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A mathematical analysis of tsunami generation in shallow water due to seabed deformation

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In numerical computations of tsunamis due to submarine earthquakes, it is frequently assumed that the initial displacement of the water surface is equal to the permanent shift of the seabed and that the initial velocity field is equal to zero and the shallow-water equations are often used to simulate the propagation of tsunamis. We give a mathematically rigorous justification of this tsunami model starting from the full water-wave problem by comparing the solution of the full problem with that of the tsunami model. We also show that, in some cases, we have to impose a non-zero initial velocity field, which arises as a nonlinear effect.

1. Introduction

Tsunamis are one of the most disastrous phenomena of water waves and are characterized by a very long wavelength. They are mainly generated by a sudden deformation of the seabed with a submarine earthquake. The motion of tsunamis can be modelled as an irrotational flow of an incompressible ideal fluid bounded from above by a free surface and bounded from below by a moving bottom under the gravitational field. The model is usually called the full water-wave problem. Due to the complexities of the model, several simplified models have been proposed and used to simulate tsunamis. One of the most common models of tsunami propagation is the shallow-water model under the assumptions that the initial displacement of the water surface is equal to the permanent shift of the seabed and that the initial velocity field is equal to zero. Namely, in numerical computations of tsunamis due to submarine earthquakes, one usually uses the shallow-water equations

$$\eta_t + \nabla\cdot((h+\eta-b_1)u) = 0, \qquad u_t + (u\cdot\nabla)u + g\nabla\eta = 0 \tag{1.1}$$

under the following particular initial conditions:

$$\eta|_{t=0} = b_1 - b_0, \qquad u|_{t=0} = 0, \tag{1.2}$$

where η is the variation of the water surface, u is the velocity of the water in the horizontal directions, g is the gravitational constant, h is the mean depth of the water, b_0 is the bottom topography before the submarine earthquake, and b_1 is the bottom topography after the earthquake. The aim of this paper is to give a mathematically rigorous justification of this shallow-water model starting from the full water-wave problem, especially, the justification of the initial conditions (1.2).

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In this paper two non-dimensional parameters δ and ε play an important role, where δ is the ratio of the water depth h to the wavelength λ and ε is the ratio of the duration t_0 of the submarine earthquake to the period of tsunami $\lambda/\sqrt{gh}$. We note that $\sqrt{gh}$ is the propagation speed of linear shallow-water waves and that the duration of the seabed deformation is very short compared with the period of tsunamis in general. Therefore, ε should be a small parameter. It is known that the shallow-water equations (1.1) are derived from the full water-wave problem in the limit $\delta \to +0$. The derivation can be traced back to Airy [1]. Friedrichs [9] then systematically derived the equations using an expansion of the solution with respect to δ^2 (see also Lamb [16] and Stoker [22]). A mathematically rigorous justification of the shallow-water approximation for two-dimensional water waves over a flat bottom was given by Ovsjannikov [19, 20] under the periodic boundary condition with respect to the horizontal spatial variable, and then by Kano and Nishida [13] in a class of analytic functions (see also [12,14]). The justification in Sobolev spaces was given by Li [18] for two-dimensional water waves over a flat bottom and by Alvarez-Samaniego and Lannes [2] and Iguchi [11] for three-dimensional water waves where non-flat bottoms were allowed. However, there is no rigorous result concerning the shallow-water approximation in the case of moving bottom, nor a justification of the initial conditions (1.2).

We will show that, under appropriate conditions on the initial data and the bottom topography, the solution of the full water-wave problem can be approximated by the solution of the tsunami model (1.1) and (1.2) in the limit $\delta, \varepsilon \to +0$ under the restriction $\delta^2/\varepsilon \to +0$. This means that if the speed of seabed deformation is fast but not too fast, then the tsunami model would be a good approximation to the full water-wave problem. Moreover, we also show that, in the critical limit $\delta, \varepsilon \to +0$ and $\delta^2/\varepsilon \to \sigma$ with a positive constant σ, the initial conditions (1.2) should be replaced by

$$\eta|_{t=0} = b_1 - b_0, \qquad u|_{t=0} = \nabla\biggl(\frac{1}{2}\int_0^{t_0} b_t(\cdot,t)^2\,\mathrm{d}t\biggr), \tag{1.3}$$

where $b = b(x,t)$ is a bottom topography during the deformation of the seabed. One of the hardest parts of the analysis is the derivation of a uniform bound of the solution with respect to small parameters δ and ε for the full water-wave problem together with its derivatives, and especially for the time interval $0 \leqslant t \leqslant \varepsilon$ when the deformation of the seabed takes place. To this end, we adopt and extend the techniques used in Iguchi [11].

It is worth mentioning here that the Korteweg–de Vries equation is also known as a model of water waves and that the applicability of this modelling to tsunami propagation was discussed, for example, by Craig [7], Segur [21], Lakshmanan [15], Constantin and Johnson [5], Constantin [4] and Stuhlmeier [23], with conflicting points of view.

We now formulate the problem mathematically. Let $x = (x_1, x_2, \dots, x_n)$ be the horizontal spatial variables and x_{n+1} the vertical spatial variable. We denote all of the spatial variables by $X = (x, x_{n+1}) = (x_1, \dots, x_n, x_{n+1})$. We will consider a water wave in an $(n+1)$-dimensional space and assume that the domain $\Omega(t)$ occupied by the water at time t, the water surface $\Gamma(t)$, and the bottom $\Sigma(t)$ are

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of the forms

$$\Omega(t) = \{X = (x, x_{n+1}) \in \mathbb{R}^{n+1};\ b(x,t) < x_{n+1} < h + \eta(x,t)\},$$
$$\Gamma(t) = \{X = (x, x_{n+1}) \in \mathbb{R}^{n+1};\ x_{n+1} = h + \eta(x,t)\},$$
$$\Sigma(t) = \{X = (x, x_{n+1}) \in \mathbb{R}^{n+1};\ x_{n+1} = b(x,t)\},$$

where h is the mean depth of the water. The functions b and η represent the bottom topography and the surface elevation, respectively. In this paper b is a given function, while η is the unknown. In fact, our main interest is the behaviour of this function η, namely, the water surface.

We assume that the water is an incompressible and inviscid fluid, and that the flow is irrotational. Then, the motion of the water is described by the velocity potential $\Phi = \Phi(X,t)$ satisfying the equation

$$\Delta_X \Phi = 0 \quad \text{in } \Omega(t), \tag{1.4}$$

where Δ_X is the Laplacian with respect to X, that is, $\Delta_X = \Delta + \partial_{n+1}^2$ and $\Delta = \partial_1^2 + \cdots + \partial_n^2$. The boundary conditions on the water surface are given by

$$\eta_t + \nabla\Phi\cdot\nabla\eta - \partial_{n+1}\Phi = 0, \qquad \Phi_t + \tfrac{1}{2}|\nabla_X\Phi|^2 + g\eta = 0 \text{ on } \Gamma(t), \tag{1.5}$$

where $\nabla = (\partial_1, \dots, \partial_n)^{\mathrm{T}}$ and $\nabla_X = (\partial_1, \dots, \partial_n, \partial_{n+1})^{\mathrm{T}}$ are the gradients with respect to $x = (x_1, \dots, x_n)$ and to $X = (x, x_{n+1})$, respectively, and g is the gravitational constant. The first equation is the kinematical condition and the second one is the restriction of Bernoulli's law on the water surface. The kinematical boundary condition on the bottom is given by

$$b_t + \nabla\Phi\cdot\nabla b - \partial_{n+1}\Phi = 0 \quad \text{on } \Sigma(t). \tag{1.6}$$

Finally, we impose the initial conditions

$$\eta = \eta_0, \quad \Phi = \Phi_0 \quad \text{at } t = 0. \tag{1.7}$$

These are the basic equations for the full water-wave problem.

Next, we rewrite the equations (1.4)–(1.6) in an appropriate non-dimensional form. Let λ be the typical wavelength and let h be the mean depth. We introduce a non-dimensional parameter δ by $\delta = h/\lambda$ and rescale the independent and dependent variables by

$$x = \lambda\tilde{x}, \quad x_{n+1} = h\tilde{x}_{n+1}, \quad t = \frac{\lambda}{\sqrt{gh}}\tilde{t}, \quad \Phi = \lambda\sqrt{gh}\tilde{\Phi}, \quad \eta = h\tilde{\eta}, \quad b = h\tilde{b}. \tag{1.8}$$

Putting these into (1.4)–(1.6) and dropping the tilde in the notation we obtain

$$\delta^2\Delta\Phi + \partial_{n+1}^2\Phi = 0 \quad \text{in } \Omega(t), \tag{1.9}$$

$$\left.\begin{aligned} \delta^2(\eta_t + \nabla\Phi\cdot\nabla\eta) - \partial_{n+1}\Phi = 0 \\ \delta^2(\Phi_t + \tfrac{1}{2}|\nabla\Phi|^2 + \eta) + \tfrac{1}{2}(\partial_{n+1}\Phi)^2 = 0 \end{aligned}\right\} \quad \text{on } \Gamma(t), \tag{1.10}$$

$$\delta^2(b_t + \nabla\Phi\cdot\nabla b) - \partial_{n+1}\Phi = 0 \quad \text{on } \Sigma(t), \tag{1.11}$$

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where

$$\Omega(t) = \{X = (x, x_{n+1}) \in \mathbb{R}^{n+1};\ b(x,t) < x_{n+1} < 1 + \eta(x,t)\},$$
$$\Gamma(t) = \{X = (x, x_{n+1}) \in \mathbb{R}^{n+1};\ x_{n+1} = 1 + \eta(x,t)\},$$
$$\Sigma(t) = \{X = (x, x_{n+1}) \in \mathbb{R}^{n+1};\ x_{n+1} = b(x,t)\}.$$

Moreover, we assume that the seabed deforms only for time interval $[0, t_0]$ in the dimensional variable t, so that the function $b = b(x,t)$, which represents the bottom topography, can be written in the form

$$b(x,t) = \beta(x, t/\varepsilon), \qquad \beta(x,\tau) = \begin{cases} b_0(x) & \text{for } \tau \leqslant 0, \\ b_1(x) & \text{for } \tau \geqslant 1 \end{cases} \tag{1.12}$$

in the non-dimensional variables, where ε is a non-dimensional parameter defined by

$$\varepsilon = \frac{t_0}{\lambda/\sqrt{gh}}. \tag{1.13}$$

In this non-dimensional time variable we note that the bottom deforms only for the short time interval $0 \leqslant t \leqslant \varepsilon$ and that $b_t = \varepsilon^{-1}\beta_\tau$. Since we are interested in asymptotic behaviour of the solution when $\delta, \varepsilon \to +0$, we always assume that $0 < \delta, \varepsilon \leqslant 1$ in the following.

As in the usual way, we transform equivalently the initial-value problem (1.9)–(1.11) and (1.7) to a problem on the water surface. To this end, we introduce a new unknown function ϕ by

$$\phi(x,t) = \Phi(x, 1 + \eta(x,t), t), \tag{1.14}$$

which is the trace of the velocity potential on the water surface. Then, we see that the initial-value problem is transformed equivalently to the following:

$$\left.\begin{aligned} \eta_t - \Lambda^{\mathrm{DN}}(\eta,b,\delta)\phi + \varepsilon^{-1}\Lambda^{\mathrm{NN}}(\eta,b,\delta)\beta_\tau = 0, \\ \phi_t + \eta + \tfrac{1}{2}|\nabla\phi|^2 \qquad\qquad\qquad\qquad\qquad\qquad \\ -\tfrac{1}{2}\delta^2(1+\delta^2|\nabla\eta|^2)^{-1}(\Lambda^{\mathrm{DN}}(\eta,b,\delta)\phi - \varepsilon^{-1}\Lambda^{\mathrm{NN}}(\eta,b,\delta)\beta_\tau + \nabla\eta\cdot\nabla\phi)^2 = 0, \end{aligned}\right\} \tag{1.15}$$

$$\eta = \eta_0, \quad \phi = \phi_0 \quad \text{at } t = 0, \tag{1.16}$$

where $\Lambda^{\mathrm{DN}} = \Lambda^{\mathrm{DN}}(\eta,b,\delta)$ and $\Lambda^{\mathrm{NN}} = \Lambda^{\mathrm{NN}}(\eta,b,\delta)$ are linear operators depending on (η,b,δ) and called the Dirichlet-to-Neumann and the Neumann-to-Neumann maps for the Laplace equation, and $\phi_0 = \Phi_0(\cdot, 1+\eta_0(\cdot))$. In § 3 we will give the definition and basic properties of these maps Λ^{DN} and Λ^{NN}. We will investigate this initial-value problem (1.15) and (1.16) mathematically rigorously in this paper.

The contents of the paper are as follows. In § 2 we formally derive the tsunami model (1.1)–(1.3) from the full water-wave problem, analyse a so-called generalized Rayleigh–Taylor sign condition and give our main results. In § 3 we define the Dirichlet-to-Neumann map Λ^{DN}, the Neumann-to-Neumann map Λ^{NN} and related operators. Then we give basic properties of the operators and derive explicit forms of their Fréchet derivatives with respect to the surface variation η and the bottom

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topography b. In § 4 we study a boundary-value problem for the scaled Laplace equation (1.9) and derive some elliptic estimates for the solution by using the techniques in [11]. Most notably, we carefully analyse the dependence of the small parameter δ. In § 5, using the estimates obtained in § 4 we derive uniform bounds of the maps Λ^{DN}, Λ^{NN}, and related operators with respect to small δ in Sobolev spaces. In § 6 we reduce the full nonlinear equations (1.15) to quasi-linear equations. Finally, in § 7, by applying energy estimates to the quasi-linear equations derived in § 6, we prove the main theorems.

1.1. Notation

For $s \in \mathbb{R}$, we denote by H^s the Sobolev space of order s on $\mathbb{R}^n$ equipped with the inner product

$$(u, v)_s = (2\pi)^{-n} \int_{\mathbb{R}^n} (1 + |\xi|)^{2s} \hat{u}(\xi) \overline{\hat{v}(\xi)} \, \mathrm{d}\xi,$$

where $\hat{u}$ is the Fourier transform of u, that is,

$$\hat{u}(\xi) = \int_{\mathbb{R}^n} u(x) \mathrm{e}^{-\mathrm{i}x\cdot\xi} \, \mathrm{d}x.$$

We set $\|u\|_s = \sqrt{(u, u)_s}$, $(u, v) = (u, v)_0$ and $\|u\| = \|u\|_0$. The norm of a Banach space X is denoted by $\|\cdot\|_X$. We set $\partial_j = \partial/\partial x_j$, $\partial_{ij} = \partial_i \partial_j$ and $\partial_{ijk} = \partial_i \partial_j \partial_k$. A pseudo-differential operator $P(D)$, $D = (D_1, \dots, D_n)$ and $D_j = -\mathrm{i}\partial_j$, with a symbol $P(\xi)$, is defined by

$$P(D)u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} P(\xi) \hat{u}(\xi) \mathrm{e}^{\mathrm{i}x\cdot\xi} \, \mathrm{d}\xi.$$

We set $J = 1 + |D|$, so that $\|u\|_s = \|J^s u\|$. For operators A and B, we denote by $[A, B] = AB - BA$ the commutator. Throughout this paper, we denote inessential constants by the same symbol C.

2. A shallow-water approximation

In this section we begin formally studying the asymptotic behaviour of the solution $(\eta^{\delta,\varepsilon}, \phi^{\delta,\varepsilon})$ to the initial-value problem (1.15) and (1.16) when $\delta, \varepsilon \to +0$. We also derive the shallow-water equations with appropriate initial conditions, whose solution approximates $(\eta^{\delta,\varepsilon}, \nabla\phi^{\delta,\varepsilon})$ in a suitable sense. Then we analyse a so-called generalized Rayleigh–Taylor sign condition that is important for the well-posedness of the initial-value problem, and give the main results of this paper.

It is known that the Dirichlet-to-Neumann map $\Lambda^{\mathrm{DN}} = \Lambda^{\mathrm{DN}}(\eta, b, \delta)$ can be approximated by the second-order differential operator up to $O(\delta^2)$ as

$$\Lambda^{\mathrm{DN}}(\eta, b, \delta)\phi = -\nabla \cdot ((1 + \eta - b)\nabla\phi) + O(\delta^2). \tag{2.1}$$

For example, we refer to [11] for the above expansion. We proceed to expand the Neumann-to-Neumann map $\Lambda^{\mathrm{NN}} = \Lambda^{\mathrm{NN}}(\eta, b, \delta)$ with respect to δ^2. For a given function β on Σ, we denote by Φ the solution of the boundary-value problem

$$\delta^2 \Delta\Phi + \partial_{n+1}^2 \Phi = 0 \text{ in } \Omega, \qquad \Phi = 0 \text{ on } \Gamma, \qquad -\partial_{n+1}\Phi + \delta^2 \nabla b \cdot \nabla\Phi = \delta^2 \beta \text{ on } \Sigma. \tag{2.2}$$

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Here and in what follows, we omit the dependence of the time t in the notation for simplicity. Then we see that

$$\begin{aligned}(\partial_{n+1}\Phi)(x,x_{n+1}) &= (\partial_{n+1}\Phi)(x,b(x)) + \int_{b(x)}^{x_{n+1}} (\partial_{n+1}^2\Phi)(x,z)\,\mathrm{d}z \\ &= -\delta^2\beta(x) + \delta^2\nabla b(x)\cdot(\nabla\Phi)(x,b(x)) - \delta^2\int_{b(x)}^{x_{n+1}}(\Delta\Phi)(x,z)\,\mathrm{d}z,\end{aligned} \tag{2.3}$$

which implies that $(\partial_{n+1}\Phi)(X) = O(\delta^2)$ and that $(\nabla\partial_{n+1}\Phi)(X) = O(\delta^2)$. This and the relation

$$(\nabla\Phi)(x,x_{n+1}) = (\nabla\Phi)(x,1+\eta(x)) + \int_{1+\eta(x)}^{x_{n+1}}(\nabla\partial_{n+1}\Phi)(x,z)\,\mathrm{d}z \tag{2.4}$$

imply that $(\nabla\Phi)(X) = (\nabla\Phi)(x,1+\eta(x)) + O(\delta^2)$. Differentiating the Dirichlet boundary condition $\Phi(x,1+\eta(x)) = 0$ on Γ, we obtain

$$(\nabla\Phi)(x,1+\eta(x)) = -(\partial_{n+1}\Phi)(x,1+\eta(x))\nabla\eta(x), \tag{2.5}$$

which is $O(\delta^2)$. Therefore, we obtain $\nabla\Phi(X) = O(\delta^2)$ so that $\Delta\Phi(X) = O(\delta^2)$. It follows from these relations and (2.3) that $(\partial_{n+1}\Phi)(X) = -\delta^2\beta(x) + O(\delta^4)$, which, together with (2.5), implies that $(\nabla\Phi)(x,1+\eta(x)) = \delta^2\beta(x)\nabla\eta(x) + O(\delta^4)$. Thus, by (2.4), we obtain

$$(\nabla\Phi)(X) = \delta^2\beta(x)\nabla\eta(x) + \delta^2(1+\eta(x)-x_{n+1})\nabla\beta(x) + O(\delta^4). \tag{2.6}$$

Particularly, it holds that

$$(\Delta\Phi)(X) = \delta^2\nabla\cdot(\beta(x)\nabla\eta(x)) + \delta^2\nabla\eta(x)\cdot\nabla\beta(x) + \delta^2(1+\eta(x)-x_{n+1})\Delta\beta(x) + O(\delta^4).$$

Therefore, by (2.3), we get

$$\begin{aligned}(\partial_{n+1}\Phi)(X) = -\delta^2\beta(x) &+ \delta^4\nabla b(x)\cdot(\beta(x)\nabla\eta(x) + (1+\eta(x)-b(x))\nabla\beta(x)) \\ &- \delta^4(x_{n+1}-b(x))(\nabla\cdot(\beta(x)\nabla\eta(x)) + \nabla\eta(x)\cdot\nabla\beta(x)) \\ &+ \tfrac{1}{2}\delta^4((1+\eta(x)-x_{n+1})^2 - (1+\eta(x)-b(x))^2)\Delta\beta(x) + O(\delta^6).\end{aligned}$$

Since the Neumann-to-Neumann map Λ^{NN} is defined by:

$$(\Lambda^{\mathrm{NN}}\beta)(x) = \delta^{-2}(\partial_{n+1}\Phi)(x,1+\eta(x)) - \nabla\eta(x)\cdot(\nabla\Phi)(x,1+\eta(x)),$$

we obtain

$$\Lambda^{\mathrm{NN}}(\eta,b,\delta)\beta = -\beta - \delta^2\nabla\cdot((1+\eta-b)(\nabla\eta)\beta + \tfrac{1}{2}(1+\eta-b)^2\nabla\beta) + O(\delta^4). \tag{2.7}$$

For the definition of the map Λ^{NN}, we refer to definition 3.1. In view of (2.1) and (2.7), we see that the equations in (1.15) can be approximated by the ordinary differential equations

$$\eta_t = \frac{1}{\varepsilon}\beta_\tau + \frac{1}{\varepsilon}O(\varepsilon+\delta^2), \qquad \phi_t = \frac{1}{2}\left(\frac{\delta}{\varepsilon}\right)^2\beta_\tau^2 + \frac{1}{\varepsilon^2}O(\varepsilon^2+\delta^4). \tag{2.8}$$

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By resolving these equations under the initial conditions (1.16), we obtain

$$\left.\begin{aligned}\eta(x,t) &= \eta_0(x) + \beta(x,t/\varepsilon) - b_0(x) + O(\varepsilon+\delta^2),\\ \phi(x,t) &= \phi_0(x) + \frac{1}{2}\frac{\delta^2}{\varepsilon}\int_0^{t/\varepsilon} \beta_\tau(x,\tau)^2\,\mathrm{d}\tau + \frac{1}{\varepsilon}O(\varepsilon^2+\delta^4)\end{aligned}\right\} \tag{2.9}$$

for the time interval $0 \leqslant t \leqslant \varepsilon$. Particularly, we get

$$\left.\begin{aligned}\eta(x,\varepsilon) &= \eta_0(x) + (b_1(x) - b_0(x)) + O(\varepsilon+\delta^2),\\ \phi(x,\varepsilon) &= \phi_0(x) + \frac{1}{2}\frac{\delta^2}{\varepsilon}\int_0^{1} \beta_\tau(x,\tau)^2\,\mathrm{d}\tau + \frac{1}{\varepsilon}O(\varepsilon^2+\delta^4).\end{aligned}\right\} \tag{2.10}$$

As $\delta, \varepsilon \to +0$ these data converge only in the case when δ^2/ε also converges to some value σ. Therefore, in this paper we will consider asymptotic behaviour of the solution $(\eta^{\delta,\varepsilon}, \phi^{\delta,\varepsilon})$ to the initial-value problem (1.15) and (1.16) in the limit

$$\delta, \varepsilon \to +0, \qquad \frac{\delta^2}{\varepsilon} \to \sigma. \tag{2.11}$$

On the other hand, noting that $\beta_\tau = 0$ and $b = b_1$ for $t > \varepsilon$, we see that the equations in (1.15) can be approximated by the partial differential equations

$$\eta_t + \nabla\cdot((1+\eta-b_1)\nabla\phi) = O(\delta^2), \qquad \phi_t + \eta + \tfrac{1}{2}|\nabla\phi|^2 = O(\delta^2) \tag{2.12}$$

for $t > \varepsilon$. Therefore, taking the limit (2.11) of (2.12) and (2.10), we obtain

$$\eta^0_t + \nabla\cdot((1+\eta^0-b_1)\nabla\phi^0) = 0, \qquad \phi^0_t + \eta^0 + \tfrac{1}{2}|\nabla\phi^0|^2 = 0$$

with initial conditions

$$\eta^0 = \eta_0 + (b_1 - b_0), \quad \phi^0 = \phi_0 + \tfrac{1}{2}\sigma\int_0^1 \beta_\tau(\cdot,\tau)^2\,\mathrm{d}\tau \quad \text{at } t = 0.$$

Finally, setting $u^0 := \nabla\phi^0$ and taking the gradient of the second equation, we are led to the shallow-water equations

$$\eta^0_t + \nabla\cdot((1+\eta^0-b_1)u^0) = 0, \qquad u^0_t + (u^0\cdot\nabla)u^0 + \nabla\eta^0 = 0 \tag{2.13}$$

with initial conditions

$$\eta^0 = \eta_0 + (b_1 - b_0), \quad u^0 = \nabla\phi_0 + \nabla\biggl(\tfrac{1}{2}\sigma\int_0^1 \beta_\tau(\cdot,\tau)^2\,\mathrm{d}\tau\biggr) \quad \text{at } t = 0. \tag{2.14}$$

Moreover, u^0 satisfies the irrotational condition

$$\operatorname{rot} u^0 = 0, \tag{2.15}$$

where $\operatorname{rot} u$ is the rotation of a vector $u = (u_1, \dots, u_n)^{\mathrm{T}}$ defined by

$$\operatorname{rot} u = (\partial_j u_i - \partial_i u_j)_{1\leqslant i,j\leqslant n}.$$

Here we note that, in the case $(\eta_0, \phi_0) = 0$, if we rewrite (2.13) and (2.14) in the dimensional variables, then we obtain (1.1) and (1.3).

We proceed to analyse a generalized Rayleigh–Taylor sign condition. It is known that the well-posedness of the initial-value problem (1.4)–(1.7) for water waves may be broken unless a generalized Rayleigh–Taylor sign condition $-\partial p/\partial N \geqslant c_0 > 0$ on the water surface is satisfied, where p is the pressure and N is the unit outward normal to the water surface (see, for example, [3]). Wu [24, 25] showed that this condition always holds for any smooth non-self-intersecting surface in the case of infinite depth. In the case with variable bottom, Lannes [17] gave a relation between this condition and the bottom topography. Constantin and Strauss [6] investigated the pressure of Stokes waves over a flat bottom and also proved that this condition holds for Stokes waves. We also mention the result of Ebin [8], where a motion close to a rigid rotation of an incompressible ideal fluid surrounded by a free surface was considered. It was shown that the corresponding initial-value problem is ill-posed. In this case, a generalized Rayleigh–Taylor sign is not satisfied. One may think that the vorticity breaks the condition but, even in the irrotational case, the condition does not hold in a certain situation. In fact, Iguchi [10] considered an irrotational circulating flow of an incompressible ideal fluid around a rigid obstacle and showed that if the circulation is stranger than the gravity, then a generalized Rayleigh–Taylor sign is not satisfied and the problem is ill-posed. In what follows we will consider this important condition in the limit (2.11).

In the dimensional variables we have the so-called Bernoulli's law

$$\Phi_t + \tfrac{1}{2}|\nabla_X \Phi|^2 + \frac{1}{\rho}(p - p_0) + g(x_{n+1} - h) = 0 \quad \text{in } \Omega(t), \tag{2.16}$$

where ρ is a constant density and p_0 is a constant atmospheric pressure. This equation is obtained by integrating the conservation of momentum, that is, the Euler equation

$$\begin{aligned} 0 &= \rho(v_t + (v \cdot \nabla_X)v) + \nabla_X p + \rho g \boldsymbol{e}_{n+1} \\ &= \rho \nabla_X \biggl(\Phi_t + \tfrac{1}{2}|\nabla_X \Phi|^2 + \frac{1}{\rho}(p - p_0) + g(x_{n+1} - h) \biggr), \end{aligned}$$

where $v = \nabla_X \Phi$ is the velocity and $\boldsymbol{e}_{n+1}$ is the unit vector in the vertical direction. We rescale the pressure p by $p = p_0 + \rho g h \tilde{p}$. Putting this and (1.8) into (2.16) and dropping the tilde in the notation, we obtain

$$-p = \Phi_t + \tfrac{1}{2}(|\nabla \Phi|^2 + \delta^{-2}(\partial_{n+1}\Phi)^2) + (x_{n+1} - 1). \tag{2.17}$$

Moreover, in the non-dimensional variables, the generalized Rayleigh–Taylor sign condition can be written in the form $a \geqslant c_0 > 0$, where

$$\begin{aligned} a &:= -(1 + \delta^2|\nabla\eta|^2)^{-1}(\partial_{n+1}p - \delta^2 \nabla\eta \cdot \nabla p)|_{\Gamma(t)} \\ &= -(\partial_{n+1}p)|_{\Gamma(t)} \\ &= 1 + \{\partial_{n+1}(\Phi_t + \tfrac{1}{2}(|\nabla\Phi|^2 + \delta^{-2}(\partial_{n+1}\Phi)^2))\}|_{\Gamma(t)} \\ &= 1 + (\partial_{n+1}\Phi_t + \nabla\Phi \cdot \nabla\partial_{n+1}\Phi - (\partial_{n+1}\Phi)\Delta\Phi)|_{\Gamma(t)}, \end{aligned} \tag{2.18}$$

where we used the relation $(\nabla Q)|_{\Gamma(t)} = \nabla(Q|_{\Gamma(t)}) - (\partial_{n+1}Q)|_{\Gamma(t)}\nabla\eta$, the boundary condition on the water surface (1.10) and the scaled Laplace equation (1.9).

We now consider the asymptotic behaviour of this function a in the limit (2.11), so that we can assume $\delta^2 = O(\varepsilon)$. We note that Φ satisfies (1.9), (1.11) and (1.14),

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and that we have (1.12). Therefore, as in the same calculation in the previous section, we see that

$$\nabla\Phi = \nabla\phi - \frac{\delta^2}{\varepsilon}\beta_\tau\nabla\eta - \frac{\delta^2}{\varepsilon}(1+\eta-x_{n+1})\nabla\beta_\tau + O(\delta^2)$$

and that

$$\begin{aligned}\partial_{n+1}\Phi = {} & \frac{\delta^2}{\varepsilon}\beta_\tau + \delta^2\nabla b\cdot\left(\nabla\phi - \frac{\delta^2}{\varepsilon}\beta_\tau\nabla\eta - \frac{\delta^2}{\varepsilon}(1+\eta-b)\nabla\beta_\tau\right)\\ & - \delta^2(x_{n+1}-b)\left(\nabla\cdot\left(\nabla\phi - \frac{\delta^2}{\varepsilon}\beta_\tau\nabla\eta\right) - \frac{\delta^2}{\varepsilon}\nabla\eta\cdot\nabla\beta_\tau\right)\\ & - \frac{\delta^2}{2}\frac{\delta^2}{\varepsilon}((1+\eta-x_{n+1})^2 - (1+\eta-b)^2)\Delta\beta_\tau + O(\delta^4).\end{aligned}$$

Here, it follows from (2.8) that

$$\eta_t = \frac{1}{\varepsilon}\beta_\tau + O(1) \qquad \phi_t = \frac{1}{2}\left(\frac{\delta}{\varepsilon}\right)^2\beta_\tau^2 + O(1), \qquad \nabla\phi_t - \frac{\delta^2}{\varepsilon}\beta_\tau\nabla\eta_t = O(1).$$

Therefore,

$$\begin{aligned}&(\partial_{n+1}\Phi_t)|_{\Gamma(t)}\\ &\quad = \left(\frac{\delta}{\varepsilon}\right)^2(1-\delta^2|\nabla\eta|^2)\beta_{\tau\tau} + \frac{\delta^2}{\varepsilon}\nabla\cdot\beta_\tau\left(\left(\nabla\phi - \frac{\delta^2}{\varepsilon}\beta_\tau\nabla\eta\right)\right) - \left(\frac{\delta^2}{\varepsilon}\right)^2\beta_\tau\nabla\eta\cdot\nabla\beta_\tau\\ &\qquad + \left(\frac{\delta^2}{\varepsilon}\right)^2\nabla\cdot((1+\eta-b)\beta_{\tau\tau}\nabla\eta + \tfrac{1}{2}(1+\eta-b)^2\nabla\beta_{\tau\tau}) + O(\delta^2).\end{aligned}$$

Putting these into (2.18), we obtain

$$\begin{aligned}a = 1 + {} & \left(\frac{\delta}{\varepsilon}\right)^2(1-\delta^2|\nabla\eta|^2)\beta_{\tau\tau} + 2\frac{\delta^2}{\varepsilon}\left(\nabla\phi - \frac{\delta^2}{\varepsilon}\beta_\tau\nabla\eta\right)\cdot\nabla\beta_\tau\\ & + \left(\frac{\delta^2}{\varepsilon}\right)^2\nabla\cdot((1+\eta-b)(\nabla\eta)\beta_{\tau\tau} + \tfrac{1}{2}(1+\eta-b)^2\nabla\beta_{\tau\tau}) + O(\delta^2).\end{aligned} \tag{2.19}$$

On the other hand, in view of (2.9) and (2.11), we define an approximate solution $(\eta^{(0)}, \phi^{(0)})$ in the fast timescale $\tau = t/\varepsilon$ by

$$\eta^{(0)}(x,\tau) := \eta_0(x) + \beta(x,\tau) - \beta(x,0), \quad \phi^{(0)}(x,\tau) := \phi_0(x) + \tfrac{1}{2}\sigma\int_0^\tau \beta_\tau(x,\tilde{\tau})^2\,\mathrm{d}\tilde{\tau}. \tag{2.20}$$

Then, we have, at least formally,

$$\eta(x,t) = \eta^{(0)}(x,t/\varepsilon) + O(\varepsilon), \quad \phi(x,t) = \phi^{(0)}(x,t/\varepsilon) + o(1)$$

for $(x,t)\in\mathbb{R}^n\times[0,\varepsilon]$. Taking this and (2.19) into account, we define a function $a^{(0)} = a^{(0)}(x,\tau)$ by

$$\begin{aligned}a^{(0)} := {} & 2(\nabla\phi^{(0)} - \sigma\beta_\tau\nabla\eta^{(0)})\cdot\nabla\beta_\tau\\ & + \sigma\nabla\cdot((1+\eta^{(0)}-\beta)(\nabla\eta^{(0)})\beta_{\tau\tau} + \tfrac{1}{2}(1+\eta^{(0)}-\beta)^2\nabla\beta_{\tau\tau}),\end{aligned} \tag{2.21}$$

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where $(\eta^{(0)}, \phi^{(0)})$ is the approximate solution defined in (2.20). We note that this function $a^{(0)}$ is explicitly written out in terms of the initial data (η_0, ϕ_0), the bottom topography β and the constant σ in the limit (2.11). Then, by (2.19), we see that

$$a(x,t) = 1 + \left(\frac{\delta}{\varepsilon}\right)^2 \left(1 - \delta^2\left(\left|\nabla\eta^{(0)}\left(x, \frac{t}{\varepsilon}\right)\right|^2 + C\right)\right)\beta_{\tau\tau}\left(x, \frac{t}{\varepsilon}\right) + \sigma\left(a^{(0)}\left(x, \frac{t}{\varepsilon}\right) + C\sigma\beta_{\tau\tau}\left(x, \frac{t}{\varepsilon}\right)\right) + o(1), \quad (2.22)$$

where $C > 0$ is an arbitrary constant. Therefore, the generalized Rayleigh–Taylor sign condition is satisfied if the following conditions are fulfilled. The conditions depend on the relations between δ and ε.

Assumption 2.1. There exist constants $C, c > 0$ such that, for any $(x, \tau) \in \mathbb{R}^n \times (0, 1)$, the following conditions are satisfied.

