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AN ANALYTICAL APPROXIMATION FORMULA FOR THE PRICING OF CREDIT DEFAULT SWAPS WITH REGIME SWITCHING

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Abstract

We derive an analytical approximation for the price of a credit default swap (CDS) contract under a regime-switching Black–Scholes model. To achieve this, we first derive a general formula for the CDS price, and establish the relationship between the unknown no-default probability and the price of a down-and-out binary option written on the same reference asset. Then we present a two-step procedure: the first step assumes that all the future information of the Markov chain is known at the current time and presents an approximation for the conditional price under a time-dependent Black–Scholes model, based on which the second step derives the target option pricing formula written in a Fourier cosine series. The efficiency and accuracy of the newly derived formula are demonstrated through numerical experiments.

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1. Introduction

Credit default swaps (CDSs) are becoming increasingly popular ever since they were first used in the 1990s, and they are one of the most widely traded credit derivatives for hedging credit risk. A CDS is a financial contract that essentially transfers the credit risk of a certain reference asset belonging to a third party from buyer to seller. When a CDS contract is entered, the buyer needs to make periodic payments to the seller until the expiry of the contract or a credit event occurs, while the seller has to compensate

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the loss of the buyer when the reference asset defaults, with the amount of payment being equal to the CDS notional face value less the recovery value.

Because of the high liquidity of CDS contracts in real markets, the accurate pricing of these contracts is very demanding. In the determination of CDS prices, the first step is usually to figure out how to appropriately model the default, and the corresponding approaches in the literature can mainly be classified into two categories: reduced-form models and structure models. The former category was established by a number of authors [1, 2, 16] and then extensively studied by many other authors, including Lando [17] and Madan and Unal [22], since this approach makes it possible to directly extract the default probability from market prices. However, this approach fails to capture the wide range of default correlations, making the latter category much more favoured, as structure models, which determine the default time with the evolution of the reference asset, are able to capture such correlation. The first structure model was proposed by Merton [23], who assumed that the default can only be triggered at expiry. However, this is obviously not realistic, and has prompted the development of one currently widely adopted model, the so-called first-passage model [5, 6], in which the default takes place as soon as the reference asset price touches or drops below the default barrier for the first time.

Another important factor that affects the accuracy in the valuation of CDSs is the model used to describe the reference asset price. At a very early stage, the log-price of the reference asset was assumed to follow a normal distribution, the same as the assumption made in the Black–Scholes (BS) model. However, this particular assumption was rejected by the empirical results, showing that the default probability obtained in this setting was significantly less than the empirical default rate [25], and it was possibly caused by the fact that the BS model is too simple to capture the main characteristics exhibited by the reference asset prices in real markets [24]. To overcome the drawbacks of the BS model, more sophisticated models have been proposed and introduced for evaluating CDS contracts. For example, the geometric Brownian motion in the BS model was completely replaced by a Poisson process in [7], while Longstaff and Schwartz [21] incorporated the stochastic interest rate, which Zhou [26] further extended by adding a jump component. Moreover, a generalized mixed fractional Brownian motion was adopted by He and Chen [13] to capture the long-range dependence observed in financial markets.

Recently, Markov-modulated models are receiving much attention from researchers as well as market practitioners [15, 18, 19], as a lot of empirical evidence has demonstrated the existence of regime switching in real markets [10, 12]. In particular, He and Chen [14] developed a semi-Monte Carlo approach for the pricing of CDSs under the regime-switching BS model. This approach, though mathematically appealing, is somewhat time-consuming as it involves simulation of the Markov chain, which requires a large number of sample paths to be generated, posing an obstacle for practical applications. To increase the computational efficiency when pricing CDSs under the regime-switching BS model, this paper presents an analytical pricing formula written in a Fourier cosine series that does not require any simulation.

The rest of the paper is organized as follows. In Section 2 we derive a general pricing formula for CDSs, containing the no-default probability as the only unknown term, which can be expressed with the price of the down-and-out binary option. In Section 3, after the brief introduction of the regime-switching BS model, the target option price is written in a Fourier cosine series, with which the CDS pricing formula is completely analytical. In Section 4 numerical examples and discussions are presented. Section 5 concludes.

2. A general pricing formula for CDSs

One of the most important factors affecting the accuracy in the determination of the CDS price is the assumption on the default model. As illustrated before, Merton's assumption [23] that the default would only occur at expiry is certainly not appropriate, and a more realistic assumption is adopted hereafter that the default can take place at any time during the lifespan of the CDS contract. With the default mechanics being determined, we can now proceed to the pricing of CDSs. Unlike most financial derivatives, whose price is defined as the value of the corresponding contract, the price of a CDS refers to the spread, that is, the regular fee that the buyer pays to the seller, and is quoted as the ratio of the face value of the reference asset. This implies that we need to analyse the cash flows between buyer and seller.

On the one hand, the CDS buyer needs to regularly pay a protection fee to the seller before the default occurs or expiry. Specifically, if we denote by c and M the target CDS spread per unit time that needs to be determined and the face value of the reference asset respectively, the buyer needs to pay $cMdt$ to the seller if there is no default before time t. According to the risk-neutral pricing rule, the current value of the cash flow made by the buyer (denoted by V_1) is the expectation of all the discounted future payments, which can be expressed as

$$V_1 = \sum_t [e^{-rt} cMp(t)\, dt] = cM \int_0^T e^{-rt} p(t)\, dt,$$

where the current time is 0, T is the expiry, r is the risk-free interest rate, and $p(t)$ represents the probability of no default before time t. On the other hand, if a default does occur, the seller has to pay $(1 - R)M$, with R being the recovery rate specified in the contract, to compensate the loss of the buyer, implying that the current value of the cash flow made by the seller (denoted by V_2) depends on the probability that the default takes place. In particular, the probability of the default taking place between time t and $t + dt$ can be directly computed as

$$[1 - p(t + dt)] - [1 - p(t)] = p(t) - p(t + dt) = -dp(t),$$

with which the risk-neutral pricing rule yields

$$V_2 = \sum_t \{e^{-rt}(1 - R)M[-dp(t)]\} = (1 - R)M \int_0^T -e^{-rt} dp(t).$$

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Thus, to be fair to both parties, the current value of the cash flow from the buyer should be equal to that from the seller, leading to

