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Surface perturbation of an elastodynamic contact problem with friction

JEAN-CLAUDE PAUMIER[1] and YVES RENARD[2]

[1] *LMC-IMAG, BP 53X, F-38041 Grenoble, France*
email: Jean-Claude.Paumier@imag.fr
[2] *INSAT, MIP, 135, avenue de Rangueil, 31077 Toulouse, France*
email: Yves.Renard@gmm.insa-tlse.fr

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We consider the dynamic process of an elastic body in unilateral frictional contact with a rigid foundation. Friction is modelled with the Coulomb law with a coefficient that depends on the slip velocity. To allow for velocity discontinuities we use the elastodynamic (hyperbolic) framework. Nevertheless, this does not lead to a well-posed problem. To remedy this, we perturb the solution of the elastodynamic problem in a thin layer next to the contact boundary. This is a generalisation of an approach previously studied in a one-dimensional case. We establish existence and uniqueness results for the perturbed and regularised problem and provide an interpretation of this perturbation.

1 Introduction

The dynamic friction of linearly elastic structures is an active subject of research from the theoretical point of view, as well as for the qualitative analysis of the behaviour of solutions. Mathematical study presents severe difficulties due principally to the weak regularity of the solutions, and the strong nonlinearity of the contact and Coulomb friction laws. Existing studies involve a certain number of regularisations, of which the more classical ones are: a penalisation of the unilateral contact condition, such as compliance law introduced by Oden & Martins [20]; a regularisation of the contact pressure by the mean of a convolution introduced by Duvaut [6], and a continuous regularisation of the friction condition (see Oden & Pires [19] and Renard [25], for example). A general discussion can be found in Kikuchi & Oden [12].

Those regularisations allow several authors [8, 11, 15, 16] to give existence and uniqueness results in the viscoelastic framework. A linear viscoelastic constitutive law 'parabolises' the problem and gives better regularity. Some precise results exist also for the quasistatic approximation (see Andersson [2] and Rocca & Cocu [27]), without any regularisations, but with the restriction of a sufficiently small coefficient of friction.

In this paper, we present an approach to the elastodynamic problem that is not in the viscoelastic or quasistatic framework, but which considers a regularisation in a layer of small thickness localised next to the contact zone. This perturbation, which we call a *surface perturbation*, keeps the local characteristic of the friction law and allows us to have a well-posed problem without the introduction of a linear viscous term. To start with, a simple geometry is considered along with regularised friction and unilateral contact conditions.

The main motivation is to try to generalise theoretical and qualitative results established in Ionescu & Paumier [10] and Renard [26] for the one-dimensional problem to the multi-dimensional case. In the one-dimensional case, the analysis of the problem shows that the use of a non-monotone slip-dependent friction coefficient in the purely elastodynamic problem introduces a multiplicity of solutions and shocks in the velocity. This non-uniqueness is still present if the friction and unilateral contact conditions are regularised. The one-dimensional case shows precisely the regularity which can be expected. In this case, it has been proved that the surface perturbation allows us to recover the uniqueness of the solution. Moreover, when the perturbation parameter goes to zero, the solution tends to a particular solution to the non-perturbed problem which is related to the 'maximum delay' criterion introduced for this problem in Ionescu & Paumier [10], and also discussed in Ionescu [9]. Unfortunately, this criterion has no clear extension to the multi-dimensional case, and the discussion is still open as to whether or not non-uniqueness can be observed with a non-decreasing friction coefficient (this is not the case in the one-dimensional case). Also, it is not clear how stick-slip motions can be observed with a constant friction coefficient (elements are presented in Renard [25], Simões & Martins [28] and Martin *et al.* [17]).

We present here a first result, where a classical regularisation of the contact and friction conditions is introduced. As a second step, we intend to generalise the result without this classical regularisation. Again, the surface perturbation allows to regain the existence and uniqueness of the solution. The proof is principally based on a fixed point method very similar to the classical one for Cauchy–Lipschitz Theorem for ordinary differential equations. An important point is that no assumption is made on the bound of the friction coefficient.

2 The surface perturbation model

The idea of the perturbation takes its origin in our contribution on dynamic sliding with friction in Campillo *et al.* [4]. There, we compared two approaches in the simplest geometrical cases: a single block and an infinite elastic slab sliding on a frictional surface. The aim was to understand the importance of the friction law for the one-dimensional problem and to identify the theoretical problems associated with nonlinearity.

Let u_N be the normal displacement on the contact boundary of a solid in frictional contact with a rigid foundation. Let also u_T be the tangential displacement, $\dot{u}_T$ the tangential velocity, $\mathbf{n}$ the unit outward normal to the domain, $F = F_N\mathbf{n} + F_T$ the reaction of the rigid foundation, and μ the friction coefficient. Assuming that there is no initial gap between the solid and the foundation and that this foundation is at rest, the contact and friction conditions for small displacements read:

$$u_N \leqslant 0, \; F_N \leqslant 0, \; u_N F_N = 0, \tag{2.1}$$

$$|F_T| \leqslant \mu|F_N|, \quad \big(|F_T| - \mu|F_N|\big)\,\dot{u}_T = 0, \tag{2.2}$$

$$\text{there exists } \gamma \geqslant 0 \text{ such that } \dot{u}_T = -\gamma F_T. \tag{2.3}$$

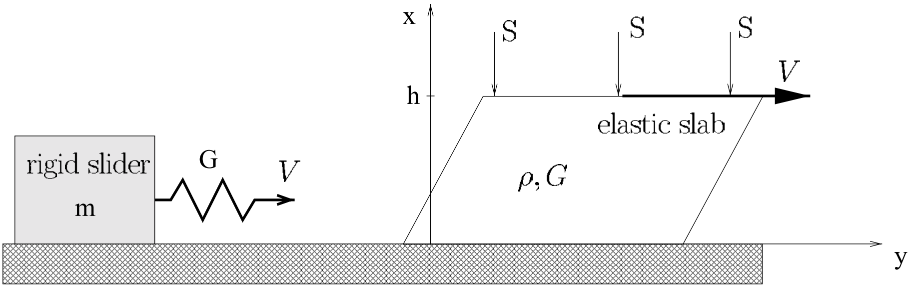

FIGURE 1. The two different models.

In this paragraph, our models are two-dimensional with persistent contact, so that F_T, u_T are scalar and $u_N = 0$.

In the *first system* we consider a rigid block called the slider which is in contact with friction on a rigid foundation and which is submitted to a traction force by means of a spring pulled at velocity V (see Figure 1). The equations of motion for this slider are

$$m\,\ddot{u}(t) - G\,(D_0 + tV - u(t)) = -\mu S\frac{\dot{u}(t)}{|\dot{u}(t)|} \quad \text{if} \quad \dot{u}(t) \neq 0, \tag{2.4}$$

$$|m\,\ddot{u}(t) - G(D_0 + tV - u(t))| \leqslant \mu S \quad \text{if} \quad \dot{u}(t) = 0, \tag{2.5}$$

$$\dot{u}(0) = 0 \quad \text{and} \quad u(0) = u_0, \tag{2.6}$$

where m is the mass of the slider, $-S < 0$ is a prescribed normal force applied on the top of the slider, u_0 is the initial position and $G\,(D_0 + tV - u(t))$ is the tension of the spring whose elastic modulus is G.