1. In the case $\delta/\varepsilon \to 0$, no conditions are satisfied;
2. in the case $\delta/\varepsilon \to \nu$, $1 + \nu^2\beta_{\tau\tau}(x,\tau) \geqslant c$;
3. in the cases where $\delta/\varepsilon \to \infty$ and $\delta^2/\varepsilon \to 0$, $\beta_{\tau\tau}(x,\tau) \geqslant 0$.
4. in the cases where $\delta/\varepsilon \to \infty$ and $\delta^2/\varepsilon \to \sigma$,

$$\beta_{\tau\tau}(x,\tau) \geqslant 0, \qquad 1 + \sigma(a^{(0)} + \sigma C\beta_{\tau\tau})(x,\tau) \geqslant c.$$

From a technical point of view, we also impose the following condition.

Assumption 2.2. For any $(x, \tau) \in \mathbb{R}^n \times (0, 1)$ the following conditions are satisfied.

1. In the case $\delta/\varepsilon \to \nu$, no conditions are satisfied.
2. In the case $\delta/\varepsilon \to \infty$, $\beta_{\tau\tau\tau}(x,\tau) \leqslant 0$.

The following theorem is one of the main results in this paper and asserts the existence of the solution to the initial-value problem for the full water-wave problem with uniform bounds of the solution independent of δ and ε on the time interval $[0, \varepsilon]$.

Theorem 2.3. *Let $M_0, c_0 > 0$, $r > \frac{1}{2}n$, and $s > \frac{1}{2}(n+9)$. Under assumptions 2.1 and 2.2, there exist constants $C_0, \delta_0, \varepsilon_0, \gamma_0 > 0$ such that, for any $\delta \in (0, \delta_0]$, $\varepsilon \in (0, \varepsilon_0]$, (η_0, ϕ_0) and b satisfying $|\delta^2/\varepsilon - \sigma| \leqslant \gamma_0$, (1.12) and*

$$\|\beta(\tau)\|_{s+9/2} + \|\beta_\tau(\tau)\|_{s+5} + \|\beta_{\tau\tau}(\tau)\|_{s+1} + \|\beta_{\tau\tau\tau}(\tau)\|_{r+2} \leqslant M_0,$$
$$\|\nabla\phi_0\|_{s+3} + \|\eta_0\|_{s+4} \leqslant M_0, 1 + \eta_0(x) - b_0(x) \geqslant c_0 \quad \text{for } (x,\tau) \in \mathbb{R}^n \times (0,1),$$

the initial-value problem (1.15), (1.16) has a unique solution $(\eta, \phi) = (\eta^{\delta,\varepsilon}, \phi^{\delta,\varepsilon})$ on the time interval $[0, \varepsilon]$ satisfying

$$\left\|\eta^{\delta,\varepsilon}(t) - \eta^{(0)}\left(\frac{t}{\varepsilon}\right)\right\|_{s+2} + \left\|\phi^{\delta,\varepsilon}(t) - \phi^{(0)}\left(\frac{t}{\varepsilon}\right)\right\|_{s+2} \leqslant C_0\left(\varepsilon + \left|\frac{\delta^2}{\varepsilon} - \sigma\right|\right),$$
$$\|\eta^{\delta,\varepsilon}(t)\|_{s+3} + \|\nabla\phi^{\delta,\varepsilon}(t)\|_{s+2} \leqslant C_0,$$
$$1 + \eta^{\delta,\varepsilon}(x,t) - b(x,t) \geqslant \tfrac{1}{2}c_0 \quad \text{for } (x,t) \in \mathbb{R}^n \times [0,\varepsilon],$$

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where $(\eta^{(0)}, \phi^{(0)})$ *is the approximate solution in the fast time variable* $\tau = t/\varepsilon$ *defined by (2.20).*

Once we have obtained this kind of existence theorem for the solution with uniform bounds, by combining the existence result obtained in [11] where the case of a fixed bottom was investigated, we can easily consider the limits $\delta, \varepsilon \to 0$ of the solution $(\eta^{\delta,\varepsilon}, \phi^{\delta,\varepsilon})$.

THEOREM 2.4. *Under the same hypothesis as theorem 2.3, there exists a time* $T > 0$ *independent of* $\delta \in (0, \delta_0]$ *and* $\varepsilon \in (0, \varepsilon_0]$ *such that the solution* $(\eta^{\delta,\varepsilon}, \phi^{\delta,\varepsilon})$ *obtained in theorem 2.3 can be extended to the time interval* $[0, T]$ *and satisfies*

$$\|\eta^{\delta,\varepsilon}(t) - \eta^0(t)\|_{s-1} + \|\nabla\phi^{\delta,\varepsilon}(t) - u^0(t)\|_{s-1} \leqslant C_0(\varepsilon + |\delta^2/\varepsilon - \sigma|) \quad \text{for } \varepsilon \leqslant t \leqslant T,$$

where (η^0, u^0) *is a unique solution of the shallow-water equations (2.13) under the initial conditions (2.14) and* u^0 *satisfies the irrotational condition (2.15).*

3. The operators Λ^{DN}, Λ^{NN}, Λ^{DD} and Λ^{ND}

Throughout this and the next sections the time t is arbitrarily fixed, so that $\Omega(t)$, $\Gamma(t)$, $\Sigma(t)$, $\eta(x,t)$ and $b(x,t)$ are simply denoted by Ω, Γ, Σ, $\eta(x)$ and $b(x)$, respectively. Introducing an $(n+1)\times(n+1)$ matrix I_δ by

$$I_\delta = \begin{pmatrix} E_n & 0 \\ 0 & \delta^{-1} \end{pmatrix},$$

where E_n is the $n \times n$ unit matrix, we consider the boundary-value problem

$$\nabla_X \cdot I_\delta^2 \nabla_X \Phi = 0 \text{ in } \Omega, \quad \Phi = \phi \text{ on } \Gamma, \quad (\nabla b, -1)^{\mathrm{T}} \cdot I_\delta^2 \nabla_X \Phi = \beta \text{ on } \Sigma. \tag{3.1}$$

We note that the first and the third equations in (3.1) with β replaced by $-\varepsilon^{-1}\beta_\tau$ are nothing but those in (1.9) and (1.11), respectively, and that the second equation in (3.1) corresponds to (1.14). Under suitable assumptions on η and b, for any functions ϕ on Γ and β on Σ in some class of functions, there exists a unique solution Φ of the boundary-value problem (3.1).

DEFINITION 3.1. The solution Φ will be denoted by $(\phi, \beta)^\hbar$. Using the solution Φ we define the linear operators $\Lambda^{\mathrm{DN}} = \Lambda^{\mathrm{DN}}(\eta, b, \delta)$, $\Lambda^{\mathrm{NN}} = \Lambda^{\mathrm{NN}}(\eta, b, \delta)$, $\Lambda^{\mathrm{DD}} = \Lambda^{\mathrm{DD}}(\eta, b, \delta)$ and $\Lambda^{\mathrm{ND}} = \Lambda^{\mathrm{ND}}(\eta, b, \delta)$ by

$$\begin{aligned} \Lambda^{\mathrm{DN}}(\eta, b, \delta)\phi + \Lambda^{\mathrm{NN}}(\eta, b, \delta)\beta &= (-\nabla\eta, 1)^{\mathrm{T}} \cdot I_\delta^2 (\nabla_X \Phi)(\cdot, 1 + \eta(\cdot)), \\ \Lambda^{\mathrm{DD}}(\eta, b, \delta)\phi + \Lambda^{\mathrm{ND}}(\eta, b, \delta)\beta &= \Phi(\cdot, b(\cdot)), \end{aligned}$$

which are called, respectively, the Dirichlet-to-Neumann (DN) map, the Neumann-to-Neumann (NN) map, the Dirichlet-to-Dirichlet (DD) map and the Neumann-to-Dirichlet (ND) map. In what follows, we write $\Lambda_0^{\mathrm{DN}} = \Lambda^{\mathrm{DN}}(0,0,\delta)$, $\Lambda_0^{\mathrm{NN}} = \Lambda^{\mathrm{NN}}(0,0,\delta)$, $\Lambda_0^{\mathrm{DD}} = \Lambda^{\mathrm{DD}}(0,0,\delta)$ and $\Lambda_0^{\mathrm{ND}} = \Lambda^{\mathrm{ND}}(0,0,\delta)$.

PROPOSITION 3.2. *We have*

$$\Lambda_0^{\mathrm{DN}} = \frac{|D|}{\delta}\tanh(\delta|D|), \quad \Lambda_0^{\mathrm{ND}} = \frac{\delta}{|D|}\tanh(\delta|D|), \quad -\Lambda_0^{\mathrm{NN}} = \Lambda_0^{\mathrm{DD}} = \frac{1}{\cosh(\delta|D|)}.$$

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Proof. In the case $(\eta, b) = 0$, the solution of (3.1) can be written explicitly in terms of Fourier multipliers as

$$\varPhi(\cdot, x_{n+1}) = \frac{\cosh(\delta|D|x_{n+1})}{\cosh(\delta|D|)}\phi + \frac{\delta\sinh(\delta|D|(1-x_{n+1}))}{|D|\cosh(\delta|D|)}\beta,$$

so that we easily obtain the desired expressions. □

Proposition 3.3. *The operators $\varLambda^{\mathrm{DN}}$ and $\varLambda^{\mathrm{ND}}$ are symmetric in L^2, and the adjoint operator of $\varLambda^{\mathrm{NN}}$ in L^2 is equal to $-\varLambda^{\mathrm{DD}}$. That is, for any $\phi, \psi \in H^1$ and any $\beta, \gamma \in L^2$, it holds that*

$$(\varLambda^{\mathrm{DN}}\phi, \psi) = (\phi, \varLambda^{\mathrm{DN}}\psi), \quad (\varLambda^{\mathrm{ND}}\beta, \gamma) = (\beta, \varLambda^{\mathrm{ND}}\gamma), \quad (\varLambda^{\mathrm{NN}}\beta, \psi) = -(\beta, \varLambda^{\mathrm{DD}}\psi).$$

Proof. Set $\varPhi := (\phi, \beta)^\hbar$ and $\varPsi := (\psi, \gamma)^\hbar$. By Green's formula we have

$$\begin{aligned} 0 &= \int_\varOmega ((\nabla_X \cdot I_\delta^2 \nabla_X \varPhi)\varPsi - \varPhi(\nabla_X \cdot I_\delta^2 \nabla_X \varPsi))\,\mathrm{d}X \\ &= \int_{\partial\varOmega} ((N \cdot I_\delta^2 \nabla_X \varPhi)\varPsi - \varPhi(N \cdot I_\delta^2 \nabla_X \varPsi))\,\mathrm{d}S \\ &= (\varLambda^{\mathrm{DN}}\phi + \varLambda^{\mathrm{NN}}\beta, \psi) - (\phi, \varLambda^{\mathrm{DN}}\psi + \varLambda^{\mathrm{NN}}\gamma) \\ &\quad + (\beta, \varLambda^{\mathrm{DD}}\psi + \varLambda^{\mathrm{ND}}\gamma) - (\varLambda^{\mathrm{DD}}\phi + \varLambda^{\mathrm{ND}}\beta, \gamma), \end{aligned}$$

where N is the unit outward normal to the boundary $\partial\varOmega$. By setting $(\beta, \gamma) = 0$, $(\phi, \psi) = 0$ and $(\phi, \gamma) = 0$ in the above equality, we obtain the respective desired identities. □

Similarly, as a simple application of Green's formula, we have the following lemma.

Lemma 3.4. *For any $\phi \in H^1$ and $\beta \in L^2$, it holds that $(\varLambda^{\mathrm{DN}}\phi, \phi) = \|I_\delta \nabla_X \varPhi\|^2_{L^2(\varOmega)}$ with $\varPhi = (\phi, 0)^\hbar$ and that $(\varLambda^{\mathrm{ND}}\beta, \beta) = \|I_\delta \nabla_X \varPsi\|^2_{L^2(\varOmega)}$ with $\varPsi = (0, \beta)^\hbar$.*

In a derivation of the linearized equations for (1.15), we need an explicit formula of the Fréchet derivatives of the operators $\varLambda^{\mathrm{DN}}$ and $\varLambda^{\mathrm{NN}}$ with respect to η. The Fréchet derivative of $\varLambda^{\mathrm{DN}}$ was given in [17] and we can generalize the formula as follows.

Theorem 3.5. *The Fréchet derivatives of $\varLambda^{\mathrm{DN}}(\eta, b, \delta)$ and $\varLambda^{\mathrm{NN}}(\eta, b, \delta)$ with respect to η have the form*

$$D_\eta \varLambda^{\mathrm{DN}}(\eta, b, \delta)[\check{\eta}]\phi + D_\eta \varLambda^{\mathrm{NN}}(\eta, b, \delta)[\check{\eta}]\beta = -\delta^2 \varLambda^{\mathrm{DN}}(\eta, b, \delta)(Z\check{\eta}) - \nabla \cdot (v\check{\eta}),$$

where

$$\left.\begin{aligned} Z &= (1 + \delta^2|\nabla\eta|^2)^{-1}(\varLambda^{\mathrm{DN}}(\eta, b, \delta)\phi + \varLambda^{\mathrm{NN}}(\eta, b, \delta)\beta + \nabla\eta \cdot \nabla\phi), \\ v &= \nabla\phi - \delta^2 Z \nabla\eta. \end{aligned}\right\} \tag{3.2}$$

Proof. First, we will give an intuitive derivation of the formula. We take $\phi, \psi, \beta \in C_0^\infty(\mathbb{R}^n)$ and set $\varPhi := (\phi, \beta)^\hbar$ and $\varPsi := (\psi, 0)^\hbar$, namely, $\varPhi$ and $\varPsi$ are the solutions of the following boundary-value problems:

$$\left.\begin{aligned} \nabla_X \cdot I_\delta^2 \nabla_X \varPhi = 0, \qquad & \nabla_X \cdot I_\delta^2 \nabla_X \varPsi = 0 \text{ in } \varOmega, \\ \varPhi = \phi, \qquad & \varPsi = \psi \text{ on } \varGamma, \\ (\nabla b, -1)^{\mathrm{T}} \cdot I_\delta^2 \nabla_X \varPhi = \beta, \quad & (\nabla b, -1)^{\mathrm{T}} \cdot I_\delta^2 \nabla_X \varPsi = 0 \text{ on } \varSigma. \end{aligned}\right\} \tag{3.3}$$

These solutions depend not only on X but also on η, so that we also denote these solutions by $\varPhi = \varPhi(X) = \varPhi(X;\eta)$ and $\varPsi = \varPsi(X) = \varPsi(X;\eta)$. Here, we note that

$$(D_\eta \varLambda^{\mathrm{DN}}(\eta, b, \delta)[\check{\eta}]\phi, \psi) = \frac{\mathrm{d}}{\mathrm{d}h}(\varLambda^{\mathrm{DN}}(\eta + h\check{\eta}, b, \delta)\phi, \psi)\bigg|_{h=0}. \tag{3.4}$$

By Green's formula and proposition 3.3, we see that

$$\begin{aligned} \int_\varOmega I_\delta \nabla_X \varPhi \cdot I_\delta \nabla_X \varPsi \,\mathrm{d}X &= \int_{\partial\varOmega} (N \cdot I_\delta^2 \nabla_X \varPhi)\varPsi \,\mathrm{d}X \\ &= (\varLambda^{\mathrm{DN}}\phi + \varLambda^{\mathrm{NN}}\beta, \psi) + (\beta, \varLambda^{\mathrm{DD}}\psi) \\ &= (\varLambda^{\mathrm{DN}}\phi, \psi), \end{aligned} \tag{3.5}$$

so that

$$\begin{aligned} &(\varLambda^{\mathrm{DN}}(\eta + h\check{\eta}, b, \delta)\phi, \psi) \\ &\quad = \int_{\mathbb{R}^n} \left(\int_{b(x)}^{1+\eta(x)+h\check{\eta}(x)} I_\delta \nabla_X \varPhi(X; \eta + h\check{\eta}) \cdot I_\delta \nabla_X \varPsi(X; \eta + h\check{\eta}) \,\mathrm{d}x_{n+1} \right) \mathrm{d}x. \end{aligned}$$

We expand formally the solutions $\varPhi(X; \eta + h\check{\eta})$ and $\varPsi(X; \eta + h\check{\eta})$ as

$$\left.\begin{aligned} \varPhi(X; \eta + h\check{\eta}) = \varPhi(X;\eta) + \varPhi_1(X)h + O(h^2), \\ \varPsi(X; \eta + h\check{\eta}) = \varPsi(X;\eta) + \varPsi_1(X)h + O(h^2). \end{aligned}\right\} \tag{3.6}$$

Then,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}h}(\varLambda^{\mathrm{DN}}(\eta + h\check{\eta}, b, \delta)\phi, \psi)\bigg|_{h=0} &= \int_\varOmega (I_\delta \nabla_X \varPhi_1 \cdot I_\delta \nabla_X \varPsi + I_\delta \nabla_X \varPhi \cdot I_\delta \nabla_X \varPsi_1) \,\mathrm{d}X \\ &\quad + \int_{\mathbb{R}^n} (I_\delta \nabla_X \varPhi \cdot I_\delta \nabla_X \varPsi)|_\varGamma \check{\eta} \,\mathrm{d}x \\ &=: J_1 + J_2. \end{aligned} \tag{3.7}$$

It follows from the boundary condition on the water surface and the expansion (3.6) that

$$\begin{aligned} \phi(x) &= \varPhi(x, 1 + \eta(x) + h\check{\eta}(x); \eta + h\check{\eta}) \\ &= \varPhi(x, 1 + \eta(x); \eta) \\ &\quad + h\{(\partial_{n+1}\varPhi)(x, 1 + \eta(x); \eta)\check{\eta}(x) + \varPhi_1(x, 1 + \eta(x))\} + O(h^2), \end{aligned}$$

which implies that $\varPhi_1|_\varGamma = -(\partial_{n+1}\varPhi)|_\varGamma \check{\eta}$. Similarly, we have $\varPsi_1|_\varGamma = -(\partial_{n+1}\varPsi)|_\varGamma \check{\eta}$. On the other hand, by taking the trace of the expansion (3.6) on the bottom $\varSigma$ and using the definition of the DD map $\varLambda^{\mathrm{DD}}$, we get

$$\varLambda^{\mathrm{DD}}(\eta + h\check{\eta}, b, \delta)\psi = \varLambda^{\mathrm{DD}}(\eta, b, \delta)\psi + h\varPsi_1|_\varSigma + O(h^2).$$

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This and proposition 3.3 imply that

$$\Psi_1|_\Sigma = D_\eta \Lambda^{\mathrm{DD}}[\check{\eta}]\psi = -(D_\eta \Lambda^{\mathrm{NN}}[\check{\eta}])^*\psi.$$

Therefore, by Green's formula we see that

$$\begin{aligned} J_1 &= -\int_\Omega (\Phi_1(\nabla_X \cdot I_\delta^2 \nabla_X \Psi) + (\nabla_X \cdot I_\delta^2 \nabla_X \Phi)\Psi_1)\,\mathrm{d}X \\ &\quad + \int_{\partial\Omega} (\Phi_1(N \cdot I_\delta^2 \nabla_X \Psi) + (N \cdot I_\delta^2 \nabla_X \Phi)\Psi_1)\,\mathrm{d}S \\ &= -\int_{\mathbb{R}^n} ((\partial_{n+1}\Phi)|_\Gamma(\Lambda^{\mathrm{DN}}\psi) + (\Lambda^{\mathrm{DN}}\phi + \Lambda^{\mathrm{NN}}\beta)(\partial_{n+1}\Psi)|_\Gamma)\check{\eta}\,\mathrm{d}x \\ &\quad - (D_\eta \Lambda^{\mathrm{NN}}[\check{\eta}]\beta, \psi). \end{aligned}$$

On the other hand, in view of the relations $(\nabla Q)|_\Gamma = \nabla(Q|_\Gamma) - (\partial_{n+1}Q)|_\Gamma \nabla\eta$ we get

$$\begin{aligned} J_2 = \int_{\mathbb{R}^n} (\nabla\phi \cdot \nabla\psi - (\partial_{n+1}\Phi)|_\Gamma \nabla\eta \cdot \nabla\psi - (\partial_{n+1}\Psi)|_\Gamma \nabla\eta \cdot \nabla\phi \\ + \delta^{-2}(1 + \delta^2|\nabla\eta|^2)(\partial_{n+1}\Psi\partial_{n+1}\Phi)|_\Gamma)\check{\eta}\,\mathrm{d}x. \end{aligned}$$

These, together with the relations

$$\begin{aligned} (\partial_{n+1}\Phi)|_\Gamma &= \delta^2(1 + \delta^2|\nabla\eta|^2)^{-1}(\Lambda^{\mathrm{DN}}\phi + \Lambda^{\mathrm{NN}}\beta + \nabla\eta \cdot \nabla\phi) = \delta^2 Z, \\ (\partial_{n+1}\Psi)|_\Gamma &= \delta^2(1 + \delta^2|\nabla\eta|^2)^{-1}(\Lambda^{\mathrm{DN}}\psi + \nabla\eta \cdot \nabla\psi), \end{aligned}$$

yield that

$$\begin{aligned} (D_\eta \Lambda^{\mathrm{DN}}[\check{\eta}]\phi + D_\eta \Lambda^{\mathrm{NN}}[\check{\eta}]\beta, \psi) &= \int_{\mathbb{R}^n} (\nabla\phi \cdot \nabla\psi - \delta^2 Z(\Lambda^{\mathrm{DN}}\psi + \nabla\eta \cdot \nabla\psi))\check{\eta}\,\mathrm{d}x \\ &= -(\delta^2 \Lambda^{\mathrm{DN}}(Z\check{\eta}) + \nabla \cdot (v\check{\eta}), \psi), \end{aligned}$$

where we used the symmetric property of Λ^{DN} stated in proposition 3.3. Since the above equality holds for any $\psi \in C_0^\infty(\mathbb{R}^n)$, we obtain the desired formula.

Next, we will justify the above formal argument. Note that the expansion (3.6) has no sense because the domains of definition of the left-hand side and the right-hand side are different. Therefore, we need to give a good definition of $\Phi_1(X)$ and $\Psi_1(X)$ in order to obtain the formula (3.7). To this end, we use a diffeomorphism $X = \Xi(Y;\eta)$ from a simple domain $\Omega_0 := \mathbb{R}^n \times (0,1)$ to the water region $\Omega = \{X \in \mathbb{R}^{n+1}; b(x) < x_{n+1} < 1 + \eta(x)\}$ defined by

$$x_j = y_j, \quad 1 \leqslant j \leqslant n, \qquad x_{n+1} = b(y) + y_{n+1}(1 + \eta(y) - b(y)).$$

For any function $f = f(X;\eta)$ defined in Ω, we set $\tilde{f}(Y;\eta) := f(\Xi(Y;\eta);\eta)$, which is a function in the fixed domain Ω_0, so that the Fréchet derivative of this function $\tilde{f}(Y;\eta)$ with respect to η has a sense. For simplicity, we write $\tilde{f}_\eta = D_\eta \tilde{f}[\check{\eta}]$. Then, we see that

$$\begin{aligned}
&\int_{\mathbb{R}^n} \left(\int_{b(x)}^{1+\eta(x)+h\check{\eta}(x)} f(X;\eta+h\check{\eta})\,\mathrm{d}x_{n+1} \right) \mathrm{d}x \\
&\quad = \int_{\mathbb{R}^n} \left(\int_{b(x)}^{1+\eta(x)+h\check{\eta}(x)} \tilde{f}(\Xi^{-1}(X;\eta+h\check{\eta});\eta+h\check{\eta})\,\mathrm{d}x_{n+1} \right) \mathrm{d}x \\
&\quad = \int_{\mathbb{R}^n} \left(\int_{b(x)}^{1+\eta(x)+h\check{\eta}(x)} \tilde{f}(\Xi^{-1}(X;\eta+h\check{\eta});\eta)\,\mathrm{d}x_{n+1} \right) \mathrm{d}x \\
&\qquad + h \int_{\mathbb{R}^n} \left(\int_{b(x)}^{1+\eta(x)+h\check{\eta}(x)} \tilde{f}_\eta(\Xi^{-1}(X;\eta+h\check{\eta}))\,\mathrm{d}x_{n+1} \right) \mathrm{d}x + O(h^2).
\end{aligned}$$

This, together with the simple identity

$$\begin{aligned}
&\int_{b(x)}^{1+\eta(x)+h\check{\eta}(x)} \tilde{f}(\Xi^{-1}(X;\eta+h\check{\eta});\eta)\,\mathrm{d}x_{n+1} \\
&\qquad = \left(1 + \frac{h\check{\eta}(x)}{1+\eta(x)-b(x)}\right) \int_{b(x)}^{1+\eta(x)} \tilde{f}(\Xi^{-1}(X;\eta);\eta)\,\mathrm{d}x_{n+1}
\end{aligned}$$

implies that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}h} \int_{\mathbb{R}^n} \left(\int_{b(x)}^{1+\eta(x)+h\check{\eta}(x)} f(X;\eta+h\check{\eta})\,\mathrm{d}x_{n+1} \right) \mathrm{d}x \bigg|_{h=0} \\
&= \int_{\mathbb{R}^n} \left(\int_{b(x)}^{1+\eta(x)} \left(\tilde{f}_\eta(\Xi^{-1}(X;\eta)) + \frac{\check{\eta}(x)}{1+\eta(x)-b(x)} \tilde{f}(\Xi^{-1}(X;\eta);\eta) \right) \mathrm{d}x_{n+1} \right) \mathrm{d}x.
\end{aligned}$$

Here, using integration by parts, we have

$$\begin{aligned}
&\int_{b(x)}^{1+\eta(x)} \frac{\check{\eta}(x)}{1+\eta(x)-b(x)} \tilde{f}(\Xi^{-1}(X;\eta);\eta)\,\mathrm{d}x_{n+1} \\
&\qquad = \check{\eta}(x)\tilde{f}(x,1;\eta) - \int_{b(x)}^{1+\eta(x)} \frac{\check{\eta}(x)(x_{n+1}-b(x))}{(1+\eta(x)-b(x))^2} (\partial_{n+1}\tilde{f})(\Xi^{-1}(X;\eta);\eta)\,\mathrm{d}x_{n+1} \\
&\qquad = \check{\eta}(x) f(x,1+\eta(x);\eta) - \int_{b(x)}^{1+\eta(x)} \frac{\check{\eta}(x)(x_{n+1}-b(x))}{1+\eta(x)-b(x)} (\partial_{n+1} f)(X;\eta)\,\mathrm{d}x_{n+1}.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}h} \int_{\mathbb{R}^n} \left(\int_{b(x)}^{1+\eta(x)+h\check{\eta}(x)} f(X;\eta+h\check{\eta})\,\mathrm{d}x_{n+1} \right) \mathrm{d}x \bigg|_{h=0} \\
&\qquad = \int_{\mathbb{R}^n} f(x,1+\eta(x);\eta)\check{\eta}(x)\,\mathrm{d}x + \int_\Omega f_1(X)\,\mathrm{d}X, \qquad (3.8)
\end{aligned}$$

where

$$f_1(X) := \tilde{f}_\eta(\Xi^{-1}(X;\eta)) - \frac{\check{\eta}(x)(x_{n+1}-b(x))}{1+\eta(x)-b(x)} (\partial_{n+1} f)(X;\eta). \qquad (3.9)$$

Now, for the functions $\Phi = \Phi(X;\eta)$ and $\Psi = \Psi(X;\eta)$ defined by (3.3), we define Φ_1 and Ψ_1 as in (3.9) and apply the formula (3.8) to the function $f(X;\eta) = I_\delta \nabla_X \Phi(X;\eta) \cdot I_\delta \nabla_X \Psi(X;\eta)$. Then, by a straightforward calculation, we see that

$$f_1(X) = I_\delta \nabla_X \Phi_1(X) \cdot I_\delta \nabla_X \Psi(X;\eta) + I_\delta \nabla_X \Phi(X;\eta) \cdot I_\delta \nabla_X \Psi_1(X),$$

so that we recover the formula (3.7). Moreover, in view of the relations $\tilde{\Phi}(\cdot,1;\eta) = \phi$, $\tilde{\Psi}(\cdot,1;\eta) = \psi$ and $\tilde{\Psi}(\cdot,0;\eta) = \Lambda^{\mathrm{DD}}(\eta,b,\delta)\psi$, we have

$$\tilde{\Phi}_\eta(\cdot,1) = 0, \qquad \tilde{\Psi}_\eta(\cdot,1) = 0, \qquad \tilde{\Psi}_\eta(\cdot,0) = D_\eta \Lambda^{\mathrm{DD}}[\check{\eta}]\psi.$$

Therefore, it follows from (3.9) that $\Phi_1|_\Gamma = -(\partial_{n+1}\Phi)|_\Gamma \check{\eta}$, $\Psi_1|_\Gamma = -(\partial_{n+1}\Psi)|_\Gamma \check{\eta}$ and $\Psi_1|_\Sigma = D_\eta \Lambda^{\mathrm{DD}}[\check{\eta}]\psi$, and that the previous formal argument is justified. □

THEOREM 3.6. *The Fréchet derivatives of* $\Lambda^{\mathrm{DN}}(\eta,b,\delta)$ *and* $\Lambda^{\mathrm{NN}}(\eta,b,\delta)$ *with respect to* b *have the form*

$$D_b \Lambda^{\mathrm{DN}}(\eta,b,\delta)[\check{b}]\phi + D_b \Lambda^{\mathrm{NN}}(\eta,b,\delta)[\check{b}]\beta = -\Lambda^{\mathrm{NN}}(\eta,b,\delta)(\nabla \cdot (w\check{b})),$$

where

$$\left.\begin{aligned} W &= (1+\delta^2|\nabla b|^2)^{-1}(-\beta + \nabla b \cdot \nabla(\Lambda^{\mathrm{DD}}\phi + \Lambda^{\mathrm{ND}}\beta)), \\ w &= \nabla(\Lambda^{\mathrm{DD}}\phi + \Lambda^{\mathrm{ND}}\beta) - \delta^2 W \nabla b. \end{aligned}\right\} \tag{3.10}$$

Proof. We will only give an intuitive derivation of the formula. The formal calculation can be justified as in the proof of the previous theorem. We take $\phi, \psi, \beta \in C_0^\infty(\mathbb{R}^n)$ and let Φ and Ψ be the solutions of the boundary-value problems (3.3). Since we are considering a variation of the maps with respect to b, we denote the solutions by $\Phi = \Phi(X) = \Phi(X;b)$ and $\Psi = \Psi(X) = \Psi(X;b)$ and expand them formally as

$$\left.\begin{aligned} \Phi(X;b+h\check{b}) &= \Phi(X;b) + \Phi_1(X)h + O(h^2), \\ \Psi(X;b+h\check{b}) &= \Psi(X;b) + \Psi_1(X)h + O(h^2). \end{aligned}\right\} \tag{3.11}$$

Then, in place of (3.7), we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}h}(\Lambda^{\mathrm{DN}}(\eta,b+h\check{b},\delta)\phi,\psi)\bigg|_{h=0} &= \int_\Omega (I_\delta \nabla_X \Phi_1 \cdot I_\delta \nabla_X \Psi + I_\delta \nabla_X \Phi \cdot I_\delta \nabla_X \Psi_1)\,\mathrm{d}X \\ &\quad - \int_{\mathbb{R}^n} (I_\delta \nabla_X \Phi \cdot I_\delta \nabla_X \Psi)|_\Sigma \check{b}\,\mathrm{d}x \\ &=: J_1 + J_2. \end{aligned} \tag{3.12}$$

By taking the trace of the expansion (3.11) on the water surface Γ and using the boundary condition, we get $\Phi_1|_\Gamma = \Psi_1|_\Gamma = 0$. By taking the trace of (3.11) on the bottom Σ and using the definition of the map Λ^{DD}, we see that

$$\begin{aligned} &(\Lambda^{\mathrm{DD}}(\eta,b+h\check{b},\delta)\psi)(x) \\ &\quad = \Psi(x,b(x)+h\check{b}(x);b+h\check{b}) \\ &\quad = (\Lambda^{\mathrm{DD}}(\eta,b,\delta)\psi)(x) + h\{(\partial_{n+1}\Psi)(x,b(x);b)\check{b}(x) + \Psi_1(x,b(x))\} + O(h^2). \end{aligned}$$

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This and proposition 3.3 imply that $\Psi_1|_\Sigma = -(D_b \Lambda^{\mathrm{NN}}[\check{b}])^* \psi - (\partial_{n+1}\Psi)|_\Sigma \check{b}$. Therefore,

$$\begin{aligned} J_1 &= \int_{\partial\Omega} (\Phi_1(N \cdot I_\delta^2 \nabla_X \Psi) + (N \cdot I_\delta^2 \nabla_X \Phi)\Psi_1)\,\mathrm{d}S \\ &= -(D_b \Lambda^{\mathrm{NN}}[\check{b}]\beta, \psi) - \int_{\mathbb{R}^n} \beta(\partial_{n+1}\Psi)|_\Sigma \check{b}\,\mathrm{d}x. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} J_2 = -\int_{\mathbb{R}^n} \{&\nabla(\Phi|_\Sigma)\cdot\nabla(\Psi|_\Sigma) - (\partial_{n+1}\Phi)|_\Sigma \nabla b \cdot \nabla(\Psi|_\Sigma) \\ &- (\partial_{n+1}\Psi)|_\Sigma \nabla b \cdot \nabla(\Phi|_\Sigma) + \delta^{-2}(1+\delta^2|\nabla b|^2)(\partial_{n+1}\Psi \partial_{n+1}\Phi)|_\Sigma\}\check{b}\,\mathrm{d}x. \end{aligned}$$

In view of the relations $\Phi|_\Sigma = \Lambda^{\mathrm{DD}}\phi + \Lambda^{\mathrm{ND}}\beta$, $\Psi|_\Sigma = \Lambda^{\mathrm{DD}}\psi$ and

$$\begin{aligned} (\partial_{n+1}\Phi)|_\Sigma &= \delta^2(1+\delta^2|\nabla b|^2)^{-1}(-\beta + \nabla b \cdot \nabla(\Lambda^{\mathrm{DD}}\phi + \Lambda^{\mathrm{ND}}\beta)) = \delta^2 W, \\ (\partial_{n+1}\Psi)|_\Sigma &= \delta^2(1+\delta^2|\nabla b|^2)^{-1}(\nabla b \cdot \nabla(\Lambda^{\mathrm{DD}}\psi)), \end{aligned}$$

we see that

$$\begin{aligned} &(D_b \Lambda^{\mathrm{DN}}[\check{b}]\phi + D_b \Lambda^{\mathrm{NN}}[\check{b}]\beta, \psi) \\ &\quad = -\int_{\mathbb{R}^n} (\nabla(\Lambda^{\mathrm{DD}}\phi + \Lambda^{\mathrm{ND}}\beta)\cdot\nabla(\Lambda^{\mathrm{DD}}\psi) - \delta^2 W(\nabla b \cdot \nabla(\Lambda^{\mathrm{DD}}\psi)))\check{b}\,\mathrm{d}x \\ &\quad = -(\Lambda^{\mathrm{NN}}(\nabla\cdot(w\check{b})), \psi), \end{aligned}$$

where we used proposition 3.3. Since the above equality holds for any $\psi \in C_0^\infty(\mathbb{R}^n)$, we obtain the desired formula. □

THEOREM 3.7. *The Fréchet derivatives of* $\Lambda^{\mathrm{DD}}(\eta, b, \delta)$ *and* $\Lambda^{\mathrm{ND}}(\eta, b, \delta)$ *with respect to* η *have the form*

$$D_\eta \Lambda^{\mathrm{DD}}(\eta, b, \delta)[\check{\eta}]\phi + D_\eta \Lambda^{\mathrm{ND}}(\eta, b, \delta)[\check{\eta}]\beta = -\delta^2 \Lambda^{\mathrm{DD}}(\eta, b, \delta)(Z\check{\eta}),$$

where Z *is given by (3.2).*

Proof. We take $\phi, \beta, \gamma \in C_0^\infty(\mathbb{R}^n)$ and set $\Phi := (\phi, \beta)^\hbar$ and $\Psi := (0, \gamma)^\hbar$. Then, in place of (3.5), we have

$$(\Lambda^{\mathrm{ND}}\beta, \gamma) = \int_\Omega I_\delta \nabla_X \Phi \cdot I_\delta \nabla_X \Psi\,\mathrm{d}X. \tag{3.13}$$

