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ON THE DIVISIBILITY OF SUMS INVOLVING APÉRY-LIKE POLYNOMIALS

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Abstract

We prove a divisibility result on sums involving the Apéry-like polynomials

$$V_n(x) = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}\binom{x}{k}\binom{x+k}{k},$$

which confirms a conjectural congruence of Z.-H. Sun. Our proof relies on some combinatorial identities and transformation formulae.

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1. Introduction

In his ingenious proof of the irrationality of $\zeta(2)$ and $\zeta(3)$, Apéry [1] introduced the numbers

$$A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \quad \text{and} \quad A_n' = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k},$$

which are now known as the Apéry numbers. Since the appearance of the Apéry numbers, some interesting arithmetic properties have been gradually discovered. For instance, Sun [8, formula (1.6)] showed that for any prime $p \geq 5$,

$$\sum_{k=0}^{p-1} (2k+1)A_k \equiv p + \frac{7}{6} p^4 B_{p-3} \pmod{p^5},$$

where B_n denotes the nth Bernoulli number. In 2012, Guo and Zeng [3, Theorem 1.3] proved that

$$\sum_{k=0}^{p-1} (2k+1)^3 A_k \equiv p^3 \pmod{2p^6}.$$

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In 2012, Sun [8] introduced the Apéry polynomials,

$$A_n(x) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 x^k,$$

and conjectured that

$$\sum_{k=0}^{n-1} \epsilon^k (2k+1) A_k(x)^m \equiv 0 \pmod{n},$$

where $\epsilon = \pm 1$ and n, m are positive integers. This conjectural divisibility result was confirmed by Pan [5, Theorem 1.1].

In 2020, Sun [7, formula (1.8)] introduced the Apéry-like polynomials

$$V_n(x) = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}\binom{x}{k}\binom{x+k}{k}.$$

Note that $V_n(n) = A_n$ and

$$V_n\Big(-\frac{1}{2}\Big) = \frac{1}{16^n} \sum_{k=0}^{n} \binom{2k}{k}^2 \binom{2n-2k}{n-k}^2.$$

Let $p \geq 5$ be a prime and, for $x \in \mathbb{Z}_p$, let $\langle x \rangle_p$ denote the least nonnegative integer a with $a \equiv x \pmod{p}$. Sun [7, Theorem 4.5] also showed that for $x \not\equiv 0, -1 \pmod{p}$ and $x' = (x - \langle x \rangle_p)/p$,

$$\sum_{k=0}^{p-1} (2k+1) V_k(x) \equiv p^3 \frac{x'(x'+1)}{x(x+1)} + 2p^4 \frac{x'(x'+1)+1}{x(x+1)} H_{\langle x \rangle_p} \pmod{p^5},$$

where $H_n = \sum_{i=1}^{n} 1/i$ denotes the nth harmonic number.

Recently, Wang [10, Theorem 1.1] proved that for any prime $p \geq 5$,

$$\sum_{k=0}^{p-1} (-1)^k (2k+1) V_n\Big(-\frac{1}{2}\Big) \equiv (-1)^{(p-1)/2} p + 3p^3 E_{p-3} \pmod{p^4},$$

$$\sum_{k=0}^{p-1} 2^k (k+1) V_n\Big(-\frac{1}{2}\Big) \equiv (-1)^{(p-1)/2} p + 5p^3 E_{p-3} \pmod{p^4},$$

which confirm two supercongruence conjectures due to Sun [9, Conjecture 33].

In his talk at the 7th Chinese National Conference on Combinatorial Number Theory, Z.-H. Sun discussed the sums of squares of the Apéry-like polynomials $V_n(x)$ and proposed the following divisibility conjecture.

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CONJECTURE 1.1. Let $p \geq 5$ be a prime. For $x \in \mathbb{Z}_p$ and $x \not\equiv -1/2 \pmod{p}$,

$$\sum_{n=0}^{p-1}(2n+1)V_n(x)^2 \equiv 0 \pmod{p^2}.$$

The aim of this note is to give a positive answer to this problem as follows.

THEOREM 1.2. *Let $p \geq 5$ be a prime. For $x \in \mathbb{Z}_p$,*

$$\sum_{n=0}^{p-1}(2n+1)V_n(x)^2 \equiv \frac{p^2}{2\langle x\rangle_p + 1} \pmod{p^2}.$$

2. Proof of Theorem 1.2

We begin with the following identity [2, formula (2.5)]:

$$\binom{x}{j}\binom{x+j}{j}\binom{x}{k}\binom{x+k}{k} = \sum_{s=0}^{j+k}\binom{j+k}{s}\binom{s}{j}\binom{s}{k}\binom{x}{s}\binom{x+s}{s}. \tag{2.1}$$

Using (2.1) twice, we obtain

$$\begin{aligned} V_n(x)^2 &= \sum_{j=0}^{n}\sum_{k=0}^{n}\binom{n}{j}\binom{n+j}{j}\binom{n}{k}\binom{n+k}{k}\binom{x}{j}\binom{x+j}{j}\binom{x}{k}\binom{x+k}{k} \\ &= \sum_{j=0}^{n}\sum_{k=0}^{n}\sum_{r=0}^{j+k}\sum_{s=0}^{j+k}\binom{n}{r}\binom{n+r}{r}\binom{x}{s}\binom{x+s}{s}\binom{j+k}{r}\binom{r}{j}\binom{r}{k}\binom{j+k}{s}\binom{s}{j}\binom{s}{k}. \end{aligned}$$

