof Consider the stabilization maps $\mathrm{s}_j : \mathrm{U}_1 \to \mathrm{U}_{abmn}$ for $j = 1, \ldots, abmn$. Let $v, \omega \in S^1$, then

$$t(\phi(v,\omega)) = \begin{pmatrix} v^{bn}\omega^{am} & 0 \\ 0 & I_{abmn-1} \end{pmatrix} \text{ and } t(v) \otimes t(\omega) = \begin{pmatrix} vt(\omega) & 0 & \cdots & 0 \\ 0 & t(\omega) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & t(\omega) \end{pmatrix}.$$

Observe that $t \circ \phi$ is equal to the composite

$$S^1 \times S^1 \xrightarrow{\Delta\times\Delta} (S^1)^{\times bn} \times (S^1)^{\times am} \xrightarrow{g} \mathrm{U}_{abmn}^{\times bn} \times \mathrm{U}_{abmn}^{\times am} \xrightarrow{\text{mult}} \mathrm{U}_{abmn}$$

where $g = (\mathrm{s}_1 \times \cdots \times \mathrm{s}_1, \mathrm{s}_1 \times \cdots \times \mathrm{s}_1)$, and $t \otimes t$ is equal to

$$S^1 \times S^1 \xrightarrow{\Delta\times\Delta} (S^1)^{\times bn} \times (S^1)^{\times am} \xrightarrow{h} \mathrm{U}_{abmn}^{\times bn} \times \mathrm{U}_{abmn}^{\times am} \xrightarrow{\text{mult}} \mathrm{U}_{abmn}.$$

where $h = (\mathrm{s}_1 \times \mathrm{s}_2 \times \cdots \times \mathrm{s}_{bn}, \mathrm{s}_1 \times \mathrm{s}_{bn+1} \times \cdots \times \mathrm{s}_{(am-1)bn+1})$. By Lemma 2.3, $t \circ \phi$ and $t \otimes t$ are pointed homotopic. ∎

3 Proof of Theorem 1.3

Proposition 3.1 *Let a, b, m and n be positive integers such that am and bn are relatively prime and am < bn. Then there exist positive integers u and v satisfying* $|vn(bn)^n - um(am)^m| = 1$, *so that there exist a positive integer N and a homomorphism* $\mathrm{T} : \mathrm{U}_{am} \times \mathrm{U}_{bn} \longrightarrow \mathrm{U}_N$ *such that*

(1) *the homomorphism* T *factors through* $\widetilde{\mathrm{T}} : \mathrm{U}_{am}/\mu_m \times \mathrm{U}_{bn}/\mu_n \longrightarrow \mathrm{U}_N$, *and*
(2) *the homomorphisms induced on homotopy groups*

$$\widetilde{\mathrm{T}}_i : \pi_i(\mathrm{U}_{am}/\mu_m) \times \pi_i(\mathrm{U}_{bn}/\mu_n) \longrightarrow \pi_i(\mathrm{U}_N)$$

are given by

$$\begin{cases} \widetilde{\mathrm{T}}_i(x,y) = um(am)^{m-1}x + vn(bn)^{n-1}y & \text{if } 1 < i < 2am, \\ \widetilde{\mathrm{T}}_i(\alpha + x, \beta + y) = um(am)^{m-1}x + vn(bn)^{n-1}y & \text{if } i = 1, \end{cases}$$

where $\alpha \in \mathbb{Z}/m$, $\beta \in \mathbb{Z}/n$ *and* $x, y \in \mathbb{Z}$.

Proof We first construct T.