In the *second system* we consider the one-dimensional shearing of an infinite linear elastic slab, with elastic Lamé coefficients λ and G, bounded by the planes $x = 0$ and $x = h$ (as in Figure 1). On the plane $x = 0$, the slab is in frictional contact with a rigid foundation. At $x = h$ the slab is pulled with a tangential velocity V from an initial position D_0 and it is compressed with an uniform normal stress $-S$. We assume that the displacement field has the value $\frac{-Sx}{\lambda+2G}$ in the x-direction. We denote by u the horizontal displacement in the y-direction, and we suppose that it depends on t and x. In this way, we get the normal and tangential stresses $\boldsymbol{\sigma}_N = -S$, $\boldsymbol{\sigma}_T = -G\frac{\partial u}{\partial x}(0,t)$ on the friction boundary $x = 0$. From (2.1)–(2.3) and the equations of elastodynamics, we get the following initial and boundary value problem:

$$\varrho\,\ddot{u}(x,t) = G\frac{\partial^2 u}{\partial x^2}(x,t), \tag{2.7}$$

$$-G\frac{\partial u}{\partial x}(0,t) = -\mu S\frac{\dot{u}(0,t)}{|\dot{u}(0,t)|} \quad \text{if} \quad \dot{u}(0,t) \neq 0, \tag{2.8}$$

$$G\left|\frac{\partial u}{\partial x}(0,t)\right| \leqslant \mu S \quad \text{if} \quad \dot{u}(0,t) = 0, \tag{2.9}$$

$$u(h,t) = D_0 + tV, \tag{2.10}$$

$$\dot{u}(x,0) = V\frac{x}{h} \quad \text{and} \quad u(x,0) = u_0 + (D_0 - u_0)\frac{x}{h}, \tag{2.11}$$

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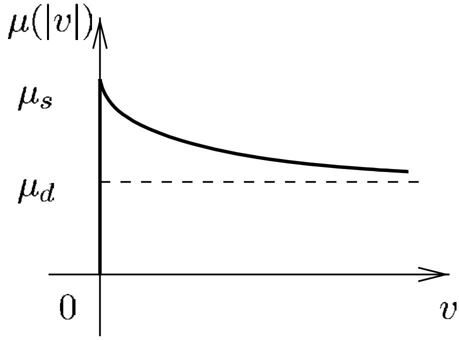

FIGURE 2. Example of variable friction coefficient.

where ϱ is the density and u_0 is the initial displacement (slip) at $x = 0$. For each of these two systems we consider the *slip weakening case*: the coefficient $\mu = \mu(|v|)$ depends upon the slip velocity v. That means $\mu = \mu\left(|\dot{u}(t)|\right)$ for the slider and $\mu = \mu\left(|\dot{u}(0,t)|\right)$ for the slab. For example, we shall consider,

$$\mu(|v|) = \mu_d + (\mu_s - \mu_d)\, e^{-|v|/V_{cr}}, \tag{2.12}$$

with $\mu_s > \mu_d > 0$, where μ_s and μ_s hold, respectively, for *static* and *dynamic* coefficients, and with $V_{cr} > 0$, which represents a critical velocity (see Figure 2).

For the *first system* (slider) there are no special mathematical difficulties for the analysis of (2.4)–(2.6), which can be viewed as an ordinary differential inclusion (see Deimling [5] for instance). This problem is well-posed.

For the *second system* (slab) the partial differential equation (2.7) is hyperbolic. The system (2.7)–(2.11) can be reduced using the method of characteristics. Indeed, for $0 \leqslant t < \frac{h}{c}$, it is easily seen from (2.7) that the quantity $A(x,t) = \dot{u}(x,t) + c\frac{\partial u}{\partial x}(x,t)$ is constant along the characteristic line $\{x + ct = \xi\}$ where $c = \sqrt{G/\rho}$. So $A(\xi - ct, t)$ is a quantity $a(\xi)$ independent of t. With $t = 0$ we get $a(\xi) = \dot{u}(\xi,0) + c\frac{\partial u}{\partial x}(\xi,0)$. Using (2.11) we get $a(\xi) = \frac{1}{h}(V\xi + c(D_0 - u_0))$. Therefore at $x = 0$ from the expression $A(0,t) = a(ct)$ we can deduce $\frac{\partial u}{\partial x}(0,t) = \frac{1}{c}(a(ct) - \dot{u}(0,t))$.

From (2.8) and (2.9), we get the following equation on the friction boundary, where the unknown is the slip rate $\dot{u}(0,t)$:

$$\beta\left(\dot{u}(0,t)\right) = \alpha(t) \quad \text{if} \quad \dot{u}(0,t) \neq 0, \tag{2.13}$$

$$|\alpha(t)| \leqslant \mu(0)\, S \quad \text{if} \quad \dot{u}(0,t) = 0, \tag{2.14}$$

where

$$\alpha(t) = \frac{G}{h}(tV + D_0 - u_0) \text{ for } 0 \leqslant t < \frac{h}{c},$$

$$\beta(v) = \sqrt{\rho G}\, v + \mu(v) S \frac{v}{|v|} \text{ for } v \neq 0.$$

From the point of view of the slip rate $v = \dot{u}(0,t)$, this is not a differential equation but an algebraic equation which may possibly have several distinct solutions $v_1(t)$, $v_2(t)$, ... This

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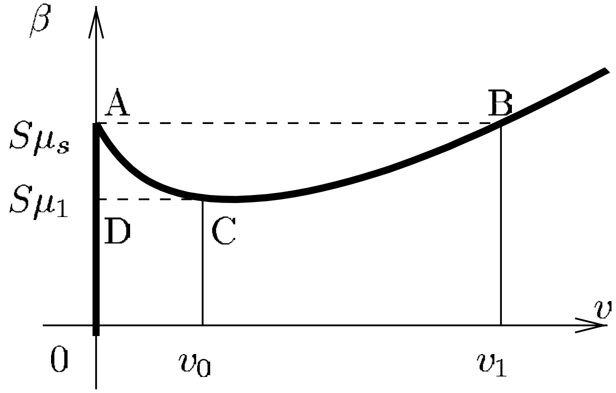

FIGURE 3. The function β which has to be inverted to solve the 1D problem.

depends on the monotonicity of the mapping $v \mapsto \beta(v)$, that is to say of the weakening of the friction coefficient $v \mapsto \mu(v)$, (see Ionescu & Paumier [10] for a complete discussion). This is the reason why the problem (2.7)–(2.11) is generally ill-posed. This non-uniqueness is still present if a simple regularisation of the contact and friction conditions is made, as in § 3.3.

For $\mu(|v|)$ given by (2.12), the mapping $v \mapsto \beta(v)$ is represented in Figure 3. We can actually have three solutions $v_1(t)$, $v_2(t), v_3(t)$ if $S\mu_1 < \alpha(t) < S\mu_s$. In this case *if the process is quasistatic,* a 'maximum delay' solution $v_{md}(t)$ can be chosen (see Ionescu & Paumier [10]). Indeed:

- on the one hand, given *growing* data $t \mapsto \alpha(t)$ from t_1 to t_2 with $\alpha(t_1) < S\mu_1$ and $\alpha(t_2) > S\mu_s$, we will take $v_{md}(t) = 0$ for $\alpha(t) < S\mu_1$ to $\alpha(t) = S\mu_s$ and the unique $v_{md}(t) > v_0$ such that $\beta\,(v_{md}(t)) = \alpha(t)$ for $\alpha(t) > S\mu_s$ (therefore we have on *AB a positive* velocity jump v_1 at $\alpha(t) = S\mu_s$);
- on the other hand, given *decreasing* data $t \mapsto \alpha(t)$ (obtained from an adapted initial condition) from t_1 to t_2 with $\alpha(t_1) > S\mu_d$ to $\alpha(t_2) < S\mu_1$, we will take the unique $v_{md}(t) > v_0$ such that $\beta\,(v_{md}(t)) = \alpha(t)$ for all $\alpha(t) > S\mu_1$ and $v_{md}(t) = 0$ for all $\alpha(t) < S\mu_1$ to $\alpha(t) = S\mu_s$ (therefore we have on CD a *negative* velocity jump $-v_0$ at $\alpha(t) = S\mu_1$).

