Feedback

On 98.10: Mark Hennings writes: When proving properties of the symmetric difference, indicator functions are somewhat simpler than truth tables (see [1]). If $\mathscr{E}$ is the universal set, then the indicator function χ_A of a subset $A \in 2^{\mathscr{E}}$ is the {0,1}-valued function $\chi_A : \mathscr{E} \to \{0, 1\}$ given by the formula

$$\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}.$$

Since indicator functions are {0,1}-valued, we see that

$$A = B \quad \Leftrightarrow \quad \chi_A = \chi_B \quad \Leftrightarrow \quad \chi_A \equiv \chi_B \pmod 2$$

and so identifying an indicator function modulo 2 is enough to identify the underlying set.

The indicator function of $A + B$ is related to the indicator functions of A and B by the identity

$$\chi_{A+B} \equiv \chi_A + \chi_B \qquad \pmod 2$$

as shown by the following table (essentially a truth table):

	$\chi_A(x)$	$\chi_B(x)$	$\chi_A(x) + \chi_B(x)$	$\chi_{A+B}(x)$
$x \in A \cap B$	1	1	2	0
$x \in A \setminus B$	1	0	1	1
$x \in B \setminus A$	0	1	1	1
$x \in (A \cup B)'$	0	0	0	0

and hence the associativity of the symmetric difference can be proved by observing that

$$\chi_{A+(B+C)} \equiv \chi_A + \chi_{B+C} \equiv (\chi_A + \chi_B) + \chi_C \equiv \chi_{A+B} + \chi_C \equiv \chi_{(A+B)+C}$$

modulo 2, so that $A + (B + C) = (A + B) + C$. This approach, together with the identity

$$\chi_{A \cap B} = \chi_A \times \chi_B$$

enables us to show that $2^{\mathscr{E}}$ is a ring, with symmetric difference + as addition and intersection $\cap$ as multiplication. For example

$$\chi_{(A+B)\cap C} \equiv \chi_{A+B}\chi_C \equiv (\chi_A + \chi_B)\chi_C \equiv \chi_A\chi_C + \chi_B\chi_C \equiv \chi_{A\cap C} + \chi_{B\cap C} \equiv \chi_{(A\cap C)+(B\cap C)}$$

modulo 2, so that $(A + B) \cap C = (A \cap C) + (B \cap C)$, establishing the distributivity of intersection over symmetric difference.

Reference

1. P. R. Halmos, Does mathematics have elements? *Math. Intelligencer* **3.4** (1981), pp. 147-153.

On 98.14: Graham Jameson and Nick Lord write: As remarked in the Feedback item [1] in the same issue, the stated result is the case $x = \frac{1}{2}$ of the identity

$$\mathrm{Li}_2(x) + \mathrm{Li}_2(1 - x) + \ln x \ln(1 - x) = \zeta(2) = \frac{\pi^2}{6},$$

where the 'dilogarithm' function Li_2 is defined by $\mathrm{Li}_2(x) = \sum_{n=1}^{\infty} x^n / n^2$. Moreover, only a slight modification of the proof given for the special case is needed to establish the general case, as follows. By termwise integration of the series

$$-\frac{\ln(1 - t)}{t} = \sum_{n=1}^{\infty} \frac{t^{n-1}}{n},$$

we have

$$\mathrm{Li}_2(x) = -\int_0^x \frac{\ln(1 - t)}{t}\, dt.$$

Substituting $t = 1 - u$, we have also, for $0 \leqslant x \leqslant 1$,

$$\mathrm{Li}_2(1 - x) = -\int_0^{1-x} \frac{\ln(1 - t)}{t}\, dt = -\int_x^1 \frac{\ln u}{1 - u}\, du.$$

In particular, $-\int_0^1 \ln u / (1 - u)\, du = \mathrm{Li}_2(1) = \zeta(2)$. Now integrating by parts, we have for $0 < x < 1$

$$\begin{aligned} -\mathrm{Li}_2(x) &= [\ln t \ln(1 - t)]_0^x + \int_0^x \frac{\ln t}{1 - t}\, dt \\ &= \ln x \ln(1 - x) + \int_0^1 \frac{\ln t}{1 - t}\, dt - \int_x^1 \frac{\ln t}{1 - t}\, dt \\ &= \ln x \ln(1 - x) - \zeta(2) + \mathrm{Li}_2(1 - x). \end{aligned}$$