Since am and bn are relatively prime, so are $m(am)^m$ and $n(bn)^n$. Hence there exist positive integers u and v such that $vn(bn)^n - um(am)^m = \pm 1$. Let N denote $u(am)^m + v(bn)^n$. We define T using the operations described in Section 2, as the composite

$$\mathrm{U}_{am} \times \mathrm{U}_{bn} \xrightarrow{(\otimes^m, \otimes^n)} \mathrm{U}_{(am)^m} \times \mathrm{U}_{(bn)^n} \xrightarrow{(\oplus^u, \oplus^v)} \mathrm{U}_{u(am)^m} \times \mathrm{U}_{v(bn)^n} \xrightarrow{\oplus} \mathrm{U}_N.$$

(1) We must show that $\mu_m \times \mu_n$ is contained in $\mathrm{Ker}(\mathrm{T})$. Let α and β be mth and nth roots of unity, respectively. Note that the element $(\alpha I_{am}, \beta I_{bn})$ is sent to $(I_{u(am)^m}, I_{v(bn)^n})$ by $(\otimes^m, \otimes^n)$, hence to the identity by the composite T defined above.

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(2) We observe that Corollaries 2.6, 2.8, and 2.13 imply

$$\begin{aligned} \mathrm{T}_i : \pi_i(\mathrm{U}_m) \times \pi_i(\mathrm{U}_n) &\longrightarrow \pi_i(\mathrm{U}_N) \\ (x, y) &\longmapsto um(am)^{m-1}x + vn(bn)^{n-1}y \end{aligned}$$

for all $i < 2am$.

From part (1) there is a map of fibrations

$$\begin{tikzcd} \mu_m \times \mu_n \arrow[r, hook] \arrow[d] & \mathrm{U}_{am} \times \mathrm{U}_{bn} \arrow[r] \arrow[d, "\mathrm{T}"] & \mathrm{U}_{am}/\mu_m \times \mathrm{U}_{bn}/\mu_n \arrow[d, "\widetilde{\mathrm{T}}"] \\ \{I_N\} \arrow[r] & \mathrm{U}_N \arrow[r, equal] & \mathrm{U}_N . \end{tikzcd}$$

Case 1. Let $i > 1$. From the long exact sequence, diagram (3.1) commutes.

$$\begin{tikzcd} \pi_i(\mathrm{U}_{am}) \times \pi_i(\mathrm{U}_{bn}) \arrow[r, "\cong"] \arrow[d, "\mathrm{T}_i"] & \pi_i(\mathrm{U}_{am}/\mu_m) \times \pi_i(\mathrm{U}_{bn}/\mu_n) \arrow[d, "\widetilde{\mathrm{T}}_i"] \\ \pi_i(\mathrm{U}_N) \arrow[r, equal] & \pi_i(\mathrm{U}_N). \end{tikzcd} \tag{3.1}$$

Thus, $\widetilde{\mathrm{T}}_i(x, y) = \mathrm{T}_i(x, y) = um(am)^{m-1}x + vn(bn)^{n-1}y$ for $1 < i < 2m$.

Case 2. Let $i = 1$. From the long exact sequence there is a homomorphism of short exact sequences

$$\begin{tikzcd} \pi_1(\mathrm{U}_{am}) \times \pi_1(\mathrm{U}_{bn}) \arrow[r, hook] \arrow[d, "\mathrm{T}_1"] & \pi_1(\mathrm{U}_{am}/\mu_m) \times \pi_1(\mathrm{U}_{bn}/\mu_n) \arrow[r, two heads] \arrow[d, "\widetilde{\mathrm{T}}_1"] & \mathbb{Z}/m \times \mathbb{Z}/n \arrow[d] \\ \pi_1(\mathrm{U}_N) \arrow[r, equal] & \pi_1(\mathrm{U}_N) \arrow[r] & 0. \end{tikzcd}$$

The top short exact sequence splits. By direct inspection, we obtain $\widetilde{\mathrm{T}}_1(\alpha + x, \beta + y) = \mathrm{T}_1(x, y) = um(am)^{m-1}x + vn(bn)^{n-1}y$. ■

3.1 A left homotopy inverse

Let a, b, m and n be positive integers. By applying the classifying-space functor to the homomorphism (2.5), we obtain a map

$$F_\otimes : \mathrm{B}\,\mathrm{U}_{am}/\mu_m \times \mathrm{B}\,\mathrm{U}_{bn}/\mu_n \to \mathrm{B}\,\mathrm{U}_{abmn}/\mu_{mn}. \tag{3.2}$$

If we take the quotient by μ_{am} and μ_{bn} in (3.2), we write $f_\otimes$ instead of $F_\otimes$.