We expand formally the solutions $\Phi(X; \eta + h\check{\eta})$ and $\Psi(X; \eta + h\check{\eta})$ as (3.6). Then, as in the previous theorems, we see that

$$\begin{aligned} (D_\eta \Lambda^{\mathrm{ND}}[\check{\eta}]\beta, \gamma) = \int_{\mathbb{R}^n} &(\Phi_1|_\Gamma \Lambda^{\mathrm{NN}}\gamma + (\Lambda^{\mathrm{DN}}\phi + \Lambda^{\mathrm{NN}}\beta)\Psi_1|_\Gamma + \Phi_1|_\Sigma \gamma + \beta\Psi_1|_\Sigma)\,\mathrm{d}x \\ &+ \int_{\mathbb{R}^n} (I_\delta \nabla_X \Phi \cdot I_\delta \nabla_X \Psi)|_\Gamma \check{\eta}\,\mathrm{d}x. \end{aligned}$$

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Here, we have

$$\Phi_1|_\Gamma = -(\partial_{n+1}\Phi)|_\Gamma \check{\eta}, \qquad \Psi_1|_\Gamma = -(\partial_{n+1}\Psi)|_\Gamma \check{\eta},$$
$$\Phi_1|_\Sigma = D_\eta \Lambda^{\mathrm{DD}}[\check{\eta}]\phi + D_\eta \Lambda^{\mathrm{ND}}[\check{\eta}]\beta, \quad \Psi_1|_\Sigma = (D_\eta \Lambda^{\mathrm{ND}}[\check{\eta}])^*\gamma,$$

so that we obtain

$$\begin{aligned}
&(D_\eta \Lambda^{\mathrm{DD}}[\check{\eta}]\phi + D_\eta \Lambda^{\mathrm{ND}}[\check{\eta}]\beta, \gamma) \\
&= \int_{\mathbb{R}^n} ((\partial_{n+1}\Phi)|_\Gamma \Lambda^{\mathrm{NN}}\gamma + (\Lambda^{\mathrm{DN}}\phi + \Lambda^{\mathrm{NN}}\beta)(\partial_{n+1}\Psi)|_\Gamma - (I_\delta \nabla_X \Phi \cdot I_\delta \nabla_X \Psi)|_\Gamma)\check{\eta}\,\mathrm{d}x \\
&= -\delta^2(\Lambda^{\mathrm{DD}}(Z\check{\eta}), \gamma).
\end{aligned}$$

Since the above equality holds for any $\gamma \in C_0^\infty(\mathbb{R}^n)$, we obtain the desired formula. □

THEOREM 3.8. *The Fréchet derivatives of* $\Lambda^{\mathrm{DD}}(\eta, b, \delta)$ *and* $\Lambda^{\mathrm{ND}}(\eta, b, \delta)$ *with respect to* b *have the form*

$$D_b \Lambda^{\mathrm{DD}}(\eta, b, \delta)[\check{b}]\phi + D_b \Lambda^{\mathrm{ND}}(\eta, b, \delta)[\check{b}]\beta = \delta^2 W\check{b} - \Lambda^{\mathrm{ND}}(\eta, b, \delta)(\nabla \cdot (w\check{b})),$$

where W *and* w *are given by (3.10).*

Proof. We take $\phi, \beta, \gamma \in C_0^\infty(\mathbb{R}^n)$ and set $\Phi := (\phi, \beta)^\hbar$ and $\Psi := (0, \gamma)^\hbar$. Then we have (3.13). We expand formally the solutions $\Phi(X; b + h\check{b})$ and $\Psi(X; b + h\check{b})$ as (3.11). Then, as in the previous theorems, we see that

$$\begin{aligned}
(D_b \Lambda^{\mathrm{ND}}[\check{b}]\beta, \gamma) = \int_{\mathbb{R}^n} (\Phi_1|_\Gamma \Lambda^{\mathrm{NN}}\gamma + (\Lambda^{\mathrm{DN}}\phi + \Lambda^{\mathrm{NN}}\beta)\Psi_1|_\Gamma + \Phi_1|_\Sigma \gamma + \beta \Psi_1|_\Sigma)\,\mathrm{d}x \\
- \int_{\mathbb{R}^n} (I_\delta \nabla_X \Phi \cdot I_\delta \nabla_X \Psi)|_\Sigma \check{b}\,\mathrm{d}x.
\end{aligned}$$

Here, we have

$$\begin{aligned}
&\Phi_1|_\Gamma = \Psi_1|_\Gamma = 0, \\
&\Phi_1|_\Sigma = -(\partial_{n+1}\Phi)|_\Sigma \check{b} + D_b \Lambda^{\mathrm{DD}}[\check{b}]\phi + D_b \Lambda^{\mathrm{ND}}[\check{b}]\beta, \\
&\Psi_1|_\Sigma = -(\partial_{n+1}\Psi)|_\Sigma \check{b} + (D_b \Lambda^{\mathrm{ND}}[\check{b}])^*\gamma,
\end{aligned}$$

so that we obtain

$$\begin{aligned}
&(D_b \Lambda^{\mathrm{DD}}[\check{b}]\phi + D_b \Lambda^{\mathrm{ND}}[\check{b}]\beta, \gamma) \\
&\quad = \int_{\mathbb{R}^n} ((\partial_{n+1}\Phi)|_\Sigma \gamma + \beta(\partial_{n+1}\Psi)|_\Sigma + (I_\delta \nabla_X \Phi \cdot I_\delta \nabla_X \Psi)|_\Sigma)\check{b}\,\mathrm{d}x \\
&\quad = (\delta^2 W\check{b} - \Lambda^{\mathrm{ND}}(\nabla \cdot (w\check{b})), \gamma).
\end{aligned}$$

Since the above equality holds for any $\gamma \in C_0^\infty(\mathbb{R}^n)$, we obtain the desired formula. □

In reducing the full nonlinear equations (1.15) to a quasi-linear system of equations, we also need explicit formulae of second-order Fréchet derivatives of the maps Λ^{DN} and Λ^{NN}, which are given in the following theorems.

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Theorem 3.9. *The second-order Fréchet derivatives of*

$$\Lambda^{\mathrm{DN}}(\eta,b,\delta) \quad and \quad \Lambda^{\mathrm{NN}}(\eta,b,\delta)$$

with respect to η *have the form*

$$\begin{aligned}
&D_\eta^2\Lambda^{\mathrm{DN}}(\eta,b,\delta)[\check\eta_1,\check\eta_2]\phi + D_\eta^2\Lambda^{\mathrm{NN}}(\eta,b,\delta)[\check\eta_1,\check\eta_2]\beta\\
&\quad = \delta^2\{\Lambda^{\mathrm{DN}}(\eta,b,\delta)((1+\delta^2|\nabla\eta|^2)^{-1}(\Delta\phi)\check\eta_1\check\eta_2)\\
&\qquad\quad - \nabla\cdot((1+\delta^2|\nabla\eta|^2)^{-1}(\Delta\phi)\check\eta_1\check\eta_2\nabla\eta) + \Delta(Z\check\eta_1\check\eta_2)\}\\
&\qquad + \delta^4\{\Lambda^{\mathrm{DN}}(\eta,b,\delta)((1+\delta^2|\nabla\eta|^2)^{-1}(\check\eta_2\Lambda^{\mathrm{DN}}(\eta,b,\delta)(Z\check\eta_1)\\
&\qquad\qquad + \check\eta_1\Lambda^{\mathrm{DN}}(\eta,b,\delta)(Z\check\eta_2) + Z\nabla\eta\cdot\nabla(\check\eta_1\check\eta_2) - \check\eta_1\check\eta_2 Z\Delta\eta))\\
&\qquad\quad - \nabla\cdot((1+\delta^2|\nabla\eta|^2)^{-1}(\check\eta_2\Lambda^{\mathrm{DN}}(\eta,b,\delta)(Z\check\eta_1) + \check\eta_1\Lambda^{\mathrm{DN}}(\eta,b,\delta)(Z\check\eta_2)\\
&\qquad\qquad + Z\nabla\eta\cdot\nabla(\check\eta_1\check\eta_2) - \check\eta_1\check\eta_2 Z\Delta\eta)\nabla\eta)\},
\end{aligned}$$

where Z *is given by (3.2).*

Proof. It follows from theorem 3.5 that

$$\begin{aligned}
&D_\eta\Lambda^{\mathrm{DN}}[\check\eta_1]\phi + D_\eta\Lambda^{\mathrm{NN}}[\check\eta_1]\beta\\
&\quad = -\delta^2\Lambda^{\mathrm{DN}}((1+\delta^2|\nabla\eta|^2)^{-1}(\Lambda^{\mathrm{DN}}\phi + \Lambda^{\mathrm{NN}}\beta + \nabla\eta\cdot\nabla\phi)\check\eta_1)\\
&\qquad - \nabla\cdot\{(\nabla\phi - \delta^2(1+\delta^2|\nabla\eta|^2)^{-1}(\Lambda^{\mathrm{DN}}\phi + \Lambda^{\mathrm{NN}}\beta + \nabla\eta\cdot\nabla\phi)\nabla\eta)\check\eta_1\}.
\end{aligned} \tag{3.14}$$

Taking the Fréchet derivative of (3.14) with respect to η once again, we obtain

$$\begin{aligned}
&D_\eta^2\Lambda^{\mathrm{DN}}[\check\eta_1,\check\eta_2]\phi + D_\eta^2\Lambda^{\mathrm{NN}}[\check\eta_1,\check\eta_2]\beta\\
&\quad = -\delta^2 D_\eta\Lambda^{\mathrm{DN}}[\check\eta_2](Z\check\eta_1)\\
&\qquad - \delta^2\Lambda^{\mathrm{DN}}\{(-2\delta^2(1+\delta^2|\nabla\eta|^2)^{-1}Z(\nabla\eta\cdot\nabla\check\eta_2)\\
&\qquad\qquad + (1+\delta^2|\nabla\eta|^2)^{-1}(D_\eta\Lambda^{\mathrm{DN}}[\check\eta_2]\phi + D_\eta\Lambda^{\mathrm{NN}}[\check\eta_2]\beta + \nabla\check\eta_2\cdot\nabla\phi))\check\eta_1\}\\
&\qquad - \nabla\cdot\{(2\delta^4(1+\delta^2|\nabla\eta|^2)^{-1}Z(\nabla\eta\cdot\nabla\check\eta_2)\nabla\eta\\
&\qquad\qquad - \delta^2(1+\delta^2|\nabla\eta|^2)^{-1}(D_\eta\Lambda^{\mathrm{DN}}[\check\eta_2]\phi + D_\eta\Lambda^{\mathrm{NN}}[\check\eta_2]\beta + \nabla\check\eta_2\cdot\nabla\phi)\nabla\eta\\
&\qquad\qquad - \delta^2 Z\nabla\check\eta_2)\check\eta_1\}.
\end{aligned}$$

Here, we again use theorem 3.5. Then, a straightforward calculation gives the desired identity. □

Theorem 3.10. *The second-order Fréchet derivatives of the DN and the NN maps with respect to* η *and* b *have the form*

$$\begin{aligned}
&D_\eta D_b\Lambda^{\mathrm{DN}}(\eta,b,\delta)[\check\eta,\check b]\phi + D_\eta D_b\Lambda^{\mathrm{NN}}(\eta,b,\delta)[\check\eta,\check b]\beta\\
&\quad = \delta^2\Lambda^{\mathrm{NN}}(\eta,b,\delta)\nabla\cdot\{\check b(\nabla(\Lambda^{\mathrm{DD}}(\eta,b,\delta)(Z\check\eta))\\
&\qquad\qquad - \delta^2(1+\delta^2|\nabla b|^2)^{-1}(\nabla b\cdot\nabla\Lambda^{\mathrm{DD}}(\eta,b,\delta)(Z\check\eta))\nabla b)\}\\
&\qquad + \delta^2\Lambda^{\mathrm{DN}}(\eta,b,\delta)(\check\eta(1+\delta^2|\nabla\eta|^2)^{-1}\Lambda^{\mathrm{NN}}(\eta,b,\delta)\nabla\cdot(w\check b))\\
&\qquad - \delta^2\nabla\cdot(\check\eta(1+\delta^2|\nabla\eta|^2)^{-1}(\Lambda^{\mathrm{NN}}(\eta,b,\delta)\nabla\cdot(w\check b))\nabla\eta),
\end{aligned}$$

where Z *and* w *are given by (3.2) and (3.10), respectively.*

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Proof. Taking the Fréchet derivative of (3.14) with respect to b, we obtain

$$\begin{aligned} D_\eta D_b \Lambda^{\mathrm{DN}}[\check\eta,\check b]\phi + D_\eta D_b \Lambda^{\mathrm{NN}}[\check\eta,\check b]\beta &= -\delta^2 D_b\Lambda^{\mathrm{DN}}[\check b](Z\check\eta) - \delta^2\Lambda^{\mathrm{DN}}((1+\delta^2|\nabla\eta|^2)^{-1}(D_b\Lambda^{\mathrm{DN}}[\check b]\phi + D_b\Lambda^{\mathrm{NN}}[\check b]\beta)\check\eta_1) \\ &\quad - \nabla\cdot\{(-\delta^2(1+\delta^2|\nabla\eta|^2)^{-1}(D_b\Lambda^{\mathrm{DN}}[\check b]\phi + D_b\Lambda^{\mathrm{NN}}[\check b]\beta)\nabla\eta)\check\eta_1\}. \end{aligned}$$

We use theorem 3.6 in the above expression. Then, a straightforward calculation gives the desired identity. □

4. Some elliptic estimates

In the next section we will give operator norms of the operators Λ^{DN}, Λ^{NN}, Λ^{DD} and Λ^{ND} in Sobolev spaces. Especially, we will analyze carefully the dependence on the small parameter δ to obtain uniform estimates with respect to δ. Since these operators depend on the unknown function η, we also have to accurately examine the dependence on regularity of η.

In order to give such estimates, we need appropriate estimates of the solution Φ of the boundary-value problem (3.1). In this section we prepare elliptic estimates of the solution, noting in particular the dependence of δ and the regularity of η. To this end, it would be convenient to transform the problem (3.1) on the water region Ω into a problem on a simple domain $\Omega_0 := \mathbb{R}^n \times (0,1)$ by using an appropriate diffeomorphism $\Theta = (\Theta_1,\dots,\Theta_n,\Theta_{n+1})\colon \overline{\Omega_0} \to \overline{\Omega}$, which is conformal in the tangential and the normal directions on the boundary in some sense. As in [11], we define such a diffeomorphism as follows. We take functions $\theta = (\theta_1,\dots,\theta_n,\theta_{n+1})$ satisfying the conditions

$$\left.\begin{aligned} \theta_j(x,0) = \theta_j(x,1) = 0, \qquad & \\ \partial_{n+1}\theta_j(x,0) = -\partial_j b(x), \quad \partial_{n+1}\theta_j(x,1) = -\partial_j\eta(x) \quad & \text{for } 1 \leqslant j \leqslant n, \\ \theta_{n+1}(x,0) = b(x), \qquad \theta_{n+1}(x,1) = \eta(x), \qquad & \\ \partial_{n+1}\theta_{n+1}(x,0) = \partial_{n+1}\theta_{n+1}(x,1) = 0, \qquad & \end{aligned}\right\} \tag{4.1}$$

and define the diffeomorphism Θ by

$$\Theta_j(X) = x_j + \delta^2\theta_j(X) \text{ for } 1 \leqslant j \leqslant n, \qquad \Theta_{n+1}(X) = x_{n+1} + \theta_{n+1}(X). \tag{4.2}$$

We set $\tilde\Phi := \Phi \circ \Theta$ and

$$P := \det\left(\frac{\partial\Theta}{\partial X}\right)\left(I_\delta^{-1}\left(\frac{\partial\Theta}{\partial X}\right)^{-1} I_\delta^2\left(\left(\frac{\partial\Theta}{\partial X}\right)^{-1}\right)^{\mathrm{T}} I_\delta^{-1}\right). \tag{4.3}$$

The matrix P has the property

$$P(x,0) = \begin{pmatrix} * & 0 \\ 0 & 1 \end{pmatrix}, \qquad P(x,1) = \begin{pmatrix} * & 0 \\ 0 & 1 \end{pmatrix}, \tag{4.4}$$

which means that the diffeomorphism Θ is conformal in the tangential and the normal directions on the boundary, so that the Neumann boundary condition on the bottom is transformed into the Neumann condition again with a very simple normal vector $N = (0, \dots, 0, -1)^{\mathrm{T}}$. Therefore, the boundary-value problem (3.1) is transformed into

$$\nabla_X \cdot I_\delta P I_\delta \nabla_X \tilde{\Phi} = 0 \text{ in } \Omega_0, \qquad \tilde{\Phi} = \phi \text{ on } \Gamma_0, \qquad -\delta^{-2}\partial_{n+1}\tilde{\Phi} = \beta \text{ on } \Sigma_0, \quad (4.5)$$

where Γ_0 and Σ_0 are upper and lower boundaries of Ω_0. Moreover, it holds that

$$\left.\begin{aligned} \Lambda^{\mathrm{DN}}(\eta, b, \delta)\phi + \Lambda^{\mathrm{NN}}(\eta, b, \delta)\beta &= \delta^{-2}(\partial_{n+1}\tilde{\Phi})(\cdot, 1), \\ \Lambda^{\mathrm{DD}}(\eta, b, \delta)\phi + \Lambda^{\mathrm{ND}}(\eta, b, \delta)\beta &= \tilde{\Phi}(\cdot, 0). \end{aligned}\right\} \quad (4.6)$$

We will impose the following conditions on the water surface and the bottom.

Assumption 4.1.

(A_1) There exists a C^1-diffeomorphism $\Theta \colon \overline{\Omega_0} \to \overline{\Omega}$ satisfying (4.1), (4.2) and the conditions

$$\det\left(\frac{\partial \Theta}{\partial X}(X)\right) \geqslant c > 0 \quad \text{and} \quad |\nabla_X \theta(X)| \leqslant M \quad \text{for } X \in \Omega_0.$$

(A_2)

$$\|\nabla_X \theta(\cdot, x_{n+1})\|_q \leqslant M \quad \text{for } 0 \leqslant x_{n+1} \leqslant 1.$$

(A_3)

$$\|J^{q+1/2}\nabla_X \theta\|_{L^2(\Omega_0)} \leqslant M.$$

(A_4)

$$\begin{aligned} \|\nabla_X(D_{(\eta,b)}\theta[\check{\eta}, \check{b}])(\cdot, x_{n+1})\|_{\check{s}} + \|J^{\check{s}+1/2}\nabla_X(D_{(\eta,b)}\theta[\check{\eta}, \check{b}])\|_{L^2(\Omega_0)} \\ \leqslant M(\|\check{\eta}\|_{\check{s}+1} + \|\check{b}\|_{\check{s}+1}) \quad \text{for } 0 \leqslant x_{n+1} \leqslant 1,\ \check{s} \in \mathbb{R}, \end{aligned}$$

and θ depends linearly on (η, b).

The construction of a diffeomorphism Θ satisfying the above conditions was given in [11]. More precisely, we have the following proposition.

Proposition 4.2. *Let $r > \frac{1}{2}n$, $c_1, M_1 > 0$ and suppose that $\eta, b \in H^{1+r}$ satisfy the conditions*

$$\|\eta\|_{1+r} + \|b\|_{1+r} \leqslant M_1, \qquad 1 + \eta(x) - b(x) \geqslant c_1 \quad \text{for } x \in \mathbb{R}^n.$$

Then, there exists a constant $\delta_1 = \delta_1(M_1, c_1, r) > 0$ such that, for any $\delta \in (0, \delta_1]$, we can construct a diffeomorphism Θ satisfying the conditions in (A_1)*. Moreover, for any $s \in \mathbb{R}$ and $k \in \mathbb{N}$, we have*

$$\left.\begin{aligned} \|J^s \nabla_X \theta\|_{L^2(\Omega_0)} &\leqslant C_1(\|\eta\|_{s+1/2} + \|b\|_{s+1/2}), \\ \sup_{0 \leqslant x_{n+1} \leqslant 1} \|\partial_{n+1}^k \theta(\cdot, x_{n+1})\|_s &\leqslant C_2(\|\eta\|_{s+k} + \|b\|_{s+k}), \end{aligned}\right\} \quad (4.7)$$

where $C_1 = C_1(c_1) > 0$ and $C_2 = C_2(c_1, k) > 0$. In the case where η and b also depend on the time t, for any $l \in \mathbb{N}$, we have

$$\left.\begin{aligned} \|J^s \nabla_X \partial_t^l \theta(t)\|_{L^2(\Omega_0)} &\leqslant C_1(\|\partial_t^l \eta(t)\|_{s+1/2} + \|\partial_t^l b(t)\|_{s+1/2}), \\ \sup_{0 \leqslant x_{n+1} \leqslant 1} \|\partial_{n+1}^k \partial_t^l \theta(\cdot, x_{n+1}, t)\|_s &\leqslant C_2(\|\partial_t^l \eta(t)\|_{s+k} + \|\partial_t^l b(t)\|_{s+k}). \end{aligned}\right\} \quad (4.8)$$

We proceed to give elliptic estimates for (4.5) in Sobolev spaces. Although standard theory for elliptic equations could provide an estimate of the solution, such an estimate depends strongly on the parameter δ and it does not give a uniform bound of the solution with respect to small δ. Therefore, we will perform an estimation of the solution with particular care on the dependence of the parameter δ and on regularity of η. The following six lemmas were slight modifications of those given in [11].

Lemma 4.3. *Under assumption* (A_1)*, there exists a constant* $C = C(M, c) \geqslant 1$ *independent of* δ *such that we have*

$$C^{-1}\|I_\delta \nabla_X \Phi\|_{L^2(\Omega)} \leqslant \|I_\delta \nabla_X \tilde{\Phi}\|_{L^2(\Omega_0)} \leqslant C\|I_\delta \nabla_X \Phi\|_{L^2(\Omega)},$$

where $\tilde{\Phi} = \Phi \circ \Theta$*.*

Lemma 4.4. *Under assumption* (A_1)*, there exists a constant* $C = C(M, c) \geqslant 1$ *independent of* δ *such that, for any* $\phi \in H^1$*, we have*

$$C^{-1}\|(\Lambda_0^{\mathrm{DN}})^{1/2}\phi\|^2 \leqslant (\Lambda^{\mathrm{DN}}\phi, \phi) \leqslant C\|(\Lambda_0^{\mathrm{DN}})^{1/2}\phi\|^2.$$

Lemma 4.5. *Let* $r > \frac{1}{2}n$*. There exists a constant* $C = C(r) > 0$ *independent of* δ *such that we have*

$$\|[(\Lambda_0^{\mathrm{DN}})^{1/2}, a]u\| \leqslant C\|\nabla a\|_r\|u\|, \qquad \|[(\Lambda_0^{\mathrm{DN}})^{1/2}, a]u\|_r \leqslant C\|\nabla a\|_r\|u\|_r.$$

Lemma 4.6. *For any* $s \in \mathbb{R}$*, we have*

$$\|(\Lambda_0^{\mathrm{DN}})^{1/2}\phi\|_s \leqslant \min\{\|\nabla\phi\|_s, \delta^{-1/2}\|\phi\|_{s+1/2}\},$$
$$\|\nabla\phi\|_s \leqslant \sqrt{2(1+\delta)}\|(\Lambda_0^{\mathrm{DN}})^{1/2}\phi\|_{s+1/2}.$$

Lemma 4.7. *For any* $s \in \mathbb{R}$ *and* $r > \frac{1}{2}n$*, there exists a constant* $C = C(s, r) > 0$ *independent of* δ *such that we have*

$$\|(\Lambda_0^{\mathrm{DN}})^{1/2}(\phi\psi)\|_s \leqslant C(\|\phi\|_r\|(\Lambda_0^{\mathrm{DN}})^{1/2}\psi\|_s + \|\phi\|_s\|(\Lambda_0^{\mathrm{DN}})^{1/2}\psi\|_r + \|(\Lambda_0^{\mathrm{DN}})^{1/2}\phi\|_s\|\psi\|_r + \|(\Lambda_0^{\mathrm{DN}})^{1/2}\phi\|_r\|\psi\|_s).$$

Lemma 4.8. *For any* $s \in \mathbb{R}$ *and* $r > \frac{1}{2}n$*, there exists a constant* $C = C(s, r) > 0$ *independent of* δ *such that we have*

$$\|(\Lambda_0^{\mathrm{DN}})^{1/2}[J^s, \psi]\nabla\phi\| \leqslant C(\|\nabla\psi\|_{r+1}\|(\Lambda_0^{\mathrm{DN}})^{1/2}\phi\|_s + \|\nabla\psi\|_s\|(\Lambda_0^{\mathrm{DN}})^{1/2}\phi\|_{r+1}).$$

Lemma 4.9. *For any function* $\tilde{\Phi}$ *defined on* Ω_0*, we have*

$$\|(\Lambda_0^{\mathrm{DN}})^{1/2}\tilde{\Phi}(\cdot, 0)\| \leqslant \|I_\delta \nabla_X \tilde{\Phi}\|_{L^2(\Omega_0)}.$$

Proof. We take $\psi \in H^0$ arbitrarily and set

$$\tilde{\Psi}(\cdot, x_{n+1}) = \frac{\cosh(\delta|D|(1 - x_{n+1}))}{\cosh(\delta|D|)}\psi,$$

which is a solution of the boundary-value problem

$$\nabla_X \cdot I_\delta^2 \nabla_X \tilde{\Psi} = 0 \text{ in } \Omega_0, \qquad \partial_{n+1}\tilde{\Psi} = 0 \text{ on } \Gamma_0, \qquad \tilde{\Psi} = \psi \text{ on } \Sigma_0.$$

Then, it holds that $-\delta^{-2}\partial_{n+1}\tilde{\Psi}(\cdot,0) = \Lambda_0^{\mathrm{DN}}\psi$. By Green's formula, we see that

$$\begin{aligned}|(\Lambda_0^{\mathrm{DN}}\tilde{\Phi}(\cdot,0),\psi)| &= |(\tilde{\Phi}(\cdot,0),\Lambda_0^{\mathrm{DN}}\psi)| \\ &= \left|\int_{\Sigma_0}\tilde{\Phi}(N\cdot I_\delta^2\nabla_X\tilde{\Psi})\,\mathrm{d}S\right| \\ &= \left|\int_{\Omega_0} I_\delta\nabla_X\tilde{\Phi}\cdot I_\delta\nabla_X\tilde{\Psi}\,\mathrm{d}X\right| \\ &\leqslant \|(\Lambda_0^{\mathrm{DN}})^{1/2}I_\delta\nabla_X\tilde{\Phi}\|_{L^2(\Omega_0)}\|(\Lambda_0^{\mathrm{DN}})^{-1/2}I_\delta\nabla_X\tilde{\Psi}\|_{L^2(\Omega_0)} \\ &= \|I_\delta\nabla_X((\Lambda_0^{\mathrm{DN}})^{1/2}\tilde{\Phi})\|_{L^2(\Omega_0)}\|\psi\|,\end{aligned}$$

which gives $\|\Lambda_0^{\mathrm{DN}}\tilde{\Phi}(\cdot,0)\| \leqslant \|I_\delta\nabla_X((\Lambda_0^{\mathrm{DN}})^{1/2}\tilde{\Phi})\|_{L^2(\Omega_0)}$. If we replace $(\Lambda_0^{\mathrm{DN}})^{1/2}\tilde{\Phi}$ by $\tilde{\Phi}$ in this inequality, then we obtain the desired estimate. □

As a preliminary step, we will consider the boundary-value problem

$$\left.\begin{aligned}\nabla_X\cdot I_\delta P I_\delta\nabla_X\tilde{\Phi} = \nabla_X\cdot I_\delta F + f \text{ in } \Omega_0, \qquad \tilde{\Phi} = 0 \text{ on } \Gamma_0,\\ -\delta^{-2}\partial_{n+1}\tilde{\Phi} = \beta + (\Lambda_0^{\mathrm{DN}})^{1/2}\gamma \text{ on } \Sigma_0,\end{aligned}\right\} \tag{4.9}$$

where the matrix P is given by (4.3).

LEMMA 4.10. *Under assumption* (A_1), *there exists a constant* $C = C(M,c) > 0$ *independent of* δ *such that the solution* $\tilde{\Phi}$ *of (4.9) with* $F_{n+1}(\cdot,0) = 0$ *satisfies*

$$\|I_\delta\nabla_X\tilde{\Phi}\|_{L^2(\Omega_0)} \leqslant C(\|F\|_{L^2(\Omega_0)} + \|J^{-1}f\|_{L^2(\Omega_0)} + \delta\|\beta\| + \|\gamma\|).$$

Proof. Taking the inner product of the first equation in (4.9) with $\tilde{\Phi}$ and using Green's formula and the boundary conditions, we see that

$$\begin{aligned}C^{-1}\|I_\delta\nabla_X\tilde{\Phi}\|_{L^2(\Omega_0)} &\leqslant \int_{\Omega_0} P I_\delta\nabla_X\tilde{\Phi}\cdot I_\delta\nabla_X\tilde{\Phi}\,\mathrm{d}X \\ &= \int_{\Omega_0}(F\cdot I_\delta\nabla_X\tilde{\Phi} - f\tilde{\Phi})\,\mathrm{d}X + \int_{\mathbb{R}^n}(\beta + (\Lambda_0^{\mathrm{DN}})^{1/2}\gamma)\tilde{\Phi}(\cdot,0)\,\mathrm{d}x \\ &\leqslant \|F\|_{L^2(\Omega_0)}\|I_\delta\nabla_X\tilde{\Phi}\|_{L^2(\Omega_0)} + \|J^{-1}f\|_{L^2(\Omega_0)}\|J\tilde{\Phi}\|_{L^2(\Omega_0)} \\ &\qquad + \|\beta\|\|\tilde{\Phi}(\cdot,0)\| + \|\gamma\|\|(\Lambda_0^{\mathrm{DN}})^{1/2}\tilde{\Phi}(\cdot,0)\|.\end{aligned}$$

Here, we easily get

$$\|\tilde{\Phi}(\cdot,x_{n+1})\| = \left\|\int_1^{x_{n+1}}(\partial_{n+1}\tilde{\Phi})(\cdot,z)\,\mathrm{d}z\right\| \leqslant \delta\|I_\delta\nabla_X\tilde{\Phi}\|_{L^2(\Omega_0)},$$

$$\|J\tilde{\Phi}\|_{L^2(\Omega_0)} \leqslant \|\tilde{\Phi}\|_{L^2(\Omega_0)} + \|\nabla\tilde{\Phi}\|_{L^2(\Omega_0)} \leqslant (\delta+1)\|I_\delta\nabla_X\tilde{\Phi}\|_{L^2(\Omega_0)}.$$

Therefore, by applying lemma 4.9 to the last term in the above estimate, we obtain the desired estimate. □

LEMMA 4.11. *Let* $s > \frac{1}{2}n + 1$. *Under assumptions* (A_1) *and* (A_2) *with* $q = s$, *there exists a constant* $C = C(M,c,s) > 0$ *independent of* δ *such that the solution* $\tilde{\Phi}$ *of*

(4.9) with $F_{n+1}(\cdot,0)=0$ *satisfies*

$$\|J^s I_\delta \nabla_X \tilde{\Phi}\|_{L^2(\Omega_0)} \leqslant C(\|J^s F\|_{L^2(\Omega_0)} + \|J^{s-1} f\|_{L^2(\Omega_0)} + \delta\|\beta\|_s + \|\gamma\|_s).$$

Proof. It is easy to see that $J^s\tilde{\Phi}$ satisfies

$$\begin{aligned} \nabla_X \cdot I_\delta P I_\delta \nabla_X J^s \tilde{\Phi} &= \nabla_X \cdot I_\delta (J^s F - [J^s, P] I_\delta \nabla_X \tilde{\Phi}) + J^s f && \text{in } \Omega_0, \\ J^s \tilde{\Phi} &= 0 && \text{on } \Gamma_0, \\ -\delta^{-2} \partial_{n+1} J^s \tilde{\Phi} &= J^s \beta + (\Lambda_0^{\mathrm{DN}})^{1/2} J^s \gamma && \text{on } \Sigma_0, \end{aligned}$$

and that $N \cdot [J^s, P] I_\delta \nabla_X \tilde{\Phi} = 0$ on $\partial\Omega_0$ thanks to (4.4). Therefore, by lemma 4.10, we obtain

$$\begin{aligned} \|J^s I_\delta \nabla_X \tilde{\Phi}\|_{L^2(\Omega_0)} \leqslant C(&\|J^s F\|_{L^2(\Omega_0)} + \|[J^s, P] I_\delta \nabla_X \tilde{\Phi}\|_{2(\Omega_0)} \\ &+ \|J^{s-1} f\|_{L^2(\Omega_0)} + \delta\|\beta\|_s + \|\gamma\|_s). \end{aligned}$$

The second term on the right-hand side can be evaluated by a commutator estimate $\|[J^s, a]u\| \leqslant C\|\nabla a\|_{s-1}\|u\|_{s-1}$, an interpolation inequality $\|u\|_{s-1} \leqslant \epsilon\|u\| + C_\epsilon\|u\|$ for $\epsilon > 0$, and lemma 4.10, so that we obtain the desired estimate. □

LEMMA 4.12. *Let* $s > \frac{1}{2}n$. *Under assumptions* (A$_1$) *and* (A$_2$) *with* $q = s+1$, *there exists a constant* $C = C(M, c, s) > 0$ *independent of* δ *such that the solution* $\tilde{\Phi}$ *of (4.9) with* $F_{n+1}(\cdot,0)=0$ *satisfies*

$$\begin{aligned} &\|J^s (\Lambda_0^{\mathrm{DN}})^{1/2} I_\delta \nabla_X \tilde{\Phi}\|_{L^2(\Omega_0)} \\ &\quad \leqslant C(\|J^s (\Lambda_0^{\mathrm{DN}})^{1/2} F\|_{L^2(\Omega_0)} + \|J^{s-1} (\Lambda_0^{\mathrm{DN}})^{1/2} f\|_{L^2(\Omega_0)} \\ &\qquad + \|\beta\|_s + \|(\Lambda_0^{\mathrm{DN}})^{1/2} \gamma\|_s + \|F\|_{L^2(\Omega_0)} + \|J^{-1} f\|_{L^2(\Omega_0)} + \|\gamma\|). \end{aligned} \tag{4.10}$$