This identity is yet another result of Euler. As the author states, the value of $\mathrm{Li}_2\left(-\frac{1}{2}\right)$ is not known in closed form. In fact, rather curiously, apart from $\frac{1}{2}$ and the obvious cases 1, -1 and 0, the only real numbers x for which the value $\mathrm{Li}_2(x)$ is known in closed form are α, $-\alpha$, α^2 and $-1 - \alpha$, where α is the golden ratio number $\frac{1}{2}\left(\sqrt{5} - 1\right)$. These values featured as exercises in Carr's *Synopsis of Pure Mathematics* which inspired Ramanujan as a boy.

Reference

1. Nick Lord, Feedback on 'A simple series representation for Apéry's constant', *Math. Gaz.* **98** (July 2014), p. 357.

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On 98.19: Peter Coast writes: The idea behind this proof is a good one, but I feel the demonstration is unduly complicated. Consider the following.

For $x = N/D \in \mathbf{Q}$,

(a) Let y be the string (N in binary notation)2(D in binary notation),

(b) Let z be the integer defined by y in ternary notation,

(c) Let $f(x) = z$.

Then f is 1-1 (by construction) and into (1 is not in the range) and therefore injective but not surjective.

A similar approach works for algebraic numbers and computable numbers as well.

On 'Two girls − the value of information': Michael Jewess writes: The concluding paragraph of the interesting and stimulating article [1] is a little dismissive of the following statement in *Wikipedia*: 'The answer [to the two-girls problem] depends on how this information comes to us − what kind of selection process brought us this knowledge.' But *Wikipedia* has a point, which complements the article, and which can be demonstrated by means of a game show.

In a game show, both the host and the contestant are mathematically skilful, and each plays to win. In accordance with the game rules, the game show host first chooses at random a family with two children, and conceals them from the contestant. The host has access to full information about the children, but initially the contestant knows only their number (two) and the fact of random selection. The host then provides some further information about the children to the contestant. Next, in the light of the further information, the contestant computes the probability x that the family contains two girls, and thereby the probability $(1 - x)$ that it does not. Finally, the contestant guesses whether or not the family has in fact two girls (in line with the higher of the computed probabilities so as to maximise his chance of receiving the prize offered for a correct guess).

Problem 3 as worked out in the article corresponds to such a game in which (i) the rules additionally prescribe that the host provides the further information only in response to a challenge as follows

Challenge A: By selecting one option within the square brackets, construct a true statement from '[At least one/neither] of the two children is a girl who was born on a Tuesday.' −

and in which (ii) the response is in fact −

Response R: 'At least one of the two children is a girl who was born on a Tuesday.'

On receiving Response R, the contestant computes, as in the article by reference to its Table 1, that $x = 13/27$.

But Table 1 also shows that, for a randomly-chosen family with two children, there is only a $27/196$ probability that Response R is a truthful response to Challenge A. If Response R is untrue (a $169/196$ probability),

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the host must give the following response:

Response S: 'Neither of the two children is a girl who was born on a Tuesday.'

A contestant receiving Response S computes from Table 1 that $x = 36/169$. (Check: $\frac{27}{196} \times \frac{13}{27} + \frac{169}{196} \times \frac{36}{169} = \frac{1}{4}$, the probability of two girls in a randomly-chosen family with two children.)

Suppose instead that the challenge in the game is the following:

Challenge B: By selecting one option within each pair of square brackets, construct a true statement from 'At least one of the two children is a [girl/boy] who was born on a [Sunday/Monday/Tuesday/Wednesday/Thursday/Friday/Saturday].'

To Challenge A there is only one truthful response for any particular family in question. But to Challenge B there exist either one or two truthful responses depending on the family in question. If there are two truthful responses, and the one is less helpful to the contestant than the other, then the host will give the less helpful one.