Let J be the map

$$\begin{aligned} J : \mathrm{B}\,\mathrm{U}_{am}/\mu_m \times \mathrm{B}\,\mathrm{U}_{bn}/\mu_n &\longrightarrow \mathrm{B}\,\mathrm{U}_{abmn}/\mu_{mn} \times \mathrm{B}\,\mathrm{U}_N \\ (x, y) &\longmapsto \big(F_\otimes(x, y), \mathrm{B}\,\widetilde{\mathrm{T}}(x, y)\big), \end{aligned}$$

where the integer N is the one provided by Proposition 3.1.

Proposition 3.2 *Let a, b, m and n be positive integers such that am and bn are relatively prime and $am < bn$. The map J is $(2am + 1)$-connected.*

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Proof We want to prove that the induced homomorphism on homotopy groups

$$\begin{tikzcd} \pi_i(\mathrm{B\,U}_{am}/\mu_m) \times \pi_i(\mathrm{B\,U}_{bn}/\mu_n) \arrow[d, "J_i"] \\ \pi_i(\mathrm{B\,U}_{abmn}/\mu_{mn}) \times \pi_i(\mathrm{B\,U}_N) \end{tikzcd} \tag{3.3}$$

is an isomorphism for all $i < 2am+1$ and an epimorphism for $i = 2am+1$.

Observe that the homotopy groups of the spaces involved are trivial in odd degrees below $2am+2$, hence it suffices to prove that J_i is an isomorphism for all i even and $i < 2am+1$.

We divide the proof into two cases.

Case 1. Let $i < 2am+1$ and $i \neq 2$. For this case computations can be done at the level of the universal covers of the groups $\mathrm{U}_{am}/\mu_m, \mathrm{U}_{bn}/\mu_n$, and $\mathrm{U}_{abmn}/\mu_{mn}$.

The homomorphism (3.3) takes the form

$$J_i : \mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{Z} \times \mathbb{Z}.$$

Propositions 2.15 and 3.1 yield

$$J_i(x,y) = \Big(bnx + amy, um(am)^{m-1}x + vn(bn)^{n-1}y\Big).$$

Thereby, the homomorphism (3.3) is represented by the matrix

$$\begin{pmatrix} bn & am \\ m(am)^{m-1}u & n(bn)^{n-1}v \end{pmatrix},$$

which is invertible. This proves J_i is an isomorphism.

Case 2. Let $i = 2$. The homomorphism (3.3) takes the form

$$J_2 : (\mathbb{Z}/m \oplus \mathbb{Z}) \times (\mathbb{Z}/n \oplus \mathbb{Z}) \longrightarrow (\mathbb{Z}/mn \oplus \mathbb{Z}) \times \mathbb{Z}.$$

Propositions 2.15 and 3.1 yield

$$J_2(x+\alpha, y+\beta) = \Big(\psi(\alpha,\beta) + (bnx + amy), um(am)^{m-1}x + vn(bn)^{n-1}y\Big).$$

Recall that $\psi : \mathbb{Z}/m \times \mathbb{Z}/n \to \mathbb{Z}/mn$ is addition where $\mathbb{Z}/m$ and $\mathbb{Z}/n$ are considered as subgroups of $\mathbb{Z}/mn$, see proof of Proposition 2.15. The homomorphism ψ is an isomorphism. From this and the invertibility of the matrix above, J_2 is an isomorphism. ∎

3.2 Factorization through $F_\otimes : \mathrm{B\,U}_{am}/\mu_m \times \mathrm{B\,U}_{bn}/\mu_b \to \mathrm{B\,U}_{abmn}/\mu_{mn}$

Theorem 3.3 *Let a, b, m and n be positive integers such that am and bn are relatively prime and $am < bn$. Let X be a topological space with the homotopy type of a finite dimensional CW complex such that $\dim(X) \leq 2am+1$.*

Every map $\mathscr{A} : X \to \mathrm{B\,U}_{abmn}/\mu_{mn}$ can be lifted to $\mathrm{B\,U}_{am}/\mu_m \times \mathrm{B\,U}_{bn}/\mu_n$ along the map $F_\otimes$ up to a pointed homotopy.