$$cM \int_0^T e^{-rt} p(t)\, dt = (1 - R)M \int_0^T -e^{-rt} dp(t),$$

which can be simplified as

$$\begin{aligned} c &= \frac{(1 - R) \int_0^T -e^{-rt} dp(t)}{\int_0^T e^{-rt} p(t)\, dt} \\ &= \frac{(1 - R)[-e^{-rt} p(t)\Big|_0^T - r \int_0^T e^{-rt} p(t)\, dt]}{\int_0^T e^{-rt} p(t)\, dt} \\ &= \frac{(1 - R)[1 - e^{-rT} p(T)]}{\int_0^T e^{-rt} p(t)\, dt} - r(1 - R). \end{aligned} \tag{2.1}$$

The general pricing formula presented in (2.1) only contains one unknown term, the no-default probability $p(t)$, solving which would yield a completely analytical solution. If S_t denotes the reference asset price at time t, and D represents the default level, $p(t)$ is clearly the probability of the minimal value of the reference asset within the time period $[0, t]$ being higher than D, or, more specifically,

$$p(t) = P(\min_{0 \le u \le t} S_u > D), \tag{2.2}$$

with $p(0) = 0$ as the CDS contract would never be entered if the default occurs at the current time. If we rewrite (2.2) as

$$p(t) = E[I_{\{\min\limits_{0 \le u \le t} S_u > D\}}],$$

with I being the indicator function, it is not difficult to find that $e^{-rt} p(t)$ is in fact the price of a down-and-out binary option having time to expiry t, which can be formulated as

$$e^{-rt} p(t) \equiv P(S, t) = e^{-rt} E[\Pi(S, t)],$$

where the payoff function is

$$\Pi(S, t) = \begin{cases} 1 & \min\limits_{0 \le u \le t} S_u > D, \\ 0 & \text{otherwise.} \end{cases}$$

In this case, the pricing formula (2.1) can be rearranged as

$$c = \frac{(1 - R)[1 - P(S, T)]}{\int_0^T P(S, t)\, dt} - r(1 - R). \tag{2.3}$$

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It is now clear that once the price of the down-and-out binary option written on the reference asset, $P(S, t)$, has been derived, we are able to determine the CDS price through (2.3). Thus, a necessary step is to first specify the dynamic of the reference asset, and then we are able to consider this particular option pricing problem. These details will be presented in the next section.

3. Analytical approximation of a down-and-out binary option

In this section the reference asset is assumed to follow a regime-switching model, and the down-and-out binary option price is then determined with a two-step process; the first step deals with the option pricing problem conditional upon all the future information of the Markov chain, based on which the target option price is written in a Fourier cosine series in the second step.

3.1. The regime-switching model As mentioned earlier, there is a lot of empirical evidence suggesting the existence of regime switching in real markets. Therefore, we would like to incorporate the effect of regime switching in the pricing of CDS contracts. It should be pointed out that the extra uncertainty introduced by regime switching makes our market incomplete. This issue has been discussed in [11], where a risk premium is introduced related to the cost of switching. We shall not repeat this argument, but directly assume that we are already working under a risk-neutral measure $\mathbb{Q}$ with zero risk premium associated with regime switching, like many other authors in the literature [4, 27], as our focus here is on how we can efficiently determine the price of a CDS contract under the regime-switching model without losing too much accuracy.

In particular, the reference asset of the CDS contract, denoted by S, is assumed to follow a regime-switching model under a risk-neutral measure $\mathbb{Q}$ as

$$\frac{dS_u}{S_u} = r\,dt + \sigma_{X_u} dW_u, \tag{3.1}$$

where r is the risk-free interest rate, W_u is a standard Brownian motion and X_u is a two-state Markov chain, independent of the Brownian motion with its state space being $\{e_1, e_2\}$, where the jth component of the column vector e_i is the Kronecker delta δ_{ij} for all $i, j = 1, 2$. It should be pointed out here that the Markov chain is assumed to contain two states for illustration purposes, but the extension to arbitrary but finite states is very straightforward. It should also be remarked that the dynamics of the Markov chain is actually generated by a transition rate matrix $A = (a_{ij})_{i,j=1,2}$ [8], where a_{ij} is the transition rate of X_t from state i to j satisfying $a_{ij} \geq 0, i \neq j$ and $\sum_{j=1}^{2} a_{ij} = 0$. In this case, if we assume that $a_{12} = \lambda_{12}$ and $a_{21} = \lambda_{21}$, then the transition rate matrix A here is defined as

$$A = \begin{pmatrix} -\lambda_{12} & \lambda_{12} \\ \lambda_{21} & -\lambda_{21} \end{pmatrix}.$$

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In this case, σ_{X_u} would have two different values, denoted by σ_1 and σ_2, according to the current state of the reference asset, and it can be represented as

$$\sigma_{X_u} = \langle \bar{\sigma}, X_u \rangle,$$

if $\langle \cdot, \cdot \rangle$ is the inner product of two vectors, and $\bar{\sigma} = (\sigma_1, \sigma_2)^T$.

3.2. The Fourier cosine series solution This subsection is devoted to deriving the analytical approximation of the down-and-out binary option under the regime-switching model $P(S, t, X_0)$, which is dependent on X_0 as we are working under the regime-switching model, where the Markov chain is also a random process. The main result is provided in the following theorem, followed by its detailed proof.