Proof. It is easy to see that $(\Lambda_0^{\mathrm{DN}})^{1/2}\tilde{\Phi}$ satisfies

$$\begin{aligned} &\nabla_X \cdot I_\delta P I_\delta \nabla_X (\Lambda_0^{\mathrm{DN}})^{1/2} \tilde{\Phi} \\ &\quad = \nabla_X \cdot I_\delta ((\Lambda_0^{\mathrm{DN}})^{1/2} F - [(\Lambda_0^{\mathrm{DN}})^{1/2}, P] I_\delta \nabla_X \tilde{\Phi}) + (\Lambda_0^{\mathrm{DN}})^{1/2} f \text{ in } \Omega_0, \\ &\qquad (\Lambda_0^{\mathrm{DN}})^{1/2} \tilde{\Phi} = 0 \text{ on } \Gamma_0, \\ &\qquad -\delta^{-2} \partial_{n+1} (\Lambda_0^{\mathrm{DN}})^{1/2} \tilde{\Phi} = (\Lambda_0^{\mathrm{DN}})^{1/2} (\beta + (\Lambda_0^{\mathrm{DN}})^{1/2} \gamma) \text{ on } \Sigma_0, \end{aligned}$$

and that $N \cdot [(\Lambda_0^{\mathrm{DN}})^{1/2}, P] I_\delta \nabla_X \tilde{\Phi} = 0$ on $\partial\Omega_0$ thanks to (4.4). Therefore, by lemma 4.11, we obtain

$$\begin{aligned} &\|J^s (\Lambda_0^{\mathrm{DN}})^{1/2} I_\delta \nabla_X \tilde{\Phi}\|_{L^2(\Omega_0)} \\ &\quad \leqslant C(\|J^s (\Lambda_0^{\mathrm{DN}})^{1/2} F\|_{L^2(\Omega_0)} + \|J^s [(\Lambda_0^{\mathrm{DN}})^{1/2}, P] I_\delta \nabla_X \tilde{\Phi}\|_{L^2(\Omega_0)} \\ &\qquad + \|J^{s-1} (\Lambda_0^{\mathrm{DN}})^{1/2} f\|_{L^2(\Omega_0)} + \|\beta\|_s + \|(\Lambda_0^{\mathrm{DN}})^{1/2} \gamma\|_s). \end{aligned}$$

Here, an interpolation inequality and lemma 4.6 imply that

$$\|u\|_s \leqslant \epsilon\|\nabla u\|_{s-1/2} + C_\epsilon\|u\| \leqslant 2\epsilon\|(\Lambda_0^{\mathrm{DN}})^{1/2} u\|_s + C_\epsilon\|u\|.$$

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Thanks to this and lemma 4.5, the second term on the right-hand side of the above estimate can be evaluated as

$$\begin{aligned}&\|J^s[(\varLambda_0^{\mathrm{DN}})^{1/2},P]I_\delta\nabla_X\tilde{\varPhi}\|_{L^2(\varOmega_0)}\\&\qquad\leqslant\epsilon\|J^s(\varLambda_0^{\mathrm{DN}})^{1/2}I_\delta\nabla_X\tilde{\varPhi}\|_{L^2(\varOmega_0)}+C_\epsilon\|I_\delta\nabla_X\tilde{\varPhi}\|_{L^2(\varOmega_0)}.\end{aligned}$$

These estimates, together with lemma 4.10 give the desired estimate. □

LEMMA 4.13. *For any $s\in\mathbb{R}$, the solution $\tilde{\varPhi}$ of (4.9) with $F_{n+1}(\cdot,0)=F_{n+1}(\cdot,1)=\gamma=0$ satisfies*

$$\begin{aligned}\delta^{-2}\|\partial_{n+1}\tilde{\varPhi}(\cdot,1)\|_s&\leqslant\|J^s(\varLambda_0^{\mathrm{DN}})^{1/2}PI_\delta\nabla_X\tilde{\varPhi}\|_{L^2(\varOmega_0)}\\&\quad+\|J^s(\varLambda_0^{\mathrm{DN}})^{1/2}F\|_{L^2(\varOmega_0)}+\|J^sf\|_{L^2(\varOmega_0)}+\|\beta\|_s.\end{aligned}$$

Proof. We take $\psi\in H^0$ arbitrarily and define $\tilde{\varPsi}$ by

$$\tilde{\varPsi}(\cdot,x_{n+1})=\frac{\cosh(\delta|D|x_{n+1})}{\cosh(\delta|D|)}\psi,$$

which is a solution of the boundary-value problem

$$\nabla_X\cdot I_\delta^2\nabla_X\tilde{\varPsi}=0\ \text{in}\ \varOmega_0,\qquad\tilde{\varPsi}=\psi\ \text{on}\ \varGamma_0,\qquad\partial_{n+1}\tilde{\varPsi}=0\ \text{on}\ \varSigma_0,$$

so that we have $\|(\varLambda_0^{\mathrm{DN}})^{-1/2}I_\delta\nabla_X\tilde{\varPsi}\|_{L^2(\varOmega_0)}=\|\psi\|$ and $\|\tilde{\varPsi}\|_{L^2(\varOmega_0)}\leqslant\|\psi\|$. By Green's formula, we see that

$$\begin{aligned}&\int_{\varOmega_0}J^sPI_\delta\nabla_X\tilde{\varPhi}\cdot I_\delta\nabla_X\tilde{\varPsi}\,\mathrm{d}X\\&\quad=-\int_{\varOmega_0}(J^s(\nabla_X\cdot I_\delta F+f))\tilde{\varPsi}\,\mathrm{d}X+\int_{\partial\varOmega_0}(N\cdot J^sI_\delta PI_\delta\nabla_X\tilde{\varPhi})\tilde{\varPsi}\,\mathrm{d}S\\&\quad=\int_{\varOmega_0}(J^sF\cdot I_\delta\nabla_X\tilde{\varPsi}-(J^sf)\tilde{\varPsi})\,\mathrm{d}X+(\delta^{-2}J^s\partial_{n+1}\tilde{\varPhi}(\cdot,1),\psi)+(J^s\beta,\varLambda_0^{\mathrm{DD}}\psi).\end{aligned}$$

Therefore, by propositions 3.2 and 3.3, we obtain

$$\begin{aligned}&|(\delta^{-2}J^s\partial_{n+1}\tilde{\varPhi}(\cdot,1),\psi)|\\&\quad=\bigg|\int_{\varOmega_0}(J^s(\varLambda_0^{\mathrm{DN}})^{1/2}(PI_\delta\nabla_X\tilde{\varPhi}-F)\cdot(\varLambda_0^{\mathrm{DN}})^{-1/2}I_\delta\nabla_X\tilde{\varPsi}+(J^sf)\tilde{\varPsi})\,\mathrm{d}X\\&\qquad\qquad+(J^s\varLambda_0^{\mathrm{NN}}\beta,\psi)\bigg|\\&\quad\leqslant(\|J^s(\varLambda_0^{\mathrm{DN}})^{1/2}PI_\delta\nabla_X\tilde{\varPhi}\|_{L^2(\varOmega_0)}+\|J^s(\varLambda_0^{\mathrm{DN}})^{1/2}F\|_{L^2(\varOmega_0)}\\&\qquad\qquad+\|J^sf\|_{L^2(\varOmega_0)}+\|\beta\|_s)\|\psi\|,\end{aligned}$$

which gives the desired estimate. □

LEMMA 4.14. *Let $s>\frac{1}{2}n$. Under assumptions* (A_1) *and* (A_2) *with $q=s+1$, there exists a constant $C=C(M,c,s)>0$ independent of δ such that the solution $\tilde{\varPhi}$ of (4.9) with $f=0$ and $F_{n+1}(\cdot,0)=F_{n+1}(\cdot,1)=\gamma=0$ satisfies*

$$\delta^{-2}\|\partial_{n+1}\tilde{\varPhi}(\cdot,1)\|_s\leqslant C(\|J^s(\varLambda_0^{\mathrm{DN}})^{1/2}F\|_{L^2(\varOmega_0)}+\|F\|_{L^2(\varOmega_0)}+\|\beta\|_s).$$

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Proof. By lemma 4.7 we have

$$\|J^s(\Lambda_0^{\mathrm{DN}})^{1/2}PI_\delta\nabla_X\tilde{\Phi}\|_{L^2(\Omega_0)} \leqslant C(\|J^s(\Lambda_0^{\mathrm{DN}})^{1/2}I_\delta\nabla_X\tilde{\Phi}\|_{L^2(\Omega_0)}+\|J^sI_\delta\nabla_X\tilde{\Phi}\|_{L^2(\Omega_0)}).$$

This and lemmas 4.11–4.13 give the desired estimate. □

Now we give estimates of the solution of the boundary-value problem (4.5).

Proposition 4.15. *Under* (A_1), *there exists a constant* $C=C(M,c)>0$ *independent of* δ *such that the solution* $\tilde{\Phi}$ *of (4.5) satisfies*

$$\|I_\delta\nabla_X\tilde{\Phi}\|_{L^2(\Omega_0)} \leqslant C(\|(\Lambda_0^{\mathrm{DN}})^{1/2}\phi\|+\delta\|\beta\|).$$

Proof. We set $\Phi_1:=(\phi,0)^\hbar$, $\Phi_2:=(0,\beta)^\hbar$, $\tilde{\Phi}_1:=\Phi_1\circ\Theta$ and $\tilde{\Phi}_2:=\Phi_2\circ\Theta$. Then, the solution can be decomposed as $\tilde{\Phi}=\tilde{\Phi}_1+\tilde{\Phi}_2$. By lemma 4.10, we have

$$\|I_\delta\nabla_X\tilde{\Phi}_2\|_{L^2(\Omega_0)} \leqslant C\delta\|\beta\|.$$

It follows from lemmas 4.3, 3.4 and 4.4 that

$$\|I_\delta\nabla_X\tilde{\Phi}_1\|_{L^2(\Omega_0)} \leqslant C\|I_\delta\nabla_X\Phi_1\|_{L^2(\Omega)} = C(\Lambda^{\mathrm{DN}}\phi,\phi)^{1/2} \leqslant C\|(\Lambda_0^{\mathrm{DN}})^{1/2}\phi\|.$$

Therefore, we obtain the desired estimate. □

Proposition 4.16. *Let* $s>\frac{1}{2}n+1$. *Under assumptions* (A_1) *and* (A_2) *with* $q=s$, *there exists a constant* $C=C(M,c,s)>0$ *independent of* δ *such that the solution* $\tilde{\Phi}$ *of (4.5) satisfies*

$$\|J^sI_\delta\nabla_X\tilde{\Phi}\|_{L^2(\Omega_0)} \leqslant C(\|(\Lambda_0^{\mathrm{DN}})^{1/2}\phi\|_s+\delta\|\beta\|_s). \tag{4.11}$$

Proof. Set $\Phi_s:=(J^s\phi,0)^\hbar$ and $\tilde{\Phi}_s:=\Phi_s\circ\Theta$. Then, we have

$$\begin{aligned} \nabla_X\cdot I_\delta PI_\delta\nabla_X(J^s\tilde{\Phi}-\tilde{\Phi}_s) &= -\nabla_X\cdot I_\delta[J^s,P]I_\delta\nabla_X\tilde{\Phi} && \text{in } \Omega_0,\\ (J^s\tilde{\Phi}-\tilde{\Phi}_s) &= 0 && \text{on } \Gamma_0,\\ -\delta^{-2}\partial_{n+1}(J^s\tilde{\Phi}-\tilde{\Phi}_s) &= J^s\beta && \text{on } \Sigma_0. \end{aligned}$$

Therefore, by lemma 4.10, we obtain

$$\begin{aligned} \|I_\delta\nabla_X(J^s\tilde{\Phi}-\tilde{\Phi}_s)\|_{L^2(\Omega_0)} &\leqslant C(\|[J^s,P]I_\delta\nabla_X\tilde{\Phi}\|_{L^2(\Omega_0)}+\delta\|\beta\|_s)\\ &\leqslant \epsilon\|J^sI_\delta\nabla_X\tilde{\Phi}\|_{L^2(\Omega_0)}+C_\epsilon(\|I_\delta\nabla_X\tilde{\Phi}\|_{L^2(\Omega_0)}+\delta\|\beta\|_s). \end{aligned} \tag{4.12}$$

On the other hand, by proposition 4.15, we have

$$\|I_\delta\nabla_X\tilde{\Phi}_s\|_{L^2(\Omega_0)} \leqslant C\|(\Lambda_0^{\mathrm{DN}})^{1/2}\phi\|_s.$$

These estimates, together with proposition 4.15, yield the desired estimate. □

Proposition 4.17. *Let* $s>\frac{1}{2}n+1$. *Under assumptions* (A_1) *and* (A_2) *with* $q=s+1$, *there exists a constant* $C=C(M,c,s)>0$ *independent of* δ *such that the solution* $\tilde{\Phi}$ *of (4.5) satisfies*

$$\|J^s(\Lambda_0^{\mathrm{DN}})^{1/2}I_\delta\nabla_X\tilde{\Phi}\|_{L^2(\Omega_0)} \leqslant C(\|\Lambda_0^{\mathrm{DN}}\phi\|_s+\|(\Lambda_0^{\mathrm{DN}})^{1/2}\phi\|_s+\|\beta\|_s).$$

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Proof. Set $\Phi_1 := ((\Lambda_0^{\mathrm{DN}})^{1/2}\phi, 0)^{\hbar}$ and $\tilde{\Phi}_1 := \Phi_1 \circ \Theta$. Then, we have

$$\begin{aligned} \nabla_X \cdot I_\delta P I_\delta \nabla_X ((\Lambda_0^{\mathrm{DN}})^{1/2}\tilde{\Phi} - \tilde{\Phi}_1) &= -\nabla_X \cdot I_\delta [(\Lambda_0^{\mathrm{DN}})^{1/2}, P] I_\delta \nabla_X \tilde{\Phi} && \text{in } \Omega_0, \\ ((\Lambda_0^{\mathrm{DN}})^{1/2}\tilde{\Phi} - \tilde{\Phi}_1) &= 0 && \text{on } \Gamma_0, \\ -\delta^{-2}\partial_{n+1}((\Lambda_0^{\mathrm{DN}})^{1/2}\tilde{\Phi} - \tilde{\Phi}_1) &= (\Lambda_0^{\mathrm{DN}})^{1/2}\beta && \text{on } \Sigma_0. \end{aligned}$$

Therefore, by lemmas 4.11 and 4.5, we obtain

$$\begin{aligned} \|J^s I_\delta \nabla_X ((\Lambda_0^{\mathrm{DN}})^{1/2}\tilde{\Phi} - \tilde{\Phi}_1)\|_{L^2(\Omega_0)} &\leqslant C(\|J^s[(\Lambda_0^{\mathrm{DN}})^{1/2}, P] I_\delta \nabla_X \tilde{\Phi}\|_{L^2(\Omega_0)} + \|\beta\|_s) \\ &\leqslant C(\|J^s I_\delta \nabla_X \tilde{\Phi}\|_{L^2(\Omega_0)} + \|\beta\|_s). \end{aligned}$$

On the other hand, it follows from proposition 4.16 that

$$\|J^s I_\delta \nabla_X \tilde{\Phi}_1\|_{L^2(\Omega_0)} \leqslant C\|\Lambda_0^{\mathrm{DN}}\phi\|_s.$$

These estimates, together with proposition 4.16, yield the desired estimate. □

We proceed to give an L^∞-estimate of $\nabla_X \tilde{\Phi}$ in order to obtain a correct order of δ and the estimate under a weaker hypothesis on the water surface and the bottom. As shown in [11], the matrix P has the form

$$P = \begin{pmatrix} (1 + \partial_{n+1}\theta_{n+1})E_n + \delta^2 P_{11} & \delta \boldsymbol{p}_{12} \\ \delta \boldsymbol{p}_{12}^{\mathrm{T}} & (1 + \partial_{n+1}\theta_{n+1})^{-1} + \delta^2 p_{22} \end{pmatrix},$$

where P_{11}, $\boldsymbol{p}_{12}$ and p_{22} are $n \times n$, $1 \times n$ and 1×1 matrices whose elements are rational functions of $\nabla_X \theta$ and whose denominators are positive definite under assumption (A_1). Moreover, $\boldsymbol{p}_{12}$ can be written in the form $\boldsymbol{p}_{12} = \boldsymbol{p}_{12}^0 + \delta^2 \tilde{\boldsymbol{p}}_{12}$, where each element of $\tilde{\boldsymbol{p}}_{12}$ is also a rational function of $\nabla_X \theta$ and

$$\boldsymbol{p}_{12}^0 = -(1 + \partial_{n+1}\theta_{n+1})^{-1}(\partial_{n+1}(\theta_1, \dots, \theta_n)^{\mathrm{T}} + (1 + \partial_{n+1}\theta_{n+1})\nabla\theta_{n+1}). \tag{4.13}$$

We note that it follows from (4.4) that

$$\boldsymbol{p}_{12}(x, 0) = \boldsymbol{p}_{12}(x, 1) = \boldsymbol{0}, \qquad p_{22}(x, 0) = p_{22}(x, 1) = 0. \tag{4.14}$$

Using this notation we can rewrite the first equation in (4.5) as

$$\begin{aligned} \partial_{n+1}((\delta^{-2}(1 + \partial_{n+1}\theta_{n+1})^{-1} + p_{22})\partial_{n+1}\tilde{\Phi}) &\\ = -\nabla \cdot (((1 + \partial_{n+1}\theta_{n+1})E_n + \delta^2 P_{11})\nabla\tilde{\Phi}) &\\ - \nabla \cdot (\boldsymbol{p}_{12}\partial_{n+1}\tilde{\Phi}) - \partial_{n+1}(\boldsymbol{p}_{12} \cdot \nabla\tilde{\Phi}). & \end{aligned}$$

Integrating this with respect to x_{n+1} and using (4.1), (4.14) and a boundary condition in (4.5), we see that

$$\begin{aligned} &((1 + \partial_{n+1}\theta_{n+1})^{-1} + \delta^2 p_{22})\partial_{n+1}\tilde{\Phi} \\ &\quad = -\delta^2\beta - \delta^2 \int_0^{x_{n+1}} \nabla \cdot (((1 + \partial_{n+1}\theta_{n+1})E_n + \delta^2 P_{11})\nabla\tilde{\Phi})\, \mathrm{d}x_{n+1} \\ &\qquad - \delta^2 \int_0^{x_{n+1}} \nabla \cdot (\boldsymbol{p}_{12}\partial_{n+1}\tilde{\Phi})\, \mathrm{d}x_{n+1} - \delta^2 \boldsymbol{p}_{12} \cdot \nabla\tilde{\Phi}. \end{aligned} \tag{4.15}$$

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We also have

$$\nabla\tilde{\Phi} = \nabla\phi - \int_{x_{n+1}}^{1} \nabla\partial_{n+1}\tilde{\Phi}\,\mathrm{d}x_{n+1}. \tag{4.16}$$

Corollary 4.18. *Let $s > \frac{1}{2}n + 1$. Under assumptions* (A_1) *and* (A_2) *with $q = s$, there exists a constant $C = C(M, c, s) > 0$ independent of δ such that the solution $\tilde{\Phi}$ of (4.5) with $\phi = 0$ satisfies*

$$\|J^s\partial_{n+1}\tilde{\Phi}\|_{L^2(\Omega_0)} + \|J^{s-1}\nabla\tilde{\Phi}\|_{L^2(\Omega_0)} \leqslant C\delta^2\|\beta\|_s.$$

Proof. It follows from proposition 4.16 that $\|J^s\partial_{n+1}\tilde{\Phi}\|_{L^2(\Omega_0)} \leqslant \delta^2\|\beta\|_s$. This and (4.16) show that

$$\|J^{s-1}\nabla\tilde{\Phi}\|_{L^2(\Omega_0)} \leqslant \|J^s\partial_{n+1}\tilde{\Phi}\|_{L^2(\Omega_0)} \leqslant \delta^2\|\beta\|_s.$$

The proof is complete. □

Corollary 4.19. *Let $s > \frac{1}{2}n + 1$. Under assumptions* (A_1) *and* (A_2) *with $q = s$, there exists a constant $C = C(M, c, s) > 0$ independent of δ such that the solution $\tilde{\Phi}$ of (4.5) with $\beta = 0$ satisfies*

$$\|J^s\nabla\tilde{\Phi}\|_{L^2(\Omega_0)} \leqslant C\|(\Lambda_0^{\mathrm{DN}})^{1/2}\phi\|_s,$$
$$\|J^{s-1}\partial_{n+1}\tilde{\Phi}\|_{L^2(\Omega_0)} \leqslant C\delta^2\|(\Lambda_0^{\mathrm{DN}})^{1/2}\phi\|_s.$$

Proof. The first estimate comes directly from proposition 4.16. It follows from (4.15) that $\|J^{s-1}\partial_{n+1}\tilde{\Phi}\|_{L^2(\Omega_0)} \leqslant C\delta^2\|J^s\nabla_X\tilde{\Phi}\|_{L^2(\Omega_0)}$. This and proposition 4.16 give the second estimate. The proof is complete. □

Proposition 4.20. *Let $r > \frac{1}{2}n$. Under assumptions* (A_1) *and* (A_2) *with $q = r + 1$, there exists a constant $C = C(M, c, r) > 0$ independent of δ such that the solution $\tilde{\Phi}$ of* (4.5) *satisfies*

$$\|\nabla\tilde{\Phi}\|_{L^\infty(\Omega_0)} \leqslant C(\|\nabla\phi\|_r + \delta\|(\Lambda_0^{\mathrm{DN}})^{1/2}\phi\|_{r+1} + \delta^2\|\beta\|_{r+1}),$$
$$\|\partial_{n+1}\tilde{\Phi}\|_{L^\infty(\Omega_0)} \leqslant C\delta^2(\|(\Lambda_0^{\mathrm{DN}})^{1/2}\phi\|_{r+1} + \|\beta\|_{r+1}).$$

Proof. Note that the assumptions imply the uniform boundedness of P_{11}, p_{22}, $\boldsymbol{p}_{12}$ and their first derivatives with respect to x. It follows from (4.16) and the Sobolev inequality that

$$\|\nabla\tilde{\Phi}\|_{L^\infty(\Omega_0)} \leqslant C(\|\nabla\phi\|_r + \delta\|J^{r+1}I_\delta\nabla_X\tilde{\Phi}\|_{L^2(\Omega_0)}),$$

which, together with proposition 4.16, implies the first estimate of the proposition. Similarly, it follows from (4.15) that

$$\|\partial_{n+1}\tilde{\Phi}\|_{L^\infty(\Omega_0)} \leqslant C\delta^2(\|\beta\|_r + \|J^{r+1}I_\delta\nabla_X\tilde{\Phi}\|_{L^2(\Omega_0)} + \|\nabla\tilde{\Phi}\|_{L^\infty(\Omega_0)}),$$

which, together with the first estimate, proposition 4.16 and lemma 4.6, gives the second estimate. The proof is complete. □

Corollary 4.21. *Let $s > \frac{1}{2}(n + 3)$. Under assumptions* (A_1)–(A_3) *with $q = s - \frac{1}{2}$, there exists a constant $C = C(M, c, s) > 0$ independent of δ such that the solution $\tilde{\Phi}$ of (4.5) satisfies (4.11).*

Proof. The proof is the same as that of proposition 4.16, except that the first term on the right-hand side of (4.12) is evaluated as

$$
\begin{aligned}
&\|[J^s, P]I_\delta \nabla_X \tilde{\Phi}\|_{L^2(\Omega_0)} \\
&\quad \leqslant C(\|\nabla P\|_{L^\infty(\Omega_0)}\|J^{s-1}I_\delta \nabla_X \tilde{\Phi}\|_{L^2(\Omega_0)} + \|J^s P\|_{L^2(\Omega_0)}\|I_\delta \nabla_X \tilde{\Phi}\|_{L^\infty(\Omega_0)}) \\
&\quad \leqslant \epsilon \|J^s I_\delta \nabla_X \tilde{\Phi}\|_{L^2(\Omega_0)} + C_\epsilon(\|I_\delta \nabla_X \tilde{\Phi}\|_{L^2(\Omega_0)} + \|I_\delta \nabla_X \tilde{\Phi}\|_{L^\infty(\Omega_0)}).
\end{aligned}
$$

Here, the last term can be evaluated by proposition 4.20 and lemma 4.6. The proof is complete. □

The solution $\tilde{\Phi}$ of the boundary-value problem (4.5) depends on (η, b) through the matrix coefficient P. Here, we will give estimates of Fréchet derivatives of the solution $\tilde{\Phi}$ with respect to (η, b).

PROPOSITION 4.22. *Let $s > \frac{1}{2}n+1$ and $m \in \mathbb{N}$. Under assumptions* (A$_1$)–(A$_4$) *with $q = s$, there exists a constant $C = C(M, c, s, m) > 0$ independent of δ such that the solution $\tilde{\Phi}$ of (4.5) satisfies*

$$
\begin{aligned}
&\|J^s I_\delta \nabla_X (D_\eta^m \tilde{\Phi}[\check{\eta}_1, \dots, \check{\eta}_m])\|_{L^2(\Omega_0)} \\
&\qquad \leqslant C\|\check{\eta}_1\|_{s+1/2} \cdots \|\check{\eta}_m\|_{s+1/2}(\|(\Lambda_0^{\mathrm{DN}})^{1/2}\phi\|_s + \delta\|\beta\|_s).
\end{aligned}
$$

A similar estimate holds for the Fréchet derivatives with respect to b.

Proof. We only show the estimate in the case $m = 1$, and the general case can be proved in the same way. For simplicity, we write $\tilde{\Phi}_\eta = D_\eta \tilde{\Phi}[\check{\eta}]$ and $P_\eta = D_\eta P[\check{\eta}]$. Taking the Fréchet derivative of (4.5), we obtain

$$
\begin{aligned}
\nabla_X \cdot I_\delta P I_\delta \nabla_X \tilde{\Phi}_\eta &= -\nabla_X \cdot I_\delta P_\eta I_\delta \nabla_X \tilde{\Phi} && \text{in } \Omega_0, \\
\tilde{\Phi}_\eta &= 0 && \text{on } \Gamma_0, \\
-\delta^{-2}\partial_{n+1}\tilde{\Phi}_\eta &= 0 && \text{on } \Sigma_0.
\end{aligned}
$$

Therefore, by lemmas 4.11 and 4.6 and propositions 4.16 and 4.20, we see that

$$
\begin{aligned}
&\|J^s I_\delta \nabla_X \tilde{\Phi}_\eta\|_{L^2(\Omega_0)} \\
&\quad \leqslant C\|J^s P_\eta I_\delta \nabla_X \tilde{\Phi}\|_{L^2(\Omega_0)} \\
&\quad \leqslant C(\|P_\eta\|_{L^\infty(\Omega_0)}\|J^s I_\delta \nabla_X \tilde{\Phi}\|_{L^2(\Omega_0)} + \|J^s P_\eta\|_{L^2(\Omega_0)}\|I_\delta \nabla_X \tilde{\Phi}\|_{L^\infty(\Omega_0)}) \\
&\quad \leqslant C\|\check{\eta}\|_{s+1/2}(\|(\Lambda_0^{\mathrm{DN}})^{1/2}\phi\|_s + \delta\|\beta\|_s),
\end{aligned}
$$

which gives the desired estimate. □

COROLLARY 4.23. *Let $s > \frac{1}{2}n+1$ and $m \in \mathbb{N}$. Under assumptions* (A$_1$)–(A$_4$) *with $q = s$, there exists a constant $C = C(M, c, s, m) > 0$ independent of δ such that the solution $\tilde{\Phi}$ of (4.5) with $\phi = 0$ satisfies*

$$
\begin{aligned}
&\|J^s \partial_{n+1}(D_\eta^m \tilde{\Phi}[\check{\eta}_1, \dots, \check{\eta}_m])\|_{L^2(\Omega_0)} + \|J^{s-1}\nabla(D_\eta^m \tilde{\Phi}[\check{\eta}_1, \dots, \check{\eta}_m])\|_{L^2(\Omega_0)} \\
&\qquad \leqslant C\delta^2\|\check{\eta}_1\|_{s+1/2} \cdots \|\check{\eta}_m\|_{s+1/2}\|\beta\|_s.
\end{aligned}
$$

A similar estimate holds for the Fréchet derivatives with respect to b.

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Proof. The estimate of the first term comes directly from proposition 4.22. On the other hand, it follows from (4.16) that

$$\nabla D_\eta^m \tilde{\Phi} = -\int_{x_{n+1}}^1 \nabla \partial_{n+1} D_\eta^m \tilde{\Phi} \, \mathrm{d}x_{n+1},$$

which, together with the estimate of the first term, gives an estimate of the second term. The proof is complete. □

Corollary 4.24. *Let $s > \frac{1}{2}n$ and $m \in \mathbb{N}$. Under assumptions* (A$_1$)–(A$_4$) *with $q = s+1$, there exists a constant $C = C(M,c,s,m) > 0$ independent of δ such that the solution $\tilde{\Phi}$ of (4.5) with $\beta = 0$ satisfies*

$$\|J^s \partial_{n+1}(D_\eta^m \tilde{\Phi}[\check{\eta}_1,\dots,\check{\eta}_m])\|_{L^2(\Omega_0)} + \|J^{s-1}\nabla(D_\eta^m \tilde{\Phi}[\check{\eta}_1,\dots,\check{\eta}_m])\|_{L^2(\Omega_0)} \leqslant C\delta^2 \|\check{\eta}_1\|_{s+3/2}\cdots\|\check{\eta}_m\|_{s+3/2}\|(\Lambda_0^{\mathrm{DN}})^{1/2}\phi\|_{s+1}.$$

A similar estimate holds for the Fréchet derivatives with respect to b.

Proof. We only show the estimate in the case $m = 1$. For simplicity, we write $\tilde{\Phi}_\eta = D_\eta \tilde{\Phi}[\check{\eta}]$. By taking the Fréchet derivative of (4.15), we see that

$$\begin{aligned}\|J^s \partial_{n+1}\tilde{\Phi}_\eta\|_{L^2(\Omega_0)} &\leqslant C\delta^2(\|J^{s+1}\nabla_X \tilde{\Phi}_\eta\|_{L^2(\Omega_0)} + \|J^{s+1}\nabla_X(D_\eta\theta[\check{\eta}])\|_{L^2(\Omega_0)}\|\nabla_X\tilde{\Phi}\|_{L^\infty(\Omega_0)} \\ &\quad + \|\nabla_X(D_\eta\theta[\check{\eta}])\|_{L^\infty(\Omega_0)}\|J^{s+1}\nabla_X\tilde{\Phi}\|_{L^2(\Omega_0)}) \\ &\leqslant C\delta^2\|\check{\eta}\|_{s+3/2}\|(\Lambda_0^{\mathrm{DN}})^{1/2}\phi\|_{s+1},\end{aligned}$$

where we used propositions 4.16, 4.20 and 4.22 and lemma 4.6. On the other hand, by (4.16), it holds that

$$\nabla\tilde{\Phi}_\eta = -\int_{x_{n+1}}^1 \nabla\partial_{n+1}\tilde{\Phi}_\eta \, \mathrm{d}x_{n+1},$$

so that

$$\|J^{s-1}\nabla\tilde{\Phi}_\eta\|_{L^2(\Omega_0)} \leqslant \|J^s\partial_{n+1}\tilde{\Phi}_\eta\|_{L^2(\Omega_0)}.$$

These estimates imply the desired estimate. □

5. Estimates of the operators

The following four propositions on the DN map $\Lambda^{\mathrm{DN}} = \Lambda^{\mathrm{DN}}(\eta, b, \delta)$ were given in [11].

Proposition 5.1. *Let $s > \frac{1}{2}n+1$. Under assumptions* (A$_1$) *and* (A$_2$) *with $q = s+1$, there exists a constant $C = C(M,c,s) > 0$ independent of δ such that we have $\|\Lambda^{\mathrm{DN}}\phi\|_s \leqslant C(\|\Lambda_0^{\mathrm{DN}}\phi\|_s + \|(\Lambda_0^{\mathrm{DN}})^{1/2}\phi\|_s)$. In particular, it holds that*

$$\|\Lambda^{\mathrm{DN}}\phi\|_s \leqslant C\delta^{-1}\|\phi\|_{s+1}.$$

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PROPOSITION 5.2. *Let $s > \frac{1}{2}n + 2$. In addition to assumptions* (A_1) *and* (A_2) *with $q = s$, we assume that $\|(\eta, b)\|_{s+1} \leqslant M$. Then, there exists a constant $C = C(M, c, s) > 0$ independent of δ such that we have*

$$\|\Lambda^{\mathrm{DN}}\phi\|_s \leqslant C\delta^{-1/2}\|(\Lambda_0^{\mathrm{DN}})^{1/2}\phi\|_{s+1/2}.$$

PROPOSITION 5.3. *Let $s > \frac{1}{2}n$. Under assumptions* (A_1)–(A_3) *with $q = s + \frac{5}{2}$, there exists a constant $C = C(M, c, s) > 0$ independent of δ such that we have*

$$\|\Lambda^{\mathrm{DN}}\phi + \nabla\cdot((1 + \eta - b)\nabla\phi)\|_s \leqslant C\delta^2(\|(\Lambda_0^{\mathrm{DN}})^{1/2}\phi\|_{s+3} + \|\nabla\phi\|_s).$$

PROPOSITION 5.4. *It holds that $|(\Lambda^{\mathrm{DN}}\phi, \psi)| \leqslant \sqrt{(\Lambda^{\mathrm{DN}}\phi, \phi)}\sqrt{(\Lambda^{\mathrm{DN}}\psi, \psi)}$.*

PROPOSITION 5.5. *Under assumption* (A_1)*, there exists a constant $C = C(M, c) > 0$ independent of δ such that we have $\|\Lambda^{\mathrm{DN}}\phi\|_{-1} \leqslant C\|\nabla\phi\|$.*

Proof. Set $\Phi := (\phi, 0)^\hbar$ and $\tilde{\Phi} := \Phi \circ \Theta$. Then, $\tilde{\Phi}$ satisfies (4.5) with $\beta = 0$ and $\delta^{-2}\partial_{n+1}\tilde{\Phi}(\cdot, 1) = \Lambda^{\mathrm{DN}}\phi$. Therefore, it follows from lemmas 4.13 and 4.6 that

$$\|\Lambda^{\mathrm{DN}}\phi\|_{-1} \leqslant \|J^{-1}(\Lambda_0^{\mathrm{DN}})^{1/2}PI_\delta\nabla_X\tilde{\Phi}\|_{L^2(\Omega_0)} \leqslant \|PI_\delta\nabla_X\tilde{\Phi}\|_{L^2(\Omega_0)} \leqslant C\|\nabla\phi\|,$$

where we also used lemmas 4.3, 3.4 and 4.4. The proof is complete. □

Now we give commutator estimates for the DN map Λ^{DN}.