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Proof Diagramatically speaking, we want to find a map

$$\mathscr{A}_m \times \mathscr{A}_n : X \to \mathrm{B\,U}_{am}/\mu_m \times \mathrm{B\,U}_{bn}/\mu_n$$

such that diagram (3.4) commutes up to homotopy

$$\begin{tikzcd} & \mathrm{B\,U}_{am}/\mu_m \times \mathrm{B\,U}_{bn}/\mu_n \arrow[d, "F_\otimes"] \\ X \arrow[ur, dotted, bend left, "\mathscr{A}_m\times\mathscr{A}_n"] \arrow[r, "\mathscr{A}"] & \mathrm{B\,U}_{abmn}/\mu_{mn}. \end{tikzcd} \tag{3.4}$$

Proposition 3.2 yields a map $J : \mathrm{B\,U}_{am}/\mu_m \times \mathrm{B\,U}_{bn}/\mu_n \longrightarrow \mathrm{B\,U}_{abmn}/\mu_{mn} \times \mathrm{B\,U}_N$ where N is some positive integer. Observe that $F_\otimes$ factors through $\mathrm{B\,U}_{abmn}/\mu_{mn} \times \mathrm{B\,U}_N$, so we can write $F_\otimes$ as the composite of J and the projection proj_1 shown in diagram (3.5).

$$\begin{tikzcd} \mathrm{B\,U}_{am}/\mu_m \times \mathrm{B\,U}_{bn}/\mu_n \arrow[r, "J"] \arrow[d, "F_\otimes"] & \mathrm{B\,U}_{abmn}/\mu_{mn} \times \mathrm{B\,U}_N \arrow[dl, "\mathrm{proj}_1"'] \\ \mathrm{B\,U}_{abmn}/\mu_{mn} & \end{tikzcd} \tag{3.5}$$

Since J is $(2am+1)$-connected and $\dim(X) \le 2am+1$, then by Whitehead's theorem

$$J_\# : [X, \mathrm{B\,U}_{am}/\mu_m \times \mathrm{B\,U}_{bn}/\mu_n] \to [X, \mathrm{B\,U}_{abmn}/\mu_{mn} \times \mathrm{B\,U}_N]$$

is a surjection, [11, Corollary 7.6.23].

Let s denote a section of proj_1. The surjectivity of $J_\#$ implies $s \circ \mathscr{A}$ has a preimage $\mathscr{A}_m \times \mathscr{A}_n : X \to \mathrm{B\,U}_{am}/\mu_m \times \mathrm{B\,U}_{bn}/\mu_n$ such that $J \circ (\mathscr{A}_m \times \mathscr{A}_n) \simeq s \circ \mathscr{A}$.

The commutativity of diagram (3.4) follows from commutativity of diagram (3.5). Thus, the result follows. ∎

3.3 Factorization through $f_\otimes : \mathrm{B\,PU}_{am} \times \mathrm{B\,PU}_{bn} \to \mathrm{B\,PU}_{abmn}$

Proposition 3.4 *Let X be a finite CW complex. Let $\alpha \in \mathrm{Br}(X)$ be a class of period m. There exists a lifting $\mathscr{A}'$ of α if and only if α is represented by a topological Azumaya algebra $\mathscr{A}$ of degree am.*

Proof Let $\alpha \in \mathrm{Br}(X)$ be a Brauer class of period m. There exists a lifting $\xi \in \mathrm{H}^2(X; \mathbb{Z}/m)$ such that $\widetilde{\beta}_m(\xi) = \alpha$. Diagrammatically,

$$\begin{tikzcd} & X \arrow[dl, dotted, bend right, "\xi"'] \arrow[d, "\alpha"] & \\ \mathrm{K}(\mathbb{Z}/m, 2) \arrow[r, "\widetilde{\beta}_m"] & \mathrm{K}(\mathbb{Z}, 3) \arrow[r, "\times m"] & \mathrm{K}(\mathbb{Z}, 3). \end{tikzcd}$$

