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# Comparison Geometry of Manifolds with Boundary under a Lower Weighted Ricci Curvature Bound

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*Abstract.* We study Riemannian manifolds with boundary under a lower weighted Ricci curvature bound. We consider a curvature condition in which the weighted Ricci curvature is bounded from below by the density function. Under the curvature condition and a suitable condition for the weighted mean curvature for the boundary, we obtain various comparison geometric results.

## 1 Introduction

We study comparison geometry of manifolds with boundary under a lower weighted Ricci curvature bound. For the lower weighted Ricci curvature bound, we consider a curvature condition in which the lower bound is controlled by the density function. We introduce a reasonable curvature condition for a lower weighted mean curvature bound for the boundary. Under these curvature conditions, we investigate comparison geometric properties and conclude twisted rigidity theorems.

For $n \geq 2$, let $(M, g)$ be an $n$-dimensional Riemannian manifold with or without boundary, and let $f\colon M \to \mathbb{R}$ be a smooth function. Let $\mathrm{Ric}_g$ denote the Ricci curvature defined by $g$. For $N \in (-\infty, \infty]$, the *$N$-weighted Ricci curvature* is defined as follows: If $N \in (-\infty, \infty)\setminus\{n\}$,

$$\mathrm{Ric}^N_f := \mathrm{Ric}_g + \mathrm{Hess}\, f - \frac{df \otimes df}{N - n}, \tag{1.1}$$

where $df$ and $\mathrm{Hess}\, f$ are the differential and the Hessian of $f$, respectively; otherwise, if $N = \infty$, then $\mathrm{Ric}^N_f := \mathrm{Ric}_g + \mathrm{Hess}\, f$; if $N = n$, and if $f$ is constant, then $\mathrm{Ric}^N_f := \mathrm{Ric}_g$; if $N = n$, and if $f$ is not constant, then $\mathrm{Ric}^N_f := -\infty$ ([2, 12]). For a smooth function $K\colon M \to \mathbb{R}$, we mean by $\mathrm{Ric}^N_{f,M} \geq K$ for every point $x \in M$, and every unit tangent vector $v$ at $x$, it holds that $\mathrm{Ric}^N_f(v) \geq K(x)$.

For manifolds without boundary whose $N$-weighted Ricci curvatures are bounded from below by constants, many comparison geometric results are already known in the usual weighted case of $N \in [n, \infty]$ (see *e.g.,* [12, 13, 15, 22]). For manifolds with boundary, the author [18] has studied such comparison geometric properties.

Recently, under a lower $N$-weighted Ricci curvature bound, Wylie [23], and Wylie and Yeroshkin [24] studied comparison geometry of manifolds without boundary in

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complementary case of $N \in (-\infty, n)$. Wylie [23] obtained a splitting theorem of Cheeger–Gromoll type under the curvature condition $\mathrm{Ric}^N_{f,M} \geq 0$ for $N \in (-\infty, 1]$. Wylie and Yeroshkin [24] introduced a curvature condition

$$\mathrm{Ric}^1_{f,M} \geq (n-1)\kappa e^{\frac{-4f}{n-1}}$$

for $\kappa \in \mathbb{R}$ from the view point of study of weighted affine connections. Under such condition, they proved a maximal diameter theorem of Cheng type for the distance induced from the metric $e^{\frac{-4f}{n-1}} g$, and a volume comparison of Bishop–Gromov type for the measure $e^{-\frac{n+1}{n-1} f} \mathrm{vol}_g$, where $\mathrm{vol}_g$ denotes the Riemannian volume measure on $(M, g)$.

In this paper, we study comparison geometry of Riemannian manifolds with boundary satisfying the curvature condition

$$\mathrm{Ric}^N_{f,M} \geq (n-1)\kappa e^{\frac{-4f}{n-1}} \tag{1.2}$$

for $\kappa \in \mathbb{R}, N \in (-\infty, \infty]$. We will also consider a curvature condition for the boundary that is compatible with (1.2). For a Riemannian manifold $M$ with boundary, let $\partial M$ stand for its boundary. For $z \in \partial M$, we denote by $u_z$ the unit inner normal vector on $\partial M$ at $z$, and by $H_z$ the mean curvature of $\partial M$ at $z$ with respect to $u_z$ (more precisely, see Subsection 2.2). The *weighted mean curvature* $H_{f,z}$ is defined as

$$H_{f,z} := H_z + g\big( (\nabla f)_z, u_z \big),$$

where $\nabla f$ is the gradient of $f$. We introduce a curvature condition

$$H_{f,\partial M} \geq (n-1)\lambda e^{\frac{-2f}{n-1}} \tag{1.3}$$

for $\lambda \in \mathbb{R}$, where (1.3) means that $H_{f,z} \geq (n-1)\lambda e^{\frac{-2f(z)}{n-1}}$ for every $z \in \partial M$. Under conditions (1.2) and (1.3) for $\kappa, \lambda \in \mathbb{R}, N \in (-\infty, 1]$, we formulate various comparison geometric results, and generalize the preceding studies by Kasue [9, 10], and by the author [17] when $f = 0$.

## 1.1 Setting

In this paper, we work in the following setting. For $n \geq 2$, let $(M, g)$ be an $n$-dimensional, connected complete Riemannian manifold with boundary, and let $f\colon M \to \mathbb{R}$ be a smooth function. For $\kappa, \lambda \in \mathbb{R}, N \in (-\infty, \infty]$ we say that a triple $(M, \partial M, f)$ has *lower* $(\kappa, \lambda, N)$*-weighted curvature bounds* if (1.2) and (1.3) hold. For $N_1 \in (n, \infty], N_2 \in (-\infty, n)$, or for $N_1, N_2 \in (-\infty, n)$ with $N_1 \leq N_2$, if $(M, \partial M, f)$ has lower $(\kappa, \lambda, N_1)$-weighted curvature bounds, then it also has lower $(\kappa, \lambda, N_2)$-weighted curvature bounds (see (1.1) and (1.2)). We mainly study a triple $(M, \partial M, f)$ with lower $(\kappa, \lambda, N)$-weighted curvature bounds for $\kappa, \lambda \in \mathbb{R}, N \in (-\infty, 1]$.

## 1.2 Splitting Theorems

For the Riemannian distance $d_M$ on $M$, let $\rho_{\partial M}\colon M \to \mathbb{R}$ stand for the distance function from the boundary $\partial M$ defined as $\rho_{\partial M}(x) := d_M(x, \partial M)$. For $z \in \partial M$, let $\gamma_z\colon [0, T) \to M$ be the geodesic with $\gamma'_z(0) = u_z$. Define functions $\tau, \tau_f\colon \partial M \to$

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$(0, \infty]$ by

$$\tau(z) := \sup\{t > 0 \mid \rho_{\partial M}(\gamma_z(t)) = t\}, \quad \tau_f(z) := \int_0^{\tau(z)} e^{\frac{-2f(\gamma_z(a))}{n-1}} da. \tag{1.4}$$

We also define a function $s_{f,z} \colon [0, \tau(z)] \to [0, \tau_f(z)]$ by

$$s_{f,z}(t) := \int_0^t e^{\frac{-2f(\gamma_z(a))}{n-1}} da. \tag{1.5}$$

Let $\mathfrak{s}_{\kappa,\lambda}(s)$ be a unique solution of the Jacobi equation $\varphi''(s) + \kappa\varphi(s) = 0$ with $\varphi(0) = 1, \varphi'(0) = -\lambda$. We also denote by $\mathfrak{s}_\kappa(s)$ the solution of the equation $\varphi''(s) + \kappa\varphi(s) = 0$ with $\varphi(0) = 0, \varphi'(0) = 1$, and note that

$$\mathfrak{s}_\kappa(s) = \begin{cases} \frac{\sin\sqrt{\kappa}s}{\sqrt{\kappa}} & \text{if } \kappa > 0, \\ s & \text{if } \kappa = 0, \\ \frac{\sinh\sqrt{|\kappa|}s}{\sqrt{|\kappa|}} & \text{if } \kappa < 0, \end{cases} \qquad \mathfrak{s}_{\kappa,\lambda}(s) = \mathfrak{s}'_\kappa(s) - \lambda\mathfrak{s}_\kappa(s).$$

For $t \in [0, \tau(z)]$, we set

$$F_{\kappa,\lambda,z}(t) := \exp\Big(\frac{f(\gamma_z(t)) - f(z)}{n-1}\Big)\mathfrak{s}_{\kappa,\lambda}(s_{f,z}(t)). \tag{1.6}$$

Let $h$ denote the induced Riemannian metric on $\partial M$. For an interval $I$, and a connected component $\partial M_1$ of $\partial M$, we denote by $I \times_{F_{\kappa,\lambda}} \partial M_1$ the twisted product Riemannian manifold $(I \times \partial M_1, dt^2 + F^2_{\kappa,\lambda,z}(t)h)$.

One of the main results is the following twisted splitting theorem.

**Theorem 1.1** *Let $\kappa \le 0$ and $\lambda := \sqrt{|\kappa|}$. For $N \in (-\infty, 1]$, assume that $(M, \partial M, f)$ has lower $(\kappa, \lambda, N)$-weighted curvature bounds. Suppose that $f$ is bounded from above. If $\tau(z_0) = \infty$ for some $z_0 \in \partial M$, then $M$ is isometric to $[0, \infty) \times_{F_{\kappa,\lambda}} \partial M$; moreover, if $N \in (-\infty, 1)$, then for every $z \in \partial M$, the function $f \circ \gamma_z$ is constant on $[0, \infty)$.*

When $\kappa = 0$ and $\lambda = 0$, Theorem 1.1 was proved by the author in the cases where $N \in [n, \infty]$ (see [18]) and $N \in (-\infty, 1]$ (see [19]). In the unweighted case of $f = 0$, Kasue [9] has proved Theorem 1.1 under the assumption that $M$ is non-compact and $\partial M$ is compact (see also Croke and Kleiner [5]), and the author [17] has proved Theorem 1.1 itself.

In Theorem 1.1, by applying a splitting theorem proved by Wylie [23] to the boundary, we obtain a multi-splitting theorem (see Subsection 5.3). We also generalize a splitting theorem studied by Kasue [9] (see also Croke and Kleiner [5] and Ichida [8]) (see Subsection 5.4).

### 1.3 Inscribed Radii

We denote by $M^n_\kappa$ the simply connected $n$-dimensional space form with constant curvature $\kappa$. We say that $\kappa$ and $\lambda$ satisfy the *ball-condition* if there exists a closed geodesic ball $B^n_{\kappa,\lambda}$ in $M^n_\kappa$ whose boundary $\partial B^n_{\kappa,\lambda}$ has constant mean curvature $(n-1)\lambda$. Notice that $\kappa$ and $\lambda$ satisfy the ball-condition if and only if either (1) $\kappa > 0$; (2) $\kappa = 0$ and $\lambda > 0$; or (3) $\kappa < 0$ and $\lambda > \sqrt{|\kappa|}$. We denote by $C_{\kappa,\lambda}$ the radius of $B^n_{\kappa,\lambda}$. We see that $\kappa$

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and $\lambda$ satisfy the ball-condition if and only if the equation $\mathfrak{s}_{\kappa,\lambda}(s) = 0$ has a positive solution; moreover, $C_{\kappa,\lambda} = \inf\{s > 0 \mid \mathfrak{s}_{\kappa,\lambda}(s) = 0\}$.

The *inscribed radius* InRad $M$ *of* $M$ is defined to be the supremum of the distance function from the boundary $\rho_{\partial M}$ over $M$. Let us consider the Riemannian metric $g_f := e^{\frac{-4f}{n-1}} g$. We denote by $\rho^{g_f}_{\partial M}$ and by $\operatorname{InRad}_{g_f} M$ the distance function from the boundary and the inscribed radius on $M$ induced from $g_f$, respectively.

Let Int $M$ be the interior of $M$. For $x \in \operatorname{Int} M$, let $U_x M$ be the unit tangent sphere at $x$ that can be identified with the $(n-1)$-dimensional standard unit sphere $(\mathbb{S}^{n-1}, ds^2_{n-1})$. For $v \in U_x M$, let $\gamma_v \colon [0, T) \to M$ be the geodesic with $\gamma'_v(0) = v$. We define $\tau_x \colon U_x M \to (0, \infty]$ by

$$\tau_x(v) := \sup\left\{ t > 0 \,\middle|\, \rho_x\big(\gamma_v(t)\big) = t, \gamma_v\big([0,t)\big) \subset \operatorname{Int} M \right\}, \tag{1.7}$$

where $\rho_x \colon M \to \mathbb{R}$ is the distance function from $x$ defined as $\rho_x(y) := d_M(x, y)$. Let $s_{f,v} \colon [0, \tau_x(v)] \to [0, \infty]$ be defined by

$$s_{f,v}(t) := \int_0^t e^{\frac{-2f(\gamma_v(a))}{n-1}} da. \tag{1.8}$$

For $t \in [0, \tau_x(v)]$ we put

$$F_{\kappa,v}(t) := \exp\left( \frac{f(\gamma_v(t)) + f(x)}{n-1} \right) \mathfrak{s}_\kappa(s_{f,v}(t)). \tag{1.9}$$

For $l > 0$, we denote by $[0, l] \times_{F_\kappa} \mathbb{S}^{n-1}$ the twisted product Riemannian manifold $\big([0, l] \times \mathbb{S}^{n-1}, dt^2 + F^2_{\kappa,v}(t) ds^2_{n-1}\big)$.

We further prove the following inscribed radius rigidity theorem.

**Theorem 1.2** *Let us assume that $\kappa$ and $\lambda$ satisfy the ball-condition. For $N \in (-\infty, 1]$, assume that $(M, \partial M, f)$ has lower $(\kappa, \lambda, N)$-weighted curvature bounds. Then we have*

$$\operatorname{InRad}_{g_f} M \leq C_{\kappa,\lambda}. \tag{1.10}$$

*If $\rho^{g_f}_{\partial M}(x_0) = C_{\kappa,\lambda}$ for some $x_0 \in M$, then $M$ is isometric to $[0, l] \times_{F_\kappa} \mathbb{S}^{n-1}$ for some $l > 0$; moreover, if $N \in (-\infty, 1)$, then $f$ is constant; in particular, $M$ is isometric to a closed ball in a space form.*

Kasue [9] proved Theorem 1.2 in the case of $f = 0$.

We will also obtain an inscribed radius rigidity theorem for InRad $M$ in the case where $f$ is bounded from above (see Theorem 6.4).

## 1.4 Volume Growths

We set $\overline{C}_{\kappa,\lambda} := C_{\kappa,\lambda}$ if $\kappa$ and $\lambda$ satisfy the ball-condition; otherwise, $\overline{C}_{\kappa,\lambda} := \infty$. We define functions $\bar{\mathfrak{s}}_{\kappa,\lambda}, \mathcal{S}_{\kappa,\lambda} \colon [0, \infty) \to \mathbb{R}$ by

$$\bar{\mathfrak{s}}_{\kappa,\lambda}(s) := \begin{cases} \mathfrak{s}_{\kappa,\lambda}(s) & \text{if } s < \overline{C}_{\kappa,\lambda}, \\ 0 & \text{if } s \geq \overline{C}_{\kappa,\lambda}, \end{cases} \quad \mathcal{S}_{\kappa,\lambda}(r) := \int_0^r \bar{\mathfrak{s}}^{n-1}_{\kappa,\lambda}(a) da. \tag{1.11}$$

For a smooth function $\phi \colon M \to \mathbb{R}$, we define $m_\phi := e^{-\phi} \operatorname{vol}_g$. For $x \in M$, we say that $z \in \partial M$ is a *foot point on $\partial M$ of $x$* if $d_M(x, z) = \rho_{\partial M}(x)$. Every point in $M$ has at

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least one foot point on $\partial M$. Let us define a function $\rho_{\partial M,f}\colon M \to \mathbb{R}$ by

$$\rho_{\partial M,f}(x) := \inf_{z} \int_0^{\rho_{\partial M}(x)} e^{\frac{-2f(\gamma_z(a))}{n-1}}\, da, \tag{1.12}$$

where the infimum is taken over all foot points $z \in \partial M$ of $x$. For $r > 0$,

$$B^f_r(\partial M) := \{x \in M \mid \rho_{\partial M,f}(x) \le r\}, \quad \operatorname{InRad}_f M := \sup_{x\in M} \rho_{\partial M,f}(x). \tag{1.13}$$

We prove absolute volume comparisons of Heintze–Karcher type [7] and relative volume comparisons (see Subsections 7.2 and 7.3).

One of the relative volume comparison theorems is the following.

***Theorem 1.3*** *For $N \in (-\infty, 1]$, assume that $(M, \partial M, f)$ has lower $(\kappa, \lambda, N)$-weighted curvature bounds. Let $\partial M$ be compact. Then for all $r, R > 0$ with $r \le R$, we have*

$$\frac{m_{\frac{n+1}{n-1}f}(B^f_R(\partial M))}{m_{\frac{n+1}{n-1}f}(B^f_r(\partial M))} \le \frac{\mathcal{S}_{\kappa,\lambda}(R)}{\mathcal{S}_{\kappa,\lambda}(r)}. \tag{1.14}$$

We provide a rigidity theorem concerning the equality case of Theorem 1.3 (see Theorem 7.11). We also present a volume growth rigidity theorem in the case where $f$ is bounded from above (see Theorem 7.13).

## 1.5 Eigenvalues

For $p \in [1, \infty)$ and a smooth function $\phi\colon M \to \mathbb{R}$, let $W^{1,p}_0(M, m_\phi)$ stand for the $(1, p)$-*Sobolev space with compact support* defined as the completion of $C^\infty_0(M)$ with respect to the standard $(1, p)$-Sobolev norm. The $(\phi, p)$-*Laplacian* is defined as

$$\Delta_{\phi,p} := -e^{\phi} \operatorname{div}\left( e^{-\phi} \|\nabla \cdot\|^{p-2} \nabla \cdot \right)$$

as a distribution on $W^{1,p}_0(M, m_\phi)$, where $\|\cdot\|$ is the standard norm, and div is the divergence with respect to $g$. A real number $\nu$ is said to be a $(\phi, p)$-*Dirichlet eigenvalue* on $M$ if there exists $\psi \in W^{1,p}_0(M, m_\phi) \setminus \{0\}$ such that $\Delta_{\phi,p}\psi = \nu |\psi|^{p-2}\psi$ holds in the distribution sense. For $\psi \in W^{1,p}_0(M, m_\phi) \setminus \{0\}$, the *Rayleigh quotient* is defined as

$$R_{\phi,p}(\psi) := \frac{\int_M \|\nabla\psi\|^p \, dm_\phi}{\int_M |\psi|^p \, dm_\phi}.$$

We study

$$\nu_{\phi,p}(M) := \inf_{\psi} R_{\phi,p}(\psi),$$

where the infimum is taken over all $\psi \in W^{1,p}_0(M, m_\phi) \setminus \{0\}$. If $M$ is compact and if $p \in (1, \infty)$, then $\nu_{\phi,p}(M)$ is equal to the infimum of the set of all $(\phi, p)$-Dirichlet eigenvalues on $M$.