THEOREM 3.1. *If the underlying price S follows the dynamic specified in (3.1), the price of a down-and-out binary option $P(S, t, X_0)$ can be approximated by a Fourier cosine series as*

$$P(S, t, X_0) = \frac{1}{2} A_0(X_0) V_0 + \sum_{k=1}^{+\infty} A_k(X_0) V_k, \tag{3.2}$$

where

$$\begin{aligned}
A_k(X_0) &= \frac{2}{b-a} \text{Real}\Big[g\Big(\frac{k\pi}{b-a} \Big) e^{-jka\pi/(b-a)} \Big], \quad k = 0, 1, 2, \ldots, \\
V_k &= \int_a^b W(h) \cos\Big(k\pi \frac{h-a}{b-a} \Big) dh, \quad k = 0, 1, 2, \ldots, \\
g(\phi) &= \langle e^{A^T \cdot t + B} X_0, I \rangle, \quad I = (1, 1)^T, \\
W(h) &= e^{-rt} N\Big[\frac{\ln(S/D) + rt - h/2}{\sqrt{h}} \Big] - e^{-rt + (h - 2rt)/h \ln(S/D)} N\Big[-\frac{\ln(S/D) - rt + h/2}{\sqrt{h}} \Big], \\
A &= \begin{pmatrix} -\lambda_{12} & \lambda_{12} \\ \lambda_{21} & -\lambda_{21} \end{pmatrix}, \quad B = \begin{pmatrix} j\phi\sigma_1^2 t & 0 \\ 0 & j\phi\sigma_2^2 t \end{pmatrix}.
\end{aligned}$$

Here, $a = \min(\sigma_1^2, \sigma_2^2)t$, $b = \max(\sigma_1^2, \sigma_2^2)t$, the current time is 0, t is the expiry, A^T denotes the transpose of the matrix A, D is the knock-out barrier, and $N(\cdot)$ is the standard normal distribution function.

PROOF. The existence of regime switching in the dynamics of the reference asset poses an obstacle in analytically evaluating the target option. To overcome this difficulty, a two-step solution procedure is applied. As a prior step, we firstly assume that all the future information of the Markov chain up to the expiry time t is known at the current time so that the regime-switching volatility becomes only a time-dependent parameter. It should be pointed out that due to this particular assumption made in the first step here, the target option price should follow a time-dependent Black–Scholes model without the effect of the Markov chain since all its future information is given. Its impact will be recovered in the second step when this assumption is relaxed. In this case, if the current time u is assumed to be within $[0, t]$, the price of the down-and-out

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binary option conditional upon all the information of the Markov chain, $U(y,\tau|X_t)$ with $y = \ln(S/D)$ and $\tau = t - u$, should satisfy the following partial differential equation (PDE) system:

$$\begin{cases} \dfrac{\partial U}{\partial \tau} = \dfrac{1}{2}\sigma_\tau^2 \dfrac{\partial^2 U}{\partial y^2} + \Big(r - \dfrac{1}{2}\sigma_\tau^2\Big)\dfrac{\partial U}{\partial y} - rU, \quad y > 0, \\ U(y,0|X_t) = I_{\{y>0\}}, \\ U(0,\tau|X_t) = 0. \end{cases} \tag{3.3}$$

Here, $\sigma_\tau = \langle \bar{\sigma}, X_\tau \rangle$ is a known value for all $\tau \in [0,t]$. Following Lo et al. [20] and He and Chen [14], we make the transformation $z = y - y^*(\tau)$, where

$$y^*(\tau) = -c_1(\tau) - \beta c_2(\tau)$$

with

$$c_1(\tau) = r\tau - \frac{1}{2}\int_0^\tau \sigma_s^2\,ds, \quad c_2(\tau) = \frac{1}{2}\int_0^\tau \sigma_s^2\,ds,$$

and β being an adjusted parameter to be determined later. If we further denote $\tilde{U}(z,\tau|X_t) = e^{r\tau}U(y,\tau|X_t)$ and $\bar{U}(z,\xi|X_t) = e^{-\beta z/2+\beta^2\xi/4}\tilde{U}(z,\tau|X_t)$ with $\xi = c_2(\tau)$, the PDE system in (3.3) can be transformed into

$$\begin{cases} \dfrac{\partial \bar{U}}{\partial \xi} = \dfrac{\partial^2 \bar{U}}{\partial z^2}, \quad z > -y^*(\tau), \\ \bar{U}(z,0|X_t) = U(z,0|X_t)e^{-\beta z/2}, \\ \bar{U}(-y^*(\tau),\xi|X_t) = 0. \end{cases} \tag{3.4}$$

Note that solving system (3.4) is still not straightforward, though its PDE is a standard heat equation, since the fixed absorbing boundary is now a moving absorbing boundary. Since such a difficulty arises from the moving absorbing boundary, one possible approach is to find an appropriate value for β, such that the moving boundary is a good simulation of a certain fixed boundary. Considering the definition of $y^*(\tau)$, what we adopt here is

$$\beta = -\frac{c_1(t)}{c_2(t)}, \tag{3.5}$$

with which we use the fixed boundary 0 to approximate $y^*(\tau)$ for all $\tau \in [0,t]$. In this case, the solution to (3.4) can be approximated by the solution to the following PDE system:

$$\begin{cases} \dfrac{\partial \bar{U}}{\partial \xi} = \dfrac{\partial^2 \bar{U}}{\partial z^2}, \quad z > 0 \\ \bar{U}(z,0|X_t) = U(z,0|X_t)e^{-\beta z/2}, \\ \bar{U}(0,\xi|X_t) = 0. \end{cases} \tag{3.6}$$

Clearly, the closer $y^*(\tau)$, $\tau \in [0,t]$ is to 0, the more accurate such an approximation will be. If the volatility becomes a constant, such an approximation will degenerate to

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the exact down-and-out binary option price under the BS model since in this case $y^*(\tau) = 0$ for all $\tau \in [0, t]$. Thus, it is of interest to calculate the magnitude of the distance between $y^*(\tau)$ and 0, which is

$$|y^*(\tau) - 0| = r\tau \frac{|\tau^{-1} \int_0^\tau \sigma_s^2 \, ds - \tau^{-1} \int_0^t \sigma_s^2 \, ds|}{\tau^{-1} \int_0^t \sigma_s^2 \, ds}$$
$$\leq r\tau\Big(\frac{\max_{0 \leq s \leq \tau} \sigma_s^2}{\min_{0 \leq s \leq \tau} \sigma_s^2} - 1\Big).$$

This is an upper bound, which indicates that the accuracy of the approximation actually depends on the extent of fluctuation of the underlying volatility. If the difference between the maximum and minimum values of the volatility during the lifetime of the contract is small, one could expect this approximation to be very accurate. Of course, it is also interesting to see whether the approximation would still provide reasonably accurate results when the difference is enlarged, which will be tested in the next section.