The map of fibrations below

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$$\begin{tikzcd}
\mu_m \arrow[r, hook] \arrow[d, hook] & \mathrm{U}_{am} \arrow[r, hook] \arrow[d, equal] & \mathrm{U}_{am}/\mu_m \arrow[d, "q"] \\
S^1 \arrow[r, hook] & \mathrm{U}_{am} \arrow[r] & \mathrm{PU}_{am}
\end{tikzcd}$$

induces a commutative diagram

$$\begin{tikzcd}
\mathrm{B}\,\mathrm{U}_{am}/\mu_m \arrow[r, "\mathrm{B}\,q"] \arrow[d] & \mathrm{B}\,\mathrm{PU}_{am} \arrow[d] \\
\mathrm{K}(\mathbb{Z}/m,2) \arrow[r, "\widetilde{\beta}_m"] & \mathrm{K}(\mathbb{Z},3).
\end{tikzcd}$$

In order to prove the proposition, we show that there exists a lifting $\mathscr{A}'$ of ξ if and only if there exists a lifting $\mathscr{A}$ of ξ, see diagram (3.6) below.

$$\begin{tikzcd}
 & \mathrm{B}\,\mathrm{U}_{am}/\mu_m \arrow[r, "\mathrm{B}\,q"] \arrow[d] & \mathrm{B}\,\mathrm{PU}_{am} \arrow[d] \\
 & \mathrm{K}(\mathbb{Z}/m,2) \arrow[r, equal] \arrow[d, equal] & \mathrm{K}(\mathbb{Z}/m,2) \arrow[d, "\widetilde{\beta}_m"] \\
X \arrow[r, "\xi"'] \arrow[uur, dotted, bend left, "\mathscr{A}'"] \arrow[uurr, dotted, "\mathscr{A}"] & \mathrm{K}(\mathbb{Z}/m,2) \arrow[r, "\widetilde{\beta}_m"'] & \mathrm{K}(\mathbb{Z},3)
\end{tikzcd} \tag{3.6}$$

If there exists a lifting $\mathscr{A}' : X \longrightarrow \mathrm{B}\,\mathrm{U}_{am}/\mu_m$, then the composite $\mathrm{B}\,q \circ \mathscr{A}'$ is a topological Azumaya algebra of degree am that represents the Brauer class α.

Conversely, suppose there exists an Azumaya algebra $\mathscr{A}$ of degree am making the outer square in the diagram below commute up to homotopy.

$$\begin{tikzcd}
X \arrow[dr, dotted, "\mathscr{A}'"] \arrow[drr, bend left, "\mathscr{A}"] \arrow[ddr, bend right, "\xi'"'] & & \\
 & \mathrm{B}\,\mathrm{U}_{am}/\mu_m \arrow[r, "\mathrm{B}\,q"] \arrow[d] & \mathrm{B}\,\mathrm{PU}_{am} \arrow[d] \\
 & \mathrm{K}(\mathbb{Z}/m,2) \arrow[r, "\widetilde{\beta}_m"] & \mathrm{K}(\mathbb{Z},3)
\end{tikzcd}$$

In the inner square, the induced map on the homotopy fibers of $\mathrm{B}\,q$ and $\widetilde{\beta}_m$ is a homotopy equivalence. An application of the five-lemma implies that the inner square is a homotopy pullback square. Therefore, there exists a lifting $\mathscr{A}'$ representing α. ■

Theorem 3.5 *Let a, b, m and n be positive integers such that am and bn are relatively prime and $am < bn$. Let X be a CW complex such that* $\dim(X) \leq 2am + 1$.

If $\mathscr{A}$ is a topological Azumaya algebra of degree $abmn$ such that $\mathrm{cl}(\mathscr{A})$ has period mn, then there exist topological Azumaya algebras $\mathscr{A}_m$ and $\mathscr{A}_n$ of degrees am and bn, respectively, such that $\mathrm{per}(\mathrm{cl}(\mathscr{A}_m)) = m$, $\mathrm{per}(\mathrm{cl}(\mathscr{A}_n)) = n$ and $\mathscr{A} \cong \mathscr{A}_m \otimes \mathscr{A}_n$.