However, following the analysis performed in Renard [26], this ill-posed problem may benefit from the well-posed property of the slider. To that end a thin rigid 'sole' is positioned on the friction surface $\{x = 0\}$ of the slab (see Figure 4). We introduce the small parameter $m > 0$ which represents the product of the thickness ε of the thin sole by its density ρ. From a balance of linear momentum, we get:

$$\varrho\, \ddot{u}(x,t) = G\frac{\partial^2 u}{\partial x^2}(x,t), \tag{2.15}$$

$$m\, \ddot{u}(0,t) - G\frac{\partial u}{\partial x}(0,t) = -\mu(\dot{u}(0,t))\, S\frac{\dot{u}(0,t)}{|\dot{u}(0,t)|} \quad \text{if} \quad \dot{u}(0,t) \neq 0, \tag{2.16}$$

$$\left| m\, \ddot{u}(0,t) - G\frac{\partial u}{\partial x}(0,t) \right| \leqslant \mu(0)\, S \quad \text{if} \quad \dot{u}(0,t) = 0, \tag{2.17}$$

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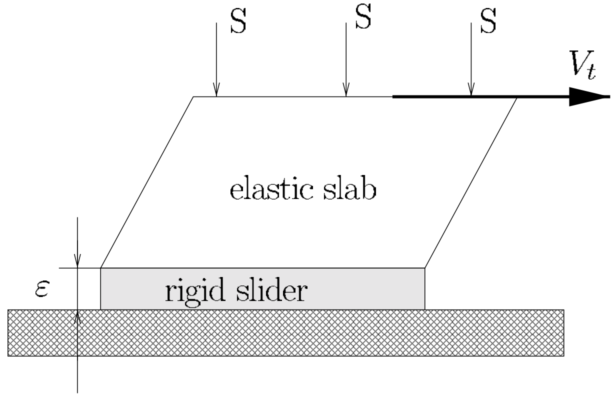

FIGURE 4. Perturbation of the 1D problem with a thin rigid sole on the contact surface.

$$u(h,t) = D_0 + tV, \tag{2.18}$$

$$\dot{u}(x,0) = V\frac{x}{h} \quad \text{and} \quad u(x,0) = u_0 + (D_0 - u_0)\frac{x}{h}, \tag{2.19}$$

As above, this system can be reduced by using the method of characteristics. In this way, we get the system

$$\varepsilon\rho\, \ddot{u}(0,t) + \beta\,(\dot{u}(0,t)) = \alpha(t) \quad \text{if} \quad \dot{u}(0,t) \neq 0, \tag{2.20}$$

$$|\varepsilon\rho\, \ddot{u}(0,t) - \alpha(t)| \leqslant \mu(0)\, S \quad \text{if} \quad \dot{u}(0,t) = 0, \tag{2.21}$$

$$\dot{u}(0,0) = 0 \quad \text{and} \quad u(0,0) = u_0. \tag{2.22}$$

This system is completely different from the system (2.13)–(2.14) because the term $\ddot{u}(0,t)$ transforms it into an initial-value differential system (in fact this system is a singular perturbation of the scalar equation (2.13)–(2.14)). Consequently this problem is well-posed, and we denote its unique solution by $v_\varepsilon(t) = u_\varepsilon(0,t)$.

A mathematical analysis of the convergence as ε vanishes can be found in Renard [25, 26]. It is shown that the unique solution $v_\varepsilon(t)$ to this Cauchy problem is close to the so-called 'maximum delay' solution $v_{md}(t)$ of the scalar equation (2.13)–(2.14). Moreover, when $\varepsilon \to 0$ the solution $v_\varepsilon(t)$ converges to $v_{md}(t)$ (see Figure 5 for a graphic representation in the (α,β) plane obtained numerically). We see in dashed line the non-monotone function β in the case (2.12). The perturbed solution, which is printed with a continuous line, is very close to the 'maximum delay' solution for $\varepsilon = 0.01$.

3 Generalisation of the perturbation to two and three dimension

3.1 Principle of the perturbation

Our goal is to adapt the perturbation introduced in the one-dimensional case to higher dimensional linear elastodynamic problems.

For the sake of simplicity, and as a first step to avoid difficulties coming from the geometry, it is assumed that the domain Ω, which represents the reference configuration

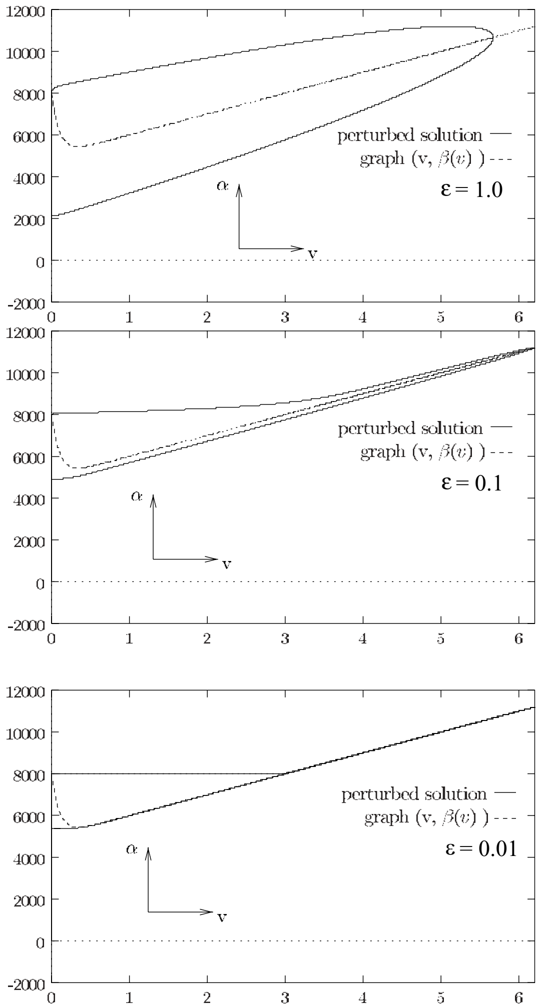

FIGURE 5. Graph $(\alpha(t), v(t))$ for three perturbed solutions. As ε goes to zero the solution tends to the solution selected by the maximum delay criterion.

of the elastic solid, is a cylinder (see Figure 6), i.e. $\Omega = \omega \times]0, D[$ where ω is a domain of $\mathbb{R}^{n-1}$ and $D > 0$ is the height. We assume that $\Gamma_D = \omega \times \{D\}$ represents that part of the boundary where the displacements are prescribed, $\Gamma_N = \partial\omega \times]0, D[$ that part of the boundary where the tractions are prescribed, and $\Gamma_C = \omega \times \{0\}$ is the contact boundary. We write $x = (x', x_n)$ where $x' \in \omega$ and $x_n \in]0, D[$.

Of course, a completely rigid layer cannot be added between the elastic body and the rigid foundation without changing drastically the behaviour of the structure because of the constancy of tangential displacements. A first approach would be to allow this layer to have free tangential displacement. This is simple to express, and mathematically, it has the same effect as the addition of a small mass to each point of the contact zone Γ_C, but it does

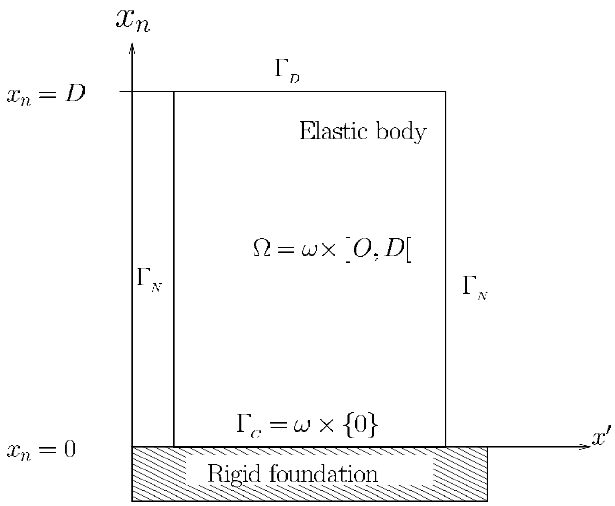

FIGURE 6. *Simplified geometry.*

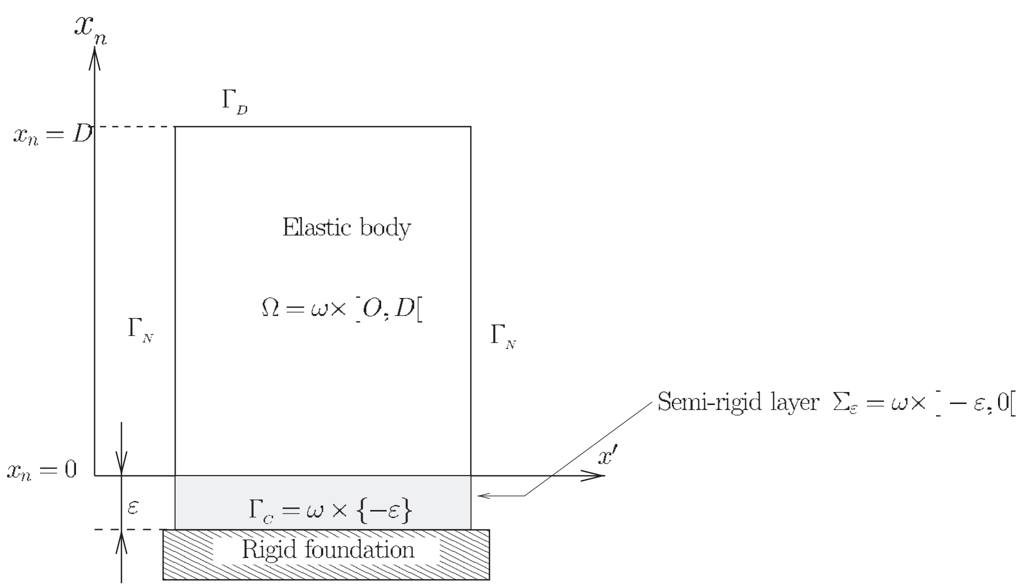

FIGURE 7. Insertion of a semi-rigid layer.

not offer many advantages for the mathematical analysis. The approach we develop here, is to insert a layer with the same elastic characteristics as the elastic body but with rigid displacements only in the normal direction (see Figure 7). We will call it a 'semi-rigid layer'.