PROPOSITION 5.6. *Let $r > \frac{1}{2}n$. Under assumptions* (A_1) *and* (A_2) *with $q = r + 1$, there exists a constant $C = C(M, c, r) > 0$ independent of δ such that we have $\|[\nabla, \Lambda^{\mathrm{DN}}]\phi\|_{-1} \leqslant C\|\nabla\phi\|$.*

Proof. Set $\Phi := (\phi, 0)^\hbar$, $\Phi_i := (\partial_i\phi, 0)^\hbar$, $\tilde{\Phi} := \Phi \circ \Theta$ and $\tilde{\Phi}_i := \Phi_i \circ \Theta$. Then, it holds that

$$\left.\begin{aligned}
\nabla_X\cdot I_\delta PI_\delta\nabla_X(\partial_i\tilde{\Phi} - \tilde{\Phi}_i) &= -\nabla_X\cdot I_\delta(\partial_i P)I_\delta\nabla_X\tilde{\Phi} && \text{in } \Omega_0,\\
(\partial_i\tilde{\Phi} - \tilde{\Phi}_i) = 0, \quad \delta^{-2}\partial_{n+1}(\partial_i\tilde{\Phi} - \tilde{\Phi}_i) &= [\partial_i, \Lambda^{\mathrm{DN}}]\phi && \text{on } \Gamma_0,\\
-\delta^{-2}\partial_{n+1}(\partial_i\tilde{\Phi} - \tilde{\Phi}_i) &= 0 && \text{on } \Sigma_0.
\end{aligned}\right\} \tag{5.1}$$

Therefore, it follows from lemmas 4.13 and 4.6 that

$$\begin{aligned}
\|[\partial_i, \Lambda^{\mathrm{DN}}]\phi\|_{-1} &\leqslant \|J^{-1}(\Lambda_0^{\mathrm{DN}})^{1/2}PI_\delta\nabla_X(\partial_i\tilde{\Phi} - \tilde{\Phi}_i)\|_{L^2(\Omega_0)}\\
&\quad + \|J^{-1}(\Lambda_0^{\mathrm{DN}})^{1/2}(\partial_i P)I_\delta\nabla_X\tilde{\Phi}\|_{L^2(\Omega_0)}\\
&\leqslant C(\|I_\delta\nabla_X(\partial_i\tilde{\Phi} - \tilde{\Phi}_i)\|_{L^2(\Omega_0)} + \|I_\delta\nabla_X\tilde{\Phi}\|_{L^2(\Omega_0)}).
\end{aligned}$$

On the other hand, by lemma 4.10, we have

$$\|I_\delta\nabla_X(\partial_i\tilde{\Phi} - \tilde{\Phi}_i)\|_{L^2(\Omega_0)} \leqslant C\|(\partial_i P)I_\delta\nabla_X\tilde{\Phi}\|_{L^2(\Omega_0)} \leqslant C\|I_\delta\nabla_X\tilde{\Phi}\|_{L^2(\Omega_0)}.$$

Hence, we obtain $\|[\partial_i, \Lambda^{\mathrm{DN}}]\phi\|_{-1} \leqslant C\|I_\delta\nabla_X\tilde{\Phi}\|_{L^2(\Omega_0)} \leqslant C\|\nabla\phi\|$, where we also used lemmas 4.3, 3.4 and 4.4. The proof is complete. □

PROPOSITION 5.7. *Let $r > \frac{1}{2}n$. Under assumptions* (A_1) *and* (A_2) *with $q = r + 1$, there exists a constant $C = C(M, c, r) > 0$ independent of δ such that we have $\|[\Lambda^{\mathrm{DN}}, a]\phi\|_{-1} \leqslant C\|\nabla a\|_{r+2}\|\phi\|$.*

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Proof. Set $\Phi := (\phi, 0)^\hbar$, $A := (a, 0)^\hbar$, $\Phi_a := (a\phi, 0)^\hbar$, $\tilde{\Phi} := \Phi \circ \Theta$, $\tilde{A} := A \circ \Theta$ and $\tilde{\Phi}_a := \Phi_a \circ \Theta$. Then it holds that

$$\begin{aligned} \nabla_X \cdot I_\delta P I_\delta \nabla_X (\tilde{\Phi}_a - \tilde{A}\tilde{\Phi}) &= -2PI_\delta \nabla_X \tilde{A} \cdot I_\delta \nabla_X \tilde{\Phi} \quad \text{in } \Omega_0, \\ (\tilde{\Phi}_a - \tilde{A}\tilde{\Phi}) = 0, \quad \delta^{-2}\partial_{n+1}(\tilde{\Phi}_a - \tilde{A}\tilde{\Phi}) &= [\Lambda^{\mathrm{DN}}, a]\phi - \phi\Lambda^{\mathrm{DN}} a \quad \text{on } \Gamma_0, \\ -\delta^{-2}\partial_{n+1}(\tilde{\Phi}_a - \tilde{A}\tilde{\Phi}) &= 0 \quad \text{on } \Sigma_0. \end{aligned}$$

Therefore, it follows from lemmas 4.13 and 4.6 that

$$\begin{aligned} &\|[\Lambda^{\mathrm{DN}}, a]\phi - \phi\Lambda^{\mathrm{DN}} a\| \\ &\qquad \leqslant \|(\Lambda_0^{\mathrm{DN}})^{1/2} PI_\delta \nabla_X (\tilde{\Phi}_a - \tilde{A}\tilde{\Phi})\|_{L^2(\Omega_0)} + 2\|PI_\delta \nabla_X \tilde{A} \cdot I_\delta \nabla_X \tilde{\Phi}\|_{L^2(\Omega_0)} \\ &\qquad \leqslant C(\|JI_\delta \nabla_X (\tilde{\Phi}_a - \tilde{A}\tilde{\Phi})\|_{L^2(\Omega_0)} + \|I_\delta \nabla_X \tilde{A} \cdot I_\delta \nabla_X \tilde{\Phi}\|_{L^2(\Omega_0)}). \end{aligned}$$

Here, by lemma 4.10, we have

$$\|I_\delta \nabla_X (\tilde{\Phi}_a - \tilde{A}\tilde{\Phi})\|_{L^2(\Omega_0)} \leqslant C\|I_\delta \nabla_X \tilde{A} \cdot I_\delta \nabla_X \tilde{\Phi}\|_{L^2(\Omega_0)}.$$

Moreover, it also holds that

$$\begin{aligned} \nabla_X \cdot I_\delta P I_\delta \nabla_X J(\tilde{\Phi}_a - \tilde{A}\tilde{\Phi}) &= -\nabla_X \cdot I_\delta [J, P] I_\delta \nabla_X (\tilde{\Phi}_a - \tilde{A}\tilde{\Phi}) \\ &\qquad - 2JPI_\delta \nabla_X \tilde{A} \cdot I_\delta \nabla_X \tilde{\Phi} \quad \text{in } \Omega_0, \\ J(\tilde{\Phi}_a - \tilde{A}\tilde{\Phi}) &= 0 \quad \text{on } \Gamma_0, \\ -\delta^{-2}\partial_{n+1} J(\tilde{\Phi}_a - \tilde{A}\tilde{\Phi}) &= 0 \quad \text{on } \Sigma_0, \end{aligned}$$

so that lemma 4.10 gives

$$\begin{aligned} &\|JI_\delta \nabla_X (\tilde{\Phi}_a - \tilde{A}\tilde{\Phi})\|_{L^2(\Omega_0)} \\ &\qquad \leqslant C(\|[J, P] I_\delta \nabla_X (\tilde{\Phi}_a - \tilde{A}\tilde{\Phi})\|_{L^2(\Omega_0)} + \|PI_\delta \nabla_X \tilde{A} \cdot I_\delta \nabla_X \tilde{\Phi}\|_{L^2(\Omega_0)}) \\ &\qquad \leqslant C(\|I_\delta \nabla_X (\tilde{\Phi}_a - \tilde{A}\tilde{\Phi})\|_{L^2(\Omega_0)} + \|I_\delta \nabla_X \tilde{A} \cdot I_\delta \nabla_X \tilde{\Phi}\|_{L^2(\Omega_0)}). \end{aligned}$$

Hence, we obtain

$$\begin{aligned} \|[\Lambda^{\mathrm{DN}}, a]\phi\| &\leqslant \|\phi\Lambda^{\mathrm{DN}} a\| + C\|I_\delta \nabla_X \tilde{A} \cdot I_\delta \nabla_X \tilde{\Phi}\|_{L^2(\Omega_0)} \\ &\leqslant C(\|\phi\|\|\Lambda^{\mathrm{DN}} a\|_r + \|I_\delta \nabla_X \tilde{A}\|_{L^\infty(\Omega_0)} \|I_\delta \nabla_X \tilde{\Phi}\|_{L^2(\Omega_0)}) \\ &\leqslant C\|\nabla a\|_{r+2}\|\phi\|_1, \end{aligned}$$

where we used propositions 5.1 and 4.20 and lemmas 4.3, 3.4, 4.4 and 4.6. Since the adjoint operator of $[\Lambda^{\mathrm{DN}}, a]$ in L^2 is equal to $-[\Lambda^{\mathrm{DN}}, a]$, the above estimate, together with the standard duality argument, shows the desired estimate. □

The following three propositions on the DN map Λ^{DN} were given in [11].

Proposition 5.8. *Let $s > \frac{1}{2}n+1$. Under assumptions* (A_1) *and* (A_2) *with $q = s+1$, there exists a constant $C = C(M, c, s) > 0$ independent of δ such that we have $\|[J^s, \Lambda^{\mathrm{DN}}]\phi\| \leqslant C\|(\Lambda_0^{\mathrm{DN}})^{1/2}\phi\|_s$.*

Proposition 5.9. *Let $r > \frac{1}{2}n$. Under assumption* (A_1)*, there exists a constant $C = C(M, c, r) > 0$ independent of δ such that we have*

$$|([\partial_t, \Lambda^{\mathrm{DN}}]\phi, \phi)| \leqslant C\|(\eta_t, b_t)\|_{r+1}(\Lambda^{\mathrm{DN}}\phi, \phi).$$

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Proposition 5.10. *Let $r > \frac{1}{2}n$. Under assumption* (A$_1$) *and* $\|(\eta, b)\|_{r+2} \leqslant M$, *there exists a constant* $C = C(M, c, r) > 0$ *independent of* δ *such that we have* $|(\Lambda^{\mathrm{DN}}\phi, v \cdot \nabla\phi)| \leqslant C\|v\|_{r+1}(\Lambda^{\mathrm{DN}}\phi, \phi)$.

We now give estimates for the other operators.

Proposition 5.11. *Let $s > \frac{1}{2}n$. Under assumptions* (A$_1$) *and* (A$_2$) *with* $q = s+1$, *there exists a constant* $C = C(M, c, s) > 0$ *independent of* δ *such that we have* $\|\Lambda^{\mathrm{NN}}\beta\|_s \leqslant C\|\beta\|_s$.

Proof. Set $\Phi := (0, \beta)^\hbar$ and $\tilde{\Phi} := \Phi \circ \Theta$. Then, $\tilde{\Phi}$ satisfies (4.5) with $\phi = 0$ and $\delta^{-2}\partial_{n+1}\tilde{\Phi}(\cdot, 1) = \Lambda^{\mathrm{NN}}\beta$. Therefore, lemma 4.14 gives the desired estimate. □

Lemma 5.12. *For any function $\tilde{\Phi}$ defined on Ω_0, we have*

$$\|\tilde{\Phi}(\cdot, 0)\| \leqslant \|(\Lambda_0^{\mathrm{ND}})^{1/2} I_\delta \nabla_X \tilde{\Phi}\|_{L^2(\Omega_0)} + \|\Lambda_0^{\mathrm{DD}}\tilde{\Phi}(\cdot, 1)\|.$$

Proof. We take $\gamma \in H^0$ arbitrarily and set

$$\tilde{\Psi}(\cdot, x_{n+1}) = \frac{\delta}{|D|}\frac{\sinh(\delta|D|(1 - x_{n+1}))}{\cosh(\delta|D|)}\gamma,$$

which is a solution of the boundary-value problem

$$\nabla_X \cdot I_\delta^2 \nabla_X \tilde{\Psi} = 0 \text{ in } \Omega_0, \qquad \tilde{\Psi} = 0 \text{ on } \Gamma_0, \qquad -\delta^{-2}\partial_{n+1}\tilde{\Psi} = \gamma \text{ on } \Sigma_0,$$

so that we have

$$\|(\Lambda_0^{\mathrm{ND}})^{-1/2} I_\delta \nabla_X \tilde{\Psi}\|_{L^2(\Omega_0)} = \|\gamma\|.$$

By Green's formula, we see that

$$\begin{aligned}(\tilde{\Phi}(\cdot, 0), \gamma) + (\tilde{\Phi}(\cdot, 1), \Lambda_0^{\mathrm{NN}}\gamma) &= \int_{\partial\Omega_0} \tilde{\Phi}(N \cdot I_\delta^2 \nabla_X \tilde{\Psi})\, \mathrm{d}S \\ &= \int_{\Omega_0} \nabla_X \cdot (\tilde{\Phi} I_\delta^2 \nabla_X \tilde{\Psi})\, \mathrm{d}X \\ &= \int_{\Omega_0} I_\delta \nabla_X \tilde{\Phi} \cdot I_\delta \nabla_X \tilde{\Psi}\, \mathrm{d}X.\end{aligned}$$

This, together with proposition 3.3, implies that

$$\begin{aligned}&|(\tilde{\Phi}(\cdot, 0), \gamma)| \\ &\quad \leqslant \|\Lambda_0^{\mathrm{DD}}\tilde{\Phi}(\cdot, 1)\|\|\gamma\| + \|(\Lambda_0^{\mathrm{ND}})^{1/2} I_\delta \nabla_X \tilde{\Phi}\|_{L^2(\Omega_0)}\|(\Lambda_0^{\mathrm{ND}})^{-1/2} I_\delta \nabla_X \tilde{\Psi}\|_{L^2(\Omega_0)} \\ &\quad = (\|\Lambda_0^{\mathrm{DD}}\tilde{\Phi}(\cdot, 1)\| + \|(\Lambda_0^{\mathrm{ND}})^{1/2} I_\delta \nabla_X \tilde{\Phi}\|_{L^2(\Omega_0)})\|\gamma\|.\end{aligned}$$

This gives the desired estimate. □

Proposition 5.13. *Let $s > \frac{1}{2}(n+5)$. Under assumptions* (A$_1$)–(A$_3$) *with* $q = s-1$, *there exists a constant* $C = C(M, c, s) > 0$ *independent of* δ *such that we have* $\|\nabla\Lambda^{\mathrm{DD}}\phi\|_{s-1} \leqslant C\|\nabla\phi\|_{s-1}$ *and* $\|\Lambda^{\mathrm{DD}}\phi\|_s \leqslant C\|\phi\|_s$.

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Proof. Set $\Phi := (\phi, 0)^\hbar$ and $\tilde{\Phi} := \Phi \circ \Theta$. Then, $\tilde{\Phi}$ satisfies (4.5) with $\beta = 0$ and $\tilde{\Phi}(\cdot, 0) = \Lambda^{\mathrm{DD}}\phi$. Therefore, by lemma 5.12 and proposition 3.2, we see that

$$\begin{aligned}\|\partial_i \Lambda^{\mathrm{DD}}\phi\|_{s-1} &\leqslant \|J^{s-1}(\Lambda_0^{\mathrm{ND}})^{1/2} I_\delta \nabla_X \partial_i \tilde{\Phi}\|_{L^2(\Omega_0)} + \|\Lambda_0^{\mathrm{DD}}\partial_i\phi\|_{s-1} \\ &\leqslant 2\delta^{1/2}\|J^{s-3/2} I_\delta \nabla_X \partial_i \tilde{\Phi}\|_{L^2(\Omega_0)} + \|\partial_i\phi\|_{s-1}.\end{aligned}$$

Here, we also set $\Phi_i := (\partial_i\phi, 0)^\hbar$ and $\tilde{\Phi}_i := \Phi_i \circ \Theta$. Then, (5.1) holds. Therefore, by lemma 4.11, we see that

$$\begin{aligned}&\|J^{s-3/2} I_\delta \nabla_X(\partial_i\tilde{\Phi} - \tilde{\Phi}_i)\|_{L^2(\Omega_0)} \\ &\quad\leqslant C\|J^{s-3/2}(\partial_i P) I_\delta \nabla_X \tilde{\Phi}\|_{L^2(\Omega_0)} \\ &\quad\leqslant C(\|J^{s-3/2}\partial_i P\|_{L^2(\Omega_0)}\|I_\delta\nabla_X\tilde{\Phi}\|_{L^\infty(\Omega_0)} + \|\partial_i P\|_{L^\infty(\Omega_0)}\|J^{s-3/2} I_\delta \nabla_X \tilde{\Phi}\|_{L^2(\Omega_0)}) \\ &\quad\leqslant C\|\nabla\phi\|_{s-3/2},\end{aligned}$$

where we used propositions 4.16 and 4.20 and lemma 4.6. Moreover, we also obtain

$$\delta^{1/2}\|J^{s-3/2} I_\delta \nabla_X \tilde{\Phi}_i\|_{L^2(\Omega_0)} \leqslant C\delta^{1/2}\|(\Lambda_0^{\mathrm{DN}})^{1/2}\partial_i\phi\|_{s-3/2} \leqslant C\|\partial_i\phi\|_{s-1}.$$

These estimates give the first estimate. A similar argument gives the second estimate. □

Proposition 5.14. *Let $s > \frac{1}{2}n + 2$. Under assumptions* (A$_1$) *and* (A$_2$) *with $q = s$, there exists a constant $C = C(M, c, s) > 0$ independent of δ such that we have $\|\Lambda^{\mathrm{ND}}\beta\|_s \leqslant C\min\{\delta^2\|\beta\|_s, \delta\|\beta\|_{s-1}\}$.*

Proof. Set $\Phi := (0, \beta)^\hbar$ and $\tilde{\Phi} := \Phi \circ \Theta$. Then, $\tilde{\Phi}$ satisfies (4.5) with $\phi = 0$ and $\tilde{\Phi}(\cdot, 0) = \Lambda^{\mathrm{ND}}\beta$. Therefore, by lemma 5.12 and propositions 3.2 and 4.16, we see that

$$\|\Lambda^{\mathrm{ND}}\beta\|_s \leqslant \|J^s(\Lambda_0^{\mathrm{ND}})^{1/2} I_\delta \nabla_X \tilde{\Phi}\|_{L^2(\Omega_0)} \leqslant \delta\|J^s I_\delta \nabla_X \tilde{\Phi}\|_{L^2(\Omega_0)} \leqslant C\delta^2\|\beta\|_s.$$

On the other hand, it follows from lemma 5.12 and propositions 3.2, 4.15, and 4.17, we see that

$$\begin{aligned}\|\Lambda^{\mathrm{ND}}\beta\|_s &\leqslant \|J^s(\Lambda_0^{\mathrm{ND}})^{1/2} I_\delta \nabla_X \tilde{\Phi}\|_{L^2(\Omega_0)} \\ &\leqslant C(\|J^{s-1}|D|(\Lambda_0^{\mathrm{ND}})^{1/2} I_\delta \nabla_X \tilde{\Phi}\|_{L^2(\Omega_0)} + \|(\Lambda_0^{\mathrm{ND}})^{1/2} I_\delta \nabla_X \tilde{\Phi}\|_{L^2(\Omega_0)}) \\ &\leqslant C\delta(\|J^{s-1}(\Lambda_0^{\mathrm{DN}})^{1/2} I_\delta \nabla_X \tilde{\Phi}\|_{L^2(\Omega_0)} + \|I_\delta \nabla_X \tilde{\Phi}\|_{L^2(\Omega_0)}) \\ &\leqslant C\delta\|\beta\|_{s-1},\end{aligned}$$

where we used the relation $|D|^2\Lambda_0^{\mathrm{ND}} = \delta^2\Lambda_0^{\mathrm{DN}}$. These two estimates give the desired estimate. □

The next two propositions are mathematically rigorous versions of the formal expansion (2.7).

Proposition 5.15. *Let $s > \frac{1}{2}n - 1$. Under assumptions* (A$_1$) *and* (A$_2$) *with $q = s + 2$, there exists a constant $C = C(M, c, s) > 0$ independent of δ such that we have $\|\Lambda^{\mathrm{NN}}\beta + \beta\|_s \leqslant C\delta^2\|\beta\|_{s+2}$.*

Proof. Set $\Phi := (0,\beta)^\hbar$ and $\tilde{\Phi} := \Phi \circ \Theta$. Then, $\tilde{\Phi}$ satisfies (4.15) and

$$\Lambda^{\mathrm{NN}}\beta = \delta^{-2}\partial_{n+1}\tilde{\Phi}(\cdot,1),$$

so that we have

$$\Lambda^{\mathrm{NN}}\beta = -\beta - \nabla\cdot\int_0^1 ((1+\partial_{n+1}\theta_{n+1})\nabla\tilde{\Phi} + \boldsymbol{p}_{12}\partial_{n+1}\tilde{\Phi} + \delta^2 P_{11}\nabla\tilde{\Phi})\,\mathrm{d}x_{n+1}, \quad (5.2)$$

where we used (4.1) and (4.14). This and corollary 4.18 give the desired estimate. □

PROPOSITION 5.16. *Let $s > \frac{1}{2}n - 2$. Under assumptions* (A_1) *and* (A_2) *with $q = s+4$, there exists a constant $C = C(M,c,s) > 0$ independent of δ such that we have*

$$\|\Lambda^{\mathrm{NN}}\beta + \beta + \delta^2\nabla\cdot((1+\eta-b)(\nabla\eta)\beta + \tfrac{1}{2}(1+\eta-b)^2\nabla\beta)\|_s \leqslant C\delta^4\|\beta\|_{s+4}.$$

Proof. Set $\Phi := (0,\beta)^\hbar$ and $\tilde{\Phi} := \Phi \circ \Theta$. It follows from (4.15) that

$$\begin{aligned}\partial_{n+1}\tilde{\Phi} &+ \delta^2(1+\partial_{n+1}\theta_{n+1})\beta \\ &= -\delta^2(1+\partial_{n+1}\theta_{n+1}) \\ &\quad\times\Big\{p_{22}\partial_{n+1}\tilde{\Phi} + \boldsymbol{p}_{12}\cdot\nabla\tilde{\Phi} \\ &\qquad + \nabla\cdot\int_0^{x_{n+1}} ((1+\partial_{n+1}\theta_{n+1})\nabla\tilde{\Phi} + \boldsymbol{p}_{12}\partial_{n+1}\tilde{\Phi} + \delta^2 P_{11}\nabla\tilde{\Phi})\,\mathrm{d}x_{n+1}\Big\},\end{aligned} \quad (5.3)$$

so that

$$\|J^{s+2}(\partial_{n+1}\tilde{\Phi} + \delta^2(1+\partial_{n+1}\theta_{n+1})\beta)\|_{L^2(\Omega_0)} \leqslant C\delta^2\|J^{s+3}\nabla_X\tilde{\Phi}\|_{L^2(\Omega_0)}.$$

In view of the relation

$$\tilde{\Phi} + \delta^2(x_{n+1}+\theta_{n+1}-1-\eta)\beta = \int_1^{x_{n+1}} (\partial_{n+1}\tilde{\Phi} + \delta^2(1+\partial_{n+1}\theta_{n+1})\beta)\,\mathrm{d}x_{n+1}, \quad (5.4)$$

we obtain

$$\|J^{s+1}\nabla(\tilde{\Phi} + \delta^2(x_{n+1}+\theta_{n+1}-1-\eta)\beta)\|_{L^2(\Omega_0)} \leqslant C\delta^2\|J^{s+3}\nabla_X\tilde{\Phi}\|_{L^2(\Omega_0)}.$$

Therefore, by corollary 4.18 we obtain

$$\|J^{s+1}\nabla_X(\tilde{\Phi} + \delta^2(x_{n+1}+\theta_{n+1}-1-\eta)\beta)\|_{L^2(\Omega_0)} \leqslant C\delta^4\|\beta\|_{s+4}. \quad (5.5)$$

On the other hand, by (4.1) and (4.13) we see that

$$\begin{aligned}\int_0^1 \{&(1+\partial_{n+1}\theta_{n+1})\nabla((1+\eta-x_{n+1}-\theta_{n+1})\beta) - \boldsymbol{p}_{12}^0(1+\partial_{n+1}\theta_{n+1})\beta\}\,\mathrm{d}x_{n+1} \\ &= \int_0^1 \partial_{n+1}\{(x_{n+1}+\theta_{n+1})\nabla((1+\eta)\beta) \\ &\qquad - \tfrac{1}{2}(x_{n+1}+\theta_{n+1})^2\nabla\beta + (\theta_1,\dots,\theta_n)^{\mathrm{T}}\beta\}\,\mathrm{d}x_{n+1} \\ &= (1+\eta-b)(\nabla\eta)\beta + \tfrac{1}{2}(1+\eta-b)^2\nabla\beta.\end{aligned}$$

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Therefore, we can rewrite (5.2) as

$$\begin{aligned}\Lambda^{\mathrm{NN}}\beta = -\beta - \delta^2\nabla\cdot((1+\eta-b)(\nabla\eta)\beta + \tfrac{1}{2}(1+\eta-b)^2\nabla\beta) \\ - \delta^2\nabla\cdot\int_0^1(\tilde{\boldsymbol{p}}_{12}\partial_{n+1}\tilde{\Phi} + P_{11}\nabla\tilde{\Phi})\,\mathrm{d}x_{n+1} \\ - \nabla\cdot\int_0^1\{(1+\partial_{n+1}\theta_{n+1})\nabla(\tilde{\Phi}+\delta^2(x_{n+1}+\theta_{n+1}-1-\eta)\beta) \\ + \boldsymbol{p}^0_{12}\partial_{n+1}(\tilde{\Phi}+\delta^2(x_{n+1}+\theta_{n+1}-1-\eta)\beta)\}\,\mathrm{d}x_{n+1}.\end{aligned} \tag{5.6}$$

This, together with corollary 4.18 and (5.5), gives the desired estimate. □

Next, we will give estimates of Fréchet derivatives. The following two propositions on the DN map $\Lambda^{\mathrm{DN}} = \Lambda^{\mathrm{DN}}(\eta, b, \delta)$ were given in [11].

Proposition 5.17. *Let $s > \frac{1}{2}n$ and $m \in \mathbb{N}$. Under assumptions* (A_1), (A_2) *and* (A_4) *with $q = s+1$, there exists a constant $C = C(M, c, s, m) > 0$ independent of δ such that we have*

$$\|D^m_\eta\Lambda^{\mathrm{DN}}[\check{\eta}_1,\dots,\check{\eta}_m]\phi\|_s \leqslant C\|\check{\eta}_1\|_{s+3/2}\cdots\|\check{\eta}_m\|_{s+3/2}\|(\Lambda^{\mathrm{DN}}_0)^{1/2}\phi\|_{s+1}.$$

A similar estimate holds for the Fréchet derivative of Λ^{DN} with respect to b.

Proposition 5.18. *Let $s > \frac{1}{2}(n+1)$ and $m \in \mathbb{N}$. Under assumptions* (A_1), (A_2) *and* (A_4) *with $q = s+\frac{1}{2}$, there exists a constant $C = C(M, c, s, m) > 0$ independent of δ such that we have*

$$\|D^m_\eta\Lambda^{\mathrm{DN}}[\check{\eta}_1,\dots,\check{\eta}_m]\phi\|_s \leqslant C\delta^{-1/2}\|\check{\eta}_1\|_{s+1}\cdots\|\check{\eta}_m\|_{s+1}\|(\Lambda^{\mathrm{DN}}_0)^{1/2}\phi\|_{s+1/2}.$$

A similar estimate holds for the Fréchet derivative of Λ^{DN} with respect to b.

In the next proposition we will modify the estimate in proposition 5.17. Specifically, we improve the norm of $(\check{\eta}_1,\dots,\check{\eta}_m)$ and the hypothesis on the regularity of the water surface and the bottom.

Proposition 5.19. *Let $s > \frac{1}{2}n+2$ and $m \in \mathbb{N}$. Under assumptions* (A_1)–(A_4) *with $q = s+\frac{1}{2}$ and $\|\eta\|_{s+1}+\|b\|_{s+3/2} \leqslant M$, there exists a constant $C = C(M, c, s, m) > 0$ independent of δ such that we have*

$$\|D^m_\eta\Lambda^{\mathrm{DN}}[\check{\eta}_1,\dots,\check{\eta}_m]\phi\|_s \leqslant C\|\check{\eta}_1\|_{s+1}\cdots\|\check{\eta}_m\|_{s+1}\|(\Lambda^{\mathrm{DN}}_0)^{1/2}\phi\|_{s+1}.$$

A similar estimate holds for the Fréchet derivative of Λ^{DN} with respect to b.

Proof. For simplicity, we only show the estimate in the case $m = 1$. It is sufficient to evaluate $\|D_\eta\Lambda^{\mathrm{DN}}[\check{\eta}]\phi\|_{s-1}$ and $\|\nabla(D_\eta\Lambda^{\mathrm{DN}}[\check{\eta}]\phi)\|_{s-1}$. By proposition 5.17 we have

$$\|D_\eta\Lambda^{\mathrm{DN}}[\check{\eta}]\phi\|_{s-1} \leqslant C\|\check{\eta}\|_{s+1/2}\|(\Lambda^{\mathrm{DN}}_0)^{1/2}\|_s.$$

Let T^j_h be a translation operator with respect to the jth spatial variable, that is,

$$(T^j_h u)(x) = u(x_1,\dots,x_{j-1},x_j+h,x_{j+1},\dots,x_n).$$

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Then, it is easy to see that $T_h^j \Lambda^{\rm DN}(\eta,b,\delta) = \Lambda^{\rm DN}(T_h^j\eta, T_h^j b,\delta)T_h^j$ and that

$$\partial_j\Lambda^{\rm DN}\phi = \Lambda^{\rm DN}\partial_j\phi + D_\eta\Lambda^{\rm DN}[\partial_j\eta]\phi + D_b\Lambda^{\rm DN}[\partial_j b]\phi.$$

Therefore, we see that

$$\begin{aligned}\partial_j(D_\eta\Lambda^{\rm DN}[\check\eta]\phi) &= D_\eta(\partial_j\Lambda^{\rm DN}\phi)[\check\eta]\\ &= D_\eta(D_\eta\Lambda^{\rm DN}[\partial_j\eta]\phi)[\check\eta] + D_\eta D_b\Lambda^{\rm DN}[\check\eta,\partial_j b]\phi + D_\eta\Lambda^{\rm DN}[\check\eta]\partial_j\phi.\end{aligned}$$

Here, by proposition 5.17 we have

$$\begin{aligned}\|D_\eta D_b\Lambda^{\rm DN}[\check\eta,\partial_j b]\phi\|_{s-1} &\leqslant C\|\check\eta\|_{s+1/2}\|\partial_j b\|_{s+1/2}\|(\Lambda_0^{\rm DN})^{1/2}\phi\|_s,\\ \|D_\eta\Lambda^{\rm DN}[\check\eta]\partial_j\phi\|_{s-1} &\leqslant C\delta^2\|\check\eta\|_{s+1/2}\|\partial_j(\Lambda_0^{\rm DN})^{1/2}\phi\|_s.\end{aligned}$$

It follows from theorem 3.5 that

$$\begin{aligned}D_\eta\Lambda^{\rm DN}[\partial_j\eta]\phi &= -\delta^2\Lambda^{\rm DN}((1+\delta^2|\nabla\eta|^2)^{-1}(\Lambda^{\rm DN}\phi+\nabla\eta\cdot\nabla\phi)(\partial_j\eta))\\ &\quad - \nabla\cdot\{(\nabla\phi-\delta^2(1+\delta^2|\nabla\eta|^2)^{-1}(\Lambda^{\rm DN}\phi+\nabla\eta\cdot\nabla\phi)\nabla\eta)(\partial_j\eta)\},\end{aligned}$$

so that

$$\begin{aligned}&D_\eta(D_\eta\Lambda^{\rm DN}[\partial_j\eta]\phi)[\check\eta]\\ &\quad= -\delta^2 D_\eta\Lambda^{\rm DN}[\check\eta]((1+\delta^2|\nabla\eta|^2)^{-1}(\Lambda^{\rm DN}\phi+\nabla\eta\cdot\nabla\phi)(\partial_j\eta))\\ &\qquad - \delta^2\Lambda^{\rm DN}\{(1+\delta^2|\nabla\eta|^2)^{-1}(\Lambda^{\rm DN}\phi+\nabla\eta\cdot\nabla\phi)(\partial_j\check\eta)\\ &\qquad\qquad + (1+\delta^2|\nabla\eta|^2)^{-1}(D_\eta\Lambda^{\rm DN}[\check\eta]\phi+\nabla\check\eta\cdot\nabla\phi)(\partial_j\eta)\\ &\qquad\qquad - 2\delta^2(1+\delta^2|\nabla\eta|^2)^{-2}(\nabla\eta\cdot\nabla\check\eta)(\Lambda^{\rm DN}\phi+\nabla\eta\cdot\nabla\phi)(\partial_j\eta)\}\\ &\qquad - \nabla\cdot\{(\nabla\phi-\delta^2(1+\delta^2|\nabla\eta|^2)^{-1}(\Lambda^{\rm DN}\phi+\nabla\eta\cdot\nabla\phi)\nabla\eta)(\partial_j\check\eta)\\ &\qquad\qquad - \delta^2(1+\delta^2|\nabla\eta|^2)^{-1}(D_\eta\Lambda^{\rm DN}[\check\eta]\phi+\nabla\check\eta\cdot\nabla\phi)(\partial_j\eta)\nabla\eta\\ &\qquad\qquad - \delta^2(1+\delta^2|\nabla\eta|^2)^{-1}(\Lambda^{\rm DN}\phi+\nabla\eta\cdot\nabla\phi)(\partial_j\eta)\nabla\check\eta\\ &\qquad\qquad + 2\delta^4(1+\delta^2|\nabla\eta|^2)^{-2}(\nabla\eta\cdot\nabla\check\eta)(\Lambda^{\rm DN}\phi+\nabla\eta\cdot\nabla\phi)(\partial_j\eta)\nabla\eta\}.\end{aligned}$$

Therefore, by propositions 5.18, 5.1, and 5.2 and lemma 4.6, we see that

$$\begin{aligned}\|D_\eta(D_\eta\Lambda^{\rm DN}[\partial_j\eta]\phi)[\check\eta]\|_{s-1} &\leqslant C\{\|\check\eta\|_{s+1}(\delta\|\Lambda^{\rm DN}\phi\|_s+\|\nabla\phi\|_s)+\delta\|D_\eta\Lambda^{\rm DN}[\check\eta]\phi\|_s\}\\ &\leqslant C\|\check\eta\|_{s+1}\|(\Lambda_0^{\rm DN})^{1/2}\phi\|_{s+1/2},\end{aligned}$$

so that we obtain $\|\nabla(D_\eta\Lambda^{\rm DN}[\check\eta]\phi)\|_{s-1}\leqslant C\|\check\eta\|_{s+1}\|(\Lambda_0^{\rm DN})^{1/2}\phi\|_{s+1}$, where we used propositions 5.17 and 5.1 and lemma 4.6. Hence, we obtain the desired estimate. □

We proceed to give estimates of the Fréchet derivatives of the NN map $\Lambda^{\rm NN} = \Lambda^{\rm NN}(\eta,b,\delta)$.

PROPOSITION 5.20. *Let* $s>\frac12(n+1)$ *and* $m\in\mathbb{N}$. *Under assumptions* (A$_1$)–(A$_4$) *with* $q=s+1$, *there exists a constant* $C=C(M,c,s,m)>0$ *independent of* δ *such that we have*

$$\|D_\eta^m\Lambda^{\rm NN}[\check\eta_1,\dots,\check\eta_m]\beta\|_s \leqslant C\delta^{1/2}\|\check\eta_1\|_{s+1}\cdots\|\check\eta_m\|_{s+1}\|\beta\|_{s+1/2}.$$

A similar estimate holds for the Fréchet derivative of $\Lambda^{\rm NN}$ *with respect to* b.