Proof In this case, we want to solve the lifting problem shown in diagram (3.7) up to homotopy, with $\mathrm{per}(\mathrm{cl}(\mathscr{A}_m)) = m$, $\mathrm{per}(\mathrm{cl}(\mathscr{A}_n)) = n$.

$$\begin{tikzcd} & \mathrm{B\,PU}_{am} \times \mathrm{B\,PU}_{bn} \arrow[d, "f_\otimes"] \\ X \arrow[ur, dotted, "\mathscr{A}_m \times \mathscr{A}_n"] \arrow[r, "\mathscr{A}"] & \mathrm{B\,PU}_{abmn}. \end{tikzcd} \tag{3.7}$$

By Proposition 3.4, there exists a map $\mathscr{A}' : X \to \mathrm{B\,U}_{abmn}/\mu_{mn}$ such that $\mathrm{per}(\mathrm{cl}(\mathscr{A}')) = \mathrm{per}(\mathrm{cl}(\mathscr{A})) = mn$. Then, by Theorem 3.3, there exists a map $\mathscr{A}'_m \times \mathscr{A}'_n : X \longrightarrow \mathrm{B\,U}_{am}/\mu_m \times \mathrm{B\,U}_{bn}/\mu_n$ such that $F_\otimes \circ (\mathscr{A}'_m \times \mathscr{A}'_n) \simeq \mathscr{A}'$.

Since $\mathrm{cl}(\mathscr{A}'_m)\,\mathrm{cl}(\mathscr{A}'_n) = \mathrm{cl}(\mathscr{A}'_{mn})$, m and n are relatively prime, and $\mathrm{per}(\mathrm{cl}(\mathscr{A}')) = mn$ then $\mathrm{per}(\mathrm{cl}(\mathscr{A}'_m)) = m$ and $\mathrm{per}(\mathrm{cl}(\mathscr{A}'_n)) = n$.

By Proposition 3.4, there exists a map $\mathscr{A}_m \times \mathscr{A}_n : X \longrightarrow \mathrm{B\,PU}_{am} \times \mathrm{B\,PU}_{bn}$ such that $\mathrm{per}(\mathrm{cl}(\mathscr{A}_m)) = \mathrm{per}(\mathrm{cl}(\mathscr{A}'_m))$ and $\mathrm{per}(\mathrm{cl}(\mathscr{A}_n)) = \mathrm{per}(\mathrm{cl}(\mathscr{A}'_n))$.

It remains to show that diagram (3.7) commutes. Consider the diagram below

$$\begin{tikzcd} \mathrm{B\,U}_{am}/\mu_m \times \mathrm{B\,U}_{bn}/\mu_n \arrow[rr, "\mathrm{B}\,q \times \mathrm{B}\,q"] \arrow[dd, "F_\otimes"'] & & \mathrm{B\,PU}_{am} \times \mathrm{B\,PU}_{bn} \arrow[dd, "f_\otimes"] \\ & X \arrow[ul, "\mathscr{A}'_m \times \mathscr{A}'_n"'] \arrow[ur, "\mathscr{A}_m \times \mathscr{A}_n"] \arrow[dl, "\mathscr{A}'"] \arrow[dr, "\mathscr{A}"'] & \\ \mathrm{B\,U}_{abmn}/\mu_{mn} \arrow[rr] & & \mathrm{B\,PU}_{abmn} \end{tikzcd} \tag{3.8}$$

Observe that the square, as well as top, bottom and left triangles, of diagram (3.8) commute. Hence, the right triangle commutes. ∎

Theorem 1.3 is a corollary of Theorem 3.5.

Theorem 3.6 *Let a, b, m and n be positive integers such that am and bn are relatively prime and $am < bn$. The map $F_\otimes : \mathrm{B\,U}_{am}/\mu_m \times \mathrm{B\,U}_{bn}/\mu_n \to \mathrm{B\,U}_{abmn}/\mu_{mn}$ does not have any section.*

Proof Suppose there exists a section σ of $F_\otimes$.