With u the kinematically admissible displacement field (u prescribed on Γ_D), the 'virtual work' formulation of the linear elastic problem can be written as

$$\int_\Omega \rho \ddot{u}.v dx + \int_\Omega \mathscr{A}\mathbf{e}(u) : \mathbf{e}(v) dx = \int_\Omega f.v dx + \int_{\Gamma_N} g.v d\Gamma + \int_{\Gamma_C} F.v d\Gamma$$

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for all kinematically admissible velocity field v that vanish on Γ_D, where $\mathbf{e}(u)$ is the small strain tensor, $\boldsymbol{\sigma}(u) = \mathscr{A}\mathbf{e}(u)$ is the stress tensor, $\mathscr{A}$ is the elastic tensor having the usual symmetry and coercivity properties, f are the body forces, g the prescribed tractions on Γ_N and F represents the friction and contact forces on Γ_C. The addition of a layer $\Sigma_\varepsilon = \omega \times]-\varepsilon, 0[$ between the elastic solid and the rigid foundation requires us to add extra terms in this formulation:

$$\int_\Omega \rho\ddot{u}.v dx + \int_{\Sigma_\varepsilon} \rho\ddot{u}.v dx + \int_\Omega \mathscr{A}\mathbf{e}(u) : \mathbf{e}(v) dx + \int_{\Sigma_\varepsilon} \mathscr{A}\mathbf{e}(u) : \mathbf{e}(v) dx$$
$$= \int_\Omega f.v dx + \int_{\Gamma_N} g.v d\Gamma + \int_{\Gamma_C} F.v d\Gamma,$$

where Γ_C is now the boundary $\Gamma_C = \omega \times \{-\varepsilon\}$, and u and v are continuous through the interface between Ω and Σ_ε (the semi-rigid layer is stuck to the elastic body), and are independent of the vertical coordinate in Σ_ε. The latter condition express the rigidity introduced in the layer Σ_ε. In Paumier & Renard [22, 23], we studied the possibility for higher degree approximations. However, even if a gain in realism can be expected, the interpretation of the additional terms is more difficult and brings nothing supplementary for the mathematical analysis. On Σ_ε, values of $\mathscr{A}$ and ρ should *a priori* be chosen in accordance with their values on Ω. The additional terms are

$$\int_{\Sigma_\varepsilon} \rho\ddot{u}.v dx + \int_{\Sigma_\varepsilon} \mathscr{A}\mathbf{e}(u) : \mathbf{e}(v) dx,$$

which can be written

$$\varepsilon \left\{ \int_\omega \rho\ddot{u}.v dx + \int_\omega \bar{\mathscr{A}}\bar{\mathbf{e}}(u) : \bar{\mathbf{e}}(v) dx \right\},$$

where $\bar{\mathbf{e}}(u)$ and $\bar{\mathscr{A}}$ have only tangential derivatives. These terms can be interpreted has a surface perturbation with ε as perturbation parameter. This corresponds to a kind of tangential elastodynamic equation.

3.2 Advantages of the perturbation

One of the main difficulties concerning the mathematical analysis of friction problems of elastic bodies is the weak regularity of the solutions and, for instance, the appearance of shocks in velocity. This weak regularity prevents us giving a clear sense to the velocity and to the stress on the contact boundary.

Here, supplementary regularity is obtained from the fact that the perturbed problem can be viewed as a Galerkin approximation of a problem on

$$\Omega_\varepsilon = \omega \times]-\varepsilon, D[.$$

Indeed, assuming that $u = 0$ on Γ_D (which is always possible using a extension operator), with the spaces

$$\bar{H}_\varepsilon = \{v \in L^2(\Omega_\varepsilon; \mathbb{R}^n)\},$$
$$\bar{V}_\varepsilon = \{v \in H^1(\Omega_\varepsilon; \mathbb{R}^n); v = 0 \text{ on } \Gamma_D\},$$

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(where $L^2(\Omega_\varepsilon;\mathbb{R}^n)$ and $H^1(\Omega_\varepsilon;\mathbb{R}^n)$ stand for the usual Sobolev spaces with vectorial values) and considering the elastodynamic problem of finding $u :]0,T] \longrightarrow \bar{V}_\varepsilon$ satisfying

$$\int_{\Omega_\varepsilon} \rho \ddot{u}.v dx + \int_{\Omega_\varepsilon} \mathscr{A}\mathbf{e}(u) : \mathbf{e}(v) dx = \int_\Omega f.v dx + \int_{\Gamma_N} g.v d\Gamma + \int_{\Gamma_C} F.v d\Gamma$$

for all $v \in \bar{V}_\varepsilon$, then the perturbed problem is nothing but a Galerkin approximation of this problem in the spaces

$$H_\varepsilon = \left\{ v \in \bar{H}_\varepsilon ; \frac{\partial v}{\partial x_n}|_{\Sigma_\varepsilon} = 0 \right\},$$
$$V_\varepsilon = \left\{ v \in \bar{V}_\varepsilon ; \frac{\partial v}{\partial x_n}|_{\Sigma_\varepsilon} = 0 \right\},$$

where $\frac{\partial v}{\partial x_n}|_{\Sigma_\varepsilon}$ is to be understood in the sense of distributions.

In these spaces, one has $u(x,x_n) = u(x,0)$ for $(x,x_n) \in \Sigma_\varepsilon$ and thus the regularity of u on Γ_C is

$$u(t) \in H^1(\Gamma_C, \mathbb{R}^n),$$

instead of the usual regularity $u \in H^{1/2}(\Gamma_C, \mathbb{R}^n)$ (see Adams [1], for instance). The velocity $\dot{u}(t)$ will be defined in H_ε, and thus the velocity on the contact boundary will have the regularity

$$\dot{u}(t) \in L^2(\Gamma_C, \mathbb{R}^n).$$

Moreover, it is elementary to see that the spaces H_ε and V_ε are closed subspaces of respectively $\bar{H}_\varepsilon$ and $\bar{V}_\varepsilon$, that V_ε is densely included in H_ε and that the following trace maps

$$\begin{array}{rcl} \gamma_\varepsilon^1 : V_\varepsilon & \longrightarrow & H^1(\Gamma_C;\mathbb{R}^n) \\ v & \longmapsto & v_{|\Gamma_C}, \end{array} \quad \text{and} \quad \begin{array}{rcl} \gamma_\varepsilon^2 : H_\varepsilon & \longrightarrow & L^2(\Gamma_C;\mathbb{R}^n) \\ v & \longmapsto & v_{|\Gamma_C}, \end{array} \tag{3.1}$$

are linear and continuous with a norm equal to $\frac{1}{\sqrt{\varepsilon}}$.

This gain in regularity is very important because it will allow us to write the contact and friction conditions in a 'strong' sense, i.e. in the sense of (2.1)–(2.3). Those conditions can be expressed also using

$$J_N(\xi) = \begin{cases} \{0\} & \text{if } \xi < 0, \\ [0,+\infty[, & \text{if } \xi = 0, \\ \emptyset, & \text{if } \xi > 0, \end{cases} \quad \text{and} \quad \mathrm{Dir}_T(v) = \begin{cases} \{\frac{v_T}{|v_T|}\}, \forall\, v \in \mathbb{R}^n, & \text{with } v_T \neq 0, \\ \{w \in \mathbb{R}^n ; |w| \leqslant 1, w_N = 0\}, & \text{if } v_T = 0. \end{cases}$$

The unilateral contact and friction conditions can be rewritten as

$$\begin{array}{l} F_N \in -J_N(u_N) \\ F_T \in F_N \mu(|\dot{u}_T|)\mathrm{Dir}_T(\dot{u}_T) \end{array} \quad \text{almost everywhere on } \Gamma_C.$$

The maps J_N and Dir_T are maximal monotone and represent the sub-gradients of respectively the indicator function of interval $]-\infty,0]$ and the function $v \longmapsto |v_T|$. See, for example, Moreau [18], Panagiotopoulos [21] and Klarbring *et al.* [13] for more details on the expression of contact and friction laws in term of generalised gradients.