Let $p \in (1,\infty)$. For $D \in (0,\overline{C}_{\kappa,\lambda}]\backslash\{\infty\}$, let $\nu_{p,\kappa,\lambda,D}$ be the positive minimum real number $\nu$ such that there exists a non-zero function $\varphi\colon [0,D] \to \mathbb{R}$ satisfying

$$(1.15)\quad \big(|\varphi'(s)|^{p-2}\varphi'(s)\big)' + (n-1)\frac{\mathfrak{s}'_{\kappa,\lambda}(s)}{\mathfrak{s}_{\kappa,\lambda}(s)}\big(|\varphi'(s)|^{p-2}\varphi'(s)\big) + \nu|\varphi(s)|^{p-2}\varphi(s) = 0, \varphi(0) = 0, \varphi'(D) = 0.$$

Let us recall the notion of the model spaces introduced by Kasue [10]. We say that $\kappa$ and $\lambda$ satisfy the *model-condition* if the equation $\mathfrak{s}'_{\kappa,\lambda}(s) = 0$ has a positive solution. Note that $\kappa$ and $\lambda$ satisfy the model-condition if and only if either (1) $\kappa > 0$ and $\lambda < 0$; (2) $\kappa = 0$ and $\lambda = 0$; or (3) $\kappa < 0$ and $\lambda \in (0,\sqrt{|\kappa|})$. Let $\kappa$ and $\lambda$ satisfy the ball-condition or the model-condition, and let $M$ be compact. For an interval $I$ and for a connected component $\partial M_1$ of $\partial M$, we denote by $I \times_{\kappa,\lambda} \partial M_1$ the warped product $(I \times \partial M_1, ds^2 + \mathfrak{s}^2_{\kappa,\lambda}(s)h)$. When $\kappa$ and $\lambda$ satisfy the model-condition, we define $D_{\kappa,\lambda}(M)$ as follows. If $\kappa = 0$ and $\lambda = 0$, then $D_{\kappa,\lambda}(M) := \operatorname{InRad} M$; otherwise, $D_{\kappa,\lambda}(M) := \inf\{s > 0 \mid \mathfrak{s}'_{\kappa,\lambda}(s) = 0\}$. We say that $M$ is a $(\kappa,\lambda)$-*equational model space* if $M$ is isometric to either (1) the closed geodesic ball $B^n_{\kappa,\lambda}$ for $\kappa$ and $\lambda$ satisfying the ball-condition; (2) the warped product $[0, 2D_{\kappa,\lambda}(M)] \times_{\kappa,\lambda} \partial M_1$ for $\kappa$ and $\lambda$ satisfying the model-condition, and for some connected component $\partial M_1$ of $\partial M$; or (3) the quotient space $([0, 2D_{\kappa,\lambda}(M)] \times_{\kappa,\lambda} \partial M)/G_\sigma$ for $\kappa$ and $\lambda$ satisfying the model-condition, and for some involutive isometry $\sigma$ of $\partial M$ without fixed points, where $G_\sigma$ denotes the isometry group on $[0, 2D_{\kappa,\lambda}(M)] \times_{\kappa,\lambda} \partial M$ whose elements consist of identity and the involute isometry $\widehat{\sigma}$ defined by $\widehat{\sigma}(s,z) := (2D_{\kappa,\lambda}(M) - s, \sigma(z))$.

We say that $f$ is $\partial M$-*radial* if there exists a smooth function $\phi_f\colon [0,\infty) \to \mathbb{R}$ such that $f = \phi_f \circ \rho_{\partial M}$ on $M$.

We establish the following theorem for the smallest eigenvalue $\nu_{\frac{n+1}{n-1}f,p}$.

***Theorem 1.4*** *Let $p \in (1,\infty)$. For $N \in (-\infty,1]$, let us assume that $(M,\partial M,f)$ has lower $(\kappa,\lambda,N)$-weighted curvature bounds. Let $M$ be compact and let $f$ be $\partial M$-radial. Suppose additionally that there exists $\delta \in \mathbb{R}$ such that $f \le (n-1)\delta$ on $M$. For $D \in (0,\overline{C}_{\kappa,\lambda}]\backslash\{\infty\}$, suppose* $\operatorname{InRad}_f M \le D$, *where* $\operatorname{InRad}_f M$ *is defined as* (1.13). *Then*

$$(1.16)\qquad \nu_{\frac{n+1}{n-1}f,p}(M) \ge \nu_{p,\kappa e^{-4\delta},\lambda e^{-2\delta},De^{2\delta}}.$$

*If the equality in* (1.16) *holds, then $M$ is a $(\kappa e^{-4\delta},\lambda e^{-2\delta})$-equational model space and $f = (n-1)\delta$ on $M$.*

In the case where $f = 0$ and $\delta = 0$, Kasue [10] proved Theorem 1.4 for $p = 2$, and the author [18] proved it for any $p \in (1,\infty)$.

We also formulate a rigidity theorem for the smallest eigenvalue $\nu_{f,p}$ in the case where $f$ is not necessarily $\partial M$-radial (see Theorem 8.8). Furthermore, we obtain a spectrum rigidity theorem for complete (not necessarily compact) manifolds with boundary (see Theorem 8.12).

### 1.6 Organization

In Section 2, we prepare some notation and recall the basic facts for Riemannian manifolds with boundary. We also recall the works by Wylie and Yeroshkin [24] (see Subsection 2.6).

In Sections 3 and 4, to prove our main theorems, we study Laplacian comparisons for the distance function from the boundary. In Section 3, we show a pointwise Laplacian comparison result (see Subsection 3.1) and a rigidity result in the equality case (see Subsection 3.2). In Section 4, we prove global Laplacian comparison inequalities in the distribution sense in the case where $f$ is bounded from above (see Subsection 4.1) and where $f$ is $\partial M$-radial (see Subsection 4.2).

In Section 5, we prove splitting theorems. In Section 6, we examine inscribed radius rigidity theorems. In Section 7, we show volume comparison theorems. In Section 8, we study eigenvalue rigidity theorems.

## 2 Preliminaries

We refer to [16] for the basics of Riemannian manifolds with boundary (see also [17, Section 2] and [18, 19]).

### 2.1 Riemannian Manifolds with Boundary

Let $M$ be a connected Riemannian manifold with boundary. For $r > 0$ and $A \subset M$, we denote by $B_r(A)$ the closed $r$-neighborhood of $A$. For $A_1, A_2 \subset M$, let $d_M(A_1, A_2)$ denote the distance between them. For an interval $I$, we say that a curve $\gamma\colon I \to M$ is a *minimal geodesic* if for all $t_1, t_2 \in I$ we have $d_M(\gamma(t_1), \gamma(t_2)) = |t_1 - t_2|$. If the metric space $(M, d_M)$ is complete, then the Hopf–Rinow theorem for length spaces (see *e.g.*, Theorem 2.5.23 in [3]) tells us that it is a proper geodesic space.

For $i = 1, 2$, let $(M_i, g_i)$ be connected Riemannian manifolds with boundary. For each $i$, the boundary $\partial M_i$ carries the induced metric $h_i$. We say that a homeomorphism $\Phi\colon M_1 \to M_2$ is a *Riemannian isometry with boundary* if $\Phi$ satisfies the following conditions:

(1) $\Phi|_{\mathrm{Int}\, M_1}\colon \mathrm{Int}\, M_1 \to \mathrm{Int}\, M_2$ is smooth, and $(\Phi|_{\mathrm{Int}\, M_1})^*(g_2) = g_1$;
(2) $\Phi|_{\partial M_1}\colon \partial M_1 \to \partial M_2$ is smooth, and $(\Phi|_{\partial M_1})^*(h_2) = h_1$.

There exists a Riemannian isometry with boundary from $M_1$ to $M_2$ if and only if $(M_1, d_{M_1})$ and $(M_2, d_{M_2})$ are isometric to each other.

### 2.2 Jacobi Fields

Let $(M, g)$ be a connected Riemannian manifold with boundary. For $x \in \mathrm{Int}\, M$, let $T_xM$ and $U_xM$ be the tangent space and the unit tangent sphere at $x$, respectively. For $z \in \partial M$ and the tangent space $T_z\partial M$ at $z$ on $\partial M$, let $T_z^{\perp}\partial M$ be its orthogonal complement in the tangent space at $z$ on $M$.

For vector fields $v_1, v_2$ on $\partial M$, the second fundamental form $S(v_1, v_2)$ is defined as the normal component of $\nabla^g_{v_1} v_2$ with respect to $\partial M$, where $\nabla^g$ is the Levi–Civita connection induced from $g$. For $u \in T_z^{\perp}\partial M$, the shape operator $A_u\colon T_z\partial M \to T_z\partial M$

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is defined as

$$g(A_u v_1, v_2) := g\big(S(v_1, v_2), u\big).$$

For the unit inner normal vector $u_z$ on $\partial M$ at $z$, the mean curvature $H_z$ of $\partial M$ at $z$ is defined as the trace of $A_{u_z}$. We say that a Jacobi field $Y$ along the geodesic $\gamma_z$ is a *$\partial M$-Jacobi field* if $Y$ satisfies

$$Y(0) \in T_z\partial M, \quad Y'(0) + A_{u_z}Y(0) \in T_z^{\perp}\partial M.$$

We say that $\gamma_z(t_0)$ is a *conjugate point* of $\partial M$ along $\gamma_z$ if there exists a non-zero $\partial M$-Jacobi field $Y$ along $\gamma_z$ such that $Y(t_0) = 0$.

## 2.3 Cut Locus for the Boundary

We recall the basic properties of the cut locus for the boundary. We refer to [17] for the proofs.

Let $(M, g)$ be a connected complete Riemannian manifold with boundary. For $x \in \operatorname{Int} M$, let $z \in \partial M$ be a foot point on $\partial M$ of $x$ (*i.e.*, $d_M(x,z) = \rho_{\partial M}(x)$). In this case, there exists a unique minimal geodesic $\gamma\colon [0,l] \to M$ from $z$ to $x$ such that $\gamma = \gamma_z|_{[0,l]}$, where $l = \rho_{\partial M}(x)$. In particular, $\gamma'(0) = u_z$ and $\gamma|_{(0,l]}$ lies in $\operatorname{Int} M$.

Let $\tau\colon \partial M \to (0,\infty]$ be the function defined as (1.4)). The supremum of $\tau$ over $\partial M$ is equal to the inscribed radius $\operatorname{InRad} M$. The function $\tau$ is continuous on $\partial M$. The continuity of $\tau$ tells us that if $\partial M$ is compact, then $\operatorname{InRad} M < \infty$ if and only if $M$ is compact.

The *cut locus for the boundary* is defined as

$$\operatorname{Cut}\partial M := \big\{\, \gamma_z(\tau(z)) \mid z \in \partial M, \tau(z) < \infty \big\}.$$

From the continuity of $\tau$ we see that $\operatorname{Cut}\partial M$ is a null set of $M$. For $x \in \operatorname{Int} M \setminus \operatorname{Cut}\partial M$, its foot point on $\partial M$ is uniquely determined.

We know the following lemma from [17].

***Lemma 2.1*** *If there exists a connected component $\partial M_0$ of $\partial M$ such that $\tau = \infty$ on $\partial M_0$, then $\partial M$ is connected and* $\operatorname{Cut}\partial M = \varnothing$.

The function $\rho_{\partial M}$ is smooth on $\operatorname{Int} M \setminus \operatorname{Cut}\partial M$. For each $x \in \operatorname{Int} M \setminus \operatorname{Cut}\partial M$, we have $\nabla\rho_{\partial M}(x) = \gamma'(l)$, where $\gamma\colon [0,l] \to M$ is the minimal geodesic from the foot point on $\partial M$ of $x$ to $x$.

For $\Omega \subset M$, we denote by $\overline{\Omega}$ its closure, and by $\partial\Omega$ its boundary. For a domain $\Omega$ in $M$ such that $\partial\Omega$ is a smooth hypersurface in $M$, we denote by $\operatorname{vol}_{\partial\Omega}$ the canonical Riemannian volume measure on $\partial\Omega$.

We recall the following fact to avoid the cut locus for the boundary (see [18, Lemma 2.6]).

***Lemma 2.2*** *Let $\Omega \subset M$ be a domain such that $\partial\Omega$ is a smooth hypersurface in $M$. Then there exists a sequence $\{\Omega_i\}$ of closed subsets of $\overline{\Omega}$ such that for every $i$, the set $\partial\Omega_i$ is a smooth hypersurface in $M$, except for a null set in $(\partial\Omega, \operatorname{vol}_{\partial\Omega})$, satisfying the following properties:*

(i) *for all $i_1, i_2$ with $i_1 < i_2$, we have $\Omega_{i_1} \subset \Omega_{i_2}$;*

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(ii) $\overline{\Omega} \setminus \mathrm{Cut}\, \partial M = \bigcup_i \Omega_i$;

(iii) *for every* $i$, *and for almost every* $x \in \partial \Omega_i \cap \partial \Omega$ *in* $(\partial \Omega, \mathrm{vol}_{\partial \Omega})$, *there exists a unique unit outer normal vector for* $\Omega_i$ *at* $x$ *that coincides with the unit outer normal vector on* $\partial \Omega$ *for* $\Omega$ *at* $x$;

(iv) *for every* $i$, *on* $\partial \Omega_i \setminus \partial \Omega$, *there exists a unique unit outer normal vector field* $u_i$ *for* $\Omega_i$ *such that* $g(u_i, \nabla \rho_{\partial M}) \geq 0$.

*Moreover, if* $\overline{\Omega} = M$, *then for every* $i$, *the set* $\partial \Omega_i$ *is a smooth hypersurface in* $M$ *and satisfies* $\partial \Omega_i \cap \partial M = \partial M$.

## 2.4 Busemann Functions and Asymptotes

Let $M$ be a connected complete Riemannian manifold with boundary. A minimal geodesic $\gamma\colon [0, \infty) \to M$ is said to be a *ray*. For a ray $\gamma\colon [0, \infty) \to M$, the *Busemann function* $b_\gamma\colon M \to \mathbb{R}$ is defined as

$$b_\gamma(x) := \lim_{t \to \infty} \big( t - d_M(x, \gamma(t)) \big).$$

We have the following lemma (see [17, Lemma 6.1]).

***Lemma 2.3*** *Suppose that for some* $z \in \partial M$, *we have* $\tau(z) = \infty$. *Take* $x \in \mathrm{Int}\, M$. *If* $b_{\gamma_z}(x) = \rho_{\partial M}(x)$, *then* $x \notin \mathrm{Cut}\, \partial M$. *Moreover, for the unique foot point* $z_x$ *on* $\partial M$ *of* $x$, *we have* $\tau(z_x) = \infty$.

Let $\gamma\colon [0, \infty) \to M$ be a ray. For $x \in M$, we say that a ray $\gamma_x\colon [0, \infty) \to M$ is an *asymptote for* $\gamma$ *from* $x$ if there exists a sequence $\{t_i\}$ with $t_i \to \infty$ such that the following holds: For each $i$, there exists a minimal geodesic $\gamma_i\colon [0, l_i] \to M$ from $x$ to $\gamma(t_i)$ such that for every $t \geq 0$ we have $\gamma_i(t) \to \gamma_x(t)$ as $i \to \infty$. Since $M$ is proper, for each $x \in M$ there exists at least one asymptote for $\gamma$ from $x$.

For asymptotes, we see the following lemma (see [17, Lemma 6.2]).

***Lemma 2.4*** *Suppose that for some* $z \in \partial M$, *we have* $\tau(z) = \infty$. *For* $l > 0$, *put* $x := \gamma_z(l)$. *Then there exists* $\epsilon > 0$ *such that for all* $y \in B_\epsilon(x)$, *all asymptotes for the ray* $\gamma_z$ *from* $y$ *lie in* $\mathrm{Int}\, M$.

## 2.5 Weighted Manifolds with Boundary

Let $(M, g)$ be a connected complete Riemannian manifold with boundary, and let $f\colon M \to \mathbb{R}$ be a smooth function. The *weighted Laplacian* $\Delta_f$ is defined by

$$\Delta_f := \Delta + g(\nabla f, \nabla \cdot),$$

where $\Delta$ is the Laplacian defined as the minus of the trace of the Hessian. Note that $\Delta_f$ coincides with the $(f, 2)$-Laplacian $\Delta_{f,2}$.

The following formula of Bochner type is well known (see *e.g.*, [21]).

***Proposition 2.5*** *For every smooth function* $\psi$ *on* $M$, *we have*

$$-\frac{1}{2} \Delta_f \|\nabla \psi\|^2 = \mathrm{Ric}_f^\infty(\nabla \psi) + \|\,\mathrm{Hess}\, \psi\|^2 - g(\nabla \Delta_f \psi, \nabla \psi),$$

*where* $\|\,\mathrm{Hess}\, \psi\|$ *is the Hilbert-Schmidt norm of* $\mathrm{Hess}\, \psi$.

For $z \in \partial M$, the value $\Delta_f \rho_{\partial M}(\gamma_z(t))$ tends to $H_{f,z}$ as $t \to 0$. For $t \in (0, \tau(z))$, and for the volume element $\theta(t,z)$ of the $t$-level surface of $\rho_{\partial M}$ at $\gamma_z(t)$, we put

$$\theta_f(t,z) := e^{-f(\gamma_z(t))}\theta(t,z). \tag{2.1}$$

For all $t \in (0, \tau(z))$, it holds that

$$\Delta_f \rho_{\partial M}(\gamma_z(t)) = -\frac{\theta_f'(t,z)}{\theta_f(t,z)}. \tag{2.2}$$

We further define a function $\overline{\theta}_f\colon [0,\infty) \times \partial M \to \mathbb{R}$ by

$$\overline{\theta}_f(t,z) := \begin{cases} \theta_f(t,z) & \text{if } t < \tau(z), \\ 0 & \text{if } t \geq \tau(z). \end{cases}$$

The following lemma was shown in [18].

***Lemma 2.6*** *If $\partial M$ is compact, then for all $r > 0$,*

$$m_f\big(B_r(\partial M)\big) = \int_{\partial M} \int_0^r \overline{\theta}_f(t,z)\, dt\, d\,\mathrm{vol}_h,$$

*where $\mathrm{vol}_h$ is the Riemannian volume measure on $\partial M$ induced from $h$.*

Let $\psi\colon M \to \mathbb{R}$ be a continuous function, and let $U$ be a domain contained in $\mathrm{Int}\, M$. For $x \in U$ and for a function $\widehat{\psi}$ defined on an open neighborhood of $x$, we say that $\widehat{\psi}$ is a *support function of* $\psi$ *at* $x$ if we have $\widehat{\psi}(x) = \psi(x)$ and $\widehat{\psi} \leq \psi$. We say that $\psi$ is *$f$-subharmonic on $U$* if for every $x \in U$, and for every $\epsilon > 0$, there exists a smooth support function $\psi_{x,\epsilon}$ of $\psi$ at $x$ such that $\Delta_f \psi_{x,\epsilon}(x) \leq \epsilon$.

We recall the following maximal principle (see *e.g.*, [4]).

***Lemma 2.7*** ([4]) *Let $U$ be a domain contained in* $\mathrm{Int}\, M$. *If an $f$-subharmonic function on $U$ takes the maximal value at a point in $U$, then it must be constant on $U$.*

## 2.6 Laplacian Comparisons from a Single Point

We recall the works done by Wylie and Yeroshkin [24]. Let $M$ be an $n$-dimensional, connected complete Riemannian manifold with boundary, and let $f\colon M \to \mathbb{R}$ be a smooth function. For the diameter $C_\kappa$ of the space form $M^n_\kappa$, we define a function $H_\kappa\colon (0, C_\kappa) \to \mathbb{R}$ by

$$H_\kappa(s) := -(n-1)\frac{\mathfrak{s}_\kappa'(s)}{\mathfrak{s}_\kappa(s)}. \tag{2.3}$$

Wylie and Yeroshkin [24] proved a Laplacian comparison inequality for the distance function from a single point (see [24, Theorem 4.4]). In our setting, the inequality holds in the following form.

***Lemma 2.8*** ([24]) *Let $x \in$ $\mathrm{Int}\, M$ and $v \in U_x M$. For $N \in (-\infty, 1]$, assume that* $\mathrm{Ric}^N_{f,M} \geq (n-1)\kappa e^{\frac{-4f}{n-1}}$. *Then for all $t \in (0, \tau_x(v))$,*

$$\Delta_f \rho_x(\gamma_v(t)) \geq H_\kappa(s_{f,v}(t))\, e^{\frac{-2f(\gamma_v(t))}{n-1}}, \tag{2.4}$$

*where $\tau_x$ and $s_{f,v}$ are defined as (1.7) and as (1.8), respectively.*

As a corollary of the Laplacian comparison inequality, Wylie and Yeroshkin have shown another Laplacian comparison inequality in the case where $f$ is bounded (see [24, Corollary 4.11]). In our setting, by using the same method for the proof, we see the following lemma.