By Proposition 2.15, the map $F_\otimes$ induces a homomorphism on homotopy groups which is given by $(x, y) \mapsto bn\,\mathrm{stab}(x) + am\,\mathrm{stab}(y)$ for $i > 2$. In degree $2am + 2$ the homomorphism $(F_\otimes)_*$ takes the form $(F_\otimes)_* : \pi_{2am+2}(\mathrm{B\,U}_{am}/\mu_m) \times \mathbb{Z} \to \mathbb{Z}$, where $\pi_{2am+2}(\mathrm{B\,U}_{am}/\mu_m) \cong \pi_{2am+2}(\mathrm{B\,U}_{am})$ and $\pi_{2am+2}(\mathrm{B\,U}_{am})$ is trivial when am is odd, and $\mathbb{Z}/2$ when am is even, see [9, p. 971]. Therefore, $(F_\otimes)_*(x, y) = am\,\mathrm{stab}(y)$. Thus $\mathrm{Im}(F_\otimes)_* = am\mathbb{Z}$.

On the other side, since σ is a section of $F_\otimes$, the composite $(F_\otimes)_* \circ \sigma_* : \pi_{2am+2}(\mathrm{B\,U}_{abmn}/\mu_{mn}) \to \pi_{2am+2}(\mathrm{B\,U}_{am}/\mu_m) \times \pi_{2am+2}(\mathrm{B\,U}_{bn}/\mu_n) \to \pi_{2am+2}(\mathrm{B\,U}_{abmn}/\mu_{mn})$ is the identity. This contradicts the fact that $\mathrm{Im}((F_\otimes)_* \circ \sigma_*) \subset am\mathbb{Z}$. ∎

In Theorem 1.3, it is proven that there exists a tensor product decomposition for topological Azumaya algebras over low dimensional CW complexes, and that such decomposition does not exist for topological Azumaya algebras over an arbitrary CW

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complex. The proof of Theorem 3.6 implies that for positive integers m and n where $m < n$, if $\mathscr{A}$ is a topological Azumaya algebra of degree mn over a finite CW complex of dimension higher than $2m + 1$, then $\mathscr{A}$ may not be decomposable as $\mathscr{A}_m \otimes \mathscr{A}_n$. In fact, consider the unit $(2m + 2)$-sphere, and let $\mathscr{S} : S^{2m+2} \to \mathrm{B\,PU}_{mn}$ be a topological Azumaya algebra of degreemn on S^{2m+2} such that $\mathscr{S}$ generates $\pi_{2m+2}(\mathrm{B\,PU}_{mn})$, then $\mathscr{S}$ cannot be decomposed as the tensor product of topological Azumaya algebras of degrees m and n.

Remark 3.7 Under the hypotheses of Theorem 1.3, the topological Azumaya algebras $\mathscr{A}_m$ and $\mathscr{A}_n$ are not neccesarily unique up to isomorphism. In order to see this, we consider the Moore–Postnikov tower for $f_\otimes$:

$$\begin{tikzcd}
 & \mathrm{B\,PU}_m \times \mathrm{B\,PU}_n \arrow[d] & \\
 & \vdots \arrow[d] & \\
\mathrm{K}\big(\pi_2 F, 2\big) \arrow[r] & Y[2] \arrow[d] & \\
\mathrm{K}\big(\pi_1 F, 1\big) \arrow[r] & Y[1] \arrow[d] \arrow[r, "k_1"] & \mathrm{K}(\pi_2 F, 3) \\
 & \mathrm{B\,PU}_{mn} \arrow[r, "k_0"] & \mathrm{K}(\pi_1 F, 2),
\end{tikzcd}$$

where F is the homotopy fiber of $f_\otimes$, and $k_{i-1} : Y[i-1] \to \mathrm{K}\big(\pi_i F, i+1\big)$ is the k-invariant that classifies the fiber sequence $Y[i] \to Y[i-1]$, for $i > 0$, [8, Theorem 4.71].