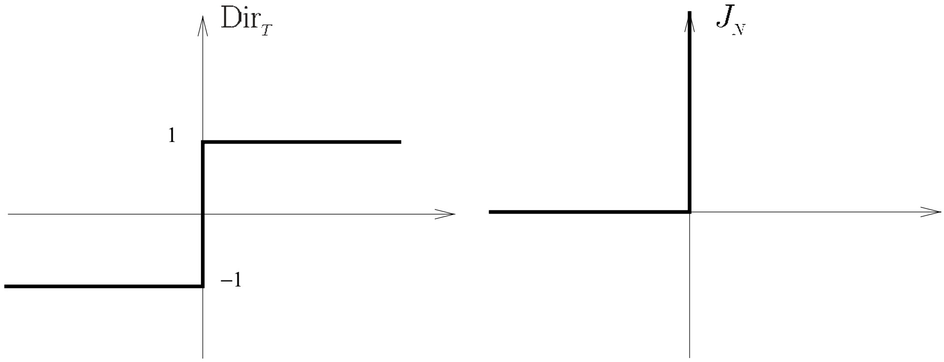

FIGURE 8. Multivalued maps J_N and Dir_T for a one-dimensional boundary.

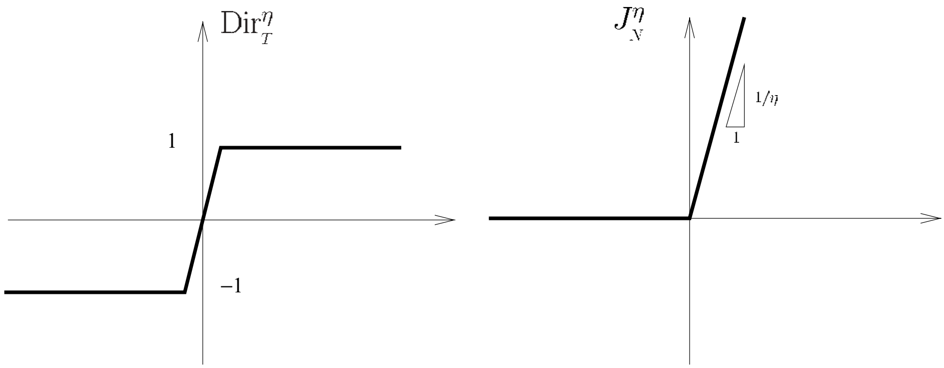

FIGURE 9. Regularisations of multivalued maps J_N and Dir_T.

3.3 An existence and uniqueness result

The existence and uniqueness result we present is established for regularised contact and friction conditions. We introduce the following regularisations of J_N and Dir_T which correspond to classical regularisations of unilateral contact and friction conditions:

$$J^\eta_N(\xi) = \begin{cases} 0 & \text{if} \quad \xi < 0, \\ \frac{\xi}{\eta} & \text{if} \quad \xi \geqslant 0. \end{cases} \quad \text{and} \quad \mathrm{Dir}^\eta_T(v) = \begin{cases} \frac{v_T}{|v_T|}, & \forall\, v \in \mathbb{R}^n, |v_T| \geqslant \eta, \\ \frac{v_T}{\eta}, & \forall\, v \in \mathbb{R}^n, |v_T| < \eta. \end{cases}$$

This corresponds to a Yosida regularisation of the two monotone maps J_N and Dir_T (see Brézis [3] and Figures 8 and 9). The functional J^η_N is also the normal compliance functional (see Oden & Martin [20]).

Writing, for brevity

$$\mu^\eta(v) = \mu(|v|)\mathrm{Dir}^\eta_T(v),$$

we introduce the regularised unilateral contact and friction conditions as:

$$F_N = -J^\eta_N(u_N),$$
$$F_T = -\hat{J}^\eta_N(u_N)\mu^\eta(\dot{u}_T),$$

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where $\hat{J}^\eta_{_N}$ is equal to $J^\eta_{_N}$ in the two-dimensional ($n = 2$) case, and is equal to

$$\hat{J}^\eta_{_N}(\xi) = \min\left(J^\eta_{_N}(\xi), \frac{1}{\eta}\right),$$

in the three-dimensional case. This avoids technical difficulties in the existence and uniqueness proof. As a remark, some experimental foundations for such a limitation of the dependence between the friction force and the contact force can be found in Strömberg *et al.* [29].

The whole problem can be written setting

$$l(u) = \int_\Omega f.v dx + \int_{\Gamma_N} g.v d\Gamma,$$

$$a_\varepsilon(u, v) = \int_{\Omega_\varepsilon} \mathscr{A}\mathbf{e}(u) : \mathbf{e}(v) dx,$$

which leads to

$$\begin{cases} \text{Find } u :]0, T] \longrightarrow V_\varepsilon \text{ such that} \\ \displaystyle\int_{\Omega_\varepsilon} \rho \ddot{u}.v dx + a_\varepsilon(u, v) = l(v) + \int_{\Gamma_C} F.v d\Gamma, \ \forall v \in V_\varepsilon, \\ F_{_N} = -J^\eta_{_N}(u_{_N}), \\ F_{_T} = -\hat{J}^\eta_{_N}(u_{_N})\mu^\eta(\dot{u}_T), \\ u(0) = \dot{u}(0) = 0, \end{cases} \tag{3.2}$$

assuming vanishing initial conditions.

For simplicity, we split the solution into two parts

$$u = \mathbf{u}^\varepsilon + w^\varepsilon,$$

with $\mathbf{u}^\varepsilon$ depending only on the data f and g, and w^ε depending only on the contact and friction forces. This means that $\mathbf{u}^\varepsilon$ is the solution to

$$\begin{cases} \text{Find } \mathbf{u}^\varepsilon :]0, T] \longrightarrow V_\varepsilon \text{ such that} \\ \displaystyle\int_{\Omega_\varepsilon} \rho \ddot{\mathbf{u}}^\varepsilon.v dx + a_\varepsilon(\mathbf{u}^\varepsilon, v) = l(v), \ \forall v \in V_\varepsilon, \\ \mathbf{u}^\varepsilon(0) = \dot{\mathbf{u}}^\varepsilon(0) = 0, \end{cases} \tag{3.3}$$

and w^ε the solution to

$$\begin{cases} \text{Find } w^\varepsilon :]0, T] \longrightarrow V_\varepsilon \text{ such that} \\ \displaystyle\int_{\Omega_\varepsilon} \rho \ddot{w}^\varepsilon.v dx + a_\varepsilon(w^\varepsilon, v) = \int_{\Gamma_C} F.v d\Gamma, \ \forall v \in V_\varepsilon \\ w^\varepsilon(0) = \dot{w}^\varepsilon(0) = 0. \end{cases} \tag{3.4}$$

Following the proof of Duvaut & Lions [7] adapted for the spaces H_ε and V_ε, Problem 3.3 has a unique solution $\mathbf{u}^\varepsilon$ which satisfies $\mathbf{u}^\varepsilon \in L^\infty(0,T;V_\varepsilon)$, $\dot{\mathbf{u}}^\varepsilon \in L^\infty(0,T;H_\varepsilon)$ and $\ddot{\mathbf{u}}^\varepsilon \in L^\infty(0,T;V'_\varepsilon)$.

We will denote by $w^\varepsilon = \mathscr{E}_\varepsilon(F)$ the application which maps F to the solution to Problem 3.4; then we have the following result.