***Lemma 2.9*** ([24]) *Let $x \in \operatorname{Int} M$ and $v \in U_x M$. For $N \in (-\infty, 1]$, assume that $\operatorname{Ric}^N_{f,M} \geq (n-1)\kappa e^{\frac{-4f}{n-1}}$. Suppose additionally that there is $\delta \in \mathbb{R}$ such that $f \leq (n-1)\delta$ on $M$. Then for all $t \in (0, \tau_x(v))$, we have*

$$\Delta_f \rho_x(\gamma_v(t)) \geq H_\kappa(e^{-2\delta} t) e^{\frac{-2f(\gamma_v(t))}{n-1}}. \tag{2.5}$$

**Proof** From $f \leq (n-1)\delta$, we deduce that $s_{f,v}(t) \geq e^{-2\delta} t$ for every $t \in (0, \tau_x(v))$. We see $H'_\kappa > 0$ on $(0, C_\kappa)$, and hence (2.4) implies

$$\Delta_f \rho_x(\gamma_v(t)) \geq H_\kappa(s_{f,v}(t)) e^{\frac{-2f(\gamma_v(t))}{n-1}} \geq H_\kappa(e^{-2\delta} t) e^{\frac{-2f(\gamma_v(t))}{n-1}}. \tag{2.6}$$

This proves (2.5). ∎

Wylie and Yeroshkin proved a rigidity result in the equality case of the Laplacian comparison inequality (see [24, Lemma 4.13]). From the argument discussed in their proof, one can derive the following lemma.

***Lemma 2.10*** ([24]) *Under the same setting as Lemma 2.8, assume that for some $t_0 \in (0, \tau_x(v))$, the equality in (2.4) holds. Choose an orthonormal basis $\{e_{v,i}\}_{i=1}^n$ of $T_x M$ with $e_{v,n} = v$. Let $\{Y_{v,i}\}_{i=1}^{n-1}$ be the Jacobi fields along $\gamma_v$ with $Y_{v,i}(0) = 0_x$, $Y'_{v,i}(0) = e_{v,i}$. Then for all $i$ we have $Y_{v,i} = F_{\kappa,v} E_{v,i}$ on $[0, t_0]$, where $F_{\kappa,v}$ is defined as (1.9), and $\{E_{v,i}\}_{i=1}^{n-1}$ are the parallel vector fields with $E_{v,i}(0) = e_{v,i}$; moreover, if $N \in (-\infty, 1)$, then $f \circ \gamma_v$ is constant on $[0, t_0]$.*

***Remark 2.11*** Under the same setting as Lemma 2.9, assume that for some $t_0 \in (0, \tau_x(v))$, the equality in (2.5) holds. Then the equalities in (2.6) hold. In particular, the equality in (2.4) holds (see Lemma 2.10), and $s_{f,v}(t_0) = e^{-2\delta} t_0$, and hence $f \circ \gamma_v = (n-1)\delta$ on $[0, t_0]$.

## 3 Laplacian Comparisons

Hereafter, let $(M, g)$ be an $n$-dimensional, connected complete Riemannian manifold with boundary, and let $f \colon M \to \mathbb{R}$ be smooth.

### 3.1 Basic Laplacian Comparisons

For the distance function from a single point, Wylie and Yeroshkin have shown an inequality of Riccati type (see [24, Lemma 4.1]). Using the same method as for the proof for the distance function from the boundary, we have the following lemma.

***Lemma 3.1*** *Let $z \in \partial M$ and $N \in (-\infty, 1]$. Then for all $t \in (0, \tau(z))$,*

$$\Big( \big(e^{\frac{2f}{n-1}} \Delta_f \rho_{\partial M}\big)(\gamma_z(t)) \Big)' \geq \operatorname{Ric}_f^N(\gamma_z'(t)) e^{\frac{2f(\gamma_z(t))}{n-1}} + \frac{\Big( \big(e^{\frac{2f}{n-1}} \Delta_f \rho_{\partial M}\big)(\gamma_z(t)) \Big)^2}{n-1} e^{\frac{-2f(\gamma_z(t))}{n-1}}. \tag{3.1}$$

**Proof** Put $f_z := f \circ \gamma_z$ and $h_{f,z} := (\Delta_f \rho_{\partial M}) \circ \gamma_z$. Applying Proposition 2.5 to the distance function $\rho_{\partial M}$, we have

$$\begin{aligned} 0 &= \operatorname{Ric}_f^\infty(\gamma_z'(t)) + \|\operatorname{Hess} \rho_{\partial M}\|^2(\gamma_z(t)) - g(\nabla \Delta_f \rho_{\partial M}, \nabla \rho_{\partial M})(\gamma_z(t)) \\ &= \Big( \operatorname{Ric}_f^N(\gamma_z'(t)) + \frac{f_z'(t)^2}{N-n} \Big) + \|\operatorname{Hess} \rho_{\partial M}\|^2(\gamma_z(t)) - h_{f,z}'(t). \end{aligned}$$

By the Cauchy–Schwarz inequality,

$$\|\operatorname{Hess} \rho_{\partial M}\|^2(\gamma_z(t)) \geq \frac{\big(\Delta \rho_{\partial M}(\gamma_z(t))\big)^2}{n-1} = \frac{\big(h_{f,z}(t) - f_z'(t)\big)^2}{n-1}. \tag{3.2}$$

Inequality (3.2) yields

$$\begin{aligned} 0 &\geq \operatorname{Ric}_f^N(\gamma_z'(t)) + \frac{f_z'(t)^2}{N-n} + \frac{(h_{f,z}(t) - f_z'(t))^2}{n-1} - h_{f,z}'(t) \\ &= \operatorname{Ric}_f^N(\gamma_z'(t)) + \frac{(1-N) f_z'(t)^2}{(n-1)(n-N)} + \frac{h_{f,z}^2(t)}{n-1} \\ &\quad - \Big( \frac{2h_{f,z}(t) f_z'(t)}{n-1} + h_{f,z}'(t) \Big). \end{aligned} \tag{3.3}$$

The last term in the right-hand side of (3.3) satisfies

$$\frac{2h_{f,z}(t) f_z'(t)}{n-1} + h_{f,z}'(t) = e^{\frac{-2f(\gamma_z(t))}{n-1}} \Big( e^{\frac{2f(\gamma_z(t))}{n-1}} h_{f,z}(t) \Big)'.$$

We put $F_z(t) := e^{\frac{2f(\gamma_z(t))}{n-1}} h_{f,z}(t)$. From $N \in (-\infty, 1]$, it follows that

$$\begin{aligned} 0 &\geq \operatorname{Ric}_f^N(\gamma_z'(t)) + \frac{(1-N) f_z'(t)^2}{(n-1)(n-N)} + \frac{h_{f,z}^2(t)}{n-1} - e^{\frac{-2f(\gamma_z(t))}{n-1}} F_z'(t) \\ &\geq \operatorname{Ric}_f^N(\gamma_z'(t)) + \frac{h_{f,z}^2(t)}{n-1} - e^{\frac{-2f(\gamma_z(t))}{n-1}} F_z'(t). \end{aligned} \tag{3.4}$$

This implies that

$$\begin{aligned} F_z'(t) &\geq e^{\frac{2f(\gamma_z(t))}{n-1}} \Big( \operatorname{Ric}_f^N(\gamma_z'(t)) + \frac{h_{f,z}^2(t)}{n-1} \Big) \\ &= \operatorname{Ric}_f^N(\gamma_z'(t)) e^{\frac{2f(\gamma_z(t))}{n-1}} + \frac{F_z^2(t)}{n-1} e^{\frac{-2f(\gamma_z(t))}{n-1}}. \end{aligned}$$

We arrive at the desired inequality (3.1). ∎

***Remark 3.2*** We assume that the equality in (3.1) holds for some $t_0 \in (0, \tau(z))$. Then the equality in the Cauchy–Schwarz inequality in (3.2) holds; in particular, there

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exists a constant $c$ such that $\operatorname{Hess}\rho_{\partial M} = cg$ on the orthogonal complement of $\nabla\rho_{\partial M}$ at $\gamma_z(t_0)$. Moreover, the equalities in (3.4) hold; in particular, $(1-N)f_z'(t_0)^2 = 0$.

Recall that $\tau_f$ and $s_{f,z}$ are defined as (1.4) and (1.5), respectively. We denote by $t_{f,z}\colon [0,\tau_f(z)] \to [0,\tau(z)]$ the inverse function of $s_{f,z}$.

We define a function $H_{\kappa,\lambda}\colon [0,\overline{C}_{\kappa,\lambda}) \to \mathbb{R}$ by

$$H_{\kappa,\lambda}(s) := -(n-1)\frac{\mathfrak{s}'_{\kappa,\lambda}(s)}{\mathfrak{s}_{\kappa,\lambda}(s)}. \tag{3.5}$$

For all $s \in [0,\overline{C}_{\kappa,\lambda})$, we see that

$$H'_{\kappa,\lambda}(s) = (n-1)\kappa + \frac{H^2_{\kappa,\lambda}(s)}{n-1}. \tag{3.6}$$

We prove the following pointwise Laplacian comparison inequality.

***Lemma 3.3*** *Let $z \in \partial M$. For $N \in (-\infty,1]$, let us assume that*

$$\operatorname{Ric}^N_f(\gamma_z'(t)) \geq (n-1)\kappa e^{\frac{-4f(\gamma_z(t))}{n-1}}$$

*for all $t \in (0,\tau(z))$, and $H_{f,z} \geq (n-1)\lambda e^{\frac{-2f(z)}{n-1}}$. Then for all*

$$s \in \left(0, \min\{\tau_f(z), \overline{C}_{\kappa,\lambda}\}\right),$$

*we have*

$$\Delta_f\rho_{\partial M}\big(\gamma_z(t_{f,z}(s))\big) \geq H_{\kappa,\lambda}(s)e^{\frac{-2f(\gamma_z(t_{f,z}(s)))}{n-1}}. \tag{3.7}$$

*In particular, for all $t \in (0,\tau(z))$ with $s_{f,z}(t) \in (0,\min\{\tau_f(z),\overline{C}_{\kappa,\lambda}\})$,*

$$\Delta_f\rho_{\partial M}(\gamma_z(t)) \geq H_{\kappa,\lambda}(s_{f,z}(t))e^{\frac{-2f(\gamma_z(t))}{n-1}}. \tag{3.8}$$

**Proof** We define a function $F_z\colon (0,\tau(z)) \to \mathbb{R}$ by

$$F_z := \left(e^{\frac{2f}{n-1}}\Delta_f\rho_{\partial M}\right)\circ\gamma_z,$$

and a function $\widehat{F}_z\colon (0,\tau_f(z)) \to \mathbb{R}$ by $\widehat{F}_z := F_z \circ t_{f,z}$. By Lemma 3.1 and the curvature assumption, for all $s \in (0,\tau_f(z))$,

$$\begin{aligned} \widehat{F}_z'(s) &= F_z'(t_{f,z}(s))e^{\frac{2f(\gamma_z(t_{f,z}(s)))}{n-1}} \\ &\geq \operatorname{Ric}^N_f(\gamma_z'(t_{f,z}(s)))e^{\frac{4f(\gamma_z(t_{f,z}(s)))}{n-1}} + \frac{F_z^2(t_{f,z}(s))}{n-1} \\ &\geq (n-1)\kappa + \frac{\widehat{F}_z^2(s)}{n-1}. \end{aligned} \tag{3.9}$$

The identity (3.6) implies that for all $s \in (0,\min\{\tau_f(z),\overline{C}_{\kappa,\lambda}\})$,

$$\widehat{F}_z'(s) - H'_{\kappa,\lambda}(s) \geq \frac{\widehat{F}_z^2(s) - H^2_{\kappa,\lambda}(s)}{n-1}. \tag{3.10}$$

Let us define a function $G_{\kappa,\lambda,z}\colon (0,\min\{\tau_f(z),\overline{C}_{\kappa,\lambda}\}) \to \mathbb{R}$ by

$$G_{\kappa,\lambda,z} := \mathfrak{s}^2_{\kappa,\lambda}\left(\widehat{F}_z - H_{\kappa,\lambda}\right).$$

From (3.10) we deduce

$$G'_{\kappa,\lambda,z} = 2\mathfrak{s}_{\kappa,\lambda}\mathfrak{s}'_{\kappa,\lambda}\left(\widehat{F}_z - H_{\kappa,\lambda}\right) + \mathfrak{s}^2_{\kappa,\lambda}\left(\widehat{F}'_z - H'_{\kappa,\lambda}\right) \geq 2\mathfrak{s}_{\kappa,\lambda}\mathfrak{s}'_{\kappa,\lambda}\left(\widehat{F}_z - H_{\kappa,\lambda}\right) + \mathfrak{s}^2_{\kappa,\lambda}\frac{\widehat{F}^2_z - H^2_{\kappa,\lambda}}{n-1} = \frac{\mathfrak{s}^2_{\kappa,\lambda}}{n-1}\left(\widehat{F}_z - H_{\kappa,\lambda}\right)^2 \geq 0. \tag{3.11}$$

Since $G_{\kappa,\lambda,z}(s)$ converges to a non-negative value $e^{\frac{2f(z)}{n-1}} H_{f,z} - (n-1)\lambda$ as $s \to 0$, the function $G_{\kappa,\lambda,z}$ is non-negative. We conclude that $\widehat{F}_z \geq H_{\kappa,\lambda}$ holds on $(0, \min\{\tau_f(z), \overline{C}_{\kappa,\lambda}\})$, and hence (3.7). ■

***Remark 3.4*** We assume that the equality in (3.7) holds for some $s_0 \in (0, \min\{\tau_f(z), \overline{C}_{\kappa,\lambda}\})$. Then we have $G_{\kappa,\lambda,z}(s_0) = 0$. From $G'_{\kappa,\lambda,z} \geq 0$ it follows that $G_{\kappa,\lambda,z} = 0$ on $[0, s_0]$; in particular, the equality in (3.7) holds on $[0, s_0]$. Since the equalities in (3.9), (3.10), (3.11) hold, the equality in (3.1) holds on $[0, t_{f,z}(s_0)]$ (see Remark 3.2).

From Lemma 3.3 we derive the following estimate for $\tau_f$.

***Lemma 3.5*** *Let $z \in \partial M$. Let $\kappa$ and $\lambda$ satisfy the ball-condition. For $N \in (-\infty, 1]$, we assume that $\mathrm{Ric}^N_f(\gamma'_z(t)) \geq (n-1)\kappa e^{\frac{-4f(\gamma_z(t))}{n-1}}$ for all $t \in (0, \tau(z))$, and $H_{f,z} \geq (n-1)\lambda e^{\frac{-2f(z)}{n-1}}$. Then we have*

$$\tau_f(z) \leq C_{\kappa,\lambda}. \tag{3.12}$$

*Moreover, if there is $\delta \in \mathbb{R}$ such that $f \circ \gamma_z \leq (n-1)\delta$ on $(0, \tau(z))$, then*

$$\tau(z) \leq C_{\kappa e^{-4\delta}, \lambda e^{-2\delta}}. \tag{3.13}$$

**Proof** The proof is by contradiction. Suppose $\tau_f(z) > C_{\kappa,\lambda}$. Then we see that $\tau(z) > t_{f,z}(C_{\kappa,\lambda})$. By (3.8), for every $t \in (0, t_{f,z}(C_{\kappa,\lambda}))$

$$\Delta_f \rho_{\partial M}(\gamma_z(t)) \geq H_{\kappa,\lambda}(s_{f,z}(t)) e^{\frac{-2f(\gamma_z(t))}{n-1}};$$

in particular, $\Delta_f \rho_{\partial M}(\gamma_z(t)) \to \infty$ as $t \to t_{f,z}(C_{\kappa,\lambda})$. This contradicts the smoothness of $\rho_{\partial M} \circ \gamma_z$ on $(0, \tau(z))$. Equation (3.12) follows. If $f \circ \gamma_z \leq (n-1)\delta$, then we have $e^{-2\delta}\tau(z) \leq \tau_f(z)$. By $e^{2\delta} C_{\kappa,\lambda} = C_{\kappa e^{-4\delta}, \lambda e^{-2\delta}}$, we arrive at the desired inequality (3.13). ■

***Remark 3.6*** Lemma 3.5 enables us to restate the conclusion of Lemma 3.3 as follows: For all $s \in (0, \tau_f(z))$, we have (3.7). In particular, for all $t \in (0, \tau(z))$, we have (3.8).

## 3.2 Equality Cases

Recall the following (see *e.g.*, [14, Theorem 2]).

***Lemma 3.7*** *Let $\rho$ be a smooth function defined on a domain in $M$ such that $\|\nabla \rho\| = 1$. Let $X$ be a parallel vector field along an integral curve of $\nabla \rho$ that is orthogonal to $\nabla \rho$.*

*Then we have*

$$g\big(R(X,\nabla\rho)\nabla\rho,X\big)=g(\nabla_{\nabla\rho}A_{\nabla\rho}X,X)-g(A_{\nabla\rho}A_{\nabla\rho}X,X),$$

*where $R$ is the curvature tensor induced from $g$, and $A_{\nabla\rho}$ is the shape operator of the level set of $\rho$ toward $\nabla\rho$. In particular, if there exists a function $\varphi$ defined on the domain of the integral curve such that $A_{\nabla\rho}X=-\varphi X$, then $g(R(X,\nabla\rho)\nabla\rho,X)=-(\varphi'+\varphi^2)\|X\|^2$.*

For the equality case of (3.7) in Lemma 3.3, we have the following lemma.

**Lemma 3.8** *Under the same setting as Lemma 3.3, assume that for some $s_0\in(0,\tau_f(z))$, the equality in (3.7) holds. Choose an orthonormal basis $\{e_{z,i}\}_{i=1}^{n-1}$ of $T_z\partial M$, and let $\{Y_{z,i}\}_{i=1}^{n-1}$ be the $\partial M$-Jacobi fields along $\gamma_z$ with $Y_{z,i}(0)=e_{z,i}$, $Y'_{z,i}(0)=-A_{u_z}e_{z,i}$. Then for all $i$, we have $Y_{z,i}=F_{\kappa,\lambda,z}E_{z,i}$ on $[0,t_{f,z}(s_0)]$, where $F_{\kappa,\lambda,z}$ is defined as (1.6), and $\{E_{z,i}\}_{i=1}^{n-1}$ are the parallel vector fields with $E_{z,i}(0)=e_{z,i}$; moreover, if $N\in(-\infty,1)$, then $f\circ\gamma_z$ is constant on $[0,t_{f,z}(s_0)]$.*

**Proof** Put $t_0:=t_{f,z}(s_0)$. Since the equality in (3.7) holds at $s_0$, the equality in (3.1) also holds on $[0,t_0]$ (see Remark 3.2). There exists a function $\varphi$ on the set $\gamma_z((0,t_0))$ such that at each point on $\gamma_z((0,t_0))$, we have $\operatorname{Hess}\rho_{\partial M}=\varphi g$ on the orthogonal complement of $\nabla\rho_{\partial M}$ (see Remark 3.4). Define $\varphi_z:=\varphi\circ\gamma_z$. For each $i$, it holds that

$$g(A_{\nabla\rho_{\partial M}}E_{z,i},E_{z,i})=-\operatorname{Hess}\rho_{\partial M}(E_{z,i},E_{z,i})=-\varphi_z;$$

in particular, $A_{\nabla\rho_{\partial M}}E_{z,i}=-\varphi_zE_{z,i}$. From Lemma 3.7 we deduce

$$R(E_{z,i},\nabla\rho_{\partial M})\nabla\rho_{\partial M}=-(\varphi_z'+\varphi_z^2)E_{z,i}=-\mathcal{F}_z''\mathcal{F}_z^{-1}E_{z,i},\tag{3.14}$$

where $\mathcal{F}_z\colon[0,t_0]\to\mathbb{R}$ is a function defined by

$$\mathcal{F}_z(t):=\exp\Big(\int_0^t\varphi_z(a)da\Big).$$

Set $f_z:=f\circ\gamma_z$ and $h_{f,z}:=(\Delta_f\rho_{\partial M})\circ\gamma_z$. By the equality assumption, $e^{\frac{2f_z}{n-1}}h_{f,z}=H_{\kappa,\lambda}\circ s_{f,z}$ on $[0,t_0]$ (see Remarks 3.2 and 3.4). Furthermore, $\operatorname{Hess}\rho_{\partial M}=\varphi g$ leads to $\Delta\rho_{\partial M}\circ\gamma_z=-(n-1)\varphi_z$. Therefore,

$$\varphi_z(t)=\frac{1}{n-1}\Big(f_z(t)-\int_0^te^{\frac{-2f(\gamma_z(a))}{n-1}}\big(H_{\kappa,\lambda}\circ s_{f,z}\big)(a)da\Big)'$$

for every $t\in[0,t_0]$. It follows that $\mathcal{F}_z=F_{\kappa,\lambda,z}$ on $[0,t_0]$. In view of (3.14), we obtain $Y_{z,i}=F_{\kappa,\lambda,z}E_{z,i}$ on $[0,t_0]$.

