Lifting of recollements and gluing of partial silting sets

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This paper focuses on recollements and silting theory in triangulated categories. It consists of two main parts. In the first part a criterion for a recollement of triangulated subcategories to lift to a torsion torsion-free triple (TTF triple) of ambient triangulated categories with coproducts is proved. As a consequence, lifting of TTF triples is possible for recollements of stable categories of repetitive algebras or self-injective finite length algebras and recollements of bounded derived categories of separated Noetherian schemes. When, in addition, the outer subcategories in the recollement are derived categories of small linear categories the conditions from the criterion are sufficient to lift the recollement to a recollement of ambient triangulated categories up to equivalence. In the second part we use these results to study the problem of constructing silting sets in the central category of a recollement generating the t-structure glued from the silting t-structures in the outer categories. In the case of a recollement of bounded derived categories of Artin algebras we provide an explicit construction for gluing classical silting objects.

1. Introduction

Recollements of triangulated categories were introduced by Beilinson, Bernstein and Deligne [**10**] as a tool to get information about the derived category of sheaves over a topological space X from the corresponding derived categories for an open subset $U \subseteq X$ and its complement $F = X \setminus U$. In the general abstract picture, when there

exists a recollement $(\mathcal{Y} \equiv \mathcal{D} \equiv \mathcal{X})$ of triangulated categories

$$
\mathcal{Y} \xrightarrow[\substack{i^* \\ i_* \\ i^!}]{} \mathcal{D} \xrightarrow[\substack{j_! \\ j^* \\ j_*}]{} \mathcal{X} \; , \tag{1.1}
$$

the properties of $\mathcal{D}$, $\mathcal{X}$ and $\mathcal{Y}$ are closely related (for unexplained terminology see §2) and the sequence $(\mathcal{Y} \equiv \mathcal{D} \equiv \mathcal{X})$ can be thought of as a short exact sequence of triangulated categories. In representation theory of finite dimensional algebras, a lot of homological invariants can be glued with respect to a recollement. For instance, given a recollement of (bounded) derived categories of finite dimensional algebras A, B and C over a field, Hochschild and cyclic homology and cohomology of A, B and C, as well as, K-theory, are related by long exact sequences [20, 25, 31, 44, 56, 58, 60]. Furthermore, global dimension, Cartan determinants and finitistic dimensions of A, B and C are related [5, 17, 22, 30, 51].

Another topic closely related to the structure of triangulated categories is silting theory. Silting theory is a new and dynamically developing topic in representation theory, it studies a special type of generators of triangulated categories, which have very nice properties but are sufficiently widespread. Silting and more generally partial silting objects in the bounded homotopy category $\mathcal{K}^b(\mathrm{proj}\text{-}A)$ of finitely generated projective modules over a finite dimensional algebra A were introduced in [29] and further developed in [1] as a completion of tilting objects under mutation. These objects parametrize bounded t-structures in $\mathcal{D}^b(\mathrm{mod}\text{-}A)$ whose heart is a length category and bounded co-t-structures in $\mathcal{K}^b(\mathrm{proj}\text{-}A)$ (see [32] and [27]), they also correspond to derived equivalences from A to a non-positive dg algebra. The notion was extended to the unbounded setting first in [59] and [6], for unbounded derived categories of algebras, and later in [50] and [47] for arbitrary triangulated categories with coproducts. Concepts like complete exceptional collections or sequences in Algebraic Geometry can be interpreted as silting sets in the derived category of a scheme (see [47, § 8]). The main feature of these generalized silting sets is that they naturally define a t-structure in the ambient triangulated category whose heart is, in many situations, a module category (see [47, § 4]). One of the results of the development of silting theory in the unbounded setting is the introduction of silting modules in [6], which have turned out to be very useful to classify homological ring epimorphisms and universal localizations (see [7] and [36]).

Traditionally, for the study of homological properties of a variety $\mathbb{X}$ over a field (or a Noetherian scheme of finite type) or a finite dimensional algebra A the bounded derived category of coherent sheaves $\mathcal{D}^b(\mathbb{X}) := \mathcal{D}^b(\mathrm{coh}(\mathbb{X}))$ or the bounded derived category $\mathcal{D}^b(\mathrm{mod}\text{-}A)$ of finitely generated A-modules are considered. Therefore, a lot of studies are concentrated on recollements of bounded derived categories and silting objects in the homotopy category of finitely generated projectives. However, the study of the structure of the unbounded derived categories is sometimes easier, since a rich arsenal of techniques of compactly generated triangulated categories with products and coproducts becomes available. The generation of recollements and t-structures also becomes more accessible at the unbounded level. Hence it is natural to look for conditions under which recollements of triangulated