With the method of images, it is not difficult to derive the solution to (3.6) as

$$\bar{U}(z, \xi | X_t) = \int_0^{+\infty} \frac{1}{\sqrt{4\pi\xi}} e^{-(z-v)^2/4\xi} \bar{U}(v, 0) \, dv - \int_{-\infty}^0 \frac{1}{\sqrt{4\pi\xi}} e^{-(z-v)^2/4\xi} \bar{U}(-v, 0) \, dv,$$

based on which we can obtain

$$U(y, \tau | X_t) = e^{-r\tau} N\Big[\frac{y + c_1(\tau)}{\sqrt{2c_2(\tau)}}\Big] - e^{-r\tau + \beta[y + c_1(\tau) + \beta c_2(\tau)]} N\Big[- \frac{y + c_1(\tau) + 2\beta c_2(\tau)}{\sqrt{2c_2(\tau)}}\Big].$$

Setting $u = 0$ finally yields $P(S, t | X_t) = U(y, t | X_t)$, and thus we have

$$P(S, t | X_t) = e^{-rt} N\Big[\frac{\ln(S/D) + c_1(t)}{\sqrt{2c_2(t)}}\Big] - e^{-rt + \beta \ln(S/D)} N\Big[- \frac{\ln(S/D) - c_1(t)}{\sqrt{2c_2(t)}}\Big], \qquad (3.7)$$

the simplification of which relies on the use of $c_1(t) + \beta c_2(t) = 0$.

Hitherto, the conditional option price has been presented in (3.7) with all the information of the Markov chain. However, one would never be able to predict the future information of the Markov chain at the current time, which forces us to conduct an extra step to work out the expectation of $P(S, t | X_t)$ with respect to the Markov chain, that is,

$$P(S, t; X_0) = E_{X_t}[P(S, t | X_t) | X_0].$$

Note that analytically evaluating the expectation in the above formula is usually not possible, given the convoluted expression for $P(S, t | X_t)$. Although He and Chen [14] developed a semi-Monte Carlo approach to obtain the value of the expectation, derivative pricing involving generating sample paths is often time-intensive, and analytical solution, if exists, is much more favoured in practice. To achieve this, the β presented in (3.5) has been chosen to be different from the one used in [14, 20], and this is actually essential, paving the way for the analytical derivation of the target expectation.

If we denote $h_t = \int_0^t \langle \bar{\sigma}^2, X_s \rangle ds$, then we have $c_1(t) = rt - h_t/2$, $c_2(t) = h_t/2$, $\beta = (h_t - 2rt)/h_t$, and thus (3.7) can be rewritten as

$$\begin{aligned} W(h_t) &\triangleq P(S, t | X_t) \\ &= e^{-rt} N\Big[\frac{\ln(S/D) + rt - h_t/2}{\sqrt{h_t}}\Big] - e^{-rt + \ln(S/D)(h_t - 2rt)/h_t} N\Big[-\frac{\ln(S/D) - rt + h_t/2}{\sqrt{h_t}}\Big]. \end{aligned}$$

By noticing that h_t is the only random variable contained in $P(S, t|X_t)$, the price of the down-and-out option can be formulated as

$$P(S, t; X_0) = E[V(h_t)|X_0] = \int_a^b W(h) f(h|X_0)\, dh, \tag{3.8}$$

where $f(h|X_0)$ is the probability density function of h_t defined on $[a, b]$, conditional upon the current state of the Markov chain. It should be pointed out here that if $f(h|X_0)$ were available in closed form, we would already obtain an analytical pricing formula for the target option. Unfortunately, this is not the case here and we have to find an alternative way to derive $P(S, t; X_0)$. In particular, $f(h|X_0)$ can be alternatively expressed as

$$f(h|X_0) = \frac{1}{2} A_0(X_0) + \sum_{k=1}^{+\infty} A_k(X_0) \cos\Big(k\pi \frac{h-a}{b-a}\Big), \tag{3.9}$$

with

$$A_k(X_0) = \frac{2}{b-a} \int_a^b f(h|X_0) \cos\Big(k\pi \frac{h-a}{b-a}\Big)\, dh. \tag{3.10}$$

Such an expansion is always possible since any real function has a Fourier cosine expansion when it is finitely supported [3], implying that its theoretical convergence is guaranteed. Thus, substituting (3.9) into (3.8) yields

$$P(S, t; X_0) = \int_a^b W(h) \Big[\frac{1}{2} A_0(X_0) + \sum_{k=1}^{+\infty} A_k(X_0) \cos\Big(k\pi \frac{h-a}{b-a}\Big)\Big]\, dh.$$

After interchanging the summation and the integration, the option price can finally be derived as

$$P(S, t, X_0) = \frac{1}{2} A_0(X_0) V_0 + \sum_{k=1}^{+\infty} A_k(X_0) V_k, \tag{3.11}$$

where

$$V_k = \int_a^b W(h)\cos\left(k\pi\frac{h-a}{b-a}\right)dh, \quad k = 0, 1, 2, \ldots. \tag{3.12}$$

Equation (3.11) is almost analytical and explicit, except that $A_k(X_0)$ defined in (3.10) still remains unknown as it involves the density function $f(h|X_0)$. However, we have managed to derive $A_k(X_0)$, by making use of the relationship between the characteristic function and the density function. If $g(\phi)$ is the characteristic function of h_t, it actually has a closed-form expression [9] given by

$$g(\phi) = \int_a^b f(h|X_0)e^{j\phi h}dh = \langle e^{A^T \cdot t + B}X_0, I\rangle, \tag{3.13}$$

from which one can easily represent $A_k(X_0)$ as

$$A_k(X_0) = \frac{2}{b-a}\text{Real}\left[g\left(\frac{k\pi}{b-a}\right)e^{-jka\pi/(b-a)}\right], \quad k = 0, 1, 2, \ldots. \tag{3.14}$$

This completes the proof. □

Clearly, the CDS pricing formula (2.3) is now completely analytical, after the substitution of the down-and-out binary option price presented in (3.2). It should particularly be emphasized that computing the option price through equation (3.2) or the CDS price with equation (2.3) does not involve the density function $f(h|X_0)$. Instead, all we need is to calculate V_k and $A_k(X_0)$ through equations (3.12) and (3.14) respectively, which are only dependent on the characteristic function $g(\phi)$ in equation (3.13).