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Proof. We only show the estimate in the case $m = 1$, and the general case can be proved in the same way. Set $\Phi := (0, \beta)^\hbar$ and $\tilde{\Phi} := \Phi \circ \Theta$. Then, $\tilde{\Phi}$ satisfies (4.5) with $\phi = 0$ and $\delta^{-2}\partial_{n+1}\tilde{\Phi}(\cdot, 1) = \Lambda^{\mathrm{NN}}\beta$. For simplicity, we write $\Lambda_\eta^{\mathrm{NN}}\beta = D_\eta \Lambda^{\mathrm{NN}}[\check{\eta}]\beta$, $\tilde{\Phi}_\eta = D_\eta \tilde{\Phi}[\check{\eta}]$ and $P_\eta = D_\eta P[\check{\eta}]$. Taking the Fréchet derivative of (4.5) with respect to η, we obtain

$$\begin{aligned} \nabla_X \cdot I_\delta P I_\delta \nabla_X \tilde{\Phi}_\eta &= -\nabla_X \cdot I_\delta P_\eta I_\delta \nabla_X \tilde{\Phi} && \text{in } \Omega_0, \\ \tilde{\Phi}_\eta = 0, \quad \delta^{-2}\partial_{n+1}\tilde{\Phi}_\eta &= \Lambda_\eta^{\mathrm{NN}}\beta && \text{on } \Gamma_0, \\ -\delta^{-2}\partial_{n+1}\tilde{\Phi}_\eta &= 0 && \text{on } \Sigma_0. \end{aligned}$$

Therefore, by lemmas 4.14 and 4.6, we obtain

$$\begin{aligned} \|\Lambda_\eta^{\mathrm{NN}}\beta\|_s &\leqslant C(\|J^s(\Lambda_0^{\mathrm{DN}})^{1/2} P_\eta I_\delta \nabla_X \tilde{\Phi}\|_{L^2(\Omega_0)} + \|P_\eta I_\delta \nabla_X \tilde{\Phi}\|_{L^2(\Omega_0)}) \\ &\leqslant C\delta^{-1/2}\|J^{s+1/2} P_\eta I_\delta \nabla_X \tilde{\Phi}\|_{L^2(\Omega_0)}. \end{aligned} \tag{5.7}$$

Here, as in the proof of proposition 4.22, we have

$$\|J^{s+1/2} P_\eta I_\delta \nabla_X \tilde{\Phi}\|_{L^2(\Omega_0)} \leqslant C\delta \|\check{\eta}\|_{s+1}\|\beta\|_{s+1/2}.$$

These estimates give the desired estimate. □

As a corollary of this proposition, we can obtain the estimate for the NN map Λ^{NN} in proposition 5.11 under a weaker hypothesis on the water surface and the bottom.

Corollary 5.21. *Let $s > \frac{1}{2}(n+3)$. In addition to assumptions* (A_1)–(A_4) *with $q = s$, we assume that $\|(\eta, b)\|_{s+1} \leqslant M$. Then, there exists a constant $C = C(M, c, s) > 0$ independent of δ such that we have $\|\Lambda^{\mathrm{NN}}\beta\|_s \leqslant C\|\beta\|_s$.*

Proof. It is sufficient to evaluate $\|\Lambda^{\mathrm{NN}}\beta\|_{s-1}$ and $\|\nabla\Lambda^{\mathrm{NN}}\beta\|_{s-1}$. We have from proposition 5.11 that $\|\Lambda^{\mathrm{NN}}\beta\|_{s-1} \leqslant C\|\beta\|_{s-1}$. Let T_h^j be a translation operator with respect to the jth spatial variable. Then, it is easy to see that

$$T_h^j \Lambda^{\mathrm{NN}}(\eta, b, \delta) = \Lambda^{\mathrm{NN}}(T_h^j\eta, T_h^j b, \delta)T_h^j$$

and that

$$\partial_j \Lambda^{\mathrm{NN}}\beta = \Lambda^{\mathrm{NN}}\partial_j\beta + D_\eta\Lambda^{\mathrm{NN}}[\partial_j\eta]\beta + D_b\Lambda^{\mathrm{NN}}[\partial_j b]\beta. \tag{5.8}$$

Hence, by propositions 5.11 and 5.20 we get

$$\|\nabla\Lambda^{\mathrm{NN}}\beta\|_{s-1} \leqslant C(\|\nabla\beta\|_{s-1} + \|(\nabla\eta, \nabla b)\|_s\|\beta\|_{s-1/2}) \leqslant C\|\beta\|_s.$$

Therefore, we obtain the desired estimate. □

Proposition 5.22. *Let $s > \frac{1}{2}n$ and $m \in \mathbb{N}$. Under assumptions* (A_1)–(A_4) *with $q = s+1$, there exists a constant $C = C(M, c, s, m) > 0$ independent of δ such that we have*

$$\|D_\eta^m \Lambda^{\mathrm{NN}}[\check{\eta}_1, \dots, \check{\eta}_m]\beta\|_s \leqslant C\delta\|\check{\eta}_1\|_{s+3/2}\cdots\|\check{\eta}_m\|_{s+3/2}\|\beta\|_{s+1}.$$

A similar estimate holds for the Fréchet derivative of Λ^{NN} with respect to b.

Proof. For simplicity, we only show the estimate in the case $m = 1$ and use the same notation as in the proof of proposition 5.20. It follows from (5.7) and lemma 4.6 that

$$\begin{aligned}\|\Lambda_\eta^{\mathrm{NN}}\beta\|_s &\leqslant C\|J^{s+1}P_\eta I_\delta \nabla_X \tilde{\Phi}\|_{L^2(\Omega_0)} \\ &\leqslant C(\|J^{s+1}P_\eta\|_{L^2(\Omega_0)}\|I_\delta \nabla_X \tilde{\Phi}\|_{L^\infty(\Omega_0)} \\ &\qquad + \|P_\eta\|_{L^\infty(\Omega_0)}\|J^{s+1}I_\delta \nabla_X \tilde{\Phi}\|_{L^2(\Omega_0)}).\end{aligned}$$

This, together with propositions 4.16 and 4.20, gives the desired estimate. □

PROPOSITION 5.23. *Let $s > \frac{1}{2}n - 1$ and $m \in \mathbb{N}$. Under assumptions* (A_1)–(A_4) *with $q = s + 2$, there exists a constant $C = C(M, c, s, m) > 0$ independent of δ such that we have*

$$\|D_\eta^m \Lambda^{\mathrm{NN}}[\check{\eta}_1, \dots, \check{\eta}_m]\beta\|_s \leqslant C\delta^2\|\check{\eta}_1\|_{s+5/2} \cdots \|\check{\eta}_m\|_{s+5/2}\|\beta\|_{s+2}.$$

A similar estimate holds for the Fréchet derivative of Λ^{NN} with respect to b.

Proof. For simplicity, we only show the estimate in the case where $m = 1$ and use the same notation as in the proof of proposition 5.20. By taking the Fréchet derivative of (5.2), we see that

$$\begin{aligned}&\|D_\eta \Lambda^{\mathrm{NN}}[\check{\eta}]\beta\|_s \\ &\quad \leqslant C(\|J^{s+1}\nabla_X \tilde{\Phi}_\eta\|_{L^2(\Omega_0)} + \|\nabla_X(D_\eta\theta[\check{\eta}])\|_{L^\infty(\Omega_0)}\|J^{s+1}\nabla_X\tilde{\Phi}\|_{L^2(\Omega_0)}) \\ &\quad \leqslant C\delta^2\|\check{\eta}\|_{s+5/2}\|\beta\|_{s+2},\end{aligned}$$

where we used corollaries 4.18 and 4.23. The proof is complete. □

In the next proposition we will modify the estimate in the above proposition. Specifically, we improve the norm of $(\check{\eta}_1, \dots, \check{\eta}_m)$ and the hypothesis on the regularity of the water surface and the bottom.

PROPOSITION 5.24. *Let $s > \frac{1}{2}n + 2$ and $m \in \mathbb{N}$. Under assumptions* (A_1)–(A_4) *with $q = s + 1$ and $\|\eta\|_{s+2} + \|b\|_{s+5/2} \leqslant M$, there exists a constant $C = C(M, c, s, m) > 0$ independent of δ such that we have*

$$\|D_\eta^m \Lambda^{\mathrm{NN}}[\check{\eta}_1, \dots, \check{\eta}_m]\beta\|_s \leqslant C\delta^2\|\check{\eta}_1\|_{s+2} \cdots \|\check{\eta}_m\|_{s+2}\|\beta\|_{s+2}.$$

A similar estimate holds for the Fréchet derivative of Λ^{NN} with respect to b.

Proof. For simplicity, we only show the estimate in the case $m = 1$. It is sufficient to evaluate $\|D_\eta \Lambda^{\mathrm{NN}}[\check{\eta}]\beta\|_{s-1}$ and $\|\nabla(D_\eta \Lambda^{\mathrm{NN}}[\check{\eta}]\beta)\|_{s-1}$. By proposition 5.23 we have

$$\|D_\eta \Lambda^{\mathrm{NN}}[\check{\eta}]\beta\|_{s-1} \leqslant C\delta^2\|\check{\eta}\|_{s+3/2}\|\beta\|_{s+1}.$$

In view of (5.8), we see that

$$\begin{aligned}\partial_j(D_\eta \Lambda^{\mathrm{NN}}[\check{\eta}]\beta) &= D_\eta(\partial_j \Lambda^{\mathrm{NN}}\beta)[\check{\eta}] \\ &= D_\eta(D_\eta \Lambda^{\mathrm{NN}}[\partial_j\eta]\beta)[\check{\eta}] + D_\eta D_b \Lambda^{\mathrm{NN}}[\check{\eta}, \partial_j b]\beta + D_\eta \Lambda^{\mathrm{NN}}[\check{\eta}]\partial_j\beta.\end{aligned}$$

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Here, by proposition 5.23, we have

$$\|D_\eta D_b \Lambda^{\rm NN}[\check\eta, \partial_j b]\beta\|_{s-1} \leqslant C\delta^2 \|\check\eta\|_{s+3/2} \|\partial_j b\|_{s+3/2} \|\beta\|_{s+1}$$

and $\|D_\eta \Lambda^{\rm NN}[\check\eta]\partial_j\beta\|_{s-1} \leqslant C\delta^2\|\check\eta\|_{s+3/2}\|\partial_j\beta\|_{s+1}$. It follows from theorem 3.5 that

$$\begin{aligned} D_\eta \Lambda^{\rm NN}[\partial_j\eta]\beta = -\delta^2\Lambda^{\rm DN}((1+\delta^2|\nabla\eta|^2)^{-1}(\partial_j\eta)\Lambda^{\rm NN}\beta) \\ + \delta^2\nabla\cdot((1+\delta^2|\nabla\eta|^2)^{-1}(\partial_j\eta)(\Lambda^{\rm NN}\beta)\nabla\eta), \end{aligned}$$

so that

$$\begin{aligned} &D_\eta(D_\eta\Lambda^{\rm NN}[\partial_j\eta]\beta)[\check\eta] \\ &\quad = -\delta^2 D_\eta\Lambda^{\rm DN}[\check\eta]((1+\delta^2|\nabla\eta|^2)^{-1}(\partial_j\eta)\Lambda^{\rm NN}\beta) \\ &\qquad - \delta^2\Lambda^{\rm DN}\{(1+\delta^2|\nabla\eta|^2)^{-1}((\partial_j\check\eta)\Lambda^{\rm NN}\beta + (\partial_j\eta)D_\eta\Lambda^{\rm NN}[\check\eta]\beta) \\ &\qquad\qquad - 2\delta^2(1+\delta^2|\nabla\eta|^2)^{-2}(\nabla\eta\cdot\nabla\check\eta)(\partial_j\eta)\Lambda^{\rm NN}\beta\} \\ &\qquad + \delta^2\nabla\cdot\{(1+\delta^2|\nabla\eta|^2)^{-1}((\partial_j\check\eta)\Lambda^{\rm NN}\beta + (\partial_j\eta)D_\eta\Lambda^{\rm NN}[\check\eta]\beta)\nabla\eta \\ &\qquad\qquad - 2\delta^2(1+\delta^2|\nabla\eta|^2)^{-2}(\nabla\eta\cdot\nabla\check\eta)(\partial_j\eta)(\Lambda^{\rm NN}\beta)\nabla\eta \\ &\qquad\qquad + (1+\delta^2|\nabla\eta|^2)^{-1}(\partial_j\eta)(\Lambda^{\rm NN}\beta)\nabla\check\eta\}. \end{aligned}$$

Therefore, by propositions 5.17, 5.1 and 5.22, lemma 4.6 and corollary 5.21, we see that

$$\begin{aligned} \|D_\eta(D_\eta\Lambda^{\rm NN}[\partial_j\eta]\beta)[\check\eta]\|_{s-1} &\leqslant C(\delta^2\|\check\eta\|_{s+2}\|\Lambda^{\rm NN}\beta\|_{s+1} + \delta\|D_\eta\Lambda^{\rm NN}[\check\eta]\beta\|_s) \\ &\leqslant C\delta^2\|\check\eta\|_{s+2}\|\beta\|_{s+1}, \end{aligned}$$

so that we get $\|\nabla(D_\eta\Lambda^{\rm NN}[\check\eta]\beta)\|_{s-1} \leqslant C\delta^2\|\check\eta\|_{s+2}\|\beta\|_{s+2}$. Hence, we obtain the desired estimate. □

As a corollary of this proposition, we can obtain the estimate for the NN map $\Lambda^{\rm NN}$ in proposition 5.15 under a weaker hypothesis on the water surface and the bottom.

Corollary 5.25. *Let* $s > \frac{1}{2}n + 3$. *In addition to assumptions* (A_1)–(A_4) *with* $q = s+1$, *we assume that* $\|(\eta, b)\|_{s+2} \leqslant M$. *Then, there exists a constant* $C = C(M, c, s) > 0$ *independent of* δ *such that we have* $\|\Lambda^{\rm NN}\beta + \beta\|_s \leqslant C\delta^2\|\beta\|_{s+2}$.

Proof. It is sufficient to evaluate $\|\Lambda^{\rm NN}\beta+\beta\|_{s-1}$ and $\|\nabla(\Lambda^{\rm NN}\beta+\beta)\|_{s-1}$. By proposition 5.15 we have $\|\Lambda^{\rm NN}\beta + \beta\|_{s-1} \leqslant C\delta^2\|\beta\|_{s+1}$. Moreover, by (5.8) and propositions 5.15 and 5.24, we get

$$\begin{aligned} &\|\partial_j(\Lambda^{\rm NN}\beta+\beta)\|_{s-1} \\ &\qquad \leqslant \|\Lambda^{\rm NN}\partial_j\beta + \partial_j\beta\|_{s-1} + \|D_\eta\Lambda^{\rm NN}[\partial_j\eta]\beta\|_{s-1} + \|D_b\Lambda^{\rm NN}[\partial_j b]\beta\|_{s-1} \\ &\qquad \leqslant C\delta^2(\|\partial_j\beta\|_{s+1} + \|(\partial_j\eta, \partial_j b)\|_{s+1}\|\beta\|_{s+1}). \end{aligned}$$

Therefore, we obtain the desired estimate. □

We end this section by giving expansions of Fréchet derivatives of the maps $\Lambda^{\rm DN}$ and $\Lambda^{\rm NN}$ with estimates of error terms.

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Proposition 5.26. *Let $s > \frac{1}{2}n - 1$. Under assumptions* (A_1)–(A_4) *with* $q = s + 3$, *there exists a constant* $C = C(M, c, s) > 0$ *independent of* δ *such that we have*

$$\|D_\eta \Lambda^{\mathrm{DN}}[\check{\eta}]\phi + D_b \Lambda^{\mathrm{DN}}[\check{b}]\phi + \nabla \cdot ((\check{\eta} - \check{b})\nabla\phi)\|_s \leqslant C\delta^2 \|(\check{\eta}, \check{b})\|_{s+7/2} \|(\Lambda_0^{\mathrm{DN}})^{1/2}\phi\|_{s+3}.$$

Proof. We only show the estimate for $D_\eta \Lambda^{\mathrm{DN}}$. The estimate for $D_b \Lambda^{\mathrm{DN}}$ can be proved in the same way. Set $\Phi := (\phi, 0)^\hbar$ and $\tilde{\Phi} := \Phi \circ \Theta$. Then, we have (4.5) with $\beta = 0$ and, in place of (5.2),

$$\begin{aligned} &\Lambda^{\mathrm{DN}}\phi + \nabla \cdot ((1 + \eta - b)\nabla\phi) \\ &= \nabla \cdot \int_0^1 \left\{ (1 + \partial_{n+1}\theta_{n+1})\nabla \int_{x_{n+1}}^1 \partial_{n+1}\tilde{\Phi}(\cdot, z)\,\mathrm{d}z - \boldsymbol{p}_{12}\partial_{n+1}\tilde{\Phi} - \delta^2 P_{11}\nabla\tilde{\Phi} \right\} \mathrm{d}x_{n+1}. \end{aligned}$$

For simplicity, we write $\tilde{\Phi}_\eta = D_\eta \tilde{\Phi}[\check{\eta}]$. Taking the Fréchet derivative of the above equation with respect to η, we obtain

$$\begin{aligned} &\|D_\eta \Lambda^{\mathrm{DN}}[\check{\eta}]\phi + \nabla \cdot (\check{\eta}\nabla\phi)\|_s \\ &\qquad \leqslant C(\|J^{s+2}\partial_{n+1}\tilde{\Phi}_\eta\|_{L^2(\Omega_0)} + \delta^2 \|J^{s+1}\nabla\tilde{\Phi}_\eta\|_{L^2(\Omega_0)}) \\ &\qquad\quad + C\|\check{\eta}\|_{s+2}(\|J^{s+2}\partial_{n+1}\tilde{\Phi}\|_{L^2(\Omega_0)} + \delta^2 \|J^{s+1}\nabla\tilde{\Phi}\|_{L^2(\Omega_0)}) \\ &\qquad \leqslant C\delta^2 \|\check{\eta}\|_{s+7/2} \|(\Lambda_0^{\mathrm{DN}})^{1/2}\phi\|_{s+3}, \end{aligned}$$

where we used corollaries 4.19 and 4.24. The proof is complete. □

Proposition 5.27. *Let $s > \frac{1}{2}n - 1$. Under assumptions* (A_1)–(A_4) *with* $q = s + 4$, *there exists a constant* $C = C(M, c, s) > 0$ *independent of* δ *such that we have*

$$\begin{aligned} &\|D_\eta \Lambda^{\mathrm{NN}}[\check{\eta}]\beta + D_b \Lambda^{\mathrm{NN}}[\check{b}]\beta \\ &\quad + \delta^2 \nabla \cdot ((1 + \eta - b)(\nabla\check{\eta})\beta + (\check{\eta} - \check{b})(\nabla\eta)\beta + (1 + \eta - b)(\check{\eta} - \check{b})\nabla\beta)\|_s \\ &\qquad \leqslant C\delta^4 \|(\check{\eta}, \check{b})\|_{s+9/2} \|\beta\|_{s+4}. \end{aligned}$$

Proof. We only show the estimate for $D_\eta \Lambda^{\mathrm{NN}}$. The estimate for $D_b \Lambda^{\mathrm{NN}}$ can be proved in the same way. We set $\Phi := (0, \beta)^\hbar$, $\tilde{\Phi} := \Phi \circ \Theta$ and $\tilde{\Phi}_\eta := D_\eta \tilde{\Phi}[\check{\eta}]$. Taking the Fréchet derivative of (5.6) with respect to η, we obtain

$$\begin{aligned} &\|D_\eta \Lambda^{\mathrm{NN}}[\check{\eta}]\beta + \delta^2 \nabla \cdot ((1 + \eta - b)(\nabla\check{\eta})\beta + \check{\eta}(\nabla\eta)\beta + (1 + \eta - b)\check{\eta}\nabla\beta)\|_s \\ &\quad \leqslant C(\delta^2 \|J^{s+1}\nabla_X \tilde{\Phi}_\eta\|_{L^2(\Omega_0)} + \|J^{s+1}\nabla_X(\tilde{\Phi}_\eta + \delta^2(D_\eta\theta_{n+1}[\check{\eta}] - \check{\eta})\beta)\|_{L^2(\Omega_0)}) \\ &\qquad + C\|\check{\eta}\|_{s+2}(\delta^2 \|J^{s+1}\nabla_X\tilde{\Phi}\|_{L^2(\Omega_0)} \\ &\qquad\qquad + \|J^{s+1}\nabla_X(\tilde{\Phi} + \delta^2(x_{n+1} + \theta_{n+1} - 1 - \eta)\beta)\|_{L^2(\Omega_0)}). \end{aligned}$$

Here, taking the Fréchet derivative of (5.3) and (5.4) with respect to η, we see that

$$\begin{aligned} &\|J^{s+1}\nabla_X(\tilde{\Phi}_\eta + \delta^2(D_\eta\theta_{n+1}[\check{\eta}] - \check{\eta})\beta)\|_{L^2(\Omega_0)} \\ &\qquad \leqslant C\delta^2(\|J^{s+3}\nabla_X\tilde{\Phi}_\eta\|_{L^2(\Omega_0)} + \|\check{\eta}\|_{s+4}\|J^{s+3}\nabla_X\tilde{\Phi}\|_{L^2(\Omega_0)}). \end{aligned}$$

By the above estimates, (5.5) and corollaries 4.18 and 4.23, we obtain the desired estimate. □

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6. Reduction to a quasi-linear system

In this section we reduce full nonlinear equations (1.15) to a quasi-linear system of equations. Suppose that (η,ϕ) is a solution of (1.15). In view of theorem 3.5, we define Z and v by

$$\left.\begin{aligned} Z &= (1+\delta^2|\nabla\eta|^2)^{-1}(\Lambda^{\mathrm{DN}}(\eta,b,\delta)\phi - \varepsilon^{-1}\Lambda^{\mathrm{NN}}(\eta,b,\delta)\beta_\tau + \nabla\eta\cdot\nabla\phi),\\ v &= \nabla\phi - \delta^2 Z\nabla\eta. \end{aligned}\right\} \tag{6.1}$$

By the same way as in [11], we differentiate the second equation in (1.15) with respect to x_i and obtain

$$\partial_i\phi_t + \partial_i\eta + v\cdot(\nabla\partial_i\phi - \delta^2 Z\nabla\partial_i\eta) - \delta^2 Z\partial_i(\Lambda^{\mathrm{DN}}\phi - \varepsilon^{-1}\Lambda^{\mathrm{NN}}\beta_\tau) = 0.$$

Differentiating this with respect to x_j and x_k, we see that

$$\begin{aligned} &\partial_{ijk}\phi_t + \partial_{ijk}\eta\\ &\quad + v\cdot\{\nabla\partial_{ijk}\phi - \delta^2(Z\nabla\partial_{ijk}\eta + (\partial_{jk}Z)\nabla\partial_i\eta + (\partial_j Z)\nabla\partial_{ki}\eta + (\partial_k Z)\nabla\partial_{ij}\eta)\}\\ &\quad + (\partial_j v)\cdot\{\nabla\partial_{ki}\phi - \delta^2(Z\nabla\partial_{ki}\eta + (\partial_k Z)\nabla\partial_i\eta)\}\\ &\quad + (\partial_k v)\cdot\{\nabla\partial_{ij}\phi - \delta^2(Z\nabla\partial_{ij}\eta + (\partial_j Z)\nabla\partial_i\eta)\} + (\partial_{jk}v)\cdot(\nabla\partial_i\phi - \delta^2 Z\nabla\partial_i\eta)\\ &\quad - \delta^2\{(\partial_j Z)\partial_{ki}(\Lambda^{\mathrm{DN}}\phi - \varepsilon^{-1}\Lambda^{\mathrm{NN}}\beta_\tau) + (\partial_k Z)\partial_{ij}(\Lambda^{\mathrm{DN}}\phi - \varepsilon^{-1}\Lambda^{\mathrm{NN}}\beta_\tau)\\ &\qquad + (\partial_{jk}Z)\partial_i(\Lambda^{\mathrm{DN}}\phi - \varepsilon^{-1}\Lambda^{\mathrm{NN}}\beta_\tau) + Z\partial_{ijk}(\Lambda^{\mathrm{DN}}\phi - \varepsilon^{-1}\Lambda^{\mathrm{NN}}\beta_\tau)\} = 0. \end{aligned}$$

Here, by the definition (6.1) of Z and v we have $\Lambda^{\mathrm{DN}}\phi - \varepsilon^{-1}\Lambda^{\mathrm{NN}}\beta_\tau = Z - v\cdot\nabla\eta$. Therefore,

$$\begin{aligned} \partial_{ki}(\Lambda^{\mathrm{DN}}\phi - \varepsilon^{-1}\Lambda^{\mathrm{NN}}\beta_\tau) &= \partial_{ki}Z - v\cdot\nabla\partial_{ki}\eta\\ &\quad - (\partial_{ki}v)\cdot\nabla\eta - (\partial_k v)\cdot\nabla\partial_i\eta - (\partial_i v)\cdot\nabla\partial_k\eta, \end{aligned}$$

so that

$$\begin{aligned} (\partial_{ijk}\phi - \delta^2 Z\partial_{ijk}\eta)_t + v\cdot\nabla(\partial_{ijk}\phi - \delta^2 Z\partial_{ijk}\eta) + (1+\delta^2 Z_t + \delta^2 v\cdot\nabla Z)\partial_{ijk}\eta&\\ = \delta^2((\partial_j Z)(\partial_{ki}Z) + (\partial_k Z)(\partial_{ij}Z) + (\partial_i Z)(\partial_{jk}Z)) + f_3^{ijk}&, \end{aligned}$$

where

$$\begin{aligned} f_3^{ijk} &= -(\partial_j v)\cdot(\nabla\partial_{ki}\phi - \delta^2 Z\nabla\partial_{ki}\eta)\\ &\quad - (\partial_k v)\cdot(\nabla\partial_{ij}\phi - \delta^2 Z\nabla\partial_{ij}\eta) - (\partial_{jk}v)\cdot(\nabla\partial_i\phi - \delta^2 Z\nabla\partial_i\eta)\\ &\quad - \delta^2\{(\partial_j Z)((\partial_{ki}v)\cdot\nabla\eta + (\partial_i v)\cdot\nabla\partial_k\eta)\\ &\qquad + (\partial_k Z)((\partial_{ij}v)\cdot\nabla\eta + (\partial_i v)\cdot\nabla\partial_j\eta) + (\partial_{jk}Z)(\partial_i v)\cdot\nabla\eta\}. \end{aligned}$$

Now, we write $u = (\eta, b)$ and denote by Λ_n^{DN} and Λ_n^{NN} the nth Fréchet derivative of the DN map Λ^{DN} and NN map Λ^{NN} with respect to u, respectively. Then, it holds that

$$
\begin{aligned}
&\partial_{ijk}(\Lambda^{\mathrm{DN}}\phi-\varepsilon^{-1}\Lambda^{\mathrm{NN}}\beta_\tau)\\
&\quad=\Lambda^{\mathrm{DN}}\partial_{ijk}\phi-\varepsilon^{-1}\Lambda^{\mathrm{NN}}\partial_{ijk}\beta_\tau+\Lambda_1^{\mathrm{DN}}[\partial_{ijk}u]\phi-\varepsilon^{-1}\Lambda_1^{\mathrm{NN}}[\partial_{ijk}u]\beta_\tau\\
&\qquad+\Lambda_1^{\mathrm{DN}}[\partial_i u]\partial_{jk}\phi+\Lambda_1^{\mathrm{DN}}[\partial_j u]\partial_{ki}\phi+\Lambda_1^{\mathrm{DN}}[\partial_k u]\partial_{ij}\phi\\
&\qquad-\varepsilon^{-1}(\Lambda_1^{\mathrm{NN}}[\partial_{ij}u]\partial_k\beta_\tau+\Lambda_1^{\mathrm{NN}}[\partial_{jk}u]\partial_i\beta_\tau+\Lambda_1^{\mathrm{NN}}[\partial_{ki}u]\partial_j\beta_\tau)\\
&\qquad+\Lambda_2^{\mathrm{DN}}[\partial_{ij}u,\partial_k u]\phi+\Lambda_2^{\mathrm{DN}}[\partial_{jk}u,\partial_i u]\phi+\Lambda_2^{\mathrm{DN}}[\partial_{ki}u,\partial_j u]\phi\\
&\qquad-\varepsilon^{-1}(\Lambda_2^{\mathrm{NN}}[\partial_{ij}u,\partial_k u]\beta_\tau+\Lambda_2^{\mathrm{NN}}[\partial_{jk}u,\partial_i u]\beta_\tau+\Lambda_2^{\mathrm{NN}}[\partial_{ki}u,\partial_j u]\beta_\tau)+f_4^{ijk},
\end{aligned}
$$

where

$$
\begin{aligned}
f_4^{ijk}&=-\varepsilon^{-1}(\Lambda_1^{\mathrm{NN}}[\partial_i u]\partial_{jk}\beta_\tau+\Lambda_1^{\mathrm{NN}}[\partial_j u]\partial_{ki}\beta_\tau+\Lambda_1^{\mathrm{NN}}[\partial_k u]\partial_{ij}\beta_\tau)\\
&\quad+\Lambda_1^{\mathrm{DN}}[\partial_{ij}u]\partial_k\phi+\Lambda_1^{\mathrm{DN}}[\partial_{jk}u]\partial_i\phi+\Lambda_1^{\mathrm{DN}}[\partial_{ki}u]\partial_j\phi\\
&\quad+\Lambda_2^{\mathrm{DN}}[\partial_i u,\partial_j u]\partial_k\phi+\Lambda_2^{\mathrm{DN}}[\partial_j u,\partial_k u]\partial_i\phi+\Lambda_2^{\mathrm{DN}}[\partial_k u,\partial_i u]\partial_j\phi\\
&\quad-\varepsilon^{-1}(\Lambda_2^{\mathrm{NN}}[\partial_i u,\partial_j u]\partial_k\beta_\tau+\Lambda_2^{\mathrm{NN}}[\partial_j u,\partial_k u]\partial_i\beta_\tau+\Lambda_2^{\mathrm{NN}}[\partial_k u,\partial_i u]\partial_j\beta_\tau)\\
&\quad+\Lambda_3^{\mathrm{DN}}[\partial_i u,\partial_j u,\partial_k u]\phi-\varepsilon^{-1}\Lambda_3^{\mathrm{NN}}[\partial_i u,\partial_j u,\partial_k u]\beta_\tau.
\end{aligned}
$$

Here, by theorem 3.5, we obtain

$$
\begin{aligned}
\Lambda^{\mathrm{DN}}\partial_{ijk}\phi+\Lambda_1^{\mathrm{DN}}[\partial_{ijk}u]\phi-\varepsilon^{-1}\Lambda_1^{\mathrm{NN}}[\partial_{ijk}u]\beta_\tau&\\
=\Lambda^{\mathrm{DN}}(\partial_{ijk}\phi-\delta^2 Z\partial_{ijk}\eta)-\nabla\cdot(v\partial_{ijk}\eta)+f_5^{ijk},&\\
\varepsilon^{-1}\Lambda_1^{\mathrm{NN}}[\partial_{ij}u]\partial_k\beta_\tau=-\varepsilon^{-1}\delta^2\Lambda^{\mathrm{DN}}((\Lambda^{\mathrm{NN}}\partial_k\beta_\tau)\partial_{ij}\eta)+f_6^{ijk},&
\end{aligned}
$$

where

$$
\begin{aligned}
f_5^{ijk}&=D_b\Lambda^{\mathrm{DN}}[\partial_{ijk}b]\phi-\varepsilon^{-1}D_b\Lambda^{\mathrm{NN}}[\partial_{ijk}b]\beta_\tau,\\
f_6^{ijk}&=\varepsilon^{-1}\delta^4\Lambda^{\mathrm{DN}}((1+\delta^2|\nabla\eta|^2)^{-1}|\nabla\eta|^2(\partial_{ij}\eta)(\Lambda^{\mathrm{NN}}\partial_k\beta_\tau))\\
&\quad+\varepsilon^{-1}\delta^2\nabla\cdot((1+\delta^2|\nabla\eta|^2)^{-1}(\partial_{ij}\eta)(\Lambda^{\mathrm{NN}}\partial_k\beta_\tau)\nabla\eta)+\varepsilon^{-1}D_b\Lambda^{\mathrm{NN}}[\partial_{ij}b]\partial_k\beta_\tau.
\end{aligned}
$$

By theorems 3.5, 3.6, 3.9 and 3.10, we see that

$$
\Lambda_1^{\mathrm{DN}}[\partial_k u]\partial_{ij}\phi+\Lambda_2^{\mathrm{DN}}[\partial_{ij}u,\partial_k u]\phi-\varepsilon^{-1}\Lambda_2^{\mathrm{NN}}[\partial_{ij}u,\partial_k u]\beta_\tau=f_7^{ijk},
$$

where

$$
\begin{aligned}
f_7^{ijk}&=D_\eta D_b\Lambda^{\mathrm{DN}}[\partial_k\eta,\partial_{ij}b]\phi-\varepsilon^{-1}D_\eta D_b\Lambda^{\mathrm{NN}}[\partial_k\eta,\partial_{ij}b]\beta_\tau\\
&\quad+D_b^2\Lambda^{\mathrm{DN}}[\partial_{ij}b,\partial_k b]\phi-\varepsilon^{-1}D_b^2\Lambda^{\mathrm{NN}}[\partial_{ij}b,\partial_k b]\beta_\tau+\Lambda_1^{\mathrm{DN}}[\partial_k u](\partial_{ij}\phi-\delta^2 Z\partial_{ij}\eta)\\
&\quad+\delta^2\Lambda^{\mathrm{DN}}((1+\delta^2|\nabla\eta|^2)^{-1}(\partial_k\eta)(\partial_{ij}\eta)\Delta\phi)+\delta^2\nabla\cdot((\partial_{ij}\eta)Z\nabla\partial_k\eta)\\
&\quad-\delta^2\nabla\cdot((1+\delta^2|\nabla\eta|^2)^{-1}(\partial_k\eta)(\partial_{ij}\eta)(\Delta\phi)\nabla\eta)\\
&\quad+\delta^4\Lambda^{\mathrm{DN}}\{(1+\delta^2|\nabla\eta|^2)^{-1}(\partial_{ij}\eta)\\
&\qquad\qquad\times(\Lambda^{\mathrm{DN}}(Z\partial_k\eta)+Z\nabla\eta\cdot\nabla\partial_k\eta-(\partial_k\eta)\nabla\cdot(Z\nabla\eta))\}
\end{aligned}
$$

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$$
\begin{aligned}
&- \delta^4 \nabla\cdot\{(1+\delta^2|\nabla\eta|^2)^{-1}(\partial_{ij}\eta) \\
&\qquad\qquad \times (\varLambda^{\mathrm{DN}}(Z\partial_k\eta) + Z\nabla\eta\cdot\nabla\partial_k\eta - (\partial_k\eta)\nabla\cdot(Z\nabla\eta))\nabla\eta\} \\
&- \delta^2\varLambda^{\mathrm{DN}}((1+\delta^2|\nabla\eta|^2)^{-1}(\partial_{ij}\eta)(D_b\varLambda^{\mathrm{DN}}[\partial_k b]\phi - \varepsilon^{-1}D_b\varLambda^{\mathrm{NN}}[\partial_k b]\beta_\tau)) \\
&+ \delta^2\nabla\cdot((1+\delta^2|\nabla\eta|^2)^{-1}(\partial_{ij}\eta)(D_b\varLambda^{\mathrm{DN}}[\partial_k b]\phi - \varepsilon^{-1}D_b\varLambda^{\mathrm{NN}}[\partial_k b]\beta_\tau)\nabla\eta).
\end{aligned}
$$

Therefore, it holds that

$$
\begin{aligned}
\partial_{ijk}(\varLambda^{\mathrm{DN}}\phi - \varepsilon^{-1}\varLambda^{\mathrm{NN}}\beta_\tau) &= \varLambda^{\mathrm{DN}}(\partial_{ijk}\phi - \delta^2 Z\partial_{ijk}\eta) - v\cdot\nabla\partial_{ijk}\eta \\
&\quad - L^{ijk}\eta + \varepsilon^{-1}\partial_{ijk}\beta_\tau + f_1^{ijk}, \qquad (6.2)
\end{aligned}
$$

where L^{ijk} is a linear operator depending on $(\eta, b, \delta, \varepsilon^{-1}\beta_\tau)$ defined by

$$
L^{ijk}\check\eta = \varepsilon^{-1}\delta^2\varLambda^{\mathrm{DN}}((\varLambda^{\mathrm{NN}}\partial_k\beta_\tau)\partial_{ij}\check\eta + (\varLambda^{\mathrm{NN}}\partial_i\beta_\tau)\partial_{jk}\check\eta + (\varLambda^{\mathrm{NN}}\partial_j\beta_\tau)\partial_{ki}\check\eta)
$$

and

$$
\begin{aligned}
f_1^{ijk} &= f_4^{ijk} - \varepsilon^{-1}(\varLambda^{\mathrm{NN}}\partial_{ijk}\beta_\tau + \partial_{ijk}\beta_\tau) - (\nabla\cdot v)\partial_{ijk}\eta \\
&\quad + f_5^{ijk} - f_6^{ijk} - f_6^{jki} - f_6^{kij} + f_7^{ijk} + f_7^{jki} + f_7^{kij}.
\end{aligned}
$$

Hence, introducing new functions ζ_{ijk} and ψ_{ijk} by

$$
\zeta_{ijk} = \partial_{ijk}\eta, \qquad \psi_{ijk} = \partial_{ijk}\phi - \delta^2 Z\partial_{ijk}\eta, \qquad (6.3)
$$

we obtain the following quasi-linear system of equations:

$$
\left.
\begin{aligned}
\partial_t\zeta_{ijk} + v\cdot\nabla\zeta_{ijk} - \varLambda^{\mathrm{DN}}\psi_{ijk} + L^{ijk}\eta &= \varepsilon^{-1}\partial_{ijk}\beta_\tau + f_1^{ijk}, \\
\partial_t\psi_{ijk} + v\cdot\nabla\psi_{ijk} + a\zeta_{ijk} &= \varepsilon^{-1}g_2^{ijk} + f_2^{ijk},
\end{aligned}
\right\} \qquad (6.4)
$$

where a, f_2^{ijk} and g_2^{ijk} are given by

$$
a = 1 + \delta^2 Z_t + \delta^2 v\cdot\nabla Z \qquad (6.5)
$$

and

$$
\begin{aligned}
f_2^{ijk} &= f_3^{ijk} + \delta^2\{(\partial_j Z)\partial_{ki}(Z - \varepsilon^{-1}\beta_\tau) + (\partial_j(Z-\varepsilon^{-1}\beta_\tau))\varepsilon^{-1}\partial_{ki}\beta_\tau \\
&\qquad + (\partial_k Z)\partial_{ij}(Z - \varepsilon^{-1}\beta_\tau) + (\partial_k(Z-\varepsilon^{-1}\beta_\tau))\varepsilon^{-1}\partial_{ij}\beta_\tau \\
&\qquad + (\partial_i Z)\partial_{jk}(Z - \varepsilon^{-1}\beta_\tau) + (\partial_i(Z-\varepsilon^{-1}\beta_\tau))\varepsilon^{-1}\partial_{jk}\beta_\tau\}, \\
g_2^{ijk} &= \varepsilon^{-1}\delta^2((\partial_j\beta_\tau)(\partial_{ki}\beta_\tau) + (\partial_k\beta_\tau)(\partial_{ij}\beta_\tau) + (\partial_i\beta_\tau)(\partial_{jk}\beta_\tau)).
\end{aligned}
$$

REMARK 6.1. The functions Z and v in (6.1) are related to the velocity potential $\varPhi$ by $\delta^2 Z = (\partial_{n+1}\varPhi)|_{\varGamma(t)}$ and $v = (\nabla\varPhi)|_{\varGamma(t)}$, so that the function a in (6.5) can be written in terms of the pressure p in (2.17) as

$$
a = -(1+\delta^2|\nabla\eta|^2)^{-1}(\partial_{n+1}p - \delta^2\nabla\eta\cdot\nabla p)|_{\varGamma(t)}.
$$

Thus, the generalized Rayleigh–Taylor sign condition ensures the positivity of this function a.