categories at the 'bounded' level lift to recollements at the 'unbounded' level. This is the first goal of this paper. The motivation for this stems from [**5**], where the authors show that recollements of bounded derived categories of finite dimensional algebras lift up to equivalence to corresponding recollements of the unbounded derived categories, which, in addition, can be extended upwards and downwards to ladders of recollements of height three. We wanted to know to what extent this relies on the context of finite dimensional algebras and to what extent this is a general phenomenon occurring in triangulated categories. As it will become apparent through the paper, our results are applicable to other areas of Mathematics, especially to Algebraic Geometry. In our general framework it turns out that lifting torsion torsion-free (TTF) triples can be guaranteed under quite general assumptions, whereas lifting of recollements is more subtle (see § 3).

Since t-structures can be glued via a recollement (see [**10**]) another natural question is the following: given a recollement (1.1) of triangulated categories and silting sets $\mathcal{T}_X$ and $\mathcal{T}_Y$ in $\mathcal{X}$ and $\mathcal{Y}$, is it possible to construct a silting set $\mathcal{T}$ in $\mathcal{D}$ corresponding to the glued t-structure? This problem was studied in the context of tilting objects in [**4**] under some restrictions and in the context of gluing with respect to co-t-structures in [**35**]. The second goal of this paper is to show that the process of gluing t-structures allows to construct partial silting sets in the central category of a recollement out of partial silting sets in its outer categories.

The paper is organized as follows. In § 2 we introduce most of the concepts and terminology used throughout the paper. In § 3 we study lifting of recollements and the associated TTF triples. In particular we prove a criterion for a recollement $(\mathcal{Y} \equiv \mathcal{D} \equiv \mathcal{X})$ of thick subcategories of compactly generated triangulated categories $\hat{\mathcal{Y}}, \hat{\mathcal{D}}$ and $\hat{\mathcal{X}}$ to lift to a TTF triple in $\hat{\mathcal{D}}$ under the assumption that the subcategories $\mathcal{Y}, \mathcal{D}, \mathcal{X}$ contain the respective subcategories of compact objects (see theorem 3.3). As a consequence of this theorem lifting of TTF triples is possible for several types of recollements, such as recollements of stable categories of repetitive algebras or self-injective finite length algebras or recollements of bounded derived categories of separated Noetherian schemes (see example 3.6). However, lifting of the recollement is more delicate and the answer to the following question seems to be unknown.

QUESTION. Does the lifting of the TTF triple corresponding to the recollement $(\mathcal{Y} \equiv \mathcal{D} \equiv \mathcal{X})$ imply the lifting of this recollement to a recollement $(\hat{\mathcal{Y}} \equiv \hat{\mathcal{D}} \equiv \hat{\mathcal{X}})$ at least up to equivalence?

It seems to be related to the problem of constructing a triangulated equivalence $\hat{\mathcal{D}} \xrightarrow{\cong} \hat{\mathcal{E}}$, having a triangulated equivalence $\hat{\mathcal{D}}^c \xrightarrow{\cong} \hat{\mathcal{E}}^c$, for compactly generated triangulated categories $\hat{\mathcal{D}}$ and $\hat{\mathcal{E}}$. Due to results of Rickard (see [**52**] and [**53**]) and Keller [**24**], such a construction is possible (even though it may not be the lift of the equivalence $\hat{\mathcal{D}}^c \xrightarrow{\cong} \hat{\mathcal{E}}^c$) when one of the categories $\hat{\mathcal{D}}$ or $\hat{\mathcal{E}}$ is the derived category of an algebra or, more generally, a small K-category. This is the reason for the following consequence of theorem 3.14.

THEOREM. *Let $\mathcal{B}$ and $\mathcal{C}$ be small K-linear categories, viewed as dg categories concentrated in zero degree, let $\mathcal{A}$ be a dg category, and suppose that there is a recollement $(\mathcal{D}_{\dagger}^{\star}(\mathcal{B}) \equiv \mathcal{D}_{\dagger}^{\star}(\mathcal{A}) \equiv \mathcal{D}_{\dagger}^{\star}(\mathcal{C}))$, where $\star \in \{\emptyset, +, -, b\}$ and $\dagger \in \{\emptyset, fl\}$ (here fl*

means 'finite length') and all the subcategories contain the respective subcategories of compact objects.

If the functors $j_!, j^*, i^*, i_*$ preserve compact objects, the given recollement lifts up to equivalence to a recollement $(\mathcal{D}(\mathcal{B}) \equiv \mathcal{D}(\mathcal{A}) \equiv \mathcal{D}(\mathcal{C}))$, which is the upper part of a ladder of recollements of height two.