Lemma 1 *Assume* $F \in L^2(0,T;L^2(\Gamma_C;\mathbb{R}^n))$, *then Problem* (3.4) *has a unique solution* $w^\varepsilon \in L^\infty(0,T;V_\varepsilon)$ *satisfying* $\dot{w}^\varepsilon \in L^\infty(0,T;H_\varepsilon)$ *and* $\ddot{w}^\varepsilon \in L^\infty(0,T;V'_\varepsilon)$. *Moreover the mapping* $\mathscr{E}_\varepsilon : F \mapsto w^\varepsilon$ *is linear continuous from* $L^2(0,T;L^2(\Gamma_C;\mathbb{R}^n))$ *in* $L^\infty(0,T;V_\varepsilon) \cap W^{1,\infty}(0,T;H_\varepsilon)$ *and the following estimate holds:*

$$\|w^\varepsilon(t)\|^2_{V_\varepsilon} + \|\dot{w}^\varepsilon(t)\|^2_{H_\varepsilon} \leqslant C\,\frac{e^{\alpha t}}{\varepsilon}\int_0^t \|F(s)\|^2_{L^2(\Gamma_C;\mathbb{R}^n)}\,ds, \quad \text{for almost all } t \in [0,T], \tag{3.5}$$

where $C > 0$ *and* $\alpha > 0$ *do not depend on* ε *and* T.

Compared to the result of Duvaut & Lions [7], this result is slightly stronger because no regularity for the derivatives of F is assumed. The price for this is the constant $\frac{1}{\varepsilon}$ in the estimation.

It is now possible to transform the initial problem into a fixed point problem. For fixed $\varepsilon > 0$ and $\eta > 0$, defining the mapping $\mathscr{H}_\eta$ as

$$\begin{aligned}\mathscr{H}_\eta : W^{1,\infty}(0,T;H_\varepsilon) \cap L^\infty(0,T;V_\varepsilon) &\longrightarrow L^\infty(0,T;L^2(\Gamma_C;\mathbb{R}^n))\\ w^\varepsilon &\longmapsto (F^{\varepsilon\eta}_T, F^{\varepsilon\eta}_N),\end{aligned}$$

with

$$\begin{aligned}F^{\varepsilon\eta}_N &= -J^\eta_N(w^\varepsilon_N + \mathbf{u}^\varepsilon_N),\\ F^{\varepsilon\eta}_T &= -\hat{J}^\eta_N(w^\varepsilon_N + \mathbf{u}^\varepsilon_N)\,\mu^\eta(\dot{w}^\varepsilon_T + \dot{\mathbf{u}}^\varepsilon_T),\end{aligned}$$

which is continuous following Krasnoselski [14], the perturbed friction problem 3.2 is equivalent to the fixed point problem for the mapping $\mathscr{G}_{\varepsilon\eta}$ defined by

$$\mathscr{G}_{\varepsilon\eta} = \mathscr{E}_\varepsilon \circ \mathscr{H}_\eta.$$

That is,

$$\begin{aligned}\text{find} \quad & w^{\varepsilon\eta} \in W^{1,\infty}(0,T;H_\varepsilon) \cap L^\infty(0,T;V_\varepsilon)\\ \text{such that} \quad & w^{\varepsilon\eta} = \mathscr{G}_{\varepsilon\eta}(w^{\varepsilon\eta}).\end{aligned}$$

We denote

$$E_\varepsilon = \{w \in W^{1,\infty}(0,T\,;\,H_\varepsilon) \cap L^\infty(0,T\,;\,V_\varepsilon)\,;\; w(0) = 0 \text{ in } H_\varepsilon\}$$

a Banach space for the norm:

$$\|w\|_{E_\varepsilon} = \operatorname{ess\,sup}_{0<t\leqslant T}\sqrt{\|w(t)\|^2_{V_\varepsilon} + \|\dot{w}(t)\|^2_{H_\varepsilon}}.$$

In this context, it is possible to establish the following existence and uniqueness result.

Theorem 1 *Whenever $v \longmapsto \mu(|v|)$ is bounded and Lispschitz continuous, for all $\varepsilon > 0$ and $\eta > 0$, the perturbed friction problem* (3.2) *has a unique solution in E_ε.*

Detailed proofs can be found in the appendices.

4 Conclusion and perspectives

The advantage of the proposed perturbation is that it ensures mathematical well-posedness in any case (in particular, for an arbitrary large coefficient of friction). A natural idea is to look at what happens when the regularisation parameters vanishes. In fact, it is possible to obtain an a priori estimate which is independent of these parameters (see Paumier & Renard [24]). But a certain number of questions are still open, in particular, the problem of giving a clear mathematical meaning to the friction force in the non-perturbed problem. Also, in the case of a sequence of solutions to the perturbed problem which tends to a solution to the non-perturbed problem, the possibility of characterising this particular solution with the same kind of idea as in the one-dimensional case (i.e. an analogue of the maximum delay criterion) is still open.

Appendix I: Proof of Lemma 1

The proof of this lemma follows Duvaut & Lions [7]. We consider a family $(V_{\varepsilon k})_{k\in\mathbb{N}}$ of finite dimensional subspaces of V_ε such that $\cup_{k\in\mathbb{N}} V_{\varepsilon k}$ is densely included in V_ε and a sequence of corresponding approximated problems:

$$\begin{cases} \text{Find } w^{\varepsilon k} :]0,T] \longrightarrow V_{\varepsilon k} \text{ such that} \\ (\ddot{w}^{\varepsilon k}(t), v) + a(w^{\varepsilon k}(t), v) = \int_{\Gamma_C} F(t).v d\sigma, \quad \forall v \in V_{\varepsilon k}, \\ w^{\varepsilon k}(0) = \dot{w}^{\varepsilon k}(0) = 0. \end{cases} \tag{4.1}$$

Obviously, for each $k \in \mathbb{N}$, solution to this problem exists and is unique in $H^2(0,T\,;V_{\varepsilon k})$. Choosing $v = \dot{w}^{\varepsilon k}(t)$ in the second equation of (4.1) and integrating in time, one has:

$$(\dot{w}^{\varepsilon k}(t), \dot{w}^{\varepsilon k}(t)) + a(w^{\varepsilon k}(t), w^{\varepsilon k}(t)) = 2\int_0^t \int_{\Gamma_C} F(s)\,.\dot{w}^{\varepsilon k}(s)\, d\sigma \; ds.$$

The left-hand side is classically minorised by

$$\beta\left(\|\dot{w}^{\varepsilon k}(t)\|^2_{H_\varepsilon} + \|w^{\varepsilon k}(t)\|^2_{V_\varepsilon} \right),$$

where $\beta > 0$ and the right-hand side is majorised by

$$\frac{2}{\varepsilon}\int_0^t \|F(s)\|^2_{L^2(\Gamma_C;\mathbb{R}^n)}\; ds + 2\varepsilon \int_0^t \|\dot{w}^{\varepsilon k}(s)\|^2_{L^2(\Gamma_C;\mathbb{R}^n)}\; ds.$$

Using the continuity of traces as in (3.1) one gets:

$$\|w^{\varepsilon k}(t)\|^2_{V_\varepsilon} + \|\dot{w}^{\varepsilon k}(t)\|^2_{H_\varepsilon} \leqslant \frac{2}{\varepsilon\beta}\int_0^t \|F(s)\|^2_{L^2(\Gamma_C;\mathbb{R}^n)} ds + \frac{2}{\beta}\int_0^t \|\dot{w}^{\varepsilon k}(s)\|^2_{H_\varepsilon}\; ds.$$

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With the Gronwall lemma, and appropriate $C > 0$ and $\alpha > 0$ one gets:

$$\|w^{\varepsilon k}(t)\|^2_{V_\varepsilon} + \|\dot{w}^{\varepsilon k}(t)\|^2_{H_\varepsilon} \leqslant C\ \frac{e^{\alpha t}}{\varepsilon} \int_0^t \|F(s)\|^2_{L^2(\Gamma_C;\mathbb{R}^n)}\, ds. \tag{4.2}$$

The proof in Duvat & Lions [7] can be continued in the same way. It shows that, for the weak-star topology, the existence of a weak limit w^ε for the sequence $(w^{\varepsilon k})_{k\in\mathbb{N}}$ which is the unique solution in the space $L^\infty(0, T; V_\varepsilon)$ of the problem. This solution satisfies $\dot{w}^\varepsilon \in L^\infty(0, T; H_\varepsilon)$ and $\ddot{w}^\varepsilon \in L^\infty(0, T; V_\varepsilon)$.

Finally, estimate (3.5) is obtained by passing to the limit inf. in (4.2) and thanks to the weak-star convergence in $L^\infty(0, T; V_\varepsilon) \cap W^{1,\infty}(0, T; H_\varepsilon)$.