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We proceed to give some estimates of the coefficients v and a, and the remainder terms $f_1 = (f_1^{ijk})$ and $f_2 = (f_2^{ijk})$. In the following we will use the notation $\partial\phi = (\partial_j\phi)$, $\partial^2\phi = (\partial_{ij}\phi)$, $\partial^3\phi = (\partial_{ijk}\phi)$, $\partial^3\phi - \delta^2 Z\partial^3\eta = (\partial_{ijk}\phi - \delta^2 Z\partial_{ijk}\eta)$ and

$$E := \|\eta\|_{s+3} + \|\nabla\phi\|_{s+2} + \|(\Lambda_0^{\mathrm{DN}})^{1/2}(\partial^3\phi - \delta^2 Z\partial^3\eta)\|_s,$$

and we will let $\delta_2 := \delta_1(M_1, c_1, s+1)$ be the constant occurring proposition 4.2.

LEMMA 6.2. *Let $s > \frac{1}{2}n + 3$, $M_1, c_1 > 0$ and suppose that*

$$\left.\begin{aligned} \|\eta\|_{s+2} + \|\nabla\phi\|_{s+1} \leqslant M_1, \quad \|b\|_{s+5} + \|\beta_\tau\|_{s+5} \leqslant M_1, \\ 1 + \eta(x) - b(x) \geqslant c_1 \quad \text{for } x \in \mathbb{R}^n. \end{aligned}\right\} \tag{6.6}$$

Then, there exists a constant $C = C(M_1, c_1, s) > 0$ such that, for any $\delta \in (0, \delta_2]$ and $\varepsilon \in (0, 1]$ satisfying $\varepsilon^{-1}\delta^2 \leqslant M_1$, we have

$$\|f_1\|_s \leqslant C(E + 1 + \delta^2\|Z\|_{s+2}).$$

Proof. By proposition 4.2, for any $\delta \in (0, \delta_2]$ we can construct a diffeomorphism Θ satisfying assumptions (A_1)–(A_4) with $q = s + 1$ and a constant M independent of δ. Therefore, we can directly evaluate f_5 and f_6 by propositions 5.1, 5.17 and 5.24, lemma 4.6 and corollary 5.21, so that

$$\|(f_5, f_6)\|_s \leqslant C(\|\eta\|_{s+3} + \|\nabla\phi\|_{s+1} + 1).$$

We can rewrite f_4 symbolically as

$$\begin{aligned} f_4 = -\varepsilon^{-1}(3\Lambda_1^{\mathrm{NN}}[\partial u]\partial^2\beta_\tau + 3\Lambda_2^{\mathrm{NN}}[\partial u, \partial u]\partial\beta_\tau + \Lambda_3^{\mathrm{NN}}[\partial u, \partial u, \partial u]\beta_\tau) \\ + 3\Lambda_1^{\mathrm{DN}}[\partial^2 u]\partial\phi + 3\Lambda_2^{\mathrm{DN}}[\partial u, \partial u]\partial\phi + \Lambda_3^{\mathrm{DN}}[\partial u, \partial u, \partial u]\phi, \end{aligned}$$

so that we have

$$\begin{aligned} \partial f_4 = -\varepsilon^{-1}(&3\Lambda_1^{\mathrm{NN}}[\partial u]\partial^3\beta_\tau + 3\Lambda_1^{\mathrm{NN}}[\partial^2 u]\partial^2\beta_\tau \\ &+ 6\Lambda_2^{\mathrm{NN}}[\partial u, \partial u]\partial^2\beta_\tau + 6\Lambda_2^{\mathrm{NN}}[\partial^2 u, \partial u]\partial\beta_\tau \\ &+ 4\Lambda_3^{\mathrm{NN}}[\partial u, \partial u, \partial u]\partial\beta_\tau + 3\Lambda_3^{\mathrm{NN}}[\partial^2 u, \partial u, \partial u]\beta_\tau \\ &\qquad + \Lambda_4^{\mathrm{NN}}[\partial u, \partial u, \partial u, \partial u]\beta_\tau) \\ &+ 3\Lambda_1^{\mathrm{DN}}[\partial^2 u]\partial^2\phi + 3\Lambda_1^{\mathrm{DN}}[\partial^3 u]\partial\phi + 3\Lambda_2^{\mathrm{DN}}[\partial^2 u, \partial u]\partial\phi \\ &+ \partial(3\Lambda_2^{\mathrm{DN}}[\partial u, \partial u]\partial\phi + \Lambda_3^{\mathrm{DN}}[\partial u, \partial u, \partial u]\phi). \end{aligned}$$

Therefore, by propositions 5.19 and 5.24 and lemma 4.6, we obtain

$$\|f_4\|_s \leqslant \|f_4\|_{s-1} + \|\nabla f_4\|_{s-1} \leqslant C(\|\eta\|_{s+3} + \|\nabla\phi\|_{s+2} + 1).$$

Concerning f_7, by proposition 5.19 and lemma 4.6, we see that

$$\begin{aligned} &\|\Lambda_1^{\mathrm{DN}}[\partial_k u](\partial_{ij}\phi - \delta^2 Z\partial_{ij}\eta)\|_s \\ &\quad \leqslant C\|(\Lambda_0^{\mathrm{DN}})^{1/2}(\partial_{ij}\phi - \delta^2 Z\partial_{ij}\eta)\|_{s+1} \\ &\quad \leqslant C(\|(\Lambda_0^{\mathrm{DN}})^{1/2}\nabla(\partial_{ij}\phi - \delta^2 Z\partial_{ij}\eta)\|_s + \|\nabla(\partial_{ij}\phi - \delta^2 Z\partial_{ij}\eta)\|_s) \\ &\quad \leqslant C(\|(\Lambda_0^{\mathrm{DN}})^{1/2}(\nabla\partial_{ij}\phi - \delta^2 Z\nabla\partial_{ij}\eta)\|_s + \delta^2\|(\nabla Z)\partial_{ij}\eta\|_{s+1} \\ &\qquad\qquad + \|\nabla\phi\|_{s+2} + \delta^2\|Z\partial_{ij}\eta\|_{s+1}). \end{aligned}$$

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Here, it holds that

$$\begin{aligned}\delta^2\|(\nabla Z)\partial_{ij}\eta\|_{s+1} &\leqslant C\delta^2(\|\nabla Z\|_{s+1}\|\partial_{ij}\eta\|_{s-1} + \|\nabla Z\|_{s-1}\|\partial_{ij}\eta\|_{s+1}) \\ &\leqslant C\delta^2(\|Z\|_{s+2} + \|Z\|_s\|\eta\|_{s+3}).\end{aligned}$$

Similarly, we have $\delta^2\|Z\partial_{ij}\eta\|_{s+1} \leqslant C\delta^2(\|Z\|_{s+1} + \|Z\|_s\|\eta\|_{s+3})$. Moreover, by the definition (6.1) of Z, propositions 5.1 and 5.11, lemma 4.6 and corollary 5.25, we have $\|Z-\varepsilon^{-1}\beta_\tau\|_s \leqslant C(\|\nabla\phi\|_{s+1}+1)$, which also yields that $\delta^2\|Z\|_s \leqslant C(\|\nabla\phi\|_{s+1}+1)$. Hence, we obtain

$$\|\Lambda_1^{\mathrm{DN}}[\partial_k u](\partial_{ij}\phi - \delta^2 Z\partial_{ij}\eta)\|_s \leqslant C(E + 1 + \delta^2\|Z\|_{s+2}).$$

The other terms in f_7 can be evaluated by propositions 5.2, 5.17 and 5.24 and lemma 4.6. For example, by proposition 5.2 and lemma 4.6, we have

$$\begin{aligned}&\delta^4\|\Lambda^{\mathrm{DN}}((1+\delta^2|\nabla\eta|^2)^{-1}(\partial_{ij}\eta)\Lambda^{\mathrm{DN}}(Z\partial_k\eta))\|_s \\ &\qquad\leqslant C\delta^3\|(\partial_{ij}\eta)\Lambda^{\mathrm{DN}}(Z\partial_k\eta)\|_{s+1} \\ &\qquad\leqslant C\delta^3(\|\partial_{ij}\eta\|_{s+1}\|\Lambda^{\mathrm{DN}}(Z\partial_k\eta)\|_{s-1} + \|\Lambda^{\mathrm{DN}}(Z\partial_k\eta)\|_{s+1}) \\ &\qquad\leqslant C\delta^2(\|\eta\|_{s+3}\|Z\partial_k\eta\|_s + \|Z\partial_k\eta\|_{s+2}) \\ &\qquad\leqslant C(\|\eta\|_{s+3} + \delta^2\|Z\|_{s+2}),\end{aligned}$$

and by proposition 5.17 we have

$$\begin{aligned}\|D_b\Lambda^{\mathrm{DN}}[\partial_k b]\phi\|_{s+1} &\leqslant \|D_b\Lambda^{\mathrm{DN}}[\partial_k b]\nabla\phi\|_s + \|D_b\Lambda^{\mathrm{DN}}[\nabla\partial_k b]\phi\|_s \\ &\qquad + \|D_u D_b\Lambda^{\mathrm{DN}}[\nabla u, \partial_k b]\phi\|_s + \|D_b\Lambda^{\mathrm{DN}}[\partial_k b]\phi\|_s \\ &\leqslant C(\|\eta\|_{s+3} + \|\nabla\phi\|_{s+2}).\end{aligned}$$

Hence, we obtain

$$\|f_7\|_s \leqslant C(E + 1 + \delta^2\|Z\|_{s+2}).$$

These estimates, together with corollary 5.25, give the desired estimate. □

Proposition 6.3. *Let $s > \frac{1}{2}(n+7)$, $M_1, c_1 > 0$ and suppose the conditions in (6.6) to hold. Then, there exists a constant $C = C(M_1, c_1, s) > 0$ such that, for any $\delta \in (0, \delta_2]$ and $\varepsilon \in (0,1]$ satisfying $\varepsilon^{-1}\delta^2 \leqslant M_1$, we have*

$$\begin{aligned}\|Z - \varepsilon^{-1}\beta_\tau\|_{s+2} + \delta\|(\Lambda_0^{\mathrm{DN}})^{1/2}(Z - \varepsilon^{-1}\beta_\tau)\|_{s+2} &\leqslant C(E+1), \\ \|v\|_{s+2} + \|(\Lambda_0^{\mathrm{DN}})^{1/2}v\|_{s+2} &\leqslant CE.\end{aligned}$$

Proof. Note that we have the diffeomorphism Θ satisfying assumptions (A_1)–(A_4) with $q = s+1$ and the estimate $\|Z - \varepsilon^{-1}\beta_\tau\|_s + \delta^2\|Z\|_s \leqslant C(\|\nabla\phi\|_{s+1} + 1)$. In order to evaluate higher derivatives of $Z - \varepsilon^{-1}\beta_\tau$, we will derive an expression of a derivative of Z. Differentiating the identity

$$(1+\delta^2|\nabla\eta|^2)Z = \Lambda^{\mathrm{DN}}\phi - \varepsilon^{-1}\Lambda^{\mathrm{NN}}\beta_\tau + \nabla\eta\cdot\nabla\phi$$

and using (6.2) and the definition (6.1) of v, we see that

$$\begin{aligned}(1+\delta^2|\nabla\eta|^2)\partial_{ijk}Z &= (\Lambda^{\mathrm{DN}}+\nabla\eta\cdot\nabla)(\partial_{ijk}\phi-\delta^2 Z\partial_{ijk}\eta) - L^{ijk}\eta + \varepsilon^{-1}\partial_{ijk}\beta_\tau \\ &\quad - \delta^2(2Z(\nabla\partial_i\eta\cdot\nabla\partial_{jk}\eta+\nabla\partial_j\eta\cdot\nabla\partial_{ki}\eta+\nabla\partial_k\eta\cdot\nabla\partial_{ij}\eta) \\ &\qquad + (\partial_i Z)\partial_{jk}|\nabla\eta|^2 + (\partial_j Z)\partial_{ki}|\nabla\eta|^2 + (\partial_k Z)\partial_{ij}|\nabla\eta|^2 \\ &\qquad + (\partial_{jk}Z)\partial_i|\nabla\eta|^2 + (\partial_{ki}Z)\partial_j|\nabla\eta|^2 + (\partial_{ij}Z)\partial_k|\nabla\eta|^2) \\ &\quad + \nabla\partial_i\eta\cdot\nabla\partial_{jk}\phi + \nabla\partial_j\eta\cdot\nabla\partial_{ki}\phi + \nabla\partial_k\eta\cdot\nabla\partial_{ij}\phi \\ &\quad + \nabla\partial_{jk}\eta\cdot\nabla\partial_i\phi + \nabla\partial_{ki}\eta\cdot\nabla\partial_j\phi + \nabla\partial_{ij}\eta\cdot\nabla\partial_k\phi \\ &\quad + \delta^2(\nabla\eta\cdot\nabla Z)\partial_{ijk}\eta + f_1^{ijk}.\end{aligned}$$

Therefore, by propositions 5.1 and 5.11, lemmas 4.6, 4.7 and 6.2 and an interpolation inequality, we obtain

$$\begin{aligned}\|\partial_{ijk}(Z-\varepsilon^{-1}\beta_\tau)\|_{s-1} &+ \delta\|(\Lambda_0^{\mathrm{DN}})^{1/2}\partial_{ijk}(Z-\varepsilon^{-1}\beta_\tau)\|_{s-1} \\ &\leqslant C(E+1+\delta^2\|(\Lambda_0^{\mathrm{DN}})^{1/2}Z\|_{s+1}+\delta^{1/2}\|f_1\|_{s-1/2}) \\ &\leqslant C(E+1+\delta^2\|(\Lambda_0^{\mathrm{DN}})^{1/2}Z\|_{s+1}+\delta^2\|Z\|_{s+3/2}) \\ &\leqslant \epsilon(\|(\Lambda_0^{\mathrm{DN}})^{1/2}(Z-\varepsilon^{-1}\beta_\tau)\|_{s+2}+\|Z-\varepsilon^{-1}\beta_\tau\|_{s+2})+C_\epsilon(E+1)\end{aligned}$$

for any $\epsilon > 0$. This gives the desired estimates for $Z-\varepsilon^{-1}\beta_\tau$, so that we also have $\delta^2\|Z\|_{s+2}+\delta^2\|(\Lambda_0^{\mathrm{DN}})^{1/2}Z\|_{s+2}\leqslant C(E+1)$. Since $v=\nabla\phi-\delta^2 Z\nabla\eta$, we easily obtain $\|v\|_{s+2}\leqslant CE$. Moreover, by lemma 4.7, it holds that

$$\begin{aligned}\|(\Lambda_0^{\mathrm{DN}})^{1/2}\partial^2 v\|_s &\leqslant \|(\Lambda_0^{\mathrm{DN}})^{1/2}(\partial^3\phi-\delta^2 Z\partial^3\eta)\|_s \\ &\qquad + \delta^2(\|(\Lambda_0^{\mathrm{DN}})^{1/2}((\partial^2 Z)(\partial\eta))\|_s + 2\|(\Lambda_0^{\mathrm{DN}})^{1/2}((\partial Z)(\partial^2\eta))\|_s) \\ &\leqslant E + C\delta^2(\|(\Lambda_0^{\mathrm{DN}})^{1/2}Z\|_{s+2}+\|Z\|_{s+2}+\|Z\|_s\|\eta\|_{s+3}).\end{aligned}$$

Therefore, we obtain the desired estimate for v. □

PROPOSITION 6.4. *Let $s>\frac{1}{2}(n+7)$, $M_1, c_1 > 0$ and suppose the conditions in (6.6) to hold. Then there exists a constant $C=C(M_1,c_1,s)>0$ such that, for any $\delta\in(0,\delta_2]$ and $\varepsilon\in(0,1]$ satisfying $\varepsilon^{-1}\delta^2\leqslant M_1$, we have*

$$\|f_1\|_s \leqslant C(E+1), \qquad \|(\Lambda_0^{\mathrm{DN}})^{1/2}f_2\|_s \leqslant C\varepsilon^{-1}\delta(E+1).$$

Proof. The estimate for f_1 is a direct consequence of lemma 6.2 and proposition 6.3. It follows from lemmas 4.6 and 4.7 and proposition 6.3 that $\|(\Lambda_0^{\mathrm{DN}})^{1/2}f_3\|_s\leqslant CE$. This and proposition 6.3 give the desired estimate for f_2. □

PROPOSITION 6.5. *Let $s>\frac{1}{2}(n+7)$ and $M_1,c_1>0$. In addition to the conditions in (6.6) we assume that $\|\beta_{\tau\tau}\|_{s+1}\leqslant M_1$ and $\|(\eta_t,\phi_t)\|_s\leqslant M_1\varepsilon^{-1}$. Then, there exists a constant $C=C(M_1,c_1,s)>0$ such that, for any $\delta\in(0,\delta_2]$ and $\varepsilon\in(0,1]$ satisfying $\varepsilon^{-1}\delta^2\leqslant M_1$, the function a defined by (6.5) satisfies*

$$\|a-1\|_{s-1}\leqslant C\varepsilon^{-1}, \qquad \|a-1\|_{s+1}\leqslant C(\varepsilon^{-1}(E+1)+\|(\eta_t,\phi_t)\|_{s+2}).$$

Proof. In view of proposition 6.3 we have $\|v\|_s+\delta^2\|Z\|_s\leqslant C$, $\|v\|_{s+2}\leqslant CE$ and $\delta^2\|Z\|_{s+2}\leqslant C(E+1)$. Differentiating the identity

$$(1+\delta^2|\nabla\eta|^2)Z = \Lambda^{\mathrm{DN}}\phi - \varepsilon^{-1}\Lambda^{\mathrm{NN}}\beta_\tau + \nabla\eta\cdot\nabla\phi$$

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we have

$$(1+\delta^2|\nabla\eta|^2)Z_t = -2\delta^2(\nabla\eta\cdot\nabla\eta_t)Z + \Lambda^{\mathrm{DN}}\phi_t - \varepsilon^{-2}\Lambda^{\mathrm{NN}}\beta_{\tau\tau} + \Lambda_1^{\mathrm{DN}}[u_t]\phi - \varepsilon^{-1}\Lambda_1^{\mathrm{NN}}[u_t]\beta_\tau + \nabla\eta\cdot\nabla\phi_t + \nabla\eta_t\cdot\nabla\phi. \tag{6.7}$$

Therefore, by propositions 5.1, 5.11, 5.19 and 5.23 and lemma 4.6, we see that $\delta^2\|Z_t\|_{s-1}\leqslant C\varepsilon^{-1}$ and that

$$\delta^2\|Z_t\|_{s+1}\leqslant\delta^2(\|\nabla Z_t\|_s+\|Z_t\|_s)\leqslant C(\varepsilon^{-1}(E+1)+\|(\eta_t,\phi_t)\|_{s+2}).$$

Since $a-1=\delta^2 v\cdot\nabla Z+\delta^2 Z_t$, we obtain the desired estimates. □

The next proposition ensures the positivity of the function a, namely, the generalized Rayleigh–Taylor sign condition. We let $\delta_3=\delta_1(M_1,c_1,r+4)$ be the constant occurring in proposition 4.2.

PROPOSITION 6.6. *Let $r>\frac{1}{2}n$, $M_1,c_1>0$ and suppose that*

$$\left.\begin{aligned} \|\beta_\tau\|_{r+9/2}+\|\beta_{\tau\tau}\|_{r+4}+\|\beta_{\tau\tau\tau}\|_{r+2}+\|(\eta,b)\|_{r+5}+\|\nabla\phi\|_{r+3}&\leqslant M_1,\\ \left\|\eta_t(t)-\varepsilon^{-1}\beta_\tau\left(\frac{t}{\varepsilon}\right)\right\|_{r+9/2}+\left\|\nabla\left(\phi_t(t)-\frac{1}{2}\left(\frac{\delta}{\varepsilon}\right)^2\beta_\tau\left(\frac{t}{\varepsilon}\right)^2\right)\right\|_{r+3}&\leqslant M_1,\\ \|\eta_{tt}\|_{r+5/2}+\|\nabla\phi_{tt}\|_{r+1}&\leqslant M_1\varepsilon^{-2},\\ 1+\eta(x,t)-b(x,t)&\geqslant c_1. \end{aligned}\right\} \tag{6.8}$$

Then, there exists a constant $C=C(M_1,c_1,r)>0$ such that, for any $\delta\in(0,\delta_3]$ and $\varepsilon\in(0,1]$ satisfying $\varepsilon^{-1}\delta^2\leqslant M_1$, the function a defined by (6.5) satisfies

$$\left.\begin{aligned} \left\|a(t)-\left(1+\left(\frac{\delta}{\varepsilon}\right)^2(1-\delta^2|\nabla\eta(t)|^2)\beta_{\tau\tau}\left(\frac{t}{\varepsilon}\right)+\sigma a^{(0)}\left(\frac{t}{\varepsilon}\right)\right)\right\|_r&\leqslant C\left(\varepsilon+\left|\frac{\delta^2}{\varepsilon}-\sigma\right|\right),\\ \left\|a_t(t)-\varepsilon^{-3}\delta^2\beta_{\tau\tau\tau}\left(\frac{t}{\varepsilon}\right)\right\|_r&\leqslant C\varepsilon^{-1}\quad\textit{for } 0<t<\varepsilon, \end{aligned}\right\} \tag{6.9}$$

where $a^{(0)}$ is the function defined by (2.21). Particularly, if we assume additionally assumptions 2.1 and 2.2, then there exist small constants $\varepsilon_0,\gamma_0>0$ such that we have

$$\tfrac{1}{2}c\leqslant a(x,t)\leqslant C\varepsilon^{-1},\qquad a_t(x,t)\leqslant C\varepsilon^{-1}$$

as long as $0<\varepsilon\leqslant\varepsilon_0$ and $|\delta^2/\varepsilon-\sigma|\leqslant\gamma_0$.

Proof. Note that under our hypothesis we have the diffeomorphism Θ satisfying assumptions (A_1)–(A_4) with $q=r+4$ and that we have $\partial_t^k b=\varepsilon^{-k}\partial_\tau^k\beta$ and

$$\left.\begin{aligned} \left\|\eta(t)-\eta^{(0)}\left(\frac{t}{\varepsilon}\right)\right\|_{r+9/2}+\left\|\nabla\left(\phi(t)-\phi^{(0)}\left(\frac{t}{\varepsilon}\right)\right)\right\|_{r+3}&\leqslant C\left(\varepsilon+\left|\frac{\delta^2}{\varepsilon}-\sigma\right|\right),\\ \|\eta_t(t)\|_{r+9/2}+\|\nabla\phi_t(t)\|_{r+3}&\leqslant C\varepsilon^{-1}\quad\text{for } 0<t<\varepsilon, \end{aligned}\right\} \tag{6.10}$$

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where $(\eta^{(0)}, \phi^{(0)})$ is the approximate solution defined by (2.20). By the definition of a, we have $a_t = \delta^2(Z_{tt} + v \cdot \nabla Z_t + v_t \cdot \nabla Z)$. In the same way as in the proof of the previous proposition, we easily obtain

$$\delta^2 \|Z\|_{r+1} + \|v\|_{r+1} \leqslant C \quad \text{and} \quad \delta^2 \|Z_t\|_{r+1} + \|v_t\|_{r+1} \leqslant C\varepsilon^{-1}.$$

Differentiating the identity

$$(1 + \delta^2 |\nabla \eta|^2) Z = \Lambda^{\mathrm{DN}} \phi - \varepsilon^{-1} \Lambda^{\mathrm{NN}} \beta_\tau + \nabla \eta \cdot \nabla \phi,$$

we have

$$\begin{aligned}
&(1 + \delta^2 |\nabla \eta|^2)(Z_{tt} - \varepsilon^{-3} \beta_{\tau\tau\tau}) \\
&\quad = -4\delta^2 (\nabla \eta \cdot \nabla \eta_t) Z_t - 2\delta^2 (\nabla \eta \cdot \nabla \eta_{tt} + |\nabla \eta_t|^2) Z - \varepsilon^{-3} \delta^2 |\nabla \eta|^2 \beta_{\tau\tau\tau} \\
&\qquad + \Lambda^{\mathrm{DN}} \phi_{tt} - \varepsilon^{-3} (\Lambda^{\mathrm{NN}} \beta_{\tau\tau\tau} + \beta_{\tau\tau\tau}) + \Lambda_1^{\mathrm{DN}} [u_{tt}] \phi - \varepsilon^{-1} \Lambda_1^{\mathrm{NN}} [u_{tt}] \beta_\tau \\
&\qquad + 2\Lambda_1^{\mathrm{DN}} [u_t] \phi_t - 2\varepsilon^{-2} \Lambda_1^{\mathrm{NN}} [u_t] \beta_{\tau\tau} + \Lambda_2^{\mathrm{DN}} [u_t, u_t] \phi - \varepsilon^{-1} \Lambda_2^{\mathrm{NN}} [u_t, u_t] \beta_\tau \\
&\qquad + \nabla \eta \cdot \nabla \phi_{tt} + \nabla \eta_{tt} \cdot \nabla \phi + 2 \nabla \eta_t \cdot \nabla \phi_t,
\end{aligned}$$

which, together with propositions 5.1, 5.17 and 5.23, implies that

$$\delta^2 \|Z_{tt} - \varepsilon^{-3} \beta_{\tau\tau\tau}\|_r \leqslant C\varepsilon^{-1}.$$

Therefore, we obtain the second estimate in (6.9). To show the first estimate, we first note that

$$\|Z - \varepsilon^{-1} \beta_\tau\|_r \leqslant C \quad \text{and} \quad \|Z_t - \varepsilon^{-2} \beta_{\tau\tau}\|_r \leqslant C\varepsilon^{-1}.$$

In view of (6.7), we can rewrite Z_t as $Z_t = Z_t^{(0)} + Z_t^{(1)}$, where

$$\begin{aligned}
\delta^2 Z_t^{(0)} &= \delta^2 \partial_t \bigg\{ -\nabla \cdot ((1 + \eta - b) \nabla \phi) + \nabla \eta \cdot \nabla \phi + \frac{1}{\varepsilon} (1 - \delta^2 |\nabla \eta|^2) \beta_\tau \\
&\qquad + \frac{\delta^2}{\varepsilon} \nabla \cdot ((1 + \eta - b)(\nabla \eta) \beta_\tau + \tfrac{1}{2} (1 + \eta - b)^2 \nabla \beta_\tau) \bigg\} \\
&= \left(\frac{\delta}{\varepsilon}\right)^2 (1 - \delta^2 |\nabla \eta|^2) \beta_{\tau\tau} + \frac{\delta^2}{\varepsilon} \left(\nabla \phi - \frac{\delta^2}{\varepsilon} \beta_\tau \nabla \eta \right) \cdot \nabla \beta_\tau \\
&\quad + \left(\frac{\delta^2}{\varepsilon}\right)^2 \nabla \cdot ((1 + \eta - b)(\nabla \eta) \beta_{\tau\tau} + \tfrac{1}{2} (1 + \eta - b)^2 \nabla \beta_{\tau\tau}) \\
&\quad + \delta^2 \bigg\{ -\nabla \cdot ((1 + \eta - b) \nabla (\phi_t - \tfrac{1}{2} \delta^2 b_t^2) + (\eta_t - b_t) \nabla \phi) \\
&\qquad + \nabla (\eta_t - b_t) \cdot \nabla \phi + \nabla \eta \cdot \nabla (\phi_t - \tfrac{1}{2} \delta^2 b_t^2) - 2 \frac{\delta^2}{\varepsilon} \beta_\tau \nabla \eta \cdot \nabla (\eta_t - b_t) \\
&\qquad + \frac{\delta^2}{\varepsilon} \nabla \cdot ((\eta_t - b_t)(\nabla \eta) \beta_\tau + (1 + \eta - b) \nabla ((\eta_t - b_t) \beta_\tau)) \bigg\}
\end{aligned}$$

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and

$$
\begin{aligned}
Z_t^{(1)} &= -\delta^2|\nabla\eta|^2(Z_t-\varepsilon^{-1}\beta_{\tau\tau}) - 2\delta^2(\nabla\eta\cdot\nabla\eta_t)(Z-\varepsilon^{-1}\beta_\tau) \\
&\quad + \{\Lambda^{\mathrm{DN}}\phi_t + \nabla\cdot((1+\eta-b)\nabla\phi_t)\} + \{\Lambda_1^{\mathrm{DN}}[u_t]\phi + \nabla\cdot((\eta_t-b_t)\nabla\phi)\} \\
&\quad - \varepsilon^{-2}\{\Lambda^{\mathrm{NN}}\beta_{\tau\tau} + \beta_{\tau\tau} + \delta^2\nabla\cdot((1+\eta-b)(\nabla\eta)\beta_{\tau\tau} + \tfrac{1}{2}(1+\eta-b)^2\nabla\beta_{\tau\tau})\} \\
&\quad - \varepsilon^{-1}\{\Lambda_1^{\mathrm{NN}}[u_t]\beta_\tau + \delta^2\nabla\cdot((1+\eta-b)(\nabla\eta_t)\beta_\tau \\
&\qquad + (\eta_t-b_t)(\nabla\eta)\beta_\tau + (1+\eta-b)(\eta_t-b_t)\nabla\beta_\tau)\}.
\end{aligned}
$$

Here, by hypothesis, we have

$$
\|\eta_t-b_t\|_{r+4} \leqslant M_1 \quad \text{and} \quad \|\nabla(\phi_t-\tfrac{1}{2}\delta^2 b_t^2)\|_{r+4} \leqslant M_1.
$$

By propositions 5.3, 5.16, 5.26 and 5.27, we also have $\|Z_t^{(1)}\|_r \leqslant C$. On the other hand, we can rewrite $\delta^2 v\cdot\nabla Z$ as

$$
\delta^2 v\cdot\nabla Z = \frac{\delta^2}{\varepsilon}\left(\nabla\phi-\frac{\delta^2}{\varepsilon}\beta_\tau\nabla\eta\right)\cdot\nabla\beta_\tau + \delta^2\left(v\cdot\nabla(Z-\varepsilon^{-1}\beta_\tau) - \frac{\delta^2}{\varepsilon}(Z-\varepsilon^{-1}\beta_\tau)\nabla\eta\cdot\nabla\beta_\tau\right).
$$

Therefore, we can obtain $\|a-(1+(\delta/\varepsilon)^2(1-\delta^2|\nabla\eta|^2)\beta_{\tau\tau}+\alpha^{(0)})\|_r \leqslant C\delta^2$, where

$$
\alpha^{(0)} := 2\frac{\delta^2}{\varepsilon}\left(\nabla\phi-\frac{\delta^2}{\varepsilon}\beta_\tau\nabla\eta\right)\cdot\nabla\beta_\tau + \frac{\delta^2}{\varepsilon}\nabla\cdot((1+\eta-b)(\nabla\eta)\beta_{\tau\tau} + \tfrac{1}{2}(1+\eta-b)^2\nabla\beta_{\tau\tau}).
$$

In view of this, (2.21) and (6.10), we easily get $\|\alpha^{(0)}-\sigma a^{(0)}\|_r \leqslant C(\varepsilon+|\delta^2/\varepsilon-\sigma|)$. These show the second estimate in (6.9). The last assertion of the proposition follows directly from (6.9) and the Sobolev inequality. The proof is complete. □

7. Proof of the main theorems

In this section we first consider a linear system of equations and give an energy estimate for the solution. Then, applying the estimate to the quasi-linear system of equations (6.4), we will derive a uniform estimate of the solution (η,ϕ) with respect to small δ and ε.