Of course, there are still several issues that need to be addressed. First of all, as the solution is written in a series form, its speed of convergence needs to be checked as this is an important factor in practice, especially in the recent trend of algorithmic trading. The accuracy of the formula should also be considered to ensure that the approximation used is appropriate and there are no algebraic errors in the derivation process. Of course, it is also of interest to demonstrate whether the introduction of regime switching into the dynamic of the reference asset would have a significant impact on the CDS price. These issues will be discussed in the next section.

4. Numerical examples and discussion

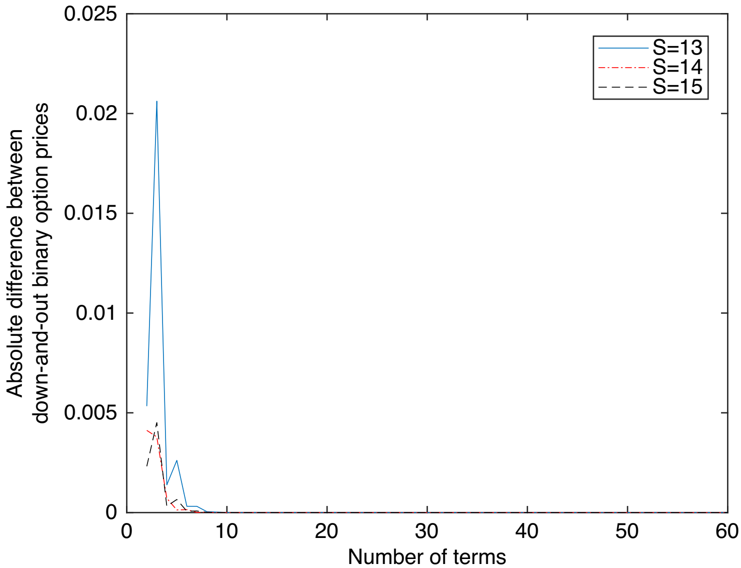
In this section numerical experiments are carried out to show the efficiency and accuracy of the approximation, after which the effect of regime switching on the CDS price is demonstrated. It should be pointed out that although any mathematical model needs to go through a calibration process before it can be applied in practice, the main purpose of this paper is to derive an efficient pricing formula for CDS contracts with a high degree of accuracy under the well-known regime-switching model that can facilitate the calibration process. Therefore, rather than calibrating the underlying model with real market data, it suffices for us to test the efficiency and accuracy of our approximation formula with artificial data.

As it was shown in the previous section that the accuracy of our approximation depends on the difference between σ_1 and σ_2, we consider two cases when conducting numerical experiments: case 1 sets σ_1 and σ_2 as 0.1 and 0.2 respectively, while case 2 uses 0.1 and 0.4 for σ_1 and σ_2 respectively. Other parameters include $\lambda_{12} = \lambda_{21} = 20$, $T = 1$, $D = 12$, $S = 15$ and $r = 0.1$. The current state is assumed to be 1.

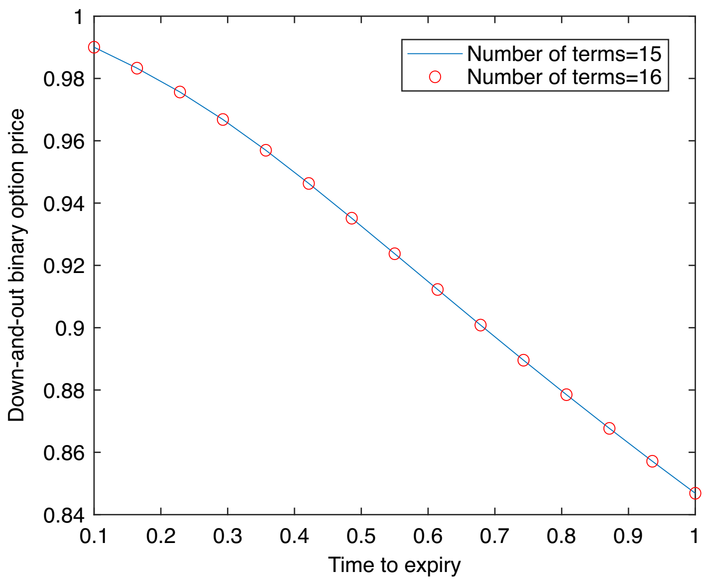
The speed of convergence for a series solution is always an important factor to demonstrate its computational efficiency. To check the efficiency, it suffices to use only the down-and-out binary options since the series solution is only introduced when deriving the option pricing formula. Thus, with N being the number of terms used in the formula, the absolute difference between the down-and-out binary option prices calculated with $(N + 1)$ terms and those produced with N terms is shown in Figures 1(a) and 2(a) corresponding to case 1 and case 2, respectively. It is clear that option prices in both figures converge very fast for different reference asset prices as the absolute difference decreases sharply when we increase the number of terms, and they are almost zero even when fewer than 10 terms are used. A clearer pattern for the two cases is respectively shown in Figures 1(b) and 2(b), where option prices produced by 15 and 16 terms are plotted against different time to expiry. Option prices in both figures are pointwise close to each other, with the maximum absolute difference being of the order of 1×10^{-7} for Figure 1(b) and 2×10^{-5} for Figure 2(b). For the convenience of the reader in repeating the experiments, numerical values in producing Figures 1 and 2 are provided in Tables 1 and 2.