If there are two boys (1/4 probability in a randomly-chosen family with two children), the host is forced to respond to Challenge B in a way that admits the existence of a boy − and to identify the birth day of one of the two boys. The contestant then knows, regardless of what birth day is given, that $x = 0$ and the contestant will win the game for certain. If there are two girls (1/4 probability in a randomly-chosen family with two children) or a boy and a girl (2/4 probability in a randomly-chosen family with two children), the host opts for 'girl' over 'boy', and identifies the birth day of one of the two girls or of the girl respectively. Regardless of whether the host's response is identical with Response R or whether it is Response R with 'Tuesday' substituted by a different day, the contestant then computes that $x = 1/(1 + 2) = 1/3$. (Check: $\frac{1}{4} \times 0 + \left(\frac{1}{4} + \frac{2}{4}\right) \times \frac{1}{3} = \frac{1}{4}$.)

Judged as games, neither the game with Challenge A nor that with Challenge B is very interesting: whatever response the host gives, the contestant's better guess is always that the family does not contain two girls − though, interestingly for the mathematician, only just (by 1/27), if Response R is given to Challenge A.

The above demonstrates that what one deduces from Response R depends on what the challenge was. A similar issue arose in a real TV game show, Monty Hall's *Let's make a deal*. In the Monty Hall game, the host knows which one (and only one) of three doors hides a prize, and the contestant has to guess which one. Marilyn vos Savant [2] imagines a little green woman who arrives from a UFO part-way through the game, in time to observe that the host has opened one door showing there is no prize there, but not in time to know the challenge to which the host has responded. Conceivably, the challenge to the host might have been: 'Open as you choose any one of the three doors;' or 'Open a particular door which I the contestant choose;' or − the actual, rather interesting case − 'Open as you choose either one of a particular pair of doors which I the contestant choose.' Unaware of what form the challenge has taken and of what choices (if any) the contestant has made, the little green woman is less able than the contestant to guess correctly where the prize is. The context in which the information has been supplied is crucial knowledge.

References

1. Keith Parramore and Joan Stephens, Two girls − the value of information, *Math. Gaz.*, **98** (July 2014), pp. 243-249.
2. Marilyn vos Savant, quoted by Paul Hoffman, *The man who loved only numbers* (Hyperion; Fourth Estate) 1998, p. 236.

On 'A non-calculator challenge: show that $\ln 2 < \frac{1}{\sqrt{2}}$': Nick Lord writes: It is great to see such a range of elegant approaches to this problem. Henry Ricardo and John Mahony place it within the more general context of the logarithmic mean inequality in either of the equivalent forms (A) $\ln\frac{b}{a} < \frac{b-a}{\sqrt{ab}}$ or (B) $\ln x < \sqrt{x} - \frac{1}{\sqrt{x}}$ with $0 < a < b$, $x = \frac{b}{a} > 1$. My original note showed that $\ln\frac{1+b}{1+a} < \sqrt{b} - \sqrt{a}$ which gives (B) on substituting $a = \frac{1}{b}$. Bob Burn's argument with abscissae a, $\sqrt{ab}$, b establishes (A), as does David Miles's with $x = \sqrt{2} - 1$ replaced by $x = \sqrt{\frac{b}{a}} - 1$.

For the specific case $\ln 2 < \frac{1}{\sqrt{2}}$, I note that Henry Ricardo's second proof gives $0 < \int_0^{\sqrt{2}} \left(1 - \frac{1}{x}\right)^2 dx = \left[x - \frac{1}{x} - 2\ln x\right]_0^{\sqrt{2}} = \frac{1}{\sqrt{2}} - \ln 2$ and John Mahony and K. B. Subramaniam's idea of using series expansions inspired this short argument: $\cosh\frac{1}{\sqrt{2}} = 1 + \frac{1}{2}\left(\frac{1}{\sqrt{2}}\right)^2 + \ldots > \frac{5}{4}$ so that $\frac{1}{\sqrt{2}} > \cosh^{-1}\frac{5}{4} = \ln 2$.

Henry Ricardo writes: I cannot resist responding to Nick Lord's challenge with several alternative proofs, the first four of which validate the left-hand member of the well-known *logarithmic mean inequality*

$$\sqrt{ab} < \frac{b-a}{\ln b - \ln a} < \frac{a+b}{2},$$

from which the desired inequality follows by letting $a = 1$, $b = 2$.