Appendix 2: Proof of Theorem 1

The principle of the proof of existence and uniqueness of this fixed point comes from the classical existence and uniqueness result for initial value problem of differential equations (Cauchy-Lipschitz theorem). We show, for an integer p large enough, that the composition p–iterated operator: $\mathscr{G}^p_{\varepsilon\eta} = \mathscr{G}_{\varepsilon\eta} \circ \cdots \circ \mathscr{G}_{\varepsilon\eta}$, is a contraction in a certain closed ball $B_{\varepsilon\eta} \subset E_\varepsilon$. This operator $\mathscr{G}^p_{\varepsilon\eta}$ has a unique fixed point $w^{\varepsilon\eta}_p$ belonging to the ball $B_{\varepsilon\eta}$. Then, if $\mathscr{G}_{\varepsilon\eta}(w^{\varepsilon\eta}_p) \in B_{\varepsilon\eta}$ we conclude that $w^{\varepsilon\eta}_p$ is also a fixed point of $\mathscr{G}_{\varepsilon\eta}$ because we have $\mathscr{G}^p_{\varepsilon\eta}(\mathscr{G}_{\varepsilon\eta}(w^{\varepsilon\eta}_p)) = \mathscr{G}_{\varepsilon\eta}(\mathscr{G}^p_{\varepsilon\eta}(w^{\varepsilon\eta}_p)) = \mathscr{G}_{\varepsilon\eta}(w^{\varepsilon\eta}_p)$ and consequently, by uniqueness in the ball $B_{\varepsilon\eta}$, we get $\mathscr{G}_{\varepsilon\eta}(w^{\varepsilon\eta}_p) = w^{\varepsilon\eta}_p$. Uniqueness of this fixed point comes from the fact that all fixed point of the mapping $\mathscr{G}_{\varepsilon\eta}$ belongs to the ball $B_{\varepsilon\eta}$ and thus is the unique fixed point of $\mathscr{G}^p_{\varepsilon\eta}$.

The essential steps of the proof consist in the two following lemmas in which we define the closed ball ($R_{\varepsilon\eta}$ is fixed in the first lemma):

$$B_{\varepsilon\eta} = \{w \in E_\varepsilon\ ;\ \|w\|_{E_\varepsilon} \leqslant R\}.$$

Lemma 2 *Whenever $v \longmapsto \mu(|v|)$ is bounded and Lispschitz continuous, there exists $p_{\varepsilon\eta} \in \mathbb{N}$ an iterated composition index and a radius $R_{\varepsilon\eta} > 0$ for the ball $B_{\varepsilon\eta}$ such that:*

- *$\mathscr{G}^p_{\varepsilon\eta}(B_{\varepsilon\eta}) \subset B_{\varepsilon\eta}$, for all $p \geqslant p_{\varepsilon\eta}$;*
- *all fixed point $w^{\varepsilon\eta} \in E_\varepsilon$ of the operator $\mathscr{G}_{\varepsilon\eta}$ belongs to $B_{\varepsilon\eta}$ and satisfies the* a priori *estimate for almost all $t \in [0, T]$:*

$$\|w^{\varepsilon\eta}(t)\|^2_{V_\varepsilon} + \|\dot{w}^{\varepsilon\eta}(t)\|^2_{H_\varepsilon} \leqslant\ \frac{C}{\varepsilon\eta^2}\ e^{T\left(\alpha + \frac{C}{\varepsilon^2\eta^2} e^{\alpha T}\right)} \int_0^t \|(\mathbf{u}^\varepsilon_N(s))_+\|^2_{L^2(\Gamma_C;\mathbb{R}^n)}\, ds,$$

where $C > 0$ and $\alpha > 0$ are constants independent on $(\varepsilon, t, T, \mathbf{u}^\varepsilon)$, and $(.)_+$ is the positive part.

Proof Let C be a generic constant which does not depend upon $w^{\varepsilon\eta}$, ε and T, and $F^{\varepsilon\eta} = \mathscr{H}_\eta(w^{\varepsilon\eta})$. Using Lemma 1, one has for almost all $t \in [0, T]$:

$$\|w^{\varepsilon\eta}(t)\|^2_{V_\varepsilon} + \|\dot{w}^{\varepsilon\eta}(t)\|^2_{H_\varepsilon} \leqslant C\ \frac{e^{\alpha t}}{\varepsilon} \int_0^t \|F^{\varepsilon\eta}(s)\|^2_{L^2(\Gamma_C;\mathbb{R}^n)}\, ds. \tag{4.3}$$

Moreover, taking into account that μ is a bounded function, a simple computation show (in the Euclidean norm of $\mathbb{R}^n$) the inequality $|F^{\varepsilon\eta}(s,x)|^2 \leqslant C\,\eta^{-2}\,|(w_N(s,x) + \mathbf{u}^\varepsilon_N(s,x))_+|^2$.

Thus, integrating on $]0, t[\times \Gamma_C$, one gets:

$$\int_0^t \|F^{\varepsilon\eta}(s)\|^2_{L^2(\Gamma_C;\mathbb{R}^n)} ds \leqslant \frac{C}{\eta^2} \int_0^t \|(w_N(s) + \mathbf{u}^\varepsilon_N(s))_+\|^2_{L^2(\Gamma_C;\mathbb{R}^n)} ds.$$

And using the continuity of traces (3.1)

$$\int_0^t \|F^{\varepsilon\eta}(s)\|^2_{L^2(\Gamma_C;\mathbb{R}^n)} ds \leqslant \frac{C}{\eta^2} \int_0^t \left(\frac{1}{\varepsilon}\|w^{\varepsilon\eta}(s)\|^2_{V_\varepsilon} + \|(\mathbf{u}^\varepsilon_N(s))_+\|^2_{L^2(\Gamma_C;\mathbb{R}^n)} \right) ds.$$

Integrating this on (4.3), by setting $\varphi(t) = \|w^{\varepsilon\eta}(t)\|^2_{V_\varepsilon} + \|\dot{w}^{\varepsilon\eta}(t)\|^2_{H_\varepsilon}$ one has:

$$\varphi(t) \leqslant C\ \frac{e^{\alpha T}}{\varepsilon^2\eta^2} \int_0^t \varphi(s)ds + C\ \frac{e^{\alpha T}}{\varepsilon\eta^2} \int_0^t \|(\mathbf{u}^\varepsilon_N(s))_+\|^2_{L^2(\Gamma_C;\mathbb{R}^n)} ds.$$

Using a Gronwall lemma, the announced *a priori* estimate is obtained.

For the first part of the lemma, setting $v = \mathscr{E}_\varepsilon(F^{\varepsilon\eta})$, we obtain as previously:

$$\|v(t)\|^2_{V_\varepsilon} + \|\dot{v}(t)\|^2_{H_\varepsilon} \leqslant C\ \frac{e^{\alpha T}}{\varepsilon^2\eta^2}\left(\int_0^t \left\{ \|w^{\varepsilon\eta}(s)\|^2_{V_\varepsilon} + \|\dot{w}^{\varepsilon\eta}(s)\|^2_{H_\varepsilon} \right\} ds + \varepsilon a_T \right), \tag{4.4}$$

where $a_T = \displaystyle\int_0^T \|(u_N(s))_+\|^2_{L^2(\Gamma_C;\mathbb{R}^n)} ds.$

With $v_0 = \sup\limits_{0<s\leqslant T} (\|w^{\varepsilon\eta}(s)\|^2_{V_\varepsilon} + \|\dot{w}^{\varepsilon\eta}(s)\|^2_{H_\varepsilon})$ one has:

$$\|v(t)\|^2_{V_\varepsilon} + \|\dot{v}(t)\|^2_{H_\varepsilon} \leqslant C\ \frac{e^{\alpha T}}{\varepsilon^2\eta^2}\ (tv_0 + \varepsilon a_T).$$