We have $(1-N)(f_z')^2=0$ on $[0,t_0]$ (see Remarks 3.2 and 3.4). If $N\in(-\infty,1)$, then $f_z'=0$ on $[0,t_0]$; in particular, $f_z$ is constant. ∎

For the equality case of (3.8), Lemma 3.8 implies the following lemma.

**Lemma 3.9** *Under the same setting as in Lemma 3.3, assume that for some $t_0\in(0,\tau(z))$ the equality in (3.8) holds. Choose an orthonormal basis $\{e_{z,i}\}_{i=1}^{n-1}$ of $T_z\partial M$, and let $\{Y_{z,i}\}_{i=1}^{n-1}$ be the $\partial M$-Jacobi fields along $\gamma_z$ with $Y_{z,i}(0)=e_{z,i}$, $Y'_{z,i}(0)=-A_{u_z}e_{z,i}$. Then for all $i$ we have $Y_{z,i}=F_{\kappa,\lambda,z}E_{z,i}$ on $[0,t_0]$, where $\{E_{z,i}\}_{i=1}^{n-1}$ are the parallel vector fields with $E_{z,i}(0)=e_{z,i}$; moreover, if $N\in(-\infty,1)$, then $f\circ\gamma_z$ is constant.*

## 4 Global Laplacian Comparisons

We start by introducing some conditions. Let us recall that $\kappa$ and $\lambda$ satisfy the *ball-condition* if and only if either (1) $\kappa > 0$; (2) $\kappa = 0$ and $\lambda > 0$; or (3) $\kappa < 0$ and $\lambda > \sqrt{|\kappa|}$. We say that $\kappa$ and $\lambda$ satisfy the *convex-ball-condition* if they satisfy the ball-condition and $\lambda \geq 0$.

Furthermore, we say that $\kappa$ and $\lambda$ satisfy the *monotone-condition* if $H_{\kappa,\lambda} \geq 0$ and $H'_{\kappa,\lambda} \geq 0$ on $[0, \overline{C}_{\kappa,\lambda})$, where $H_{\kappa,\lambda}$ is defined as (3.5). We see that $\kappa$ and $\lambda$ satisfy the monotone-condition if and only if either (1) $\kappa$ and $\lambda$ satisfy the convex-ball-condition; or (2) $\kappa \leq 0$ and $\lambda = \sqrt{|\kappa|}$. For $\kappa$ and $\lambda$ satisfying the monotone-condition, if $\kappa = 0$ and $\lambda = 0$, then $H_{\kappa,\lambda} = 0$ on $[0, \infty)$; otherwise, $H_{\kappa,\lambda} > 0$ on $(0, \overline{C}_{\kappa,\lambda})$.

We also say that $\kappa$ and $\lambda$ satisfy the *weakly-monotone-condition* if $H'_{\kappa,\lambda} \geq 0$ on $[0, \overline{C}_{\kappa,\lambda})$. Notice that $\kappa$ and $\lambda$ satisfy the weakly-monotone-condition if and only if either (1) $\kappa \geq 0$; or (2) $\kappa < 0$ and $|\lambda| \geq \sqrt{|\kappa|}$. In particular, if $\kappa$ and $\lambda$ satisfy the ball-condition, then they also satisfy the weakly-monotone-condition. For $\kappa$ and $\lambda$ satisfying the weakly-monotone-condition, if $\kappa \leq 0$ and $|\lambda| = \sqrt{|\kappa|}$, then $H_{\kappa,\lambda} = (n-1)\lambda$ on $[0, \infty)$; otherwise, $H'_{\kappa,\lambda} > 0$ on $[0, \overline{C}_{\kappa,\lambda})$.

### 4.1 Bounded Cases

If $f$ is bounded from above, then we have the following lemma.

***Lemma 4.1*** *Let $z \in \partial M$. Let $\kappa$ and $\lambda$ satisfy the weakly-monotone-condition. For $N \in (-\infty, 1]$, let us assume that*

$$\operatorname{Ric}_f^N(\gamma'_z(t)) \geq (n-1)\kappa e^{\frac{-4f(\gamma_z(t))}{n-1}}$$

*for all $t \in (0, \tau(z))$, and $H_{f,z} \geq (n-1)\lambda e^{\frac{-2f(z)}{n-1}}$. Suppose additionally that there exists $\delta \in \mathbb{R}$ such that $f \circ \gamma_z \leq (n-1)\delta$ on $(0, \tau(z))$. Then for all $t \in (0, \tau(z))$ we have*

$$\Delta_f \rho_{\partial M}(\gamma_z(t)) \geq H_{\kappa,\lambda}(e^{-2\delta} t) e^{\frac{-2f(\gamma_z(t))}{n-1}}. \tag{4.1}$$

*Moreover, if $\kappa$ and $\lambda$ satisfy the monotone-condition, then*

$$\Delta_f \rho_{\partial M}(\gamma_z(t)) \geq H_{\kappa,\lambda}(e^{-2\delta} t) e^{-2\delta}. \tag{4.2}$$

**Proof** By $f \circ \gamma_z \leq (n-1)\delta$, it holds that $s_{f,z}(t) \geq e^{-2\delta} t$ and $e^{\frac{-2f(\gamma_z(t))}{n-1}} \geq e^{-2\delta}$ for all $t \in (0, \tau(z))$. Inequality (3.8) and $H'_{\kappa,\lambda} \geq 0$ tell us that

$$\Delta_f \rho_{\partial M}(\gamma_z(t)) \geq H_{\kappa,\lambda}(s_{f,z}(t)) e^{\frac{-2f(\gamma_z(t))}{n-1}} \geq H_{\kappa,\lambda}(e^{-2\delta} t) e^{\frac{-2f(\gamma_z(t))}{n-1}} \tag{4.3}$$

for all $t \in (0, \tau(z))$, and hence (4.1) holds. Moreover, if $\kappa$ and $\lambda$ satisfy the monotone-condition, then (4.1) and $H_{\kappa,\lambda} \geq 0$ lead to

$$\Delta_f \rho_{\partial M}(\gamma_z(t)) \geq H_{\kappa,\lambda}(e^{-2\delta} t) e^{\frac{-2f(\gamma_z(t))}{n-1}} \geq H_{\kappa,\lambda}(e^{-2\delta} t) e^{-2\delta}. \tag{4.4}$$

This proves (4.2). ∎

***Remark 4.2*** Assume that for some $t_0 \in (0, \tau(z))$, the equality in (4.1) holds. Then the equalities in (4.3) hold, and the equality in (3.8) also holds (see Lemma 3.9). Moreover, if either (1) $\kappa > 0$; or (2) $\kappa \le 0$ and $|\lambda| > \sqrt{|\kappa|}$, then $H'_{\kappa,\lambda} > 0$ on $[0, \overline{C}_{\kappa,\lambda})$, and hence $s_{f,z}(t_0) = e^{-2\delta} t_0$; in particular, $f \circ \gamma_z = (n-1)\delta$ on $[0, t_0]$.

***Remark 4.3*** Assume that for some $t_0 \in (0, \tau(z))$, the equality in (4.2) holds. Then the equalities in (4.4) hold, and the equality in (4.1) holds (see Remark 4.2). Moreover, if either (1) $\kappa$ and $\lambda$ satisfy the convex-ball-condition; or (2) $\kappa < 0$ and $\lambda = \sqrt{|\kappa|}$, then $H_{\kappa,\lambda} > 0$ on $(0, \overline{C}_{\kappa,\lambda})$, and hence $e^{\frac{-2f(\gamma_z(t_0))}{n-1}} = e^{-2\delta}$; in particular, $(f \circ \gamma_z)(t_0) = (n-1)\delta$.

Lemma 4.1 implies the following lemma.

***Lemma 4.4*** *Let $z \in \partial M$ and $p \in (1, \infty)$. Let $\kappa$ and $\lambda$ satisfy the monotone-condition. For $N \in (-\infty, 1]$, we assume that*

$$\mathrm{Ric}_f^N(\gamma_z'(t)) \ge (n-1)\kappa e^{\frac{-4f(\gamma_z(t))}{n-1}}$$

*for all $t \in (0, \tau(z))$, and $H_{f,z} \ge (n-1)\lambda e^{\frac{-2f(z)}{n-1}}$. Suppose additionally that there exists $\delta \in \mathbb{R}$ such that $f \circ \gamma_z \le (n-1)\delta$ on $(0, \tau(z))$. We define $\rho_{\partial M,\delta} := e^{-2\delta}\rho_{\partial M}$. Let $\varphi\colon [0, \infty) \to \mathbb{R}$ be a monotone increasing smooth function. Then for all $t \in (0, \tau(z))$,*

$$(4.5)\quad \Delta_{f,p}\big(\varphi \circ \rho_{\partial M,\delta}\big)(\gamma_z(t)) \ge \\ -e^{-2p\delta}\Big\{\big(((\varphi')^{p-1})' - H_{\kappa,\lambda}(\varphi')^{p-1}\big) \circ \rho_{\partial M,\delta}\Big\}(\gamma_z(t)).$$

**Proof** Set $\Phi := \varphi \circ \rho_{\partial M,\delta}$, and define $\varphi_\delta(t) := \varphi(e^{-2\delta} t)$. We see $\Phi = \varphi_\delta \circ \rho_{\partial M}$. By (4.2), for every $t \in (0, \tau(z))$,

$$(4.6)\quad \begin{aligned} \Delta_{f,p}\Phi(\gamma_z(t)) &= -\big((\varphi_\delta')^{p-1}\big)'(t) + \Delta_{f,2}\rho_{\partial M}(\gamma_z(t))(\varphi_\delta')^{p-1}(t) \\ &\ge -\big((\varphi_\delta')^{p-1}\big)'(t) + H_{\kappa,\lambda}(e^{-2\delta}t)e^{-2\delta}(\varphi_\delta')^{p-1}(t). \end{aligned}$$

Since $(\varphi_\delta')^{p-1}(t) = e^{-2(p-1)\delta}(\varphi')^{p-1}(e^{-2\delta}t)$ and

$$\big((\varphi_\delta')^{p-1}\big)'(t) = e^{-2p\delta}\big((\varphi')^{p-1}\big)'(e^{-2\delta}t),$$

the right-hand side of (4.6) is equal to that of (4.5). ∎

***Remark 4.5*** The equality case of Lemma 4.4 results in that of (4.2) (see Remark 4.3).

We now prove the following global Laplacian comparison inequality.

***Proposition 4.6*** *Let $p \in (1, \infty)$. Let $\kappa$ and $\lambda$ satisfy the monotone-condition. For $N \in (-\infty, 1]$, assume that the triple $(M, \partial M, f)$ has lower $(\kappa, \lambda, N)$-weighted curvature bounds. We suppose additionally that there exists $\delta \in \mathbb{R}$ such that $f \le (n-1)\delta$ on $M$. We define $\rho_{\partial M,\delta} := e^{-2\delta}\rho_{\partial M}$. Let $\varphi\colon [0, \infty) \to \mathbb{R}$ be a monotone increasing smooth*

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*function. Then we have*

$$\Delta_{f,p}(\varphi \circ \rho_{\partial M,\delta}) \geq -e^{-2p\delta}\Big( ((\varphi')^{p-1})' - H_{\kappa,\lambda}(\varphi')^{p-1}\Big) \circ \rho_{\partial M,\delta}$$

*in the following distribution sense on $M$: For every non-negative function $\psi \in C_0^\infty(M)$, we have*

$$(4.7)\quad \int_M \|\nabla(\varphi \circ \rho_{\partial M,\delta})\|^{p-2} g\big(\nabla\psi, \nabla(\varphi \circ \rho_{\partial M,\delta})\big)\, dm_f \geq - e^{-2p\delta}\int_M \psi\Big\{\Big( ((\varphi')^{p-1})' - H_{\kappa,\lambda}(\varphi')^{p-1}\Big) \circ \rho_{\partial M,\delta}\Big\}\, dm_f.$$

**Proof** By Lemma 2.2, there exists a sequence $\{\Omega_i\}$ of closed subsets of $M$ such that for every $i$, the set $\partial\Omega_i$ is a smooth hypersurface in $M$, and satisfying the following: (1) for all $i_1, i_2$ with $i_1 < i_2$, we have $\Omega_{i_1} \subset \Omega_{i_2}$; (2) $M \setminus \operatorname{Cut}\partial M = \bigcup_i \Omega_i$; (3) $\partial\Omega_i \cap \partial M = \partial M$ for all $i$; (4) for each $i$, on $\partial\Omega_i \setminus \partial M$, there exists a unique unit outer normal vector field $u_i$ for $\Omega_i$ with $g(u_i, \nabla\rho_{\partial M}) \geq 0$.

For the canonical volume measure $\mathrm{vol}_i$ on $\partial\Omega_i \setminus \partial M$, put $m_{f,i} := e^{-f}\,\mathrm{vol}_i$. Set $\Phi := \varphi \circ \rho_{\partial M,\delta}$. By integration by parts, we see

$$\int_{\Omega_i} \|\nabla\Phi\|^{p-2} g(\nabla\psi, \nabla\Phi)\, dm_f = \int_{\Omega_i} \psi \Delta_{f,p}\Phi\, dm_f + \int_{\partial\Omega_i \setminus \partial M} \|\nabla\Phi\|^{p-2} \psi g(u_i, \nabla\Phi)\, dm_{f,i}.$$

From (4.5) and $g(u_i, \nabla\rho_{\partial M,\delta}) \geq 0$, it follows that the right-hand side of the above equality is at least

$$-e^{-2p\delta}\int_{\Omega_i} \psi\Big\{\Big( ((\varphi')^{p-1})' - H_{\kappa,\lambda}(\varphi')^{p-1}\Big) \circ \rho_{\partial M,\delta}\Big\}\, dm_f.$$

Letting $i \to \infty$, we obtain (4.7). ■

***Remark 4.7*** In Proposition 4.6, assume that the equality in (4.7) holds. In this case, the equality in (4.5) also holds on $\operatorname{supp}\psi \setminus (\partial M \cup \operatorname{Cut}\partial M)$, where $\operatorname{supp}\psi$ denotes the support of $\psi$. The equality case of Proposition 4.6 results in that of (4.5) (see Remark 4.5).

***Remark 4.8*** The argument in the proof of Proposition 4.6 also tells us the following (see also Remark 4.7). Under the same setting as Proposition 4.6, if $M$ is compact, then the inequality (4.7) holds for every non-negative function $\psi \in C^1(M)$ with $\psi|_{\partial M} = 0$. Moreover, if the equality in (4.7) holds for some $\psi$, then the equality in (4.5) holds on $\operatorname{supp}\psi \setminus (\partial M \cup \operatorname{Cut}\partial M)$ (see Remark 4.5).

## 4.2 Radial Cases

Suppose that $f$ is $\partial M$-radial. Then there exists a smooth function $\phi_f\colon [0,\infty) \to \mathbb{R}$ such that $f = \phi_f \circ \rho_{\partial M}$ on $M$. Define a function $s_f\colon [0,\infty] \to [0,\infty]$ by

$$(4.8)\quad s_f(t) := \int_0^t e^{\frac{-2\phi_f(a)}{n-1}}\, da.$$

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For every $z \in \partial M$, we see $s_{f,z} = s_f$ on $[0, \tau(z)]$, where $s_{f,z}$ is defined as (1.5). Furthermore, $\rho_{\partial M,f} = s_f \circ \rho_{\partial M}$, where $\rho_{\partial M,f}$ is defined as (1.12).

If $f$ is $\partial M$-radial, then we have the following comparison inequality.

***Lemma 4.9*** *Let $z \in \partial M$ and $p \in (1,\infty)$. For $N \in (-\infty,1]$, suppose that*

$$\mathrm{Ric}_f^N(\gamma_z'(t)) \geq (n-1)\kappa e^{\frac{-4f(\gamma_z(t))}{n-1}}$$

*for all $t \in (0,\tau(z))$, and $H_{f,z} \geq (n-1)\lambda e^{\frac{-2f(z)}{n-1}}$. Suppose that $f$ is $\partial M$-radial. Let $\varphi\colon [0,\infty) \to \mathbb{R}$ be a monotone increasing smooth function. Then for all $t \in (0,\tau(z))$,*

$$\begin{aligned}(4.9)\quad \Delta_{\frac{n+1-2p}{n-N}f,p}\big(\varphi \circ \rho_{\partial M,f}\big)(\gamma_z(t)) \geq \\ -e^{\frac{-2pf}{n-1}}\Big\{\Big(((\varphi')^{p-1})' - H_{\kappa,\lambda}(\varphi')^{p-1}\Big) \circ \rho_{\partial M,f}\Big\}(\gamma_z(t)).\end{aligned}$$

**Proof** Set $\Phi := \varphi \circ \rho_{\partial M,f}$. For the function $s_f$ defined as (4.8), if we put $\varphi_f := \varphi \circ s_f$, then we have $\Phi = \varphi_f \circ \rho_{\partial M}$. For each $t \in (0,\tau(z))$, the left-hand side of (4.9) can be written as

$$-\Big((\varphi_f')^{p-1}\Big)'(t) + \Big(\Delta_f \rho_{\partial M}(\gamma_z(t)) - \frac{2(p-1)}{n-1}\phi_f'(t)\Big)(\varphi_f')^{p-1}(t).$$

By using (3.8), $s_{f,z}(t) = s_f(t)$, and $e^{\frac{-2f(\gamma_z(t))}{n-1}} = s_f'(t)$, we have

$$\begin{aligned}\Delta_{\frac{n+1-2p}{n-1}f,p}\Phi(\gamma_z(t)) \geq -\Big((\varphi_f')^{p-1}\Big)'(t) + H_{\kappa,\lambda}(s_f(t))s_f'(t)(\varphi_f')^{p-1}(t) \\ - \frac{2(p-1)}{n-1}\phi_f'(t)(\varphi_f')^{p-1}(t).\end{aligned}$$

Notice that $(\varphi_f')^{p-1}(t) = (\varphi')^{p-1}(s_f(t))(s_f')^{p-1}(t)$ and

$$\Big((\varphi_f')^{p-1}\Big)'(t) = \Big((\varphi')^{p-1}\Big)'(s_f(t))(s_f')^p(t) - \frac{2(p-1)}{n-1}\phi_f'(t)(\varphi_f')^{p-1}(t).$$

These equalities tell us that the left-hand side of (4.9) is at least

$$-(s_f')^p(t)\Big(\Big((\varphi')^{p-1}\Big)'(s_f(t)) - H_{\kappa,\lambda}(s_f(t))(\varphi')^{p-1}(s_f(t))\Big).$$

Since $\rho_{\partial M,f} = s_f \circ \rho_{\partial M}$, this is equal to the right-hand side of (4.9). ∎

We further yield the following global comparison inequality.