I. First we can give an easy geometric proof by noting that the area under

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the curve $y = 1/x$, $0 < a \leqslant x \leqslant b$, is less than the sum of the areas of the trapezia with vertices $\{(a, 0), (\sqrt{ab}, 0), (\sqrt{ab}, 1/\sqrt{ab}), (a, 1/a)\}$ and $\{(\sqrt{ab}, 0), (b, 0), (b, 1/b), (\sqrt{ab}, 1/\sqrt{ab})\}$. This comparison yields

$$\ln b - \ln a < \frac{ab - a^2}{2a\sqrt{ab}} + \frac{b^2 - ab}{2b\sqrt{ab}} = \frac{b - a}{\sqrt{ab}}.$$

II. Consider the function $f(x) = x - 1/x - 2\ln x$. Since $f(1) = 0$ and $f'(x) = 1 + 1/x^2 - 2/x = (x - 1)^2/x^2 > 0$ for $x > 1$, we see that $f(x) > 0$ for $x > 1$, or $\ln x < \dfrac{x^2 - 1}{2x}$. Assuming $0 < a < b$ and $x = \sqrt{b/a}$ in the last inequality, we get

$$\ln b - \ln a = \ln\frac{b}{a} < \frac{\frac{b}{a} - 1}{\sqrt{\frac{b}{a}}} = \frac{b - a}{\sqrt{ab}}.$$

III. For $t > 0$, the AM-GM inequality gives us

$$\frac{1}{t} = \sqrt{\frac{1}{\sqrt{t}}\cdot\frac{1}{t\sqrt{t}}} \leqslant \frac{1}{2\sqrt{t}} + \frac{1}{2t\sqrt{t}}.$$

Then

$$\ln\frac{b}{a} = \int_1^{b/a}\frac{dt}{t} \leqslant \int_1^{b/a}\frac{1}{2\sqrt{t}}\,dt + \int_1^{b/a}\frac{1}{2t\sqrt{t}}\,dt = \left(\sqrt{\frac{b}{a}} - 1\right) + \left(-\sqrt{\frac{a}{b}} + 1\right)$$
$$= \sqrt{\frac{b}{a}} - \sqrt{\frac{a}{b}} = \frac{b - a}{\sqrt{ab}}.$$

IV. Applying the Cauchy-Schwarz-Bunyakovsky inequality to the functions $f(x) = 1/x$ and $g(x) = 1$ on the interval $[a, b]$, $0 < a < b$, yields

$$\left(\int_a^b \frac{1}{x}\,dx\right)^2 < \left(\int_a^b \frac{1}{x^2}\,dx\right)\left(\int_a^b 1\,dx\right),$$

or

$$(\ln b - \ln a)^2 < (b - a)\left(\frac{1}{a} - \frac{1}{b}\right) = \frac{(b - a)^2}{ab},$$

giving us $\ln b - \ln a < \dfrac{b - a}{\sqrt{ab}}$.

In addition to the four proofs just given, we can show that $\ln 2 < 1/\sqrt{2}$ via some known inequalities whose proofs are given on pages 272-273 of [1]:

A If $x > 0$ and $x \neq 1$, then $\dfrac{\ln x}{x - 1} \leqslant \dfrac{1}{\sqrt{x}}$. Let $x = 2$.

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B If $x > 0$ and $x \neq 1$, then $\dfrac{\ln x}{x - 1} \leqslant \dfrac{1 + \sqrt[3]{x}}{x + \sqrt[3]{x}}$. Let $x = 2$. We show that $\dfrac{1 + \sqrt[3]{2}}{2 + \sqrt[3]{2}} < \dfrac{1}{\sqrt{2}}$. Setting $c = \sqrt[6]{2}$, we have

$$\frac{1}{\sqrt{2}} - \frac{1 + \sqrt[3]{2}}{2 + \sqrt[3]{2}} = \frac{1}{c^3} - \frac{1 + c^2}{c^6 + c^2} = \frac{c^6 + c^2 - c^3(1 + c^2)}{c^3(c^6 + c^2)}$$

$$= \frac{c^2(c - 1)^2(c^2 + c + 1)}{c^5(c^4 + 1)} > 0.$$

C For $x > 0$, $\ln\left(1 + \dfrac{1}{x}\right) \leqslant \dfrac{1}{\sqrt{x^2 + x}}$. Let $x = 1$.