Having demonstrated the rapid speed of convergence for equation (3.2), the accuracy of our approach is another aspect that needs to be checked as approximation is involved in the derivation process. Thus, CDS prices calculated with our approach (our prices), are compared with those obtained using the approach in [14] (He–Chen prices) and those from Monte Carlo simulation (Monte Carlo prices) in Figures 3 and 4 corresponding to case 1 and case 2, respectively. It can be easily observed from Figures 3(a) and 4(a) that our approximation is almost the same as the He–Chen approximation, both of which are very close to the Monte Carlo price. Figures 3(b) and 4(b) exhibit the relative errors between the two approximation prices and Monte Carlo prices, and the error of our price is slightly lower than that of the He–Chen price. Furthermore, our maximum relative error being less than 0.91% for both cases is certainly evidence demonstrating the accuracy of our approach. One should also note that although enlarging the difference between σ_1 and σ_2 increases the relative error between our price and the Monte Carlo price, which confirms what we have shown in the previous section, our approximation is still quite accurate even when

$$\frac{\max_{0\le s\le\tau}\sigma_s^2}{\min_{0\le s\le\tau}\sigma_s^2} = 16.$$



(a) Absolute difference between $(N+1)$-term price and N-term price



(b) Option prices with $N = 15, 16$

FIGURE 1. Speed of convergence for equation (3.2) with $\sigma_1 = 0.1$, $\sigma_2 = 0.2$.

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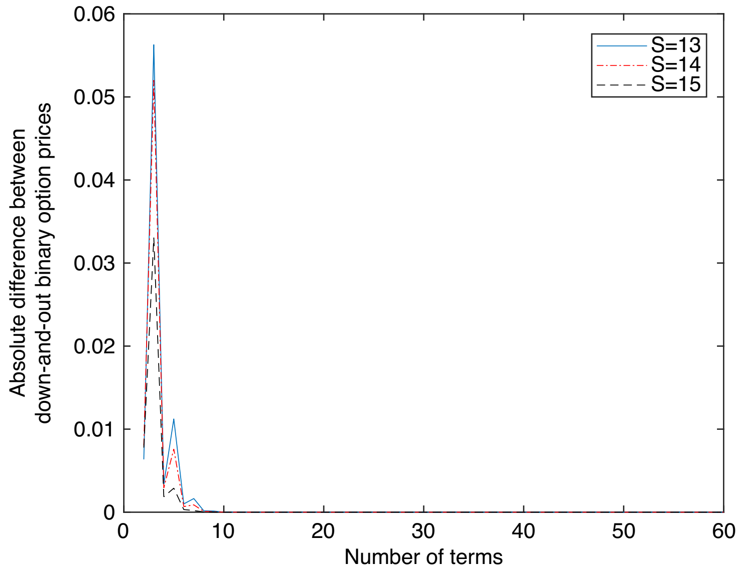
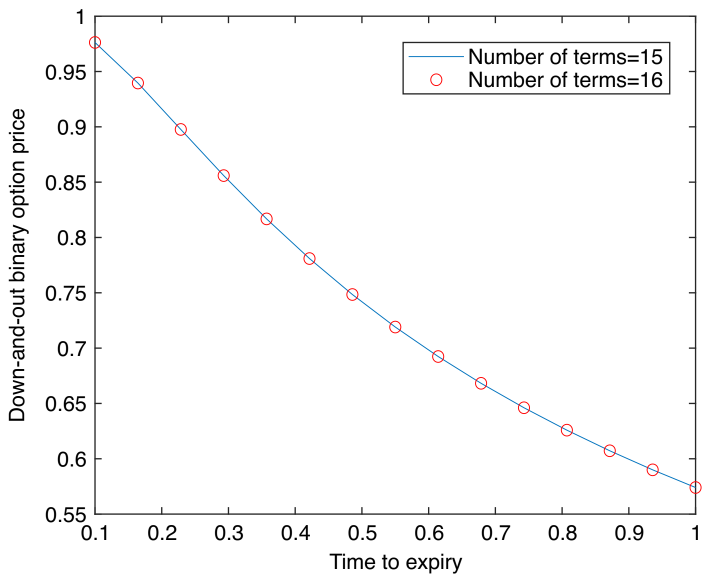


(a) Absolute difference between $(N + 1)$-term price and N-term price



(b) Option prices with $N = 15, 16$

FIGURE 2. Speed of convergence for equation (3.2) with $\sigma_1 = 0.1$, $\sigma_2 = 0.4$.

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TABLE 1. Numerical values for Figures 1(a) and 2(a).

N	1	11	21	31	41	51
			Case 1: $\sigma_1 = 0.1,\ \sigma_2 = 0.2$			
$S = 13$	5.3e-3	6.3e-8	9.5e-13	1.7e-13	1.3e-13	9.0e-14
$S = 14$	4.1e-3	2.9e-8	4.3e-13	7.8e-14	1.6e-15	1.1e-15
$S = 15$	2.3e-3	1.4e-8	2.1e-13	3.9e-14	3.3e-14	2.3e-14
			Case 2: $\sigma_1 = 0.1,\ \sigma_2 = 0.4$			
$S = 13$	6.4e-3	2.5e-7	4.0e-12	1.0e-12	8.8e-13	5.1e-13
$S = 14$	8.0e-3	1.1e-7	1.4e-12	2.4e-13	2.1e-13	1.6e-13
$S = 15$	7.8e-3	3.3e-8	4.0e-13	7.3e-14	1.7e-14	1.1e-14

TABLE 2. Numerical values for Figures 1(b) and 2(b).

τ	0.1000	0.2929	0.4857	0.6786	0.8714
		Case 1: $\sigma_1 = 0.1,\ \sigma_2 = 0.2$			
15 terms	0.9900	0.9668	0.9351	0.9008	0.8677
16 terms	0.9900	0.9668	0.9351	0.9008	0.8677
		Case 2: $\sigma_1 = 0.1,\ \sigma_2 = 0.4$			
15 terms	0.9762	0.8559	0.7484	0.6682	0.6072
16 terms	0.9762	0.8559	0.7484	0.6682	0.6072

Again, for the convenience of the reader in repeating the numerical experiments, Table 3 presents the numerical values in producing Figures 3 and 4.