***Proposition 4.10*** *Let $p \in (1,\infty)$. For $N \in (-\infty,1]$, assume that $(M,\partial M,f)$ has lower $(\kappa,\lambda,N)$-weighted curvature bounds. Suppose that $f$ is $\partial M$-radial. Let $\varphi\colon [0,\infty) \to \mathbb{R}$ be a monotone increasing smooth function. Then we have*

$$\Delta_{\frac{n+1-2p}{n-1}f,p}(\varphi \circ \rho_{\partial M,f}) \geq -e^{\frac{-2pf}{n-1}}\Big(\Big((\varphi')^{p-1}\Big)' - H_{\kappa,\lambda}(\varphi')^{p-1}\Big) \circ \rho_{\partial M,f}$$

*in the following distribution sense on $M$: For every non-negative function $\psi \in C_0^\infty(M)$, we have*

$$\begin{aligned}(4.10)\quad \int_M \|\nabla(\varphi\circ\rho_{\partial M,f})\|^{p-2} g(\nabla\psi,\nabla(\varphi\circ\rho_{\partial M,f}))\,dm_{\frac{n+1-2p}{n-1}f} \geq \\ -\int_M \psi\Big\{\big(((\varphi')^{p-1})' - H_{\kappa,\lambda}(\varphi')^{p-1}\big)\circ\rho_{\partial M,f}\Big\}\,dm_{\frac{n+1}{n-1}f}.\end{aligned}$$

**Proof** The proof is similar to that of Proposition 4.6. Similarly, we first take a sequence $\{\Omega_i\}$ of closed subsets of $M$ in Lemma 2.2. Let $u_i$ be the unit outer normal vector on $\partial\Omega_i\backslash\partial M$ for $\Omega_i$. We define $\widehat{f} := (n+1-2p)(n-1)^{-1}f$. For the canonical volume $\mathrm{vol}_i$ on $\partial\Omega_i\backslash\partial M$, we put $m_{\widehat{f},i} := e^{-\widehat{f}}\,\mathrm{vol}_i$. We set $\Phi := \varphi\circ\rho_{\partial M,f}$. By integration by parts (with respect to $m_{\widehat{f}}$) and by Lemma 4.9 and $g(u_i,\nabla\rho_{\partial M,f}) \geq 0$, we get

$$\begin{aligned}\int_{\Omega_i} \|\nabla\Phi\|^{p-2} g(\nabla\psi,\nabla\Phi)\,dm_{\widehat{f}} \geq \\ -\int_{\Omega_i} \psi e^{\frac{-2pf}{n-1}}\Big\{\big(((\varphi')^{p-1})' - H_{\kappa,\lambda}(\varphi')^{p-1}\big)\circ\rho_{\partial M,f}\Big\}\,dm_{\widehat{f}}.\end{aligned}$$

Using $e^{\frac{-2pf}{n-1}} m_{\widehat{f}} = m_{\frac{n+1}{n-1}f}$, we complete the proof by letting $i\to\infty$. ■

***Remark 4.11*** The argument in the proof of Proposition 4.10 also leads us to the following. Under the same setting as Proposition 4.10, if $M$ is compact, then inequality (4.10) holds for every non-negative $\psi \in C^1(M)$ with $\psi|_{\partial M} = 0$.

# 5 Splitting Theorems

## 5.1 Main Splitting Theorems

**Proof of Theorem 1.1** Let $\kappa \leq 0$ and $\lambda := \sqrt{|\kappa|}$. For $N \in (-\infty,1]$, assume that $(M,\partial M,f)$ has lower $(\kappa,\lambda,N)$-weighted curvature bounds. Suppose that $f$ is bounded from above. Let $z_0 \in \partial M$ satisfy $\tau(z_0) = \infty$. Recall that our goal is to show that $M$ is isometric to $[0,\infty)\times_{F_{\kappa,\lambda}}\partial M$; moreover, if $N \in (-\infty,1)$, then $f\circ\gamma_z$ is constant for every $z \in \partial M$.

Let $\partial M_0$ be the connected component of $\partial M$ with $z_0 \in \partial M_0$. We define a closed subset $\Omega$ of $\partial M_0$ by

$$\Omega := \{z \in \partial M_0 \mid \tau(z) = \infty\}.$$

We show that $\Omega$ is open in $\partial M_0$. Fix $z_1 \in \Omega$. Take $l > 0$, and put $x_0 := \gamma_{z_1}(l)$. There exists an open neighborhood $U$ of $x_0$ contained in $\mathrm{Int}\,M\backslash\mathrm{Cut}\,\partial M$. Taking $U$ smaller, we can assume that for each $x \in U$ the unique foot point on $\partial M$ of $x$ belongs to $\partial M_0$. By Lemma 2.4, there exists $\epsilon > 0$ such that for all $x \in B_\epsilon(x_0)$, all asymptotes for $\gamma_{z_1}$ from $x$ lie in $\mathrm{Int}\,M$. We can assume that $U \subset B_\epsilon(x_0)$. Fix $x_1 \in U$, and take an asymptote $\gamma_{x_1}: [0,\infty)\to M$ for $\gamma_{z_1}$ from $x_1$. For $t > 0$, define a function $b_{\gamma_{z_1},t}: M\to\mathbb{R}$ by

$$b_{\gamma_{z_1},t}(x) := b_{\gamma_{z_1}}(x_1) + t - d_M(x,\gamma_{x_1}(t)).$$

We see that $b_{\gamma_{z_1},t} - \rho_{\partial M}$ is a support function of $b_{\gamma_{z_1}} - \rho_{\partial M}$ at $x_1$. Since $\gamma_{x_1}$ lie in $\mathrm{Int}\, M$, for every $t > 0$ the function $b_{\gamma_{z_1},t}$ is smooth on a neighborhood of $x_1$. From Lemma 2.9 we deduce

$$\Delta_f b_{\gamma_{z_1},t}(x_1) \leq -H_\kappa\Big( e^{\frac{-2\sup f}{n-1}}\, t\Big)\, e^{\frac{-2f(x_1)}{n-1}},$$

where $H_\kappa$ is defined as (2.3). Note that $H_\kappa(s)$ tends to $-(n-1)\sqrt{|\kappa|}$ as $s \to \infty$. Furthermore, $\rho_{\partial M}$ is smooth on $U$, and by (3.8) we have

$$\Delta_f \rho_{\partial M} \geq (n-1)\sqrt{|\kappa|}\, e^{\frac{-2f}{n-1}}$$

on $U$. Hence $b_{\gamma_{z_1}} - \rho_{\partial M}$ is $f$-subharmonic on $U$. Now, $b_{\gamma_{z_1}} - \rho_{\partial M}$ takes the maximal value 0 at $x_1$. Lemma 2.7 implies $b_{\gamma_{z_1}} = \rho_{\partial M}$ on $U$. By Lemma 2.3, the set $\Omega$ is open in $\partial M_0$.

The connectedness of $\partial M_0$ leads to $\Omega = \partial M_0$. By Lemma 2.1, $\partial M$ is connected and $\mathrm{Cut}\, \partial M = \emptyset$. The equality in (3.8) holds on $\mathrm{Int}\, M$. For each $z \in \partial M$, choose an orthonormal basis $\{e_{z,i}\}_{i=1}^{n-1}$ of $T_z \partial M$. Let $\{Y_{z,i}\}_{i=1}^{n-1}$ be the $\partial M$-Jacobi fields along $\gamma_z$ with $Y_{z,i}(0) = e_{z,i}$, $Y'_{z,i}(0) = -A_{u_z} e_{z,i}$. By Lemma 3.9, for all $i$ we see $Y_{z,i} = F_{\kappa,\lambda,z} E_{z,i}$ on $[0,\infty)$, where $\{E_{z,i}\}_{i=1}^{n-1}$ are the parallel vector fields with $E_{z,i}(0) = e_{z,i}$. Moreover, if $N \in (-\infty, 1)$, then $f \circ \gamma_z$ is constant on $[0,\infty)$. We define a diffeomorphism $\Phi\colon [0,\infty) \times \partial M \to M$ by $\Phi(t,z) := \gamma_z(t)$. The rigidity of Jacobi fields implies that $\Phi$ is a Riemannian isometry with boundary from $[0,\infty) \times_{F_{\kappa,\lambda}} \partial M$ to $M$. We complete the proof of Theorem 1.1. ∎

***Remark 5.1*** The author [19] has concluded that under the same setting as Theorem 1.1, if $\kappa = 0$, then $M$ is isometric to a warped product (see [19, Corollary 1.4]). The author does not know whether the same conclusion holds when $\kappa < 0$.

## 5.2 Weighted Ricci Curvature on the Boundary

We next recall the following formula (see *e.g.*, [19, Lemma 5.5]).

***Lemma 5.2*** *Let $z \in \partial M$, and take a unit vector $v$ in $T_z \partial M$. Choose an orthonormal basis $\{e_{z,i}\}_{i=1}^{n-1}$ of $T_z \partial M$ with $e_{z,1} = v$. Then*

$$\begin{aligned}(5.1)\qquad \mathrm{Ric}^{N-1}_{f|_{\partial M}}(v) = \mathrm{Ric}^N_f(v) &+ g((\nabla f)_z, u_z) g(S(v,v), u_z) \\ &- K_g(u_z, v) + \mathrm{trace}\, A_{S(v,v)} - \sum_{i=1}^{n-1} \|S(v, e_{z,i})\|^2\end{aligned}$$

*for all $N \in (-\infty, \infty]$ (when $N = \infty$, we interpret $N-1$ in the left hand side as $\infty$), where $K_g(u_z, v)$ denotes the sectional curvature of the 2-plane at $z$ spanned by $u_z$ and $v$.*

***Remark 5.3*** In [19], the author has presented (5.1) only for $N \in (-\infty, \infty)$ (see [19, Lemma 5.5]). The calculation in the proof also tells us that it can be formulated for $N = \infty$ as in Lemma 5.2.

Using Lemma 5.2, we show the following lemma.

***Lemma 5.4*** *Take $z \in \partial M$, and take a unit vector $v$ in $T_z\partial M$. If $M$ is isometric to $[0,\infty) \times_{F_{\kappa,\lambda}} \partial M$, then for all $N \in (-\infty,\infty]$, we have*

$$\mathrm{Ric}^{N-1}_{f|_{\partial M}}(v) = \mathrm{Ric}^N_f(v) + (n-1)\lambda^2 e^{\frac{-4f(z)}{n-1}} - \kappa e^{\frac{-4f(z)}{n-1}} - \lambda g((\nabla f)_z, u_z) e^{\frac{-2f(z)}{n-1}} + \frac{\mathrm{Hess}\, f(u_z,u_z)}{n-1},$$

*where when $N=\infty$, we interpret $N-1$ in the left hand side as $\infty$.*

**Proof** We choose an orthonormal basis $\{e_{z,i}\}_{i=1}^{n-1}$ of $T_z\partial M$ with $e_{z,1}=v$. Let $\{Y_{z,i}\}_{i=1}^{n-1}$ denote the $\partial M$-Jacobi fields along $\gamma_z$ with $Y_{z,i}(0)=e_{z,i}$ and $Y'_{z,i}(0) = -A_{u_z} e_{z,i}$. By the rigidity assumption, $Y_{z,i} = F_{\kappa,\lambda,z} E_{z,i}$, where $\{E_{z,i}\}_{i=1}^{n-1}$ are the parallel vector fields with $E_{z,i}(0)=e_{z,i}$. Therefore, for all $i$ it holds that

$$A_{u_z} e_{z,i} = -Y'_{z,i}(0) = -\Big(\frac{g((\nabla f)_z,u_z)}{n-1} - \lambda e^{\frac{-2f(z)}{n-1}}\Big) e_{z,i}. \tag{5.2}$$

From (5.2), we deduce that $S(v,e_{z,i}) = 0_z$ for all $i \neq 1$, and we also deduce

$$S(v,v) = -\Big(\frac{g((\nabla f)_z,u_z)}{n-1} - \lambda e^{\frac{-2f(z)}{n-1}}\Big) u_z, \tag{5.3}$$

$$\mathrm{trace}\, A_{S(v,v)} = (n-1)\Big(\frac{g((\nabla f)_z,u_z)}{n-1} - \lambda e^{\frac{-2f(z)}{n-1}}\Big)^2. \tag{5.4}$$

The sectional curvature $K_g(u_z,v)$ is equal to $-g(Y''_{z,1}(0),v)$, and hence

$$K_g(u_z,v) = -\frac{\mathrm{Hess}\, f(u_z,u_z)}{n-1} - \Big(\frac{g((\nabla f)_z,u_z)}{n-1}\Big)^2 + \kappa e^{\frac{-4f(z)}{n-1}}. \tag{5.5}$$

Lemma 5.2 together with (5.3), (5.4), (5.5) yields the desired result. ∎

## 5.3 Multi-splitting

On a connected complete Riemannian manifold $M_0$ (without boundary), a minimal geodesic $\gamma\colon \mathbb{R} \to M_0$ is said to be a *line*. Wylie [23] has proved the following splitting theorem of Cheeger–Gromoll type (see [23, Theorem 1.2 and Corollary 1.3]).

***Theorem 5.5*** ([23]) *Let $M_0$ be a connected complete Riemannian manifold, and let $f_0\colon M_0 \to \mathbb{R}$ be a smooth function bounded from above. For $N \in (-\infty,1]$, suppose $\mathrm{Ric}^N_{f_0,M_0} \geq 0$. If $M_0$ contains a line, then there exists a Riemannian manifold $\widetilde{M}_0$ such that $M_0$ is isometric to a warped product space over $\mathbb{R} \times \widetilde{M}_0$; moreover, if $N \in (-\infty,1)$, then $M_0$ is isometric to the standard product $\mathbb{R} \times \widetilde{M}_0$.*

From Theorem 5.5 we derive the following corollary of Theorem 1.1.

***Corollary 5.6*** *Let $\kappa \leq 0$ and $\lambda := \sqrt{|\kappa|}$. For $N \in (-\infty,1)$, assume that $(M,\partial M,f)$ has lower $(\kappa,\lambda,N)$-weighted curvature bounds. Suppose that $f$ is bounded from above. If for some $z_0 \in \partial M$ we have $\tau(z_0)=\infty$, then there exist an integer $k \in \{0,\dots,n-1\}$ and an $(n-1-k)$-dimensional Riemannian manifold $\widetilde{\partial M}$ containing no line such that $\partial M$ is isometric to $\mathbb{R}^k \times \widetilde{\partial M}$; in particular, $M$ is isometric to $[0,\infty) \times_{F_{\kappa,\lambda}} (\mathbb{R}^k \times \widetilde{\partial M})$.*

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**Proof** Due to Theorem 1.1, $M$ is isometric to $[0,\infty)\times_{F_{\kappa,\lambda}}\partial M$, and for each $z\in\partial M$, the function $f\circ\gamma_z$ is constant on $[0,\infty)$. In particular, $g((\nabla f)_z,u_z)=0$ and $\operatorname{Hess} f(u_z,u_z)=0$. By Lemma 5.4 and by $\kappa\le 0$ and $\lambda=\sqrt{|\kappa|}$, for every unit vector $v$ in $T_z\partial M$, we have

$$\begin{aligned}\operatorname{Ric}^{N-1}_{f|_{\partial M}}(v)&=\operatorname{Ric}^N_f(v)+(n-1)\lambda^2 e^{\frac{-4f(z)}{n-1}}-\kappa e^{\frac{-4f(z)}{n-1}}\\&\ge \operatorname{Ric}^N_f(v)+(n-1)\lambda^2 e^{\frac{-4f(z)}{n-1}}\\&\ge (n-1)\kappa e^{\frac{-4f(z)}{n-1}}+(n-1)\lambda^2 e^{\frac{-4f(z)}{n-1}}=0.\end{aligned}$$

It follows that $\operatorname{Ric}^{N-1}_{f|_{\partial M},\partial M}\ge 0$. Now, $N-1$ is smaller than 1, and $f|_{\partial M}$ is bounded from above. Therefore, by applying Theorem 5.5 to $\partial M$ inductively, we complete the proof. ∎

## 5.4 Variants of Splitting Theorems

We study generalizations of rigidity results of Kasue [9], Croke and Kleiner [5], and Ichida [8] for manifolds with boundary whose boundaries are disconnected.

Wylie [23] proved the following (see [23, Theorem 5.1]).

***Theorem 5.7*** ([23]) *For $N\in(-\infty,1]$, assume that $(M,\partial M,f)$ has lower $(0,0,N)$-weighted curvature bounds. Let $\partial M$ be disconnected, and let $\{\partial M_i\}_{i=1,2,\dots}$ denote the connected components of $\partial M$. Let $\partial M_1$ be compact, and put $D:=\inf_{i=2,3,\dots} d_M(\partial M_1,\partial M_i)$. Then $M$ is isometric to $[0,D]\times_{F_{0,0}}\partial M_1$, and $\operatorname{Ric}^N_f(\gamma'_z(t))=0$ for all $z\in\partial M_1$ and $t\in[0,D]$.*

For $\kappa>0$ and $\lambda<0$, put $D_{\kappa,\lambda}:=\inf\{s>0\,|\,\mathfrak{s}'_{\kappa,\lambda}(s)=0\}$. By using Theorem 5.7, we obtain the following splitting.

***Theorem 5.8*** *Let $\kappa>0$. For $N\in(-\infty,1]$, assume that $(M,\partial M,f)$ has lower $(\kappa,\lambda,N)$-weighted curvature bounds. Let $\partial M$ be disconnected, and let $\{\partial M_i\}_{i=1,2,\dots}$ denote the connected components of $\partial M$. Let $\partial M_1$ be compact, and put $D:=\inf_{i=2,3,\dots} d_M(\partial M_1,\partial M_i)$. Suppose additionally that there exists $\delta\in\mathbb{R}$ such that $f\le(n-1)\delta$ on $M$. Then*

$$\lambda<0,\qquad D\le 2e^{2\delta}D_{\kappa,\lambda}.$$

*Moreover, if $D=2e^{2\delta}D_{\kappa,\lambda}$, then $M$ is isometric to $[0,D]\times_{F_{\kappa,\lambda}}\partial M_1$, and $f=(n-1)\delta$ on $M$.*

**Proof** If we have $\lambda\ge 0$, then Theorem 5.7 tells us that $M$ is isometric to $[0,D]\times_{F_{0,0}}\partial M_1$, and $\operatorname{Ric}^N_f(\gamma'_z(t))=0$ for all $z\in\partial M_1, t\in[0,D]$. This contradicts $\kappa>0$, and hence $\lambda<0$.

Let us prove that if $D\ge 2e^{2\delta}D_{\kappa,\lambda}$, then $M$ is isometric to the twisted product $[0,2e^{2\delta}D_{\kappa,\lambda}]\times_{F_{\kappa,\lambda}}\partial M_1$, and $f=(n-1)\delta$ on $M$. Suppose $D\ge 2e^{2\delta}D_{\kappa,\lambda}$. There exists

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a connected component $\partial M_2$ of $\partial M$ such that $d_M(\partial M_1, \partial M_2) = D$ (*cf.* [9, Lemma 1.6]). For each $i = 1, 2$, let $\rho_{\partial M_i}\colon M \to \mathbb{R}$ be the function defined by $\rho_{\partial M_i}(x) := d_M(x, \partial M_i)$. Set

$$\Omega := \{x \in \operatorname{Int} M \mid \rho_{\partial M_1}(x) + \rho_{\partial M_2}(x) = D\}.$$

We show that $\Omega$ is open in $\operatorname{Int} M$. Fix $x \in \Omega$. For each $i = 1, 2$, we take a foot point $z_{x,i} \in \partial M_i$ on $\partial M_i$ of $x$ such that $d_M(x, z_{x,i}) = \rho_{\partial M_i}(x)$. From the triangle inequality, we derive $d_M(z_{x,1}, z_{x,2}) = D$. The minimal geodesic $\gamma\colon [0, D] \to M$ from $z_{x,1}$ to $z_{x,2}$ is orthogonal to $\partial M$ at $z_{x,1}$ and at $z_{x,2}$. Furthermore, $\gamma|_{(0,D)}$ lies in $\operatorname{Int} M$ and passes through $x$. There exists an open neighborhood $U$ of $x$ such that $\rho_{\partial M_i}$ is smooth on $U$. In view of (4.1), for all $y \in U$, we see

$$\begin{aligned}(5.6)\quad -&\frac{\Delta_f(\rho_{\partial M_1} + \rho_{\partial M_2})(y)}{(n-1)e^{\frac{-2f(y)}{n-1}}}\\ &\le \frac{\mathfrak{s}'_{\kappa,\lambda}}{\mathfrak{s}_{\kappa,\lambda}}(\rho_{\partial M_1,\delta}(y)) + \frac{\mathfrak{s}'_{\kappa,\lambda}}{\mathfrak{s}_{\kappa,\lambda}}(\rho_{\partial M_2,\delta}(y))\\ &= \frac{\mathfrak{s}'_{\kappa,\lambda}(\rho_{\partial M_1,\delta}(y) + \rho_{\partial M_2,\delta}(y)) - \lambda\mathfrak{s}_{\kappa,\lambda}(\rho_{\partial M_1,\delta}(y) + \rho_{\partial M_2,\delta}(y))}{\mathfrak{s}_{\kappa,\lambda}(\rho_{\partial M_1,\delta}(y))\mathfrak{s}_{\kappa,\lambda}(\rho_{\partial M_2,\delta}(y))},\end{aligned}$$

where $\rho_{\partial M_i,\delta} := e^{-2\delta}\rho_{\partial M_i}$. Since $\kappa > 0$, the function $\mathfrak{s}'_{\kappa,\lambda}/\mathfrak{s}_{\kappa,\lambda}$ is monotone decreasing on $(0, C_{\kappa,\lambda})$, and satisfies $\mathfrak{s}'_{\kappa,\lambda}(2D_{\kappa,\lambda})/\mathfrak{s}_{\kappa,\lambda}(2D_{\kappa,\lambda}) = \lambda$. By $D \ge 2e^{2\delta}D_{\kappa,\lambda}$ and the triangle inequality, $\rho_{\partial M_1,\delta} + \rho_{\partial M_2,\delta} \ge 2D_{\kappa,\lambda}$ on $U$. Inequality (5.6) tells us that $-(\rho_{\partial M_1} + \rho_{\partial M_2})$ is $f$-subharmonic on $U$. By Lemma 2.7, $\Omega$ is open in $\operatorname{Int} M$.