Since the map $f_\otimes$ induces an isomorphism on π_2, and π_{2i+1} for $0 < i < m$, it follows that $\mathrm{B\,PU}_{mn} \simeq Y[i]$ for $i = 1, 2, 3$, and $Y[2i] \simeq Y[2i+1]$ for $1 < i < m$.

The long exact sequence of $F \to \mathrm{B\,PU}_m \times \mathrm{B\,PU}_n \to \mathrm{B\,PU}_{mn}$ yields

$$\pi_i F \cong \begin{cases} 0 & \text{if } i = 2 \text{ or } i \text{ is odd and } i < 2m+1, \\ \mathbb{Z} & \text{if } i \neq 2, i \text{ is even and } i < 2m+1. \end{cases}$$

Hence, the Moore–Postnikov tower of $f_\otimes$ takes the form

$$\begin{tikzcd}
 & \mathrm{B\,PU}_m \times \mathrm{B\,PU}_n \arrow[d] & \\
 & \vdots \arrow[d] & \\
\mathrm{K}(\mathbb{Z}, 6) \arrow[r] & Y[6] \arrow[d] & \\
\mathrm{K}(\mathbb{Z}, 4) \arrow[r] & Y[4] \arrow[d] \arrow[r, "k_5"] & \mathrm{K}(\mathbb{Z}, 7) \\
 & \mathrm{B\,PU}_{mn} \arrow[r, "k_3"] & \mathrm{K}(\mathbb{Z}, 5).
\end{tikzcd}$$

Let X be a CW complex of $\dim(X) \leq 6$. Let m and n be as in the hypothesis of Theorem 1.3, and $m > 3$. Let $\mathscr{A}$ be a topological Azumaya algebra of degree mn.

Observe that the k-invariant k_3 is trivial because $\mathrm{H}^5(\mathrm{B}\,\mathrm{PU}_{mn};\mathbb{Z})$ is trivial, [1, Proposition 4.1]. Hence, there is no obstruction to lift $\mathscr{A}$ to $Y[4]$. Similarly, we can lift the identity map $\mathrm{id}_{\mathrm{B}\,\mathrm{PU}_{mn}}$ to $Y[4]$, in this case, we obtain the splitting $Y[4] \simeq \mathrm{B}\,\mathrm{PU}_{mn} \times \mathrm{K}(\mathbb{Z},4)$. Then the lifting of $\mathscr{A}$ takes the form $(\mathscr{A},\xi) : X \to \mathrm{B}\,\mathrm{PU}_{mn} \times \mathrm{K}(\mathbb{Z},4)$.

The cohomology groups of X vanish for all degrees greater than 6, given that X is six-dimensional. Thus $(\mathscr{A},\xi)$ can be lifted up the Moore–Postnikov tower to $\mathrm{B}\,\mathrm{PU}_m \times \mathrm{B}\,\mathrm{PU}_n$. See diagram (3.9).

$$
\begin{tikzcd}
 & \mathrm{B}\,\mathrm{PU}_m \times \mathrm{B}\,\mathrm{PU}_n \arrow[d] & \\
 & \vdots \arrow[d] & \\
 & Y[4] \simeq \mathrm{B}\,\mathrm{PU}_{mn} \times \mathrm{K}(\mathbb{Z},4) \arrow[r, "k_5"] \arrow[d] & \mathrm{K}(\mathbb{Z},7) \\
X \arrow[uuur, bend left, "\mathscr{A}_m \times \mathscr{A}_n"] \arrow[ur, bend left, "(\mathscr{A},\xi)"] \arrow[r, "\mathscr{A}"] & \mathrm{B}\,\mathrm{PU}_{mn} \arrow[r, "k_3"] & \mathrm{K}(\mathbb{Z},5)
\end{tikzcd}
\tag{3.9}
$$

This proves that $\mathscr{A}$ can be decomposed as $\mathscr{A}_m \otimes \mathscr{A}_n$. The lifting $(\mathscr{A},\xi)$ is not necessarily unique. In fact, every cohomology class $\xi \in \mathrm{H}^4(X;\mathbb{Z})$ gives rise to a lifting $(\mathscr{A},\xi)$.

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