Now we consider the following system of linear equations:

$$
\left.
\begin{aligned}
\partial_t\zeta_{ijk} + v\cdot\nabla\zeta_{ijk} - \Lambda^{\mathrm{DN}}\psi_{ijk} + L^{ijk}\eta = \varepsilon^{-1}\partial_{ijk}g_1 + f_1^{ijk}, \quad \zeta_{ijk} = \partial_{ijk}\eta, \\
\partial_t\psi_{ijk} + v\cdot\nabla\psi_{ijk} + a\zeta_{ijk} = \varepsilon^{-1}g_2^{ijk} + f_2^{ijk},
\end{aligned}
\right\} \tag{7.1}
$$

where a, $v=(v_1,\dots,v_n)^{\mathrm{T}}$, $f_1=(f_1^{ijk})$, $f_2=(f_2^{ijk})$ are given functions of x and t and may depend on δ and ε, whereas g_1 and $g_2=(g_2^{ijk})$ are given functions of x and $\tau=t/\varepsilon$, $\Lambda^{\mathrm{DN}}=\Lambda^{\mathrm{DN}}(\eta,b,\delta)$ is the DN map, and L^{ijk} are linear operators defined by

$$
L^{ijk}\eta = \Lambda^{\mathrm{DN}}(p_k\partial_{ij}\eta + p_i\partial_{jk}\eta + p_j\partial_{ki}\eta),
$$

where $p=(p_1,\dots,p_n)$ are given functions of x and t and may depend on δ and ε. The above system in the case where $p=0$ and $(g_1,g_2)=0$ was investigated in [11].

REMARK 7.1. It follows from proposition 5.1 that $\|L^{ijk}\eta\|_s \leqslant C\delta^{-1}\|p\|_{s+1}\|\eta\|_{s+3}$, so we can regard $L^{ijk}\eta$ in (7.1) as a lower-order term and put it into the right-hand side if we fix the parameter δ. However, in order to derive a uniform estimate of the solution with respect to small δ, we have to use the estimate $\|L^{ijk}\eta\|_s \leqslant C\|p\|_{s+2}\|\eta\|_{s+4}$, so that $L^{ijk}\eta$ cannot be regarded as a lower-order term. Note that the norm $\|\eta\|_{s+4}$ in the last estimate is optimal because Λ^{DN} converges a second-order differential operator as δ goes to zero. This is the reason why we have to treat $L^{ijk}\eta$ as one of the principal terms.

Proposition 7.2. *Let $r > \frac{1}{2}n$. In addition to assumptions* (A_1) *and* (A_2) *with $q = r+1$, we assume that*

$$\left.\begin{aligned} \|(\eta,b)\|_{r+2} \leqslant M, \quad \|(\eta_t,b_t)\|_{r+1} \leqslant M\varepsilon^{-1}, \quad \|v\|_{r+1} \leqslant M, \quad \|p\|_{r+3} \leqslant M, \\ M^{-1} \leqslant a(x,t) \leqslant M\varepsilon^{-1}, \quad a_t(x,t) \leqslant M\varepsilon^{-1}, \quad \|\nabla a\|_{r+2} \leqslant M\varepsilon^{-1}. \end{aligned}\right\} \tag{7.2}$$

Then, there exists a constant $C = C(M,c,r) > 0$ independent of δ and ε such that, for any smooth solution (η,ζ,ψ) of (7.1), we have

$$\begin{aligned} &\|\zeta(t)\|^2 + \|(\Lambda_0^{\mathrm{DN}})^{1/2}\psi(t)\|^2 \\ &\quad \leqslant C\mathrm{e}^{Ct/\varepsilon}\biggl\{\|\eta(0)\|_4^2 + \|(\Lambda_0^{\mathrm{DN}})^{1/2}\psi(0)\|^2 \\ &\qquad + \biggl(\int_0^{t/\varepsilon}(\|g_1(\tau)\|_4 + \|(\Lambda_0^{\mathrm{DN}})^{1/2}g_2(\tau)\|)\,\mathrm{d}\tau\biggr)^2 \\ &\qquad + \int_0^t \mathrm{e}^{-C\tilde{t}/\varepsilon}\biggl(\frac{1}{\varepsilon}(\|f_1(\tilde{t})\|^2 + \|\eta(\tilde{t})\|^2) + \varepsilon\|(\Lambda_0^{\mathrm{DN}})^{1/2}f_2(\tilde{t})\|^2\biggr)\,\mathrm{d}\tilde{t}\biggr\}. \end{aligned}$$

Proof. First, we will consider the case where $(g_1,g_2) = 0$ and $\eta|_{t=0} = 0$, so that we also have $\zeta|_{t=0} = 0$. Let (η,ζ,ψ) be a smooth solution of (7.1) and define an energy function $E(t)$ by

$$E(t) = (a\zeta(t),\zeta(t)) + (\Lambda^{\mathrm{DN}}\psi(t),\psi(t)).$$

Then it holds that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}E(t) &= (a_t\zeta,\zeta) + 2(a\zeta,\zeta_t) + ([\partial_t,\Lambda^{\mathrm{DN}}]\psi,\psi) + 2(\Lambda^{\mathrm{DN}}\psi,\psi_t) \\ &= (a_t\zeta,\zeta) + ((\nabla\cdot(av))\zeta,\zeta) + 2(a\zeta,f_1) - 2(a\zeta,L\eta) \\ &\quad + ([\partial_t,\Lambda^{\mathrm{DN}}]\psi,\psi) - 2(\Lambda^{\mathrm{DN}}\psi,v\cdot\nabla\psi) + 2(\Lambda^{\mathrm{DN}}\psi,f_2). \end{aligned} \tag{7.3}$$

Here, by the definition we have

$$\begin{aligned} (a\zeta,L\eta) = \sum_{ijk=1}^n \{&(a\partial_{ijk}\eta,\Lambda^{\mathrm{DN}}(p_k\partial_{ij}\eta)) \\ &+ (a\partial_{ijk}\eta,\Lambda^{\mathrm{DN}}(p_i\partial_{jk}\eta)) + (a\partial_{ijk}\eta,\Lambda^{\mathrm{DN}}(p_j\partial_{ki}\eta))\}. \end{aligned}$$

By proposition 3.3 and using integration by parts, we see that

$$\begin{aligned} &2(a\partial_{ijk}\eta,\Lambda^{\mathrm{DN}}(p_k\partial_{ij}\eta)) \\ &\quad = -(\partial_{ij}\eta,(\partial_k a)\Lambda^{\mathrm{DN}}(p_k\partial_{ij}\eta) + a[\partial_k,\Lambda^{\mathrm{DN}}](p_k\partial_{ij}\eta) \\ &\qquad + a\Lambda^{\mathrm{DN}}((\partial_k p_k)\partial_{ij}\eta) + [a,\Lambda^{\mathrm{DN}}](p_k\partial_{ijk}\eta) + [\Lambda^{\mathrm{DN}},p_k](a\partial_{ijk}\eta)), \end{aligned}$$

so that

$$
\begin{aligned}
&2|(a\partial_{ijk}\eta, \Lambda^{\mathrm{DN}}(p_k\partial_{ij}\eta))| \\
&\quad \leqslant \sqrt{(\Lambda^{\mathrm{DN}}((\partial_k a)\partial_{ij}\eta), (\partial_k a)\partial_{ij}\eta)}\sqrt{(\Lambda^{\mathrm{DN}}(p_k\partial_{ij}\eta), p_k\partial_{ij}\eta)} \\
&\qquad + \|a\partial_{ij}\eta\|_1\|[\partial_k, \Lambda^{\mathrm{DN}}](p_k\partial_{ij}\eta)\|_{-1} \\
&\qquad + \sqrt{(\Lambda^{\mathrm{DN}}(a\partial_{ij}\eta), a\partial_{ij}\eta)}\sqrt{(\Lambda^{\mathrm{DN}}((\partial_k p_k)\partial_{ij}\eta), (\partial_k p_k)\partial_{ij}\eta)} \\
&\qquad + \|\partial_{ij}\eta\|_1(\|[a, \Lambda^{\mathrm{DN}}](p_k\partial_{ijk}\eta)\|_{-1} + \|[\Lambda^{\mathrm{DN}}, p_k](a\partial_{ijk}\eta)\|_{-1}) \\
&\quad \leqslant C(\|\nabla a\|_{r+2} + \|a\|_{L^\infty(\mathbb{R}^n)})\|p\|_{r+3}\|\eta\|_3^2,
\end{aligned}
$$

where we used propositions 5.4, 5.6 and 5.7 and lemmas 4.4 and 4.6. The other terms on the right-hand side of (7.3) can be evaluated by propositions 5.9, 5.10 and 5.4 and lemma 4.4, so that we obtain

$$
\frac{\mathrm{d}}{\mathrm{d}t}E(t) \leqslant C\varepsilon^{-1}E(t) + C(\varepsilon^{-1}(\|f_1(t)\|^2 + \|\eta(t)\|^2) + \varepsilon\|(\Lambda_0^{\mathrm{DN}})^{1/2}f_2(t)\|^2).
$$

This, together with Gronwall's inequality and the relations

$$
\|\zeta(t)\|^2 + \|(\Lambda_0^{\mathrm{DN}})^{1/2}\psi(t)\|^2 \leqslant CE(t), \qquad E(0) \leqslant C\|(\Lambda_0^{\mathrm{DN}})^{1/2}\psi(0)\|^2,
$$

gives

$$
\begin{aligned}
&\|\zeta(t)\|^2 + \|(\Lambda_0^{\mathrm{DN}})^{1/2}\psi\|^2 \\
&\quad \leqslant C\mathrm{e}^{Ct/\varepsilon}\bigg\{\|(\Lambda_0^{\mathrm{DN}})^{1/2}\psi(0)\|^2 \\
&\qquad\qquad \times \int_0^t \mathrm{e}^{-C\tilde{t}/\varepsilon}(\varepsilon^{-1}(\|f_1(\tilde{t})\|^2 + \|\eta(\tilde{t})\|^2) + \varepsilon\|(\Lambda_0^{\mathrm{DN}})^{1/2}f_2(\tilde{t})\|^2)\,\mathrm{d}\tilde{t}\bigg\}.
\end{aligned} \tag{7.4}
$$

Next we will consider the general case. Let (η, ζ, ψ) be a smooth solution of (7.1) and define $(\eta^{(0)}, \zeta^{(0)})$ and $(\bar{\eta}, \bar{\zeta})$ by

$$
\eta^{(0)}(x,t) := \eta(x,0) + \int_0^{t/\varepsilon} g_1(x,\tau)\,\mathrm{d}\tau, \qquad \zeta_{ijk}^{(0)} := \partial_{ijk}\eta^{(0)}
$$

and $\bar{\eta} := \eta - \eta^{(0)}$, $\bar{\zeta} := \zeta - \zeta^{(0)}$. Then, it holds that

$$
\begin{gathered}
\partial_t\bar{\zeta}_{ijk} + v\cdot\nabla\bar{\zeta}_{ijk} - \Lambda^{\mathrm{DN}}\psi_{ijk} + L^{ijk}\bar{\eta} = \bar{f}_1^{ijk}, \quad \bar{\zeta}_{ijk} = \partial_{ijk}\bar{\eta}, \\
\partial_t\psi_{ijk} + v\cdot\nabla\psi_{ijk} + a\bar{\zeta}_{ijk} = \bar{f}_2^{ijk},
\end{gathered}
$$

and $\bar{\eta}|_{t=0} = 0$, where

$$
\bar{f}_1^{ijk} = f_1^{ijk} - v\cdot\nabla\zeta_{ijk}^{(0)} - L^{ijk}\eta^{(0)}, \qquad \bar{f}_2^{ijk} = \varepsilon^{-1}g_2^{ijk} + f_2^{ijk} - a\zeta_{ijk}^{(0)}.
$$

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Therefore, applying the estimate obtained in the previous case, we obtain

$$\begin{aligned}\|\bar{\zeta}(t)\|^2 + \|(\Lambda_0^{\mathrm{DN}})^{1/2}\psi(t)\|^2 &\leqslant C\mathrm{e}^{Ct/\varepsilon}\biggl\{\|(\Lambda_0^{\mathrm{DN}})^{1/2}\psi(0)\|^2 \\ &\quad + \int_0^t \mathrm{e}^{-C\tilde{t}/\varepsilon}(\varepsilon^{-1}(\|\bar{f}_1(\tilde{t})\|^2 + \|\bar{\eta}(\tilde{t})\|^2) + \varepsilon\|(\Lambda_0^{\mathrm{DN}})^{1/2}\bar{f}_2(\tilde{t})\|^2)\,\mathrm{d}\tilde{t}\biggr\}.\end{aligned}$$

It is easy to see that

$$\|\zeta(t)\| \leqslant \|\bar{\zeta}(t)\| + \|\eta(0)\|_3 + \int_0^{t/\varepsilon}\|g_1(\tau)\|_3\,\mathrm{d}\tau$$

and

$$\begin{aligned}&\varepsilon^{-1}(\|\bar{f}_1(t)\|^2 + \|\bar{\eta}(t)\|^2) + \varepsilon\|(\Lambda_0^{\mathrm{DN}})^{1/2}\bar{f}_2(t)\|^2 \\ &\quad\leqslant C\biggl\{\varepsilon^{-1}(\|f_1(t)\|^2 + \|\eta(t)\|^2 + \|\eta(0)\|_4^2) + \varepsilon\|(\Lambda_0^{\mathrm{DN}})^{1/2}f_2(t)\|^2 \\ &\qquad + \varepsilon^{-1}\biggl(\int_0^{t/\varepsilon}(\|g_1(\tau)\|_4 + \|(\Lambda_0^{\mathrm{DN}})^{1/2}g_2(\tau)\|)\,\mathrm{d}\tau\biggr)^2\biggr\}.\end{aligned}$$

To summarize the above estimates, we obtain the desired estimate. □

Let (η, ϕ) be the solution of (1.15) and (1.16) and set

$$\mathcal{E}(t)^2 := \|\eta(t)\|_{s+3}^2 + \|\nabla\phi(t)\|_{s+2}^2 + \|(\Lambda_0^{\mathrm{DN}})^{1/2}(\partial^3\phi(t) - \delta^2 Z\partial^3\eta(t))\|_s^2,$$

where Z is determined by (6.1). Suppose that the solution (η, ϕ) satisfies

$$\left.\begin{aligned}\mathcal{E}(t) \leqslant N_1, \quad \|\eta(t)\|_{s+2} + \|\nabla\phi(t)\|_{s+1} \leqslant N_2, \\ 1 + \eta(x,t) - b(x,t) \geqslant \tfrac{1}{2}c_0 \quad \text{for } x \in \mathbb{R}^n,\ 0 \leqslant t \leqslant \varepsilon,\ 0 < \delta \leqslant \delta_0,\end{aligned}\right\} \tag{7.5}$$

where positive constants N_1, N_2 and δ_0 will be determined later. Then, by proposition 4.1, there exists a constant $\delta_1 = \delta_1(M_0, N_2, c_0, s)$ independent of N_1 such that, for any $\delta \in (0, \delta_1]$, we can construct a diffeomorphism Θ satisfying assumptions (A_1)–(A_4) with $r = s + 1$ and a constant M independent of δ and N_1 but depending on N_2. Set $\delta_0 := \min\{\delta_1, \delta_2, \delta_3\}$, where $\delta_2, \delta_3 > 0$ are the constants occurring in propositions 6.3–6.6. In the following we simply write the constants depending only on (M_0, N_1, c_0, s) and (M_0, N_2, c_0, s) by C_1 and C_2, respectively. It follows from (1.15) that

$$\left.\begin{aligned}\eta_t - \varepsilon^{-1}\beta_\tau &= (Z - \varepsilon^{-1}\beta_\tau) + \delta^2|\nabla\eta|^2 Z - \nabla\eta\cdot\nabla\phi, \\ \phi_t - \frac{1}{2}\biggl(\frac{\delta}{\varepsilon}\biggr)^2\beta_\tau^2 &= -\eta - \tfrac{1}{2}|\nabla\phi|^2 + \tfrac{1}{2}(\delta^4|\nabla\eta|^2 Z^2 + \delta^2(Z + \varepsilon^{-1}\beta_\tau)(Z - \varepsilon^{-1}\beta_\tau)).\end{aligned}\right\} \tag{7.6}$$

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By proposition 5.1, lemma 4.6 and corollary 5.25 we obtain $\|Z-\varepsilon^{-1}\beta_\tau\|_s \leqslant C_2$, so that

$$\left\|\eta_t(t)-\varepsilon^{-1}\beta_\tau\left(\frac{t}{\varepsilon}\right)\right\|_s+\left\|\phi_t(t)-\frac{1}{2}\left(\frac{\delta}{\varepsilon}\right)^2\beta_\tau\left(\frac{t}{\varepsilon}\right)^2\right\|_s \leqslant C_2,$$
$$\|\eta_t(t)\|_s+\|\phi_t(t)\|_s \leqslant C_2\varepsilon^{-1}.$$

Moreover, it holds that

$$\eta_{tt}=\Lambda^{\mathrm{DN}}\phi_t+\Lambda_1^{\mathrm{DN}}[u_t]\phi-\varepsilon^{-2}\Lambda^{\mathrm{NN}}\beta_{\tau\tau}-\varepsilon^{-1}\Lambda_1^{\mathrm{NN}}[u_t]\beta_\tau,$$
$$\phi_{tt}=\delta^2Z\eta_{tt}-\eta_t-(\nabla\phi-\delta^2Z\nabla\eta)\cdot(\nabla\phi_t-\delta^2Z\nabla\eta_t),$$

which, together with the previous estimates, propositions 5.1, 5.19, 5.11 and 5.20 and lemma 4.6, easily yields that $\|\eta_{tt}(t)\|_{s-1}+\|\phi_{tt}(t)\|_{s-1} \leqslant C_2\varepsilon^{-2}$. Therefore, by propositions 6.5 and 6.6, there exist small constants $\varepsilon_0,\gamma_0>0$ such that the function a defined by (6.5) satisfies $\|\nabla a(t)\|_{s-2} \leqslant C_2\varepsilon^{-1}$, $\frac{1}{2}c \leqslant a(x,t) \leqslant C_2\varepsilon^{-1}$ and $a_t(x,t) \leqslant C_2\varepsilon^{-1}$ as long as $0<\varepsilon \leqslant \varepsilon_0$ and $|\delta^2/\varepsilon-\sigma| \leqslant \gamma_0$. It is easy to see that $\|(v(t),p(t))\|_s \leqslant C_2$, where $p=\varepsilon^{-1}\delta^2\Lambda^{\mathrm{NN}}(\nabla\beta_\tau)$. Hence, we have checked all of the conditions in proposition 7.2.

Now, introducing new variables $\zeta=(\zeta_{ijk})$ and $\psi=(\psi_{ijk})$ by (6.3), we obtain the quasi-linear system of equations (6.4). Applying the operator J^s to the equations in (6.4), we have

$$\left.\begin{aligned}\partial_t(J^s\zeta)_{ijk}+v\cdot\nabla(J^s\zeta)_{ijk}-\Lambda^{\mathrm{DN}}(J^s\psi)_{ijk}+L^{ijk}(J^s\eta)&=\varepsilon^{-1}\partial_{ijk}(J^s\beta_\tau)+\tilde f_1^{ijk},\\ \partial_t(J^s\psi)_{ijk}+v\cdot\nabla(J^s\psi)_{ijk}+a(J^s\zeta)_{ijk}&=\varepsilon^{-1}(J^sg_2)^{ijk}+\tilde f_2^{ijk},\end{aligned}\right\}\tag{7.7}$$

where

$$\begin{aligned}\tilde f_1^{ijk}&=J^sf_1^{ijk}-[J^s,v]\cdot\nabla\zeta_{ijk}+[J^s,\Lambda^{\mathrm{DN}}]\psi_{ijk}\\&\quad-[J^s,\Lambda^{\mathrm{DN}}](p_k\partial_{ij}\eta+p_i\partial_{jk}\eta+p_j\partial_{ki}\eta)\\&\quad-\Lambda^{\mathrm{DN}}([J^s,p_k]\partial_{ij}\eta+[J^s,p_i]\partial_{jk}\eta+[J^s,p_j]\partial_{ki}\eta),\\ \tilde f_2^{ijk}&=J^sf_2^{ijk}-[J^s,v]\cdot\nabla\psi_{ijk}-[J^s,a]\zeta_{ijk}.\end{aligned}$$

Here, it follows from propositions 5.1 and 5.8 and lemmas 4.6 and 4.8 that

$$\|\tilde f_1(t)\| \leqslant C_2(\mathcal{E}(t)+\|v(t)\|_s+\|p(t)\|_{s+2}+\|f_1(t)\|_s),$$
$$\|(\Lambda_0^{\mathrm{DN}})^{1/2}\tilde f_2(t)\| \leqslant C_2(\mathcal{E}(t)+\|v(t)\|_{s+1}+\|\nabla a(t)\|_s+\|(\Lambda_0^{\mathrm{DN}})^{1/2}f_2(t)\|_s).$$

By propositions 6.3–6.5, we can evaluate the right-hand sides of the above estimates except the term $\|p\|_{s+2}$. Since $p=\varepsilon^{-1}\delta^2\Lambda^{\mathrm{NN}}(\nabla\beta_\tau)$, we have

$$\begin{aligned}\partial_{jk}p_i=\varepsilon^{-1}\delta^2(\Lambda^{\mathrm{NN}}\partial_{ijk}\beta_\tau&+\Lambda_1^{\mathrm{NN}}[\partial_ku]\partial_{ij}\beta_\tau+\Lambda_1^{\mathrm{NN}}[\partial_ju]\partial_{ki}\beta_\tau\\&+\Lambda_1^{\mathrm{NN}}[\partial_{jk}u]\partial_i\beta_\tau+\Lambda_2^{\mathrm{NN}}[\partial_ju,\partial_ku]\partial_i\beta_\tau),\end{aligned}$$

so that, by propositions 5.11 and 5.20, we can also evaluate $\|p\|_{s+2}$ and obtain that

$$\varepsilon^{-1}\|\tilde f_1(t)\|^2+\varepsilon\|(\Lambda_0^{\mathrm{DN}})^{1/2}\tilde f_2(t)\|^2 \leqslant C_2(\varepsilon^{-1}\mathcal{E}(t)^2+1).$$

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Therefore, applying the basic energy estimate in proposition 7.2 to (7.7), we obtain

$$\|\zeta(t)\|_s^2 + \|(\Lambda_0^{\mathrm{DN}})^{1/2}\psi(t)\|_s^2 \leqslant C_2 + \frac{C_2}{\varepsilon}\int_0^t \mathcal{E}(\tilde{t})^2 \,\mathrm{d}\tilde{t} \quad \text{for } 0 \leqslant t \leqslant \varepsilon.$$

It is easy to see that

$$\begin{aligned}\|\nabla\phi(t)\|_{s+2} &\leqslant \|\nabla\phi(t)\|_{s+1} + \|\nabla(\partial^3\phi(t) - \delta^2 Z\partial^3\eta(t))\|_{s-1} + \delta^2\|Z\partial^3\eta(t)\|_s \\ &\leqslant C_2(1 + \|(\Lambda_0^{\mathrm{DN}})^{1/2}\psi(t)\|_{s-1/2} + \|\zeta(t)\|_s).\end{aligned}$$

By the above two estimates, we have

$$\mathcal{E}(t)^2 \leqslant C_2 + \frac{C_2}{\varepsilon}\int_0^t \mathcal{E}(\tilde{t})^2 \,\mathrm{d}\tilde{t} \quad \text{for } 0 \leqslant t \leqslant \varepsilon,$$

so that Gronwall's inequality gives

$$\mathcal{E}(t) \leqslant C_2 \quad \text{for } 0 \leqslant t \leqslant \varepsilon. \tag{7.8}$$

On the other hand, in view of (7.6) and proposition 6.3, we have

$$\left\|\eta_t(t) - \varepsilon^{-1}\beta_\tau\left(\frac{t}{\varepsilon}\right)\right\|_{s+2} + \left\|\phi_t(t) - \frac{1}{2}\left(\frac{\delta}{\varepsilon}\right)^2\beta_\tau\left(\frac{t}{\varepsilon}\right)^2\right\|_{s+2} \leqslant C_1.$$

Let $(\eta^{(0)}, \phi^{(0)})$ be the approximate solution defined by (2.20). Then, we see that

$$\begin{aligned}&\left\|\eta(t) - \eta^{(0)}\left(\frac{t}{\varepsilon}\right)\right\|_{s+2} + \left\|\phi(t) - \phi^{(0)}\left(\frac{t}{\varepsilon}\right)\right\|_{s+2} \\ &\qquad\leqslant \int_0^t \left(\left\|\eta_t(\tilde{t}) - \varepsilon^{-1}\beta_\tau\left(\frac{\tilde{t}}{\varepsilon}\right)\right\|_{s+2} + \left\|\phi_t(t) - \frac{\sigma}{\varepsilon}\beta_\tau\left(\frac{\tilde{t}}{\varepsilon}\right)^2\right\|_{s+2}\right)\mathrm{d}\tilde{t} \\ &\qquad\leqslant C_1\left(t + \frac{t}{\varepsilon}\left|\frac{\delta^2}{\varepsilon} - \sigma\right|\right) \\ &\qquad\leqslant C_1\left(\varepsilon + \left|\frac{\delta^2}{\varepsilon} - \sigma\right|\right) \quad \text{for } 0 \leqslant t \leqslant \varepsilon.\end{aligned} \tag{7.9}$$

In particular, we obtain

$$\begin{aligned}&\|\eta(t)\|_{s+2} + \|\nabla\phi(t)\|_{s+1} \\ &\qquad\leqslant \max_{0\leqslant\tau\leqslant 1}(\|\eta^{(0)}(\tau)\|_{s+2} + \|\nabla\phi^{(0)}(\tau)\|_{s+1}) + C_1(\varepsilon + |\delta^2/\varepsilon - \sigma|)\end{aligned} \tag{7.10}$$

for $0 \leqslant t \leqslant \varepsilon$. Moreover, we see that

$$\begin{aligned}&1 + \eta(x,t) - b(x,t) \\ &\qquad= 1 + \eta_0(x) - b_0(x) + \int_0^t \left(\eta_t(x,\tilde{t}) - \varepsilon^{-1}\beta_\tau\left(x, \frac{\tilde{t}}{\varepsilon}\right)\right)\mathrm{d}\tilde{t} \\ &\qquad\geqslant c_0 - C\int_0^t \left\|\eta_t(\tilde{t}) - \varepsilon^{-1}\beta_\tau\left(\frac{\tilde{t}}{\varepsilon}\right)\right\|_{s+2}\mathrm{d}\tilde{t} \\ &\qquad\geqslant c_0 - C_1 t \geqslant c_0 - C_1\varepsilon \quad \text{for } 0 \leqslant t \leqslant \varepsilon.\end{aligned} \tag{7.11}$$

In view of (7.8), (7.10) and (7.11), we define the constants N_1, N_2, ε_0 and γ_0 by

$$N_2 := 2 \max_{0\leqslant\tau\leqslant 1}(\|\eta^{(0)}(\tau)\|_{s+2} + \|\nabla\phi^{(0)}(\tau)\|_{s+1}), \qquad N_1 := C_2,$$

$$\varepsilon_0 := (2C_1)^{-1}\min\{c_0, N_2\}, \qquad \gamma_0 := (2C_1)^{-1}N_2.$$

Then, we see that the estimates in (7.5) hold. Therefore, by (7.9), we obtain the error estimate. The proof of theorem 2.3 is complete.

We proceed to prove theorem 2.4. By theorem 2.3, we have

$$\|\eta^{\delta,\varepsilon}(\varepsilon)\|_{s+3} + \|\nabla\phi^{\delta,\varepsilon}(\varepsilon)\|_{s+2} \leqslant C_0,$$
$$1 + \eta^{\delta,\varepsilon}(x,\varepsilon) - b_1(x) \geqslant \tfrac{1}{2}c_0 \quad \text{for } x \in \mathbb{R}^n$$

and

$$\|\eta^{\delta,\varepsilon}(\varepsilon) - \eta^{(0)}(1)\|_{s+2} + \|\nabla\phi^{\delta,\varepsilon}(\varepsilon) - \nabla\phi^{(0)}(1)\|_{s+1} \leqslant C_0(\varepsilon + |\delta^2/\varepsilon - \sigma|). \tag{7.12}$$

Since $b(x,t) = b_1(x)$ for $t \geqslant \varepsilon$, the results in [11] imply that the solution $(\eta^{\delta,\varepsilon}, \phi^{\delta,\varepsilon})$ obtained in theorem 2.3 can be extended to a time interval $[0,T]$ independent of $\delta \in (0,\delta_0]$ and $\varepsilon \in (0,\varepsilon_0]$ and satisfies

$$\left.\begin{aligned} \|\eta^{\delta,\varepsilon}(t) - \eta^{\varepsilon}(t)\|_{s-1} + \|\nabla\phi^{\delta,\varepsilon}(t) - u^{\varepsilon}(t)\|_{s-1} &\leqslant C\delta^2, \\ \|\eta^{\delta,\varepsilon}(t)\|_{s+2} + \|\nabla\phi^{\delta,\varepsilon}(t)\|_{s+1} &\leqslant C \quad \text{for } \varepsilon \leqslant t \leqslant T, \end{aligned}\right\} \tag{7.13}$$

where $(\eta^{\varepsilon}, u^{\varepsilon}) \in C([-T,T]; H^{s+2})$ is a unique solution of the shallow-water equations

$$\eta^{\varepsilon}_t + \nabla\cdot((1+\eta^{\varepsilon} - b_1)u^{\varepsilon}) = 0, \qquad u^{\varepsilon}_t + (u^{\varepsilon}\cdot\nabla)u^{\varepsilon} + \nabla\eta^{\varepsilon} = 0$$

under the initial conditions $\eta^{\varepsilon} = \eta^{\delta,\varepsilon}(\cdot,\varepsilon)$, $u^{\varepsilon} = \nabla\phi^{\delta,\varepsilon}(\cdot,\varepsilon)$ at $t = \varepsilon$, and satisfies

$$\|(\eta^{\varepsilon}(t), u^{\varepsilon}(t))\|_{s+2} + \|(\eta^{\varepsilon}_t(t), u^{\varepsilon}_t(t))\|_{s+1} \leqslant C \quad \text{for } -T \leqslant t \leqslant T.$$

In particular, we have

$$\|\eta^{\varepsilon}(\varepsilon) - \eta^{\varepsilon}(0)\|_{s+1} + \|u^{\varepsilon}(\varepsilon) - u^{\varepsilon}(0)\|_{s+1} \leqslant C\varepsilon. \tag{7.14}$$

Now, let (η^0, u^0) be the unique solution to the initial-value problem for the shallow-water equations (2.13) and (2.14). Since $\eta^{(0)}(1) = \eta^0(0)$ and $\nabla\phi^{(0)}(1) = u^0(0)$, equation (7.12) implies that

$$\|\eta^{\varepsilon}(\varepsilon) - \eta^0(0)\|_{s+2} + \|u^{\varepsilon}(\varepsilon) - u^0(0)\|_{s+1} \leqslant C_0(\varepsilon + |\delta^2/\varepsilon - \sigma|),$$

which, together with (7.14), yields that

$$\|\eta^{\varepsilon}(0) - \eta^0(0)\|_{s+1} + \|u^{\varepsilon}(0) - u^0(0)\|_{s+1} \leqslant C(\varepsilon + |\delta^2/\varepsilon - \sigma|).$$

Since $(\eta^{\varepsilon}, u^{\varepsilon})$ and (η^0, u^0) satisfy the same shallow-water equations and their initial data satisfy the above estimate, we obtain

$$\|\eta^{\varepsilon}(t) - \eta^0(t)\|_{s+1} + \|u^{\varepsilon}(t) - u^0(t)\|_{s+1} \leqslant C(\varepsilon + |\delta^2/\varepsilon - \sigma|) \quad \text{for } -T \leqslant t \leqslant T,$$

which, together with (7.13), yields the desired estimate. The proof of theorem 2.4 is complete.

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References

1 G. B. Airy. Tides and waves. In *Encyclopaedia metropolitana*, vol. 3 (1845), 241–396.

2 B. Alvarez-Samaniego and D. Lannes. Large time existence for 3D water-waves and asymptotics. *Invent. Math.* **171** (2008), 485–541.

3 J. T. Beale, T. Y. Hou and J. S. Lowengrub. Growth rates for the linearized motion of fluid interfaces away from equilibrium. *Commun. Pure Appl. Math.* **46** (1993), 1269–1301.

4 A. Constantin. On the relevance of soliton theory to tsunami modelling. *Wave Motion* **46** (2009), 420–426.

5 A. Constantin and R. S. Johnson. Modelling tsunamis. *J. Phys.* A **39** (2006), 215–217.

6 A. Constantin and W. Strauss. Pressure beneath a Stokes wave. *Commun. Pure Appl. Math.* **63** (2010), 533–557.

7 W. Craig. Surface water waves and tsunamis. *J. Dynam. Diff. Eqns* **18** (2006), 525–549.

8 D. G. Ebin. The equations of motion of a perfect fluid with free boundary are not well posed. *Commun. PDEs* **12** (1987), 1175–1201.

9 K. O. Friedrichs. On the derivation of the shallow water theory. Appendix to: "The formulation of breakers and bores", J. J. Stoker. *Commun. Pure Appl. Math.* **1** (1948), 1–87.

10 T. Iguchi. On the irrotational flow of incompressible ideal fluid in a circular domain with free surface. *Publ. RIMS* **34** (1998), 525–565.

11 T. Iguchi. A shallow water approximation for water waves. *J. Math. Kyoto Univ.* **49** (2009), 13–55.

12 T. Kano. Une théorie trois-dimensionnelle des ondes de surface de l'eau et le développement de Friedrichs. *J. Math. Kyoto Univ.* **26** (1986), 101–175.

13 T. Kano and T. Nishida. Sur les ondes de surface de l'eau avec une justification mathématique des équations des ondes en eau peu profonde. *J. Math. Kyoto Univ.* **19** (1979), 335–370.

14 T. Kano and T. Nishida. Water waves and Friedrichs expansion. In *Recent topics in nonlinear PDE*, North-Holland Mathamatics Studies 98, pp. 39–57 (Amsterdam: North-Holland, 1984).

15 M. Lakshmanan. Integrable nonlinear wave equations and possible connections to tsunami dynamics. In *Tsunami and nonlinear waves*, pp. 31–49 (Springer, 2007).

16 H. Lamb. *Hydrodynamics*, 6th edn (Cambridge University Press, 1994).

17 D. Lannes. Well-posedness of the water-waves equations. *J. Am. Math. Soc.* **18** (2005), 605–654.

18 Y. A. Li. A shallow-water approximation to the full water wave problem. *Commun. Pure Appl. Math.* **59** (2006), 1225–1285.

19 L. V. Ovsjannikov. To the shallow water theory foundation. *Arch. Mech.* **26** (1974), 407–422.

20 L. V. Ovsjannikov. Cauchy problem in a scale of Banach spaces and its application to the shallow water theory justification. In *Applications of methods of functional analysis to problems in mechanics*, Lecture Notes in Mathamatics 503, pp. 426–437 (Springer, 1976).

21 H. Segur. Waves in shallow water, with emphasis on the tsunami of 2004. In *Tsunami and nonlinear waves*, ed. A. Kundu, pp. 3–29 (Springer, 2007).

22 J. J. Stoker. Water waves: the mathematical theory with application (Wiley/Interscience, 1992).

23 R. Stuhlmeier. KdV theory and the Chilean tsunami of 1960. *Discrete Contin. Dynam. Syst.* B **12** (2009), 623–632.

24 S. Wu. Well-posedness in Sobolev spaces of the full water wave problem in 2-D. *Invent. Math.* **130** (1997), 39–72.

25 S. Wu. Well-posedness in Sobolev spaces of the full water wave problem in 3-D. *J. Am. Math. Soc.* **12** (1999), 445–495.

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