The connectedness of $\operatorname{Int} M$ implies that $\operatorname{Int} M = \Omega$. The equality in (4.1) holds. For each $z \in \partial M_1$, choose an orthonormal basis $\{e_{z,i}\}_{i=1}^{n-1}$ of $T_z\partial M$. Let $\{Y_{z,i}\}_{i=1}^{n-1}$ be the $\partial M$-Jacobi fields along $\gamma_z$ with $Y_{z,i}(0) = e_{z,i}$, $Y'_{z,i}(0) = -A_{u_z}e_{z,i}$. For all $i$, we see that $Y_{z,i} = F_{\kappa,\lambda,z}E_{z,i}$ on $[0, D]$, where $\{E_{z,i}\}_{i=1}^{n-1}$ are the parallel vector fields with $E_{z,i}(0) = e_{z,i}$. Moreover, $f \circ \gamma_z = (n-1)\delta$ on $[0, D]$ (see Remark 4.2). We see $D = 2e^{2\delta}D_{\kappa,\lambda}$. By the rigidity of Jacobi fields, a map $\Phi\colon [0, D] \times \partial M_1 \to M$ defined by $\Phi(t, z) := \gamma_z(t)$ is a desired Riemannian isometry with boundary. ■

## 6 Inscribed Radii

We denote by $L_{g_f}$, $d_M^{g_f}$, $\rho_{\partial M}^{g_f}$, and $\operatorname{InRad}_{g_f} M$ the length, the Riemannian distance, the distance function from the boundary and the inscribed radius on $M$ induced from the Riemannian metric $g_f := e^{\frac{-4f}{n-1}}g$.

### 6.1 Inscribed Radius Comparisons

We first show the following lemma.

***Lemma 6.1*** *Let $\kappa$ and $\lambda$ satisfy the ball-condition. For $N \in (-\infty, 1]$, assume that $(M, \partial M, f)$ has lower $(\kappa, \lambda, N)$-weighted curvature bounds. Then we have* $\operatorname{InRad}_{g_f} M \le C_{\kappa,\lambda}$.

**Proof** Take $x \in M$, and a foot point $z_x$ on $\partial M$ of $x$. Then we have

$$\rho^{g_f}_{\partial M}(x) \le L_{g_f}(\gamma_{z_x}|_{[0,l]}) = \int_0^l e^{\frac{-2f(\gamma_{z_x}(a))}{n-1}} da \le \tau_f(z_x) \le \sup_{z\in\partial M} \tau_f(z),$$

where $l := \rho_{\partial M}(x)$. Lemma 3.5 implies the desired inequality. ∎

From Lemma 3.5 and $\operatorname{InRad} M = \sup_{z\in\partial M} \tau(z)$, we also derive the following lemma.

***Lemma 6.2*** *Let $\kappa$ and $\lambda$ satisfy the ball-condition. For $N \in (-\infty, 1]$, assume that $(M, \partial M, f)$ has lower $(\kappa, \lambda, N)$-weighted curvature bounds. Suppose additionally that there is $\delta \in \mathbb{R}$ such that $f \le (n-1)\delta$ on $M$. Then we have* $\operatorname{InRad} M \le C_{\kappa e^{-4\delta}, \lambda e^{-2\delta}}$.

## 6.2 Inscribed Radius Rigidity

**Proof of Theorem 1.2** Let $\kappa$ and $\lambda$ satisfy the ball-condition. For $N \in (-\infty, 1]$, assume that $(M, \partial M, f)$ has lower $(\kappa, \lambda, N)$-weighted curvature bounds. According to Lemma 6.1, we have (1.10). Now, let $x_0 \in M$ satisfy $\rho^{g_f}_{\partial M}(x_0) = C_{\kappa,\lambda}$. Recall that our goal is to prove that $M$ is isometric to $[0, l] \times_{F_\kappa} \mathbb{S}^{n-1}$ for some $l > 0$; moreover, if $N \in (-\infty, 1)$, then $f$ is constant. In particular, $M$ is isometric to a closed ball in a space form. Note that $x_0$ will become the center of $M$.

Put $l := \rho_{\partial M}(x_0)$ and

$$\Omega := \left\{ x \in \operatorname{Int} M \backslash \{x_0\} \;\middle|\; \rho_{\partial M}(x) + \rho_{x_0}(x) = l \right\}.$$

We show that $\Omega$ is open in $\operatorname{Int} M\backslash\{x_0\}$. Fix $x \in \Omega$ and take a foot point $z_x$ on $\partial M$ of $x$. Note that $z_x$ is also a foot point on $\partial M$ of $x_0$. Let $\gamma\colon [0, l] \to M$ be the minimal geodesic from $z_x$ to $x_0$. Then $\gamma|_{(0,l)}$ passes through $x$. There exists an open neighborhood $U$ of $x$ such that the distance functions $\rho_{x_0}$ and $\rho_{\partial M}$ are smooth on $U$, and for every $y \in U$ there exists a unique minimal geodesic in $M$ from $x_0$ to $y$ that lies in $\operatorname{Int} M$. By Lemma 2.8 and (3.8), for each $y \in U$, we have

$$\begin{aligned}
(6.1) \qquad -\frac{\Delta_f(\rho_{\partial M} + \rho_{x_0})(y)}{(n-1)e^{\frac{-2f(y)}{n-1}}} &\le \frac{\mathfrak{s}'_{\kappa,\lambda}}{\mathfrak{s}_{\kappa,\lambda}}\left(s_{f,z_y}(\rho_{\partial M}(y))\right) + \frac{\mathfrak{s}'_{\kappa}}{\mathfrak{s}_{\kappa}}\left(s_{f,v_y}(\rho_{x_0}(y))\right)\\
&= \frac{\mathfrak{s}_{\kappa,\lambda}(s_{f,z_y}(\rho_{\partial M}(y)) + s_{f,v_y}(\rho_{x_0}(y)))}{\mathfrak{s}_{\kappa,\lambda}(s_{f,z_y}(\rho_{\partial M}(y)))\mathfrak{s}_{\kappa}(s_{f,v_y}(\rho_{x_0}(y)))},
\end{aligned}$$

where $z_y$ is a unique foot point on $\partial M$ of $y$, and $v_y$ is the initial velocity vector of the unique minimal geodesic from $x_0$ to $y$. Let us define $\rho^{g_f}_{x_0} := d^{g_f}_M(\cdot, x_0)$. The triangle inequality for $d^{g_f}_M$ leads us to

$$\begin{aligned}
(6.2) \quad s_{f,z_y}(\rho_{\partial M}(y)) + s_{f,v_y}(\rho_{x_0}(y)) &= L_{g_f}(\gamma_{z_y}|_{[0,\rho_{\partial M}(y)]}) + L_{g_f}(\gamma_{v_y}|_{[0,\rho_{x_0}(y)]})\\
&\ge \rho^{g_f}_{\partial M}(y) + \rho^{g_f}_{x_0}(y) \ge \rho^{g_f}_{\partial M}(x_0) = C_{\kappa,\lambda}.
\end{aligned}$$

By (6.1) and (6.2), we have $\Delta_f(\rho_{\partial M} + \rho_{x_0})(y) \ge 0$. Lemma 2.7 tells us that $U \subset \Omega$, and $\Omega$ is open.

Since $\operatorname{Int} M\backslash\{x_0\}$ is connected, we have $\Omega = \operatorname{Int} M\backslash\{x_0\}$, and hence $\rho_{\partial M} + \rho_{x_0} = l$ on $M$. This implies that $M = B_l(x_0)$. For each $v \in U_{x_0}M$, we have $\tau_{x_0}(v) = l$, and $\gamma_v$ is orthogonal to $\partial M$ at $l$. The equality in (2.4) holds on $\operatorname{Int} M\backslash\{x_0\}$. Choose an orthonormal basis $\{e_{v,i}\}_{i=1}^n$ of $T_{x_0}M$ with $e_{v,n} = v$. Let $\{Y_{v,i}\}_{i=1}^{n-1}$ be the Jacobi fields

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along $\gamma_v$ with $Y_{v,i}(0) = 0_{x_0}$, $Y'_{v,i}(0) = e_{v,i}$. By Lemma 2.10, for all $i$ we have $Y_{v,i} = F_{\kappa,v} E_{v,i}$ on $[0,l]$, where $\{E_{v,i}\}_{i=1}^{n-1}$ are the parallel vector fields with $E_{v,i}(0) = e_{v,i}$; moreover, if $N \in (-\infty, 1)$, then $f \circ \gamma_v$ is constant on $[0,l]$. Since the equalities in (6.2) hold, we have $s_{f,v}(l) = C_{\kappa,\lambda}$ and $F_{\kappa,v}(l) > 0$; in particular, we have no conjugate point of $x_0$ along $\gamma_v$. Thus, a map $\Phi\colon [0,l] \times U_{x_0} M \to M$ defined by $\Phi(t,v) := \gamma_v(t)$ is a Riemannian isometry with boundary from $[0,l] \times_{F_\kappa} \mathbb{S}^{n-1}$ to $M$.

Assume that $N \in (-\infty, 1)$. Then $f = (n-1)\delta$ for some constant $\delta \in \mathbb{R}$; in particular, $F_{\kappa,v}(t) = e^{2\delta} \mathfrak{s}_\kappa\big(e^{-2\delta} t\big) = \mathfrak{s}_{\kappa e^{-4\delta}}(t)$ and $l = e^{2\delta} C_{\kappa,\lambda} = C_{\kappa e^{-4\delta}, \lambda e^{-2\delta}}$. Hence $[0,l] \times_{F_\kappa} \mathbb{S}^{n-1}$ can be written as $B^n_{\kappa e^{-4\delta}, \lambda e^{-2\delta}}$. This completes the proof of Theorem 1.2. ∎

***Remark 6.3*** From the argument discussed in the proof of [24, Proposition 4.15], one can also conclude the following: Under the same setting as in Theorem 1.2, if $\rho^{g_f}_{\partial M}(x_0) = C_{\kappa,\lambda}$ for some $x_0 \in M$, then we have $\nabla f = g(\nabla f, \nabla \rho_{x_0}) \nabla \rho_{x_0}$ on $M$; in particular, $M$ is a warped product.

If $f$ is bounded from above, then we obtain the following theorem.

***Theorem 6.4*** *Let us assume that $\kappa$ and $\lambda$ satisfy the ball-condition. For $N \in (-\infty, 1]$, assume that $(M, \partial M, f)$ has lower $(\kappa, \lambda, N)$-weighted curvature bounds. Suppose additionally that there exists $\delta \in \mathbb{R}$ such that $f \le (n-1)\delta$ on $M$. Then we have*

$$\operatorname{InRad} M \le C_{\kappa e^{-4\delta}, \lambda e^{-2\delta}}. \tag{6.3}$$

*If $\rho_{\partial M}(x_0) = C_{\kappa e^{-4\delta}, \lambda e^{-2\delta}}$ for some $x_0 \in M$, then $M$ is isometric to $B^n_{\kappa e^{-4\delta}, \lambda e^{-2\delta}}$, and $f = (n-1)\delta$ on $M$.*

**Proof** Inequality (6.3) follows from Lemma 6.2. Let $x_0$ satisfy $\rho_{\partial M}(x_0) = C_{\kappa e^{-4\delta}, \lambda e^{-2\delta}}$, which will become the center of $M$. Put

$$l := \rho_{\partial M}(x_0) \quad \text{and} \quad \Omega := \big\{ x \in \operatorname{Int} M \setminus \{x_0\} \bigm| \rho_{\partial M}(x) + \rho_{x_0}(x) = l \big\}.$$

We prove that $\Omega$ is open in $\operatorname{Int} M \setminus \{x_0\}$. For a fixed point $x \in \Omega$, there exists an open neighborhood $U$ of $x$ such that $\rho_{x_0}$ and $\rho_{\partial M}$ are smooth on $U$, and for every $y \in U$ there exists a unique minimal geodesic in $M$ from $x_0$ to $y$ that lies in $\operatorname{Int} M$. Let us define $\rho_{\partial M,\delta} := e^{-2\delta} \rho_{\partial M}$ and $\rho_{x_0,\delta} := e^{-2\delta} \rho_{x_0}$. By Lemma 2.9 and (4.1), for each $y \in U$,

$$\begin{aligned} -\frac{\Delta_f(\rho_{\partial M} + \rho_{x_0})(y)}{(n-1) e^{\frac{-2f(y)}{n-1}}} &\le \frac{\mathfrak{s}'_{\kappa,\lambda}}{\mathfrak{s}_{\kappa,\lambda}}(\rho_{\partial M,\delta}(y)) + \frac{\mathfrak{s}'_\kappa}{\mathfrak{s}_\kappa}(\rho_{x_0,\delta}(y)) \\ &= \frac{\mathfrak{s}_{\kappa,\lambda}(\rho_{\partial M,\delta}(y) + \rho_{x_0,\delta}(y))}{\mathfrak{s}_{\kappa,\lambda}(\rho_{\partial M,\delta}(y)) \mathfrak{s}_\kappa(\rho_{x_0,\delta}(y))} \le 0. \end{aligned}$$

Lemma 2.7 implies that $U \subset \Omega$. Hence, $\Omega$ is open.

From the connectedness of $\operatorname{Int} M \setminus \{x_0\}$, we deduce that $\Omega = \operatorname{Int} M \setminus \{x_0\}$, and hence $\rho_{\partial M} + \rho_{x_0} = l$ on $M$. This implies $M = B_l(x_0)$. For each $v \in U_{x_0} M$, we have $\tau_{x_0}(v) = l$, and $\gamma_v$ is orthogonal to $\partial M$ at $l$. The equality in (2.4) holds on $\operatorname{Int} M \setminus \{x_0\}$. Choose an orthonormal basis $\{e_{v,i}\}_{i=1}^n$ of $T_{x_0} M$ with $e_{v,n} = v$. Let $\{Y_{v,i}\}_{i=1}^{n-1}$ be the Jacobi fields along $\gamma_v$ with $Y_{v,i}(0) = 0_{x_0}$, $Y'_{v,i}(0) = e_{v,i}$. For all $i$, we have $Y_{v,i} = F_{\kappa,v} E_{v,i}$ on $[0,l]$, where $F_{\kappa,v}$ is defined as (1.9), and $\{E_{v,i}\}_{i=1}^{n-1}$ are the parallel vector fields with $E_{v,i}(0) = e_{v,i}$; moreover, $f \circ \gamma_v = (n-1)\delta$ on $[0,l]$ (see Remark 2.11 and Lemma 2.10).

We see $F_{\kappa,\nu} = \mathfrak{s}_{\kappa e^{-4\delta}}$ on $[0,l]$. Therefore, a map $\Phi\colon [0,l] \times U_{x_0}M \to M$ defined by $\Phi(t,v) := \gamma_v(t)$ gives a desired Riemannian isometry with boundary. ■

# 7 Volume Growths

## 7.1 Volume Elements

We first recall that $\tau_f$ is defined as (1.4). For $z \in \partial M$ and $s \in (0, \tau_f(z))$, we define

$$\widehat{\theta}_f(s,z) := \theta_f(t_{f,z}(s), z), \tag{7.1}$$

where $\theta_f(t,z)$ is defined as (7.1), and $t_{f,z}$ is the inverse function of the function $s_{f,z}$ defined as (1.5).

We show the following volume element comparison inequality.

***Lemma 7.1*** *Let $z \in \partial M$. For $N \in (-\infty, 1]$, let us assume that*

$$\mathrm{Ric}_f^N(\gamma_z'(t)) \ge (n-1)\kappa e^{\frac{-4f(\gamma_z(t))}{n-1}}$$

*for all $t \in (0, \tau(z))$, and $H_{f,z} \ge (n-1)\lambda e^{\frac{-2f(z)}{n-1}}$. Then for all $s_1, s_2 \in [0, \tau_f(z))$ with $s_1 \le s_2$,*

$$\frac{\widehat{\theta}_f(s_2,z)}{\widehat{\theta}_f(s_1,z)} \le \frac{\mathfrak{s}_{\kappa,\lambda}^{n-1}(s_2)}{\mathfrak{s}_{\kappa,\lambda}^{n-1}(s_1)}.$$

*In particular, for all $s \in [0, \tau_f(z))$,*

$$\widehat{\theta}_f(s,z) \le e^{-f(z)} \mathfrak{s}_{\kappa,\lambda}^{n-1}(s). \tag{7.2}$$

**Proof** By (2.2) and (3.7), for all $s \in (0, \tau_f(z))$, we see

$$\frac{d}{ds} \log \frac{\widehat{\theta}_f(s,z)}{\mathfrak{s}_{\kappa,\lambda}^{n-1}(s)} = -\left( e^{\frac{2f}{n-1}} \Delta_f \rho_{\partial M} \right) \left( \gamma_z(t_{f,z}(s)) \right) + H_{\kappa,\lambda}(s) \le 0,$$

where $H_{\kappa,\lambda}$ is defined as (3.5). This implies the lemma. ■

***Remark 7.2*** Assume that for some $s_0 \in (0, \tau_f(z))$ the equality in (7.2) holds. Then the equality in (7.2) holds on $[0, s_0]$; in particular, the equality in (3.7) holds on $[0, s_0]$ (see Lemma 3.8).

If $f$ is bounded from above, then we have the following lemma.

***Lemma 7.3*** *Let $z \in \partial M$. Let $\kappa$ and $\lambda$ satisfy the monotone-condition. For $N \in (-\infty, 1]$, assume that $\mathrm{Ric}_f^N(\gamma_z'(t)) \ge (n-1)\kappa e^{\frac{-4f(\gamma_z(t))}{n-1}}$ for all $t \in (0, \tau(z))$, and $H_{f,z} \ge (n-1)\lambda e^{\frac{-2f(z)}{n-1}}$. We suppose additionally that there exists $\delta \in \mathbb{R}$ such that $f \circ \gamma_z \le (n-1)\delta$ on $(0, \tau(z))$. Then for all $t_1, t_2 \in [0, \tau(z))$ with $t_1 \le t_2$, we have*

$$\frac{\theta_f(t_2,z)}{\theta_f(t_1,z)} \le \frac{\mathfrak{s}_{\kappa e^{-4\delta}, \lambda e^{-2\delta}}^{n-1}(t_2)}{\mathfrak{s}_{\kappa e^{-4\delta}, \lambda e^{-2\delta}}^{n-1}(t_1)}.$$

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*In particular, for all* $t \in [0, \tau(z))$, *we have*

$$\theta_f(t,z) \le e^{-f(z)} \mathfrak{s}^{n-1}_{\kappa e^{-4\delta}, \lambda e^{-2\delta}}(t). \tag{7.3}$$

**Proof** From (2.2) and (4.2) we deduce

$$\frac{d}{dt} \log \frac{\theta_f(t,z)}{\mathfrak{s}^{n-1}_{\kappa,\lambda}(e^{-2\delta}t)} = -\Delta_f \rho_{\partial M}(\gamma_z(t)) + H_{\kappa,\lambda}(e^{-2\delta}t)e^{-2\delta} \le 0.$$

Since $\mathfrak{s}_{\kappa e^{-4\delta}, \lambda e^{-2\delta}}(t) = \mathfrak{s}_{\kappa,\lambda}(e^{-2\delta}t)$, we obtain the desired inequality. ■

***Remark 7.4*** Assume that for some $t_0 \in (0, \tau(z))$, the equality in (7.3) holds. Then the equality in (7.3) holds on $[0, t_0]$; in particular, the equality in (4.2) holds on $[0, t_0]$ (see Remark 4.3).

## 7.2 Absolute Comparisons

We define $\check{\theta}_f : [0,\infty) \times \partial M \to \mathbb{R}$ by

$$\check{\theta}_f(s,z) := \begin{cases} \widehat{\theta}_f(s,z) & \text{if } s < \tau_f(z), \\ 0 & \text{if } s \ge \tau_f(z). \end{cases} \tag{7.4}$$

To prove our volume comparison theorems, we need the following lemma.