Finally, Problem 1584 in *Crux Mathematicorum* [2] states that

$$\left(\frac{\ln \lambda}{\lambda - 1}\right)^3 < \frac{2}{\lambda(\lambda + 1)} \qquad \text{for } \lambda > 1.$$

Letting $\lambda = 2$, we find that $\ln 2 < \dfrac{1}{\sqrt[3]{3}}$. But $\dfrac{1}{\sqrt[3]{3}} < \dfrac{1}{\sqrt{2}} \Leftrightarrow \sqrt{2} < \sqrt[3]{3}$ $\Leftrightarrow (\sqrt{2})^6 < (\sqrt[3]{3})^6$, or $2^3 < 3^2$.

References

1. D. S. Mitrinović and P. M. Vasić, *Analytic Inequalities*, Springer-Verlag, New York (1970).
2. I. Bluskov, Solution to Problem 1584, *Crux Mathematicorum* **17:10** (1991), pp. 311-312.

John Mahony writes: Using the standard series expansions, for $|x| < 1$, we have

$$-\ln(1 - x) = x \sum_{n=0}^{\infty} \frac{1}{n + 1} x^n$$

$$\text{and} \qquad \frac{x}{\sqrt{1 - x}} = x \sum_{n=0}^{\infty} \frac{1}{4^n} \binom{2n}{n} x^n.$$

Since, by induction, $\dfrac{1}{n + 1} \leqslant \dfrac{1}{4^n}\dbinom{2n}{n}$ for all $n \geqslant 0$ (strict for $n \geqslant 2$), term-by-term comparison of the two series establishes that $-\ln(1 - x) < x(1 - x)^{-1/2}$ for $0 < x < 1$.

Substituting $x = 1 - \frac{1}{u}$ gives $\ln u < \left(1 - \dfrac{1}{u}\right)\left(\dfrac{1}{u}\right)^{-1/2} = \sqrt{u} - \dfrac{1}{\sqrt{u}}$ for $u > 1$ so that $\ln 2 < \sqrt{2} - \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}$. Replacing u by $\frac{1}{u}$ also shows that $\ln u > \sqrt{u} - \dfrac{1}{\sqrt{u}}$ for $0 < u < 1$.

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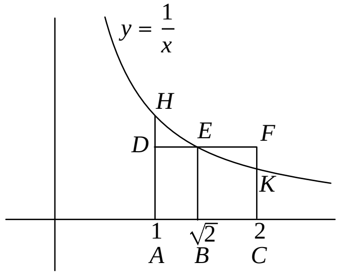
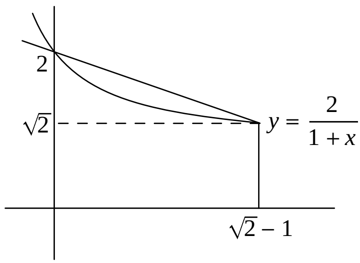
Bob Burn writes: The figure shows the graph of $y = \frac{1}{x}$ with the abscissae 1, $\sqrt{2}$ and 2. The hyperbolic area $AHKC = \ln 2$. The rectangular area $ADFC = \frac{1}{\sqrt{2}}$. The overlap $ADEKC$ is considerable. The required inequality is equivalent to the area of the curvilinear triangle EFK being greater than that of the curvilinear triangle HDE. However the two rectilinear triangles EFK and HDE have the same area $(3 - 2\sqrt{2})/(2\sqrt{2})$ so the convex triangle EFK has a greater area than the concave HDE.



David Miles writes: This result can be immediately established by comparing the area under the curve with the area of the trapezium in the diagram

$$\int_0^{\sqrt{2}-1} \frac{2}{1+x}\,dx < \frac{(2+\sqrt{2})(\sqrt{2}-1)}{2}.$$



K. B. Subramaniam writes: This note demonstrates a proof for the same without using a calculator. In fact, not even calculus is used. We have

$$\begin{aligned} e^{0.7} &= 1 + 0.7 + \frac{0.7^2}{2!} + \frac{0.7^3}{3!} + \dots \\ &> 1 + 0.7 + \frac{0.49}{2} + \frac{0.343}{6} \\ &> 1 + 0.7 + 0.245 + 0.057 \\ &= 2.002 \\ &> 2 \end{aligned}$$

$$\Rightarrow \ln 2 < \frac{7}{10} = \sqrt{\frac{49}{100}} < \frac{1}{\sqrt{2}}.$$

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