Setting $v_p = \mathscr{G}_{\varepsilon\eta}{}^p(w^{\varepsilon\eta})$ and $v_p = \sup\limits_{0<s\leqslant T} (\|v_p(s)\|^2_{V_\varepsilon} + \|\dot{v}_p(s)\|^2_{H_\varepsilon})$, and using the fact that $\|v_1(t)\|^2_{V_\varepsilon} + \|\dot{v}_1(t)\|^2_{H_\varepsilon} \leqslant C\ e^{\alpha T}\varepsilon^{-2}\eta^{-2}\ (tv_0 + \varepsilon a_T)$ and (4.4) with $w^{\varepsilon\eta} = v_1$ and $v = v_2$ one has:

$$\|v_2(t)\|^2_{V_\varepsilon} + \|\dot{v}_2(t)\|^2_{H_\varepsilon} \leqslant \frac{(Ct\varepsilon^{-2}\eta^{-2}e^{\alpha T})^2}{2\,!}\ v_0 + a_T\ C\ \frac{e^{\alpha T}}{\varepsilon\eta^2}\left(Ct\varepsilon^{-2}\eta^{-2}e^{\alpha T} + 1\right).$$

Thus $v_2 \leqslant \frac{(CT\varepsilon^{-2}\eta^{-2}e^{\alpha T})^2}{2\,!}\ v_0 + a_T\ C\ \frac{e^{\alpha T}}{\varepsilon\eta^2}\left(CT\varepsilon^{-2}\eta^{-2}e^{\alpha T} + 1\right)$. Now, by recursion on p one has:

$$v_p \leqslant \frac{(CT\varepsilon^{-2}\eta^{-2}e^{\alpha T})^p}{p\,!}\ v_0 + a_T\ C\ \frac{e^{\alpha T}}{\varepsilon\eta^2} \sum_{j=0}^{p-1} \frac{(CT\varepsilon^{-2}\eta^{-2}e^{\alpha T})^j}{j\,!},$$

$$\text{and then} \quad v_p \leqslant X_p v_0 + a_T\ \frac{C}{\varepsilon\eta^2}\ e^{T(\alpha + Ce^{\alpha T}\varepsilon^{-2}\eta^{-2})},$$

where $X_p = \frac{(CT\varepsilon^{-2}\eta^{-2}e^{\alpha T})^p}{p\,!}$. Because the sequence (X_p) converges toward zero, it is possible to find $p_{\varepsilon\eta} \in \mathbb{N}$ such that for all $p \geqslant p_{\varepsilon\eta}$ one has $X_p < \frac{1}{2}$. Then defining $R_{\varepsilon\eta}$ as

$$R_{\varepsilon\eta} = \sqrt{2\ a_T\ \frac{C}{\varepsilon\eta^2}\ e^{T(\alpha + Ce^{\alpha T}/\varepsilon^{-2}\eta^{-2})}},$$

for $p \geqslant p_{\varepsilon\eta}$ one has $v_p \leqslant \frac{v_0}{2} + \frac{R^2_{\varepsilon\eta}}{2}$. Thus if $v_0 \leqslant R^2_{\varepsilon\eta}$ then $v_p \leqslant R^2_{\varepsilon\eta}$ for all $p \geqslant p_{\varepsilon\eta}$. □

Lemma 3 *Under the same assumptions as Lemma 2, there exists an integer* $p \geqslant p_{\varepsilon\eta}$ *and a constant* $\Lambda \in]0,1[$ *such that for all* $w_1, w_2 \in B_{\varepsilon\eta}$

$$\|\mathscr{G}_{\varepsilon\eta}^p(w_1) - \mathscr{G}_{\varepsilon\eta}^p(w_2)\|_{E_\varepsilon} \leqslant \Lambda \, \|w_1 - w_2\|_{E_\varepsilon}.$$

Proof Let w_1, w_2 be in $B_{\varepsilon\eta}$. Thus, for almost all $t \in]0,T]$, one has $w_1(t)_{|\Gamma_C}, w_2(t)_{|\Gamma_C} \in H^1(\omega)$.

In the case $n = 2$ the inclusion $H^1(\omega) \subset L^\infty(\omega)$ holds. Thus $w_1(t)_{|\Gamma_C}, w_2(t)_{|\Gamma_C}$ are bounded. In the case $n = 3$, this result does not hold, but the function $\hat{J}_N^\eta$ is itself bounded by $1/\eta$.

Setting $F_i = \mathscr{H}_\eta(w_i)$ and $v_i = \mathscr{E}_\varepsilon(F_i) = \mathscr{G}_{\varepsilon\eta}(w_i)$ for $i = 1, 2$, and using Lemma 1 one has:

$$\|v_1(t) - v_2(t)\|_{V_\varepsilon}^2 + \|\dot{v}_1(t) - \dot{v}_2(t)\|_{H_\varepsilon}^2 \leqslant C \, \frac{e^{\alpha t}}{\varepsilon} \int_0^t \|F_1(s) - F_2(s)\|_{L^2(\Gamma_C;\mathbb{R}^n)}^2 \, ds. \tag{4.5}$$

In the following, K denotes a generic constant independent of t, w_1 and w_2 but dependent on ε, T et $\mathbf{u}_N^\varepsilon$. Taking into account that μ is a Lipschitz function, a simple computation shows that:

$$|F_1(s,x) - F_2(s,x)|^2 \leqslant K \, (|w_1(s,x) - w_2(s,x)|^2 + |\dot{w}_1(s,x) - \dot{w}_2(s,x)|^2),$$

where K depends upon ε, η and T because it depends upon $R_{\varepsilon\eta}$. Integrating on $]0,t[\times\Gamma_C$ one has:

$$\int_0^t \|F_1(s) - F_2(s)\|_{L^2(\Gamma_C;\mathbb{R}^n)}^2 ds$$
$$\leqslant K \int_0^t \left(\|w_1(s) - w_2(s)\|_{L^2(\Gamma_C;\mathbb{R}^n)}^2 + \|\dot{w}_1(s) - \dot{w}_2(s)\|_{L^2(\Gamma_C;\mathbb{R}^n)}^2 \right) ds.$$

Using the continuity of traces (3.1),

$$\int_0^t \|F_1(s) - F_2(s)\|_{L^2(\Gamma_C;\mathbb{R}^n)}^2 ds \leqslant K \int_0^t \left(\|w_1(s) - w_2(s)\|_{V_\varepsilon}^2 + \|\dot{w}_1(s) - \dot{w}_2(s)\|_{H_\varepsilon}^2 \right) ds.$$

Putting this expression into (4.5) one obtains:

$$\|v_1(t) - v_2(t)\|_{V_\varepsilon}^2 + \|\dot{v}_1(t) - \dot{v}_2(t)\|_{H_\varepsilon}^2 \leqslant K \int_0^t \left(\|w_1(s) - w_2(s)\|_{V_\varepsilon}^2 + \|\dot{w}_1(s) - \dot{w}_2(s)\|_{H_\varepsilon}^2 \right) ds. \tag{4.6}$$

Thus

$$\|v_1(t) - v_2(t)\|_{V_\varepsilon}^2 + \|\dot{v}_1(t) - \dot{v}_2(t)\|_{H_\varepsilon}^2 \leqslant K \; t \, \|w_1 - w_2\|_{E_\varepsilon}^2. \tag{4.7}$$

Now $v_i^p = \mathscr{G}_\varepsilon^p(w_i)$ for $i = 1, 2$ and $p \geqslant 2$. Using (4.7) and (4.6) one obtains:

$$\|v_1^2(t) - v_2^2(t)\|_{V_\varepsilon}^2 + \|\dot{v}_1^2(t) - \dot{v}_2^2(t)\|_{H_\varepsilon}^2 \leqslant \frac{K^2 t^2}{2} \, \|w_1 - w_2\|_{E_\varepsilon}^2.$$

Then, with the same recursion on p, we obtain for all $p \in \mathbb{N}$:

$$\|v_1^p(t) - v_2^p(t)\|_{V_\varepsilon}^2 + \|\dot{v}_1^p(t) - \dot{v}_2^p(t)\|_{H_\varepsilon}^2 \leqslant \frac{K^p t^p}{p\,!} \, \|w_1 - w_2\|_{E_\varepsilon}^2.$$

and thus, with $Z_p = \dfrac{(KT)^p}{p\,!}$ one has $\|\mathscr{G}^p_{\varepsilon\eta}(w_1) - \mathscr{G}^p_{\varepsilon\eta}(w_2)\|_{E_\varepsilon} \leqslant \sqrt{Z_p}\,\|w_1 - w_2\|_{E_\varepsilon}$.

Because the sequence (Z_p) converges toward zero, the result of the lemma is obtained with $\Lambda = \sqrt{Z_p}$ for $p \geqslant p_{\varepsilon\eta}$, a sufficiently large integer such that $Z_p < 1$. □

Finally, from the two previous lemmas, Theorem 1 holds.

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