***Lemma 7.5*** *Let $\partial M$ be compact. Then for all $r > 0$,*

$$m_{\frac{n+1}{n-1}f}\left(B^f_r(\partial M)\right) = \int_{\partial M} \int_0^r \check{\theta}_f(s,z)\, ds\, d\,\mathrm{vol}_h,$$

*where $B^f_r(\partial M)$ is defined as (1.13), and* $\mathrm{vol}_h$ *is the Riemannian volume measure on $\partial M$ determined by the induced metric $h$.*

**Proof** We give an outline of the proof. For $r > 0$, we set

$$U^f_r := \{ z \in \partial M \mid \tau_f(z) \le r \}, \quad \widehat{U}^f_r := \bigcup_{z \in U^f_r} \{ \gamma_z(t) \mid t \in [0, \tau(z)) \},$$

$$V^f_r := \{ z \in \partial M \mid \tau_f(z) > r \}, \quad \widehat{V}^f_r := \bigcup_{z \in V^f_r} \{ \gamma_z(t) \mid t \in [0, t_{f,z}(r)] \}.$$

For all $z \in \partial M$ and $t \in [0, \tau(z))$, we see $\rho_{\partial M, f}(\gamma_z(t)) = s_{f,z}(t)$. Hence, by a straightforward argument, one can verify $B^f_r(\partial M) \setminus \mathrm{Cut}\, \partial M = \widehat{U}^f_r \sqcup \widehat{V}^f_r$. In virtue of the coarea formula and the Fubini theorem,

$$\begin{aligned} m_{\frac{n+1}{n-1}f}\left(\widehat{U}^f_r\right) &= \int_{U^f_r} \int_0^{\tau(z)} e^{\frac{-(n+1)f(\gamma_z(t))}{n-1}} \theta(t,z)\, dt\, d\,\mathrm{vol}_h \\ &= \int_{U^f_r} \int_0^r \check{\theta}_f(s,z)\, ds\, d\,\mathrm{vol}_h, \\ m_{\frac{n+1}{n-1}f}\left(\widehat{V}^f_r\right) &= \int_{V^f_r} \int_0^{t_{f,z}(r)} e^{\frac{-(n+1)f(\gamma_z(t))}{n-1}} \theta(t,z)\, dt\, d\,\mathrm{vol}_h \\ &= \int_{V^f_r} \int_0^r \check{\theta}_f(s,z)\, ds\, d\,\mathrm{vol}_h. \end{aligned}$$

Since Cut $\partial M$ is a null set, we conclude the lemma. ∎

We set $m_{f,\partial M} := e^{-f}\,\mathrm{vol}_h$ .
We prove the following absolute volume comparison inequality.

***Lemma 7.6*** *For $N \in (-\infty, 1]$, assume that $(M, \partial M, f)$ has lower $(\kappa, \lambda, N)$-weighted curvature bounds. Let $\partial M$ be compact. Then for all $r > 0$ we have*

$$m_{\frac{n+1}{n-1}f}\left(B^f_r(\partial M)\right) \le \mathcal{S}_{\kappa,\lambda}(r)\, m_{f,\partial M}(\partial M). \tag{7.5}$$

**Proof** By Lemma 7.1, $\check{\theta}_f(s,z) \le e^{-f(z)} \bar{\mathfrak{s}}^{n-1}_{\kappa,\lambda}(s)$ for all $s \ge 0$, where $\bar{\mathfrak{s}}_{\kappa,\lambda}$ is defined as (1.11). Integrate both sides over $[0, r]$ with respect to $s$, and over $\partial M$ with respect to $z$. Lemma 7.5 implies the lemma. ∎

One can also prove the following by replacing the role of Lemmas 7.1 and 7.5 with Lemmas 7.3 and 2.6 in the proof of Lemma 7.6.

***Lemma 7.7*** *Let $\kappa$ and $\lambda$ satisfy the monotone-condition. For $N \in (-\infty, 1]$, assume that $(M, \partial M, f)$ has lower $(\kappa, \lambda, N)$-weighted curvature bounds. Let $\partial M$ be compact. Suppose additionally that there is $\delta \in \mathbb{R}$ such that $f \le (n-1)\delta$ on $M$. Then for all $r > 0$ we have*

$$m_f\left(B_r(\partial M)\right) \le \mathcal{S}_{\kappa e^{-4\delta},\lambda e^{-2\delta}}(r)\, m_{f,\partial M}(\partial M).$$

## 7.3 Relative Comparisons

**Proof of Theorem 1.3** For $N \in (-\infty, 1]$, assume that $(M, \partial M, f)$ has lower $(\kappa, \lambda, N)$-weighted curvature bounds. Let $\partial M$ be compact. Let us prove the desired inequality (1.14).

By Lemma 7.1, for all $s_1, s_2 \ge 0$ with $s_1 \le s_2$,

$$\check{\theta}_f(s_2, z)\, \bar{\mathfrak{s}}^{n-1}_{\kappa,\lambda}(s_1) \le \check{\theta}_f(s_1, z)\, \bar{\mathfrak{s}}^{n-1}_{\kappa,\lambda}(s_2).$$

We integrate the both sides over $[0, r]$ with respect to $s_1$, and over $[r, R]$ with respect to $s_2$. It follows that

$$\frac{\int_r^R \check{\theta}_f(s_2, z)\, ds_2}{\int_0^r \check{\theta}_f(s_1, z)\, ds_1} \le \frac{\mathcal{S}_{\kappa,\lambda}(R) - \mathcal{S}_{\kappa,\lambda}(r)}{\mathcal{S}_{\kappa,\lambda}(r)}.$$

Lemma 7.5 implies that

$$\frac{m_{\frac{n+1}{n-1}f}(B^f_R(\partial M))}{m_{\frac{n+1}{n-1}f}(B^f_r(\partial M))} \le 1 + \frac{\mathcal{S}_{\kappa,\lambda}(R) - \mathcal{S}_{\kappa,\lambda}(r)}{\mathcal{S}_{\kappa,\lambda}(r)} = \frac{\mathcal{S}_{\kappa,\lambda}(R)}{\mathcal{S}_{\kappa,\lambda}(r)},$$

and hence (1.14). We complete the proof of Theorem 1.3. ∎

***Remark 7.8*** Suppose that there exists $R \in (0, \overline{C}_{\kappa,\lambda}] \setminus \{\infty\}$ such that for every $r \in (0, R]$ the equality in (1.14) holds. Then we see $\tau_f \ge R$ on $\partial M$ (*cf.* [19, Lemma 4.3]).

We can also prove the following volume comparison by using Lemmas 7.3 and 2.6 instead of Lemmas 7.1 and 7.5 in the proof of Theorem 1.3.

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***Theorem 7.9*** *Let $\kappa$ and $\lambda$ satisfy the monotone-condition. For $N \in (-\infty, 1]$, assume that $(M, \partial M, f)$ has lower $(\kappa, \lambda, N)$-weighted curvature bounds. Let $\partial M$ be compact. Suppose additionally that there is $\delta \in \mathbb{R}$ such that $f \leq (n-1)\delta$ on $M$. Then for all $r, R > 0$ with $r \leq R$,*

$$\frac{m_f(B_R(\partial M))}{m_f(B_r(\partial M))} \leq \frac{\mathcal{S}_{\kappa e^{-4\delta}, \lambda e^{-2\delta}}(R)}{\mathcal{S}_{\kappa e^{-4\delta}, \lambda e^{-2\delta}}(r)}. \tag{7.6}$$

***Remark 7.10*** Assume that there exists $R \in (0, \overline{C}_{\kappa e^{-4\delta}, \lambda e^{-2\delta}}] \setminus \{\infty\}$ such that for every $r \in (0, R]$ the equality in (7.6) holds. Then one can verify that $\tau \geq R$ on $\partial M$ (*cf.* [19, Lemma 4.3]).

## 7.4 Volume Growth Rigidity

Now, let us prove the following theorem.

***Theorem 7.11*** *Suppose that $\kappa$ and $\lambda$ do not satisfy the ball-condition. For $N \in (-\infty, 1]$, assume that $(M, \partial M, f)$ has lower $(\kappa, \lambda, N)$-weighted curvature bounds. Let $\partial M$ be compact. If we have*

$$\liminf_{r \to \infty} \frac{m_{\frac{n+1}{n-1} f}(B_r^f(\partial M))}{\mathcal{S}_{\kappa, \lambda}(r)} \geq m_{f, \partial M}(\partial M), \tag{7.7}$$

*then $M$ is isometric to $[0, \infty) \times_{F_{\kappa,\lambda}} \partial M$; moreover, if $N \in (-\infty, 1)$, then for every $z \in \partial M$, the function $f \circ \gamma_z$ is constant.*

**Proof** Lemma 7.6 and Theorem 1.3 imply that for every $R > 0$ and for every $r \in (0, R]$, the equality in (1.14) holds. Then $\tau_f = \infty$ on $\partial M$ (see Remark 7.8). In particular, we have $\tau = \infty$ on $\partial M$.

Fix $z \in \partial M$. For all $s \geq 0$, we see $\check{\theta}_f(s, z) = e^{-f(z)} \mathfrak{s}_{\kappa,\lambda}^{n-1}(s)$. Choose an orthonormal basis $\{e_{z,i}\}_{i=1}^{n-1}$ of $T_z \partial M$, and let $\{Y_{z,i}\}_{i=1}^{n-1}$ be the $\partial M$-Jacobi fields along $\gamma_z$ with $Y_{z,i}(0) = e_{z,i}$, $Y'_{z,i}(0) = -A_{u_z} e_{z,i}$. For all $i$, we have $Y_{z,i} = F_{\kappa,\lambda,z} E_{z,i}$, where $\{E_{z,i}\}_{i=1}^{n-1}$ are the parallel vector fields $E_{z,i}(0) = e_{z,i}$. Moreover, if $N \in (-\infty, 1)$, then $f \circ \gamma_z$ is constant (see Remark 7.2). By the rigidity of Jacobi fields, a map $\Phi \colon [0, \infty) \times \partial M \to M$ defined by $\Phi(t, z) = \gamma_z(t)$ gives a desired isometry. ■

***Remark 7.12*** If $\kappa$ and $\lambda$ satisfy the ball-condition, then the author does not know whether a similar result to Theorem 7.11 holds. In this case, under the same setting as in Theorem 7.11, Lemma 3.5 implies $\tau_f = C_{\kappa,\lambda}$ on $\partial M$ (see Remark 7.8). Since $\tau(z)$ can be either finite or infinite for each $z \in \partial M$, it seems to be difficult to conclude any rigidity results.

Next, we prove the following volume growth rigidity theorem.

***Theorem 7.13*** *Let $\kappa$ and $\lambda$ satisfy the monotone-condition. For $N \in (-\infty, 1]$, assume that $(M, \partial M, f)$ has lower $(\kappa, \lambda, N)$-weighted curvature bounds. Let $\partial M$ be compact.*

*Suppose additionally that there exists $\delta \in \mathbb{R}$ such that $f \leq (n-1)\delta$ on $M$. If*

$$\liminf_{r\to\infty} \frac{m_f(B_r(\partial M))}{\mathcal{S}_{\kappa e^{-4\delta},\lambda e^{-2\delta}}(r)} \geq m_{f,\partial M}(\partial M),$$

*then the following hold.*

(i) *If $\kappa$ and $\lambda$ satisfy the convex-ball-condition, then $M$ is isometric to $B^n_{\kappa e^{-4\delta},\lambda e^{-2\delta}}$, and $f = (n-1)\delta$ on $M$.*

(ii) *If $\kappa \leq 0$ and $\lambda = \sqrt{|\kappa|}$, then $M$ is isometric to $[0,\infty) \times_{F_{\kappa,\lambda}} \partial M$; moreover, the following hold:*

 (a) *if $\kappa = 0$ and $N \in (-\infty, 1)$, then for every $z \in \partial M$ the function $f \circ \gamma_z$ is constant on $[0,\infty)$;*

 (b) *if $\kappa < 0$, then $f = (n-1)\delta$ on $M$.*

**Proof** In view of Lemma 7.7 and Theorem 7.9, we see the following. If $\kappa$ and $\lambda$ satisfy the convex-ball-condition, then for $R = C_{\kappa e^{-4\delta},\lambda e^{-2\delta}}$, and for every $r \in (0, R]$ the equality in (7.6) holds; in particular, $\tau = C_{\kappa e^{-4\delta},\lambda e^{-2\delta}}$ on $\partial M$ (see Remark 7.10). If $\kappa \leq 0$ and $\lambda = \sqrt{|\kappa|}$, then for every $R > 0$ and for every $r \in (0, R]$ the equality in (7.6) holds; in particular, $\tau = \infty$ (see Remark 7.10). Hence, $\tau = \overline{C}_{\kappa e^{-4\delta},\lambda e^{-2\delta}}$ on $\partial M$.

If $\kappa$ and $\lambda$ satisfy the convex-ball-condition, then due to Theorem 6.4, $M$ is isometric to $B^n_{\kappa e^{-4\delta},\lambda e^{-2\delta}}$, and $f = (n-1)\delta$.

If $\kappa \leq 0$ and $\lambda = \sqrt{|\kappa|}$, then $\mathrm{Cut}\,\partial M = \emptyset$. Theorem 1.1 tells us that $M$ is isometric to $[0,\infty) \times_{F_{\kappa,\lambda}} \partial M$; moreover, if $N \in (-\infty, 1)$, then for every $z \in \partial M$ the function $f \circ \gamma_z$ is constant. In the case of $\kappa < 0$, for all $t \geq 0, z \in \partial M$ we see $\theta_f(t,z) = e^{-f(z)} \mathfrak{s}^{n-1}_{\kappa e^{-4\delta},\lambda e^{-2\delta}}(t)$; in particular, $f = (n-1)\delta$ (see Remark 7.4). We complete the proof. ∎

# 8 Eigenvalues

## 8.1 Lower Bounds

We recall the following inequality of Picone type (see [1, Theorem 1.1] and [18, Lemma 7.1]).

***Lemma 8.1*** ([1, 18]) *Let $p \in (1,\infty)$. Let $\phi > 0$ and $\psi \geq 0$ be two $C^1$-functions on a domain $U \subset M$. Then we have*

$$\|\nabla\psi\|^p \geq \|\nabla\phi\|^{p-2} g\big(\nabla(\psi^p\phi^{1-p}), \nabla\phi\big). \tag{8.1}$$

*If the equality in (8.1) holds on $U$, then $\psi = c\phi$ for some $c \neq 0$ on $U$.*

We prove the inequality (1.16) in Theorem 1.4.

***Lemma 8.2*** *Let $p \in (1,\infty)$. For $N \in (-\infty, 1]$, assume that $(M, \partial M, f)$ has lower $(\kappa,\lambda,N)$-weighted curvature bounds. Let $M$ be compact, and let $f$ be $\partial M$-radial. Suppose additionally that there is $\delta \in \mathbb{R}$ such that $f \leq (n-1)\delta$ on $M$. For $D \in (0, \overline{C}_{\kappa,\lambda}] \setminus \{\infty\}$, suppose $\mathrm{InRad}_f\, M \leq D$, where $\mathrm{InRad}_f\, M$ is defined as (1.13). Then we have (1.16).*

**Proof** First, we notice $\nu_{p,\kappa e^{-4\delta},\lambda e^{-2\delta},De^{2\delta}} = e^{-2p\delta}\nu_{p,\kappa,\lambda,D}$. Let $\widehat{\varphi}$ be a non-zero function satisfying (1.15) for $\nu = \nu_{p,\kappa,\lambda,D}$. We may assume $\widehat{\varphi}|_{(0,D]} > 0$. The equation (1.15) can be written in the form

$$\left(|\varphi'(s)|^{p-2}\varphi'(s)\mathfrak{s}^{n-1}_{\kappa,\lambda}(s)\right)' + \nu|\varphi(s)|^{p-2}\varphi(s)\mathfrak{s}^{n-1}_{\kappa,\lambda}(s) = 0,$$
$$\varphi(0) = 0, \quad \varphi'(D) = 0.$$

Hence, we see that $\widehat{\varphi}'|_{[0,D)} > 0$. Put $\Phi := \widehat{\varphi}\circ\rho_{\partial M,f}$, and fix a non-negative, non-zero function $\psi \in C_0^\infty(M)$. From Lemma 8.1, we deduce that

$$\|\nabla\psi\|^p \geq \|\nabla\Phi\|^{p-2} g\left(\nabla(\psi^p\Phi^{1-p}), \nabla\Phi\right) \tag{8.2}$$

on $\operatorname{Int} M\setminus \operatorname{Cut}\partial M$. Here we put

$$\widehat{f} := (n+1)(n-1)^{-1}f \quad \text{and} \quad \check{f} := (n+1-2p)(n-1)^{-1}f.$$

By $f \leq (n-1)\delta$ and (8.2),

$$\begin{aligned} e^{2p\delta}\int_M \|\nabla\psi\|^p\, dm_{\widehat{f}} &\geq \int_M e^{\frac{2pf}{n-1}}\|\nabla\psi\|^p\, dm_{\widehat{f}} = \int_M \|\nabla\psi\|^p\, dm_{\check{f}} \\ &\geq \int_M \|\nabla\Phi\|^{p-2} g\left(\nabla(\psi^p\Phi^{1-p}), \nabla\Phi\right) dm_{\check{f}}. \end{aligned} \tag{8.3}$$

Further, (8.3) and (4.10) tell us that $e^{2p\delta}\int_M \|\nabla\psi\|^p\, dm_{\widehat{f}}$ is at least

$$-\int_M \psi^p\Phi^{1-p}\left\{\left(((\widehat{\varphi}')^{p-1})' - H_{\kappa,\lambda}(\widehat{\varphi}')^{p-1}\right)\circ\rho_{\partial M,f}\right\} dm_{\widehat{f}}$$

that is equal to $\nu_{p,\kappa,\lambda,D}\int_M \psi^p\, dm_{\widehat{f}}$ by the definition of $H_{\kappa,\lambda}$ (see (1.15) and (3.5)). Therefore, $R_{\widehat{f},p}(\psi) \geq e^{-2p\delta}\nu_{p,\kappa,\lambda,D}$. This implies (1.16). ∎

***Remark 8.3*** From the argument in the proof of Lemma 8.2, one can also verify the following. Under the same setting as in Lemma 8.2, we have $R_{\widehat{f},p}(\psi) \geq \nu_{p,\kappa e^{-4\delta},\lambda e^{-2\delta},De^{2\delta}}$ for every non-negative, non-zero $\psi \in C^1(M)$ with $\psi|_{\partial M} = 0$ (*cf.* Remark 4.11). Moreover, if the equality holds for some $\psi$, then the equalities in (8.3) hold, and hence $f = (n-1)\delta$ on the set where $\nabla\psi \neq 0$, and $\psi = c\Phi$ for some $c \neq 0$ (see Lemma 8.1); in particular, we can conclude $\nabla\psi \neq 0$ on $M\setminus\operatorname{Cut}\partial M$, and $f = (n-1)\delta$ on $M$.

We next prove the following comparison inequality.