In certain circumstances, one can guarantee that the functors preserve compact objects. For example, this holds in the situation of a recollement $\mathcal{Y} \equiv \mathcal{D} \equiv \mathcal{X}$ of homologically non-positive homologically finite length dg algebras A, B and C. This allows us to generalize some of the results from [**5**] (see proposition 3.19).

In §4 we define partial silting sets and objects in arbitrary triangulated categories, give some examples, and study when partial silting sets are uniquely determined by the associated t-structure. In §5 we revise the connection between the construction of (pre)envelopes, t-structures and co-t-structures. It is seen, in particular, that the bijection between silting objects and bounded co-t-structures of [**1**, **32**, **38**] extends to any small triangulated category with split idempotents, more generally, one can consider one-sided bounded co-t-structures in the bijection (see proposition 5.9). The study of envelopes is later used in the last section to give an explicit construction of a classical silting object glued with respect to a recollement of bounded derived categories of finite length algebras.

In §6 is devoted to the construction of partial silting sets in arbitrary triangulated categories by gluing t-structures via recollements. Our results on gluing partial silting sets are based on a technical criterion (theorem 6.3), which allows to glue partial silting sets $\mathcal{T}_X$ and $\mathcal{T}_Y$ in triangulated categories $\mathcal{X}$ and $\mathcal{Y}$ with respect to a recollement $(\mathcal{Y} \equiv \mathcal{D} \equiv \mathcal{X})$. Conditions of theorem 6.3 are easier to check when $\mathcal{T}_X$ and $\mathcal{T}_Y$ consist of compact objects and some of the functors in the recollement preserve compact objects. We refer the reader to a more general theorem 6.10, which has the following consequence. Note that when A, B and C are finite length algebras the corollary can be applied, replacing $\mathcal{D}_{fl}^b(-)$ by the equivalent category $\mathcal{D}^b(\mathrm{mod}-)$.

COROLLARY (see corollary 6.12, 6.13). *Let A, B and C be homologically finite length dg algebras, the first of which is homologically non-positive. Let $(\mathcal{D}_{fl}^b(B) \equiv \mathcal{D}_{fl}^b(A) \equiv \mathcal{D}_{fl}^b(C))$ be a recollement, let T_C and T_B be silting objects in $\mathcal{D}^c(C)$ and $\mathcal{D}^c(B)$, respectively, with the associated t-structures $(\mathcal{X}^{\leqslant 0}, \mathcal{X}^{\geqslant 0})$ and $(\mathcal{Y}^{\leqslant 0}, \mathcal{Y}^{\geqslant 0})$. There exists a triangle $\tilde{T}_B \longrightarrow i_*(T_B) \longrightarrow U_{T_B}[1] \overset{+}{\longrightarrow}$ in $\mathcal{D}^c(A)$ such that $U_{T_B} \in j_!(\mathcal{X}^{\leqslant 0})$ and $\tilde{T}_B \in {}^{\perp}j_!(\mathcal{X}^{\leqslant 0})[1]$. In particular $T = j_!(T_C) \oplus \tilde{T}_B$ is a silting object in $\mathcal{D}^c(A)$, uniquely determined up to add-equivalence, which generates the glued t-structure $(\mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant 0})$ in $\mathcal{D}_{fl}^b(A)$.*

We finish the paper, comparing our results on gluing silting objects in the particular context of finite dimensional algebras over a field with the results of [**35**]. As mentioned before, our methods provide an explicit inductive construction of the glued silting object in this case, we illustrate how to apply this explicit inductive construction with an example.

2. Preliminaries

All categories considered in this paper are K-categories over some commutative ring K and all functors are assumed to be K-linear. Unless explicitly said otherwise, the categories which appear will be either triangulated K-categories with split idempotents or their subcategories, and all of them are assumed to have Hom-sets. All subcategories will be full and closed under isomorphisms. Coproducts and products are always small (i.e. set-indexed). The expression '$\mathcal{D}$ has coproducts (resp. products)' will mean that $\mathcal{D}$ has arbitrary set-indexed coproducts (resp. products). When $\mathcal{S} \subset \mathrm{Ob}(\mathcal{D})$ is a class of objects, we shall denote by $\mathrm{add}_{\mathcal{A}}(\mathcal{S})$ (resp. $\mathrm{Add}_{\mathcal{A}}(\mathcal{S})$) the subcategory of $\mathcal{D}$ consisting of the objects which are direct summands of finite (resp. arbitrary) coproducts of objects in $\mathcal{S}$.

Let $\mathcal{D}$ be a triangulated category, we will denote by $[1] : \mathcal{D} \longrightarrow \mathcal{D}$ the *suspension functor*, $[k]$ will denote the k-th power of