It follows that

$$\begin{aligned} \sum_{n=0}^{p-1}(2n+1)V_n(x)^2 = &\sum_{j=0}^{p-1}\sum_{k=0}^{p-1}\sum_{r=0}^{j+k}\sum_{s=0}^{j+k}\binom{x}{s}\binom{x+s}{s}\binom{j+k}{r}\binom{r}{j}\binom{r}{k}\binom{j+k}{s}\binom{s}{j}\binom{s}{k} \\ &\times\sum_{n=0}^{p-1}(2n+1)\binom{n}{r}\binom{n+r}{r}. \end{aligned} \tag{2.2}$$

Since

$$\sum_{n=0}^{N-1}(2n+1)\binom{n}{r}\binom{n+r}{r} = \frac{N^2}{r+1}\binom{N+r}{r}\binom{N-1}{r},$$

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which can be easily proved by induction on N, we deduce that for $0 \le r \le 2p-2$,

$$\sum_{n=0}^{p-1}(2n+1)\binom{n}{r}\binom{n+r}{r} = \frac{p^2}{r+1}\binom{p+r}{r}\binom{p-1}{r}$$
$$= \frac{p^2(p^2-1^2)\cdots(p^2-r^2)}{(r+1)r!^2}$$
$$\equiv \frac{(-1)^r p^2}{r+1} \pmod{p^2}. \tag{2.3}$$

Substituting (2.3) into the right-hand side of (2.2) gives

$$\sum_{n=0}^{p-1}(2n+1)V_n(x)^2 \equiv p^2 \sum_{j=0}^{p-1}\sum_{k=0}^{p-1}\sum_{s=0}^{j+k}\binom{x}{s}\binom{x+s}{s}\binom{j+k}{s}\binom{s}{j}\binom{s}{k}$$
$$\times \sum_{r=0}^{j+k}\frac{(-1)^r}{r+1}\binom{j+k}{r}\binom{r}{j}\binom{r}{k} \pmod{p^2}. \tag{2.4}$$

Using the identity,

$$\sum_{r=0}^{j+k}\frac{(-1)^r}{r+1}\binom{r}{j}\binom{j}{r-k} = \frac{(-1)^{j+k}}{(j+k+1)\binom{j+k}{j}},$$

from [4, formula (2.2)], we get

$$\sum_{r=0}^{j+k}\frac{(-1)^r}{r+1}\binom{j+k}{r}\binom{r}{j}\binom{r}{k} = \binom{j+k}{j}\sum_{r=0}^{j+k}\frac{(-1)^r}{r+1}\binom{r}{j}\binom{j}{r-k} = \frac{(-1)^{j+k}}{j+k+1}. \tag{2.5}$$

Combining (2.4) and (2.5), we obtain

$$\sum_{n=0}^{p-1}(2n+1)V_n(x)^2 \equiv p^2 \sum_{j=0}^{p-1}\sum_{k=0}^{p-1}\sum_{s=0}^{j+k}\frac{(-1)^{j+k}}{j+k+1}\binom{x}{s}\binom{x+s}{s}\binom{j+k}{s}\binom{s}{j}\binom{s}{k} \pmod{p^2}$$
$$= p^2 \sum_{s=0}^{2p-2}\sum_{j=0}^{p-1}\sum_{k=0}^{p-1}\frac{(-1)^{j+k}}{j+k+1}\binom{x}{s}\binom{x+s}{s}\binom{j+k}{s}\binom{s}{j}\binom{s}{k}. \tag{2.6}$$

Since the summands on the right-hand side of (2.6) are congruent to 0 modulo p^2 except for $j+k=p-1$, we conclude that

$$\sum_{n=0}^{p-1}(2n+1)V_n(x)^2 \equiv p\sum_{s=0}^{p-1}\binom{x}{s}\binom{x+s}{s}\binom{p-1}{s}\sum_{j+k=p-1}\binom{s}{j}\binom{s}{k} \pmod{p^2}$$
$$= p\sum_{s=0}^{p-1}\binom{x}{s}\binom{x+s}{s}\binom{p-1}{s}\binom{2s}{p-1},$$

where we have used the Chu–Vandermonde identity in the last step.

Furthermore, we have $\binom{p-1}{s} \equiv (-1)^s \pmod{p}$ and $\binom{2s}{p-1} \equiv p/(2s+1) \pmod{p}$ for $0 \le s \le p-1$. It follows that

$$\sum_{n=0}^{p-1}(2n+1)V_n(x)^2 \equiv p^2 \sum_{s=0}^{p-1} \frac{(-1)^s}{2s+1}\binom{x}{s}\binom{x+s}{s}$$

$$\equiv p^2 \sum_{s=0}^{\langle x\rangle_p} \frac{(-1)^s}{2s+1}\binom{\langle x\rangle_p}{s}\binom{\langle x\rangle_p+s}{s} \pmod{p^2}. \qquad (2.7)$$

Next, we recall the identity [6, formula (2.1)],

$$\sum_{s=0}^{n} \frac{(-1)^s}{z+s}\binom{n}{s}\binom{n+s}{s} = \frac{(1-z)_n}{z(1+z)_n}. \qquad (2.8)$$

The case $z = 1/2$ in (2.8) reads

$$\sum_{s=0}^{n} \frac{(-1)^s}{2s+1}\binom{n}{s}\binom{n+s}{s} = \frac{1}{2n+1}. \qquad (2.9)$$

Finally, combining (2.7) and (2.9), we obtain

$$\sum_{n=0}^{p-1}(2n+1)V_n(x)^2 \equiv \frac{p^2}{2\langle x\rangle_p+1} \pmod{p^2},$$

as desired.

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