As our approach provides similar accuracy to the He–Chen approach, it is of interest to compare the computational efficiency of the two approaches, and the CPU time cost to compute one CDS price is listed in Table 4. It is expected that Monte Carlo simulation, as the benchmark, is the most time-intensive approach of the three, as it needs to simulate both the Brownian motion and the Markov chain. The He–Chen approach has an advantage over Monte Carlo simulation, since it avoids generating sample paths for the Brownian motion. Our approach, on the other hand, involving no simulations, is the most efficient one, and it only requires around 0.13 seconds to derive one price, almost 100 times less than the time cost by the He–Chen approach, even when 100 terms are used for the series solution.

Having established the advantages of our approach, we are now ready to study how regime switching would affect the CDS prices. Depicted in Figure 5 is the change in CDS prices under the regime-switching model with different transition rates, with the prices under the BS model as a benchmark. Firstly, our price of state 1 (2) is exactly the same as the BS price, with its volatility being equal to σ_1 (σ_2) when the transition rates are 0 for both cases, which is expected as there is no actual regime switching and the volatility value will remain as its initial value. With the increase in the transition rates,

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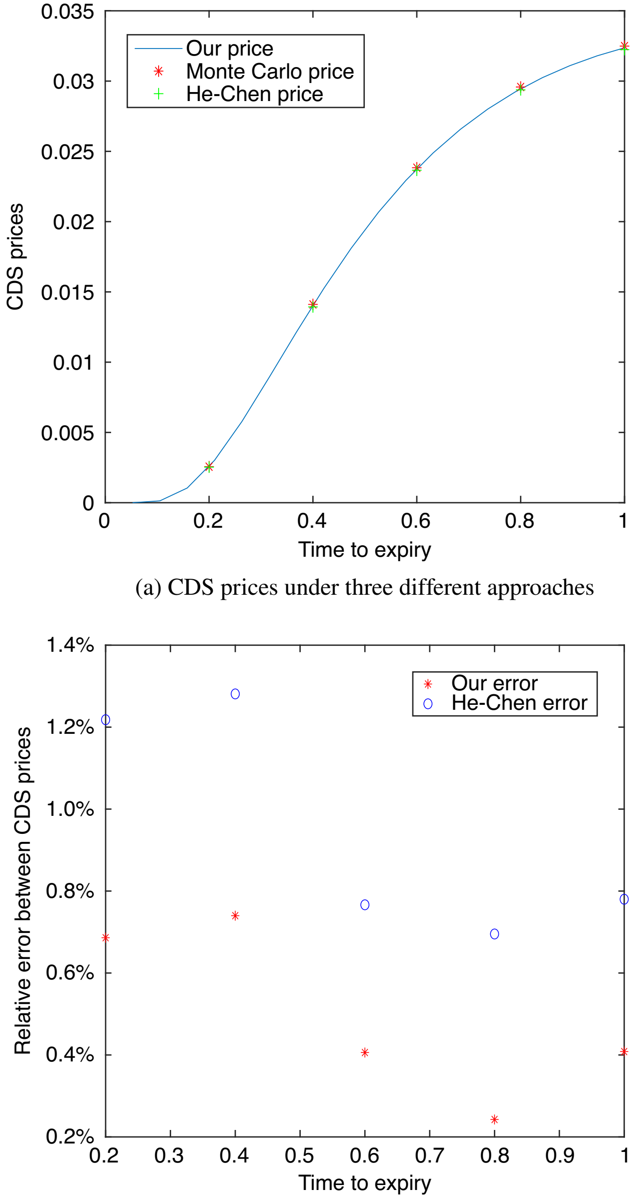

(a) CDS prices under three different approaches

(b) Relative errors

FIGURE 3. Accuracy with $\sigma_1 = 0.1$, $\sigma_2 = 0.2$.

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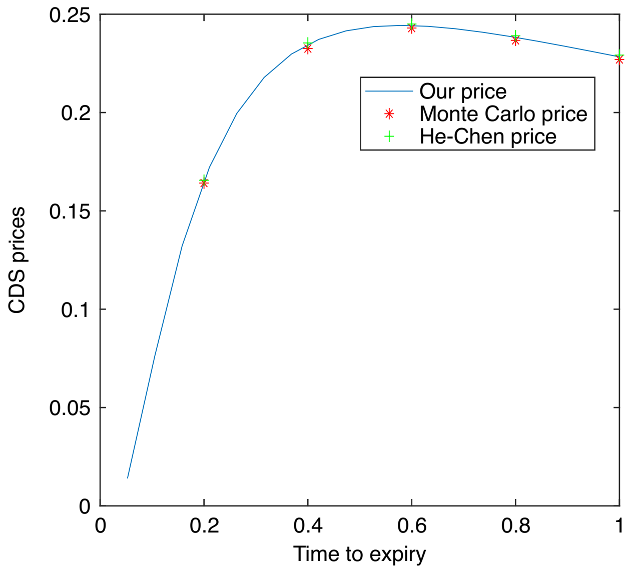
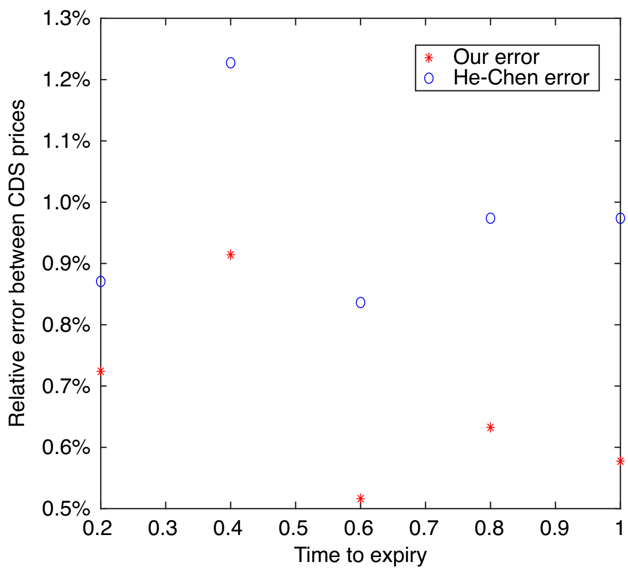


(a) CDS prices under three different approaches



(b) Relative errors

FIGURE 4. Accuracy with $\sigma_1 = 0.1,\ \sigma_2 = 0.4$.

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TABLE 3. Numerical values for Figures 3 and 4.

τ	0.2	0.4	0.6	0.8	1.0
			Case 1: $\sigma_1 = 0.1,\ \sigma_2 = 0.2$		
Our price	0.0025	0.0140	0.0237	0.0294	0.0324
He–Chen price	0.0026	0.0141	0.0238	0.0296	0.0325
Monte Carlo price	0.0025	0.0139	0.0237	0.0294	0.0322
Our error (%)	0.69	0.74	0.41	0.24	0.41
He–Chen error (%)	1.22	1.28	0.77	0.70	0.78
			Case 2: $\sigma_1 = 0.1,\ \sigma_2 = 0.4$		
Our price	0.1653	0.2346	0.2442	0.2382	0.2283
He–Chen price	0.1656	0.2354	0.2350	0.2390	0.2292
Monte Carlo price	0.1641	0.2325	0.2430	0.2367	0.2270
Our error (%)	0.72	0.91	0.52	0.63	0.58
He–Chen error (%)	0.87	1.23	0.84	0.97	0.97

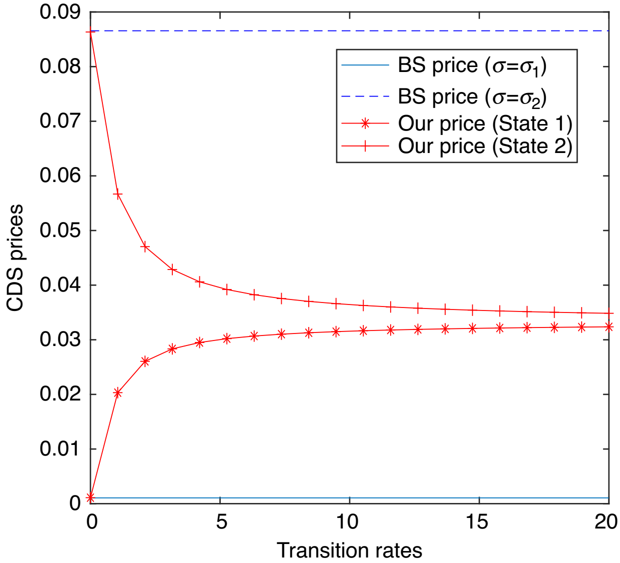
TABLE 4. Comparison of CPU times (seconds).

	Ours (N = 20)	Ours (N = 50)	Ours (N = 100)	He–Chen	Monte Carlo
Time	0.0399	0.0781	0.1327	11.7192	15.9004

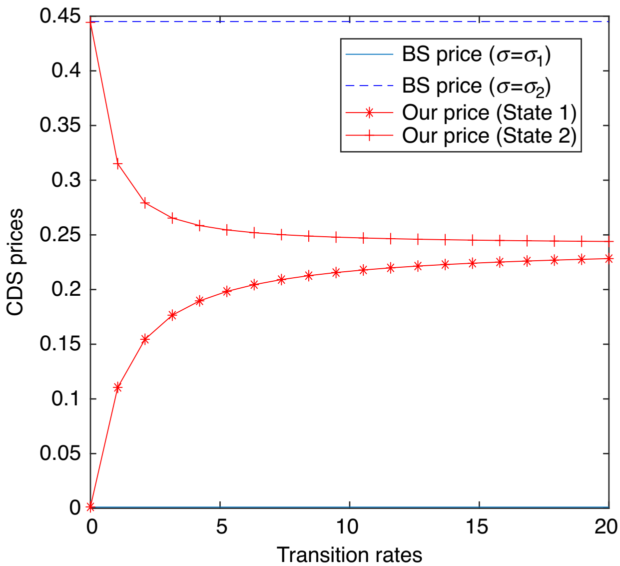
TABLE 5. Numerical values for Figure 5.

Transition rates	0	5.2631	10.5263	15.7895	20
			Case 1: $\sigma_1 = 0.1,\ \sigma_2 = 0.2$		
Our price(state 1)	0.0011	0.0302	0.0317	0.0322	0.0324
Our price(state 2)	0.0863	0.0392	0.0363	0.0353	0.0348
BS price ($\sigma = \sigma_1$)	0.0011	0.0011	0.0011	0.0011	0.0011
BS price ($\sigma = \sigma_2$)	0.0863	0.0863	0.0863	0.0863	0.0863
			Case 2: $\sigma_1 = 0.1,\ \sigma_2 = 0.4$		
Our price(state 1)	0.0011	0.1983	0.2179	0.2252	0.2283
Our price(state 2)	0.4443	0.2545	0.2471	0.2449	0.2440
BS price ($\sigma = \sigma_1$)	0.0011	0.0011	0.0011	0.0011	0.0011
BS price ($\sigma = \sigma_2$)	0.4443	0.4443	0.4443	0.4443	0.4443

the transition of the volatility between the two states would become more frequent, increasing (decreasing) the average of the volatility of the reference asset starting in state 1 (2), and this leads to a higher (lower) risk, as a result of which both prices of the two states are moving away from the corresponding BS prices while at the same time



(a) $\sigma_1 = 0.1$, $\sigma_2 = 0.2$



(b) $\sigma_1 = 0.1$, $\sigma_2 = 0.4$

FIGURE 5. Effect of regime switching.

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getting closer to each other. The corresponding numerical values to produce Figure 5 are shown in Table 5.

5. Conclusion

In this paper the pricing of CDSs is investigated under a regime-switching BS model. By first establishing the relationship between the unknown no-default probability and the price of a down-and-out binary option written on the same reference asset, the CDS pricing problem reduces to the valuation of this particular option. After assuming that all the future information of the Markov chain is given, we obtain an approximation formula for the conditional option price, taking the expectation of which yields the target unconditional price. The solution is written in a Fourier cosine series, and its computational efficiency and accuracy are demonstrated through numerical experiments.

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