***Lemma 8.4*** *Let $p \in (1,\infty)$. Let $\kappa$ and $\lambda$ satisfy the convex-ball-condition. For $N \in (-\infty, 1]$, let us assume that $(M, \partial M, f)$ has lower $(\kappa,\lambda,N)$-weighted curvature bounds. Let $M$ be compact. Suppose additionally that there exists $\delta \in \mathbb{R}$ such that $f \leq (n-1)\delta$ on $M$. Then we have*

$$\nu_{f,p}(M) \geq \nu_{0,p}(B^n_{\kappa e^{-4\delta},\lambda e^{-2\delta}}).$$

**Proof** We first note that $\nu_{0,p}(B^n_{\kappa e^{-4\delta},\lambda e^{-2\delta}}) = e^{-2p\delta}\nu_{p,\kappa,\lambda,C_{\kappa,\lambda}}$. Let $\widehat{\varphi}\colon [0, C_{\kappa,\lambda}] \to \mathbb{R}$ be a non-zero function satisfying (1.15) for $\nu = \nu_{p,\kappa,\lambda,C_{\kappa,\lambda}}$, and let $\widehat{\varphi}|_{(0,C_{\kappa,\lambda}]} > 0$. We see that $\widehat{\varphi}'|_{[0,C_{\kappa,\lambda})} > 0$. Define functions $\rho_{\partial M,\delta} := e^{-2\delta}\rho_{\partial M}$ and $\Phi := \widehat{\varphi} \circ \rho_{\partial M,\delta}$. We fix a non-negative, non-zero function $\psi \in C_0^\infty(M)$. Then Lemma 8.1 leads us to

$$\|\nabla\psi\|^p \geq \|\nabla\Phi\|^{p-2} g\big(\nabla(\psi^p\Phi^{1-p}), \nabla\Phi\big) \tag{8.4}$$

on $\operatorname{Int} M \setminus \operatorname{Cut}\partial M$. Notice that $\kappa$ and $\lambda$ satisfy the model-condition. From (8.4) and (4.7), it follows that

$$\begin{aligned} &e^{2p\delta}\int_M \|\nabla\psi\|^p\, dm_f \\ &\geq e^{2p\delta}\int_M \|\nabla\Phi\|^{p-2} g\big(\nabla(\psi^p\Phi^{1-p}), \nabla\Phi\big)\, dm_f \\ &\geq -\int_M \psi^p\Phi^{1-p}\Big\{\Big(((\widehat{\varphi}')^{p-1})' - H_{\kappa,\lambda}(\widehat{\varphi}')^{p-1}\Big) \circ \rho_{\partial M,\delta}\Big\}\, dm_f. \end{aligned}$$

The right-hand side is equal to $\nu_{p,\kappa,\lambda,C_{\kappa,\lambda}} \int_M \psi^p\, dm_f$. Therefore, we obtain $R_{f,p}(\psi) \geq e^{-2p\delta}\nu_{p,\kappa,\lambda,C_{\kappa,\lambda}}$. This proves the lemma. ∎

***Remark 8.5*** From the argument in the proof of Lemma 8.4, we can also conclude the following. Under the same setting as Lemma 8.4, we have $R_{f,p}(\psi) \geq \nu_{0,p}(B^n_{\kappa e^{-4\delta},\lambda e^{-2\delta}})$ for every non-negative, non-zero $\psi \in C^1(M)$ with $\psi|_{\partial M} = 0$ (*cf.* Remark 4.8). Moreover, if the equality holds for some $\psi$, then the equalities in (8.4) and (4.7) hold (see Lemma 8.1 and Remark 4.8); in particular, $\psi = c\Phi$ for some $c \neq 0$ on $M$.

## 8.2 Equality Cases

We recall the following fact for eigenfunctions of the weighted $p$-Laplacian (see *e.g.*, [20]).

***Proposition 8.6*** ([20]) *Let $p \in (1,\infty)$. Let $\phi\colon M \to \mathbb{R}$ be a smooth function. Let $M$ be compact. Then there exists a non-negative, non-zero function $\psi \in W_0^{1,p}(M, m_\phi)$ such that $R_{\phi,p}(\psi) = \nu_{\phi,p}(M)$. Moreover, $\psi \in C^{1,\alpha}(M)$ for some $\alpha \in (0,1)$.*

By using Proposition 8.6, we prove Theorem 1.4.

**Proof of Theorem 1.4** Let $p \in (1,\infty)$. For $N \in (-\infty, 1]$, assume that $(M, \partial M, f)$ has lower $(\kappa, \lambda, N)$-weighted curvature bounds. Let $M$ be compact, and let $f$ be $\partial M$-radial. Suppose additionally that there is $\delta \in \mathbb{R}$ such that $f \leq (n-1)\delta$ on $M$. For $D \in (0, \overline{C}_{\kappa,\lambda}]\setminus\{\infty\}$, suppose $\operatorname{InRad}_f M \leq D$. Lemma 8.2 yields (1.16). Now, we assume that the equality in (1.16) holds. Recall that our goal is to show that $M$ is a $(\kappa e^{-4\delta}, \lambda e^{-2\delta})$-equational model space, and $f = (n-1)\delta$ on $M$.

By applying Proposition 8.6 to $(n+1)(n-1)^{-1}f$, there exists a non-negative, non-zero $\psi \in W_0^{1,p}(M, m_{\frac{n+1}{n-1}f}) \cap C^{1,\alpha}(M)$ with

$$R_{\frac{n+1}{n-1}f,p}(\psi) = \nu_{p,\kappa e^{-4\delta},\lambda e^{-2\delta},De^{2\delta}}.$$

Then $f = (n-1)\delta$ on $M$ (see Remark 8.3). Theorem 1.4 is already known when $f$ is constant (see [18, Theorem 1.6]). Thus, we complete the proof of Theorem 1.4. ∎

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***Remark 8.7*** Kasue [10] obtained an explicit lower bound for $\mu_{2,\kappa,\lambda,D}$ (see [10, Lemma 1.3]). Due to the estimate, under the same setting as in Theorem 1.4 with $p = 2$, we have an explicit bound for $\nu_{\frac{n+1}{n-1}f,2}(M)$.

We also formulate the following eigenvalue rigidity theorem.

***Theorem 8.8*** *Let $p \in (1,\infty)$. Let $\kappa$ and $\lambda$ satisfy the convex-ball-condition. For $N \in (-\infty,1]$, let us assume that $(M,\partial M,f)$ has lower $(\kappa,\lambda,N)$-weighted curvature bounds. Let $M$ be compact. Suppose additionally that there is $\delta \in \mathbb{R}$ such that $f \leq (n-1)\delta$ on $M$. Then*

$$\nu_{f,p}(M) \geq \nu_{0,p}(B^n_{\kappa e^{-4\delta},\lambda e^{-2\delta}}). \tag{8.5}$$

*If the equality in (8.5) holds, then $M$ is isometric to $B^n_{\kappa e^{-4\delta},\lambda e^{-2\delta}}$, and $f = (n-1)\delta$ on $M$.*

**Proof** By Lemma 8.4, we have (8.5). Assume that the equality holds. Applying Proposition 8.6 to $f$, we have a non-negative, non-zero $\psi \in W^{1,p}_0(M,m_f)\cap C^{1,\alpha}(M)$ with $R_{f,p}(\psi) = \nu_{0,p}(B^n_{\kappa e^{-4\delta},\lambda e^{-2\delta}})$. Let $\widehat{\varphi}$ be a non-zero function satisfying (1.15) for $\nu = \nu_{p,\kappa,\lambda,C_{\kappa,\lambda}}$, and let $\widehat{\varphi}|_{(0,C_{\kappa,\lambda}]} > 0$. Define $\rho_{\partial M,\delta} := e^{-2\delta}\rho_{\partial M}$ and $\Phi := \widehat{\varphi}\circ\rho_{\partial M,\delta}$ (*cf.* Lemma 6.2). Then $\Phi = c\psi$ for some $c \neq 0$; in particular, $\operatorname{supp}\psi = M$ and $\Phi \in C^{1,\alpha}(M)$. The equality in (4.7) also holds (see Remark 8.5).

Since $\operatorname{supp}\psi = M$, the equality in (4.5) holds on $\operatorname{Int} M\setminus \operatorname{Cut}\partial M$ (see Remark 4.8). Fix $z \in \partial M$. Choose an orthonormal basis $\{e_{z,i}\}_{i=1}^{n-1}$ of $T_z\partial M$. Let $\{Y_{z,i}\}_{i=1}^{n-1}$ be the $\partial M$-Jacobi fields along $\gamma_z$ with $Y_{z,i}(0) = e_{z,i}$, $Y'_{z,i}(0) = -A_{u_z}e_{z,i}$. For all $i$, we see $Y_{z,i} = F_{\kappa,\lambda,z}E_{z,i}$ on $[0,\tau(z)]$, where $\{E_{z,i}\}_{i=1}^{n-1}$ are the parallel vector fields with $E_{z,i}(0) = e_{z,i}$. Moreover, $f\circ\gamma_z = (n-1)\delta$ on $[0,\tau(z)]$.

By Theorem 6.4, it suffices to show that $\operatorname{InRad} M = C_{\kappa e^{-4\delta},\lambda e^{-2\delta}}$. Let us suppose that $\operatorname{InRad} M < C_{\kappa e^{-4\delta},\lambda e^{-2\delta}}$. Take $x_0 \in M$ with $\rho_{\partial M}(x_0) = \operatorname{InRad} M$. Note that $x_0 \in \operatorname{Cut}\partial M$. By $\rho_{\partial M}(x_0) < C_{\kappa e^{-4\delta},\lambda e^{-2\delta}}$, and by the rigidity of the Jacobi fields, $x_0$ is not the first conjugate point along $\gamma_{z_0}$, where $z_0$ is a foot point of $x_0$. Hence $\rho_{\partial M,\delta}$ is not differentiable at $x_0$. From $\Phi \in C^{1,\alpha}(M)$, we deduce that $\widehat{\varphi}'(\rho_{\partial M,\delta}(x_0)) = 0$. This contradicts $\widehat{\varphi}'|_{[0,C_{\kappa,\lambda})} > 0$. Thus, we complete the proof of Theorem 8.8. ∎

## 8.3 Spectrum Rigidity

Let $\Omega$ be a relatively compact domain in $M$ such that its boundary is a smooth hypersurface in $M$ with $\partial\Omega\cap\partial M = \emptyset$. For the canonical measure $\operatorname{vol}_{\partial\Omega}$ on $\partial\Omega$, put $m_{f,\partial\Omega} := e^{-f}\operatorname{vol}_{\partial\Omega}$.

Let us prove the following area estimate.

***Lemma 8.9*** *Let $\kappa$ and $\lambda$ satisfy the monotone-condition. For $N \in (-\infty,1]$, assume that $(M,\partial M,f)$ has lower $(\kappa,\lambda,N)$-weighted curvature bounds. Suppose additionally that there exists $\delta \in \mathbb{R}$ such that $f \leq (n-1)\delta$ on $M$. Define $\rho_{\partial M,\delta} := e^{-2\delta}\rho_{\partial M}$. Let $\Omega$ be a relatively compact domain in $M$ such that $\partial\Omega$ is a smooth hypersurface in $M$ satisfying $\partial\Omega\cap\partial M = \emptyset$. Set*

$$D_{\delta,1}(\Omega) := \inf_{x\in\Omega}\rho_{\partial M,\delta}(x) \quad \textit{and} \quad D_{\delta,2}(\Omega) := \sup_{x\in\Omega}\rho_{\partial M,\delta}(x).$$

*Then we have*

$$m_f(\Omega) \le e^{2\delta} \sup_{s \in (D_{\delta,1}(\Omega), D_{\delta,2}(\Omega))} \frac{\int_s^{D_{\delta,2}(\Omega)} \mathfrak{s}_{\kappa,\lambda}^{n-1}(a)\,da}{\mathfrak{s}_{\kappa,\lambda}^{n-1}(s)} m_{f,\partial\Omega}(\partial\Omega).$$

**Proof** Define a function $\widehat{\varphi}\colon [D_{\delta,1}(\Omega), D_{\delta,2}(\Omega)] \to \mathbb{R}$ by

$$\widehat{\varphi}(s) := \int_{D_{\delta,1}(\Omega)}^{s} \frac{\int_a^{D_{\delta,2}(\Omega)} \mathfrak{s}_{\kappa,\lambda}^{n-1}(b)\,db}{\mathfrak{s}_{\kappa,\lambda}^{n-1}(a)}\,da.$$

Put $\Phi := \widehat{\varphi} \circ \rho_{\partial M,\delta}$. By Lemma 4.4, on $\Omega \setminus \operatorname{Cut} \partial M$, we have

$$\Delta_f \Phi \ge -e^{-4\delta}(\widehat{\varphi}'' - H_{\kappa,\lambda}\widehat{\varphi}') \circ \rho_{\partial M,\delta} = e^{-4\delta}. \tag{8.6}$$

By Lemma 2.2, there exists a sequence $\{\Omega_i\}$ of compact subsets of the closure $\overline{\Omega}$ such that for every $i$, the boundary $\partial\Omega_i$ is a smooth hypersurface in $M$ except for a null set in $(\partial\Omega, m_{f,\partial\Omega})$, and satisfying the following: (1) for all $i_1, i_2$ with $i_1 < i_2$, we have $\Omega_{i_1} \subset \Omega_{i_2}$; (2) $\overline{\Omega} \setminus \operatorname{Cut} \partial M = \bigcup_i \Omega_i$: (3) for every $i$, and for almost every point $x \in \partial\Omega_i \cap \partial\Omega$ in $(\partial\Omega, m_{f,\partial\Omega})$, there exists a unique unit outer normal vector for $\Omega_i$ at $x$ that coincides with the unit outer normal vector $u_{\partial\Omega}$ on $\partial\Omega$ for $\Omega$; (4) for every $i$, on $\partial\Omega_i \setminus \partial\Omega$, there exists a unique unit outer normal vector field $u_i$ for $\Omega_i$ such that $g(u_i, \nabla\rho_{\partial M}) \ge 0$.

For the canonical measure $\mathrm{vol}_i$ on $\partial\Omega_i \setminus \partial\Omega$, put $m_{f,i} := e^{-f}\,\mathrm{vol}_i$. By integrating the both sides of (8.6) on $\Omega_i$, and by integration by parts,

$$\begin{aligned} e^{-4\delta} m_f(\Omega_i) &\le \int_{\Omega_i} \Delta_f \Phi\, dm_f \\ &= -\int_{\partial\Omega_i \setminus \partial\Omega} g(u_i, \nabla\Phi)\, dm_{f,i} - \int_{\partial\Omega_i \cap \partial\Omega} g(u_{\partial\Omega}, \nabla\Phi)\, dm_{f,\partial\Omega}. \end{aligned}$$

The Cauchy–Schwarz inequality and $g(u_i, \nabla\Phi) \ge 0$ tell us that

$$\begin{aligned} e^{-4\delta} m_f(\Omega_i) &\le -\int_{\partial\Omega_i \cap \partial\Omega} g(u_{\partial\Omega}, \nabla\Phi)\, dm_{f,\partial\Omega} \\ &\le \int_{\partial\Omega_i \cap \partial\Omega} (\widehat{\varphi}' \circ \rho_{\partial M,\delta}) |g(u_{\partial\Omega}, \nabla\rho_{\partial M,\delta})|\, dm_{f,\partial\Omega} \\ &\le e^{-2\delta} \sup_{s \in (D_{\delta,1}(\Omega), D_{\delta,2}(\Omega))} \widehat{\varphi}'(s) m_{f,\partial\Omega}(\partial\Omega). \end{aligned}$$

By letting $i \to \infty$, we complete the proof. ∎

Kasue [11] has proved Lemma 8.9 when $f = 0$ and $\delta = 0$.

For $\alpha > 0$, the $(f,\alpha)$*-Dirichlet isoperimetric constant* is defined as

$$DI_\alpha(M, m_f) := \inf_{\Omega} \frac{m_{f,\partial\Omega}(\partial\Omega)}{(m_f(\Omega))^{1/\alpha}},$$

where the infimum is taken over all relatively compact domains $\Omega$ in $M$ such that $\partial\Omega$ are smooth hypersurfaces in $M$ satisfying $\partial\Omega \cap \partial M = \varnothing$. The $(f,\alpha)$*-Dirichlet Sobolev constant* is defined as

$$DS_\alpha(M, m_f) := \inf_{\phi \in W_0^{1,1}(M, m_f) \setminus \{0\}} \frac{\int_M \|\nabla\phi\|\, dm_f}{(\int_M |\phi|^\alpha dm_f)^{1/\alpha}}.$$

Let us recall the following relation between the constants.

***Proposition 8.10*** ([6]) *For all $\alpha > 0$, $DI_\alpha(M, m_f) = DS_\alpha(M, m_f)$.*

For $D \in (0, \overline{C}_{\kappa,\lambda}]$, we put

$$C(\kappa, \lambda, D) := \sup_{s\in[0,D)} \frac{\int_s^D \mathfrak{s}_{\kappa,\lambda}^{n-1}(a)da}{\mathfrak{s}_{\kappa,\lambda}^{n-1}(s)}. \tag{8.7}$$

Notice that $C(\kappa, \lambda, \infty)$ is finite if and only if $\kappa < 0$ and $\lambda = \sqrt{|\kappa|}$; in this case, we have $C(\kappa, \lambda, D) = ((n-1)\lambda)^{-1}(1 - e^{-(n-1)\lambda D})$.

From Lemma 8.9 we derive the following lemma.

***Lemma 8.11*** *Let $p \in (1, \infty)$. Let $\kappa$ and $\lambda$ satisfy the monotone-condition. For $N \in (-\infty, 1]$, assume that the triple $(M, \partial M, f)$ has lower $(\kappa, \lambda, N)$-weighted curvature bounds. Suppose additionally that there exists $\delta \in \mathbb{R}$ such that $f \le (n-1)\delta$ on $M$. For $D \in (0, \overline{C}_{\kappa,\lambda}]$, suppose* InRad $M \le e^{2\delta} D$. *Then we have*

$$\nu_{f,p}(M) \ge \left(pe^{2\delta} C(\kappa, \lambda, D)\right)^{-p}.$$

**Proof** Let $\Omega$ be a relatively compact domain in $M$ such that $\partial\Omega$ is a smooth hypersurface in $M$ with $\partial\Omega \cap \partial M = \varnothing$. Set $C_\delta := e^{2\delta} C(\kappa, \lambda, D)$. Lemma 8.9 implies that $m_f(\Omega) \le C_\delta m_{f,\partial\Omega}(\partial\Omega)$. By Proposition 8.10, we obtain $DS_1(M, m_f) \ge C_\delta^{-1}$. For all $\phi \in W_0^{1,1}(M, m_f)$, we have

$$\int_M |\phi| dm_f \le C_\delta \int_M \|\nabla\phi\| dm_f. \tag{8.8}$$

Let $\psi$ be a non-zero function in $W_0^{1,p}(M, m_f)$. Put $q := p(1-p)^{-1}$. In (8.8), by replacing $\phi$ with $|\psi|^p$, and by the Hölder inequality, we see

$$\begin{aligned}\int_M |\psi|^p dm_f &\le pC_\delta \int_M |\psi|^{p-1} \|\nabla\psi\| dm_f \\ &\le pC_\delta \Big(\int_M |\psi|^p dm_f\Big)^{1/q} \Big(\int_M \|\nabla\psi\|^p dm_f\Big)^{1/p}.\end{aligned}$$

Considering the Rayleigh quotient $R_{f,p}(\psi)$, we complete the proof. ∎

Finally, we prove the following spectrum rigidity theorem.

***Theorem 8.12*** *Let $p \in (1, \infty)$. Let $\kappa < 0$ and $\lambda := \sqrt{|\kappa|}$. For $N \in (-\infty, 1]$, assume that $(M, \partial M, f)$ has lower $(\kappa, \lambda, N)$-weighted curvature bounds. Let $\partial M$ be compact. Suppose additionally that there exists $\delta \in \mathbb{R}$ such that $f \le (n-1)\delta$ on $M$. Then*

$$\nu_{f,p}(M) \ge e^{-2p\delta} \Big(\frac{(n-1)\lambda}{p}\Big)^p. \tag{8.9}$$

*If the equality in (8.9) holds, then $M$ is isometric to $[0, \infty) \times_{F_{\kappa,\lambda}} \partial M$; moreover, if $N \in (-\infty, 1)$, then $f \circ \gamma_z$ is constant for every $z \in \partial M$.*

**Proof** For $D > 0$, we see that $C(\kappa, \lambda, D) = ((n-1)\lambda)^{-1}(1-e^{-(n-1)\lambda D})$. Note that the right-hand side is monotone increasing as $D \to \infty$. Put $D_\delta := e^{-2\delta}\,\mathrm{InRad}\,M$. From Lemma 8.11, we conclude

$$\nu_{f,p}(M) \geq e^{-2p\delta}\big(pC(\kappa, \lambda, D_\delta)\big)^{-p} \geq e^{-2p\delta}\Big(\frac{(n-1)\lambda}{p}\Big)^{p}.$$

Assume that the equality holds in (8.9). Then the monotonicity of $C(\kappa, \lambda, D)$ with respect to $D$ implies $D_\delta = \infty$; in particular, we have $\mathrm{InRad}\,M = \infty$. Since $\partial M$ is compact, $\tau(z_0) = \infty$ for some $z_0 \in \partial M$. Theorem 1.1 completes the proof of Theorem 8.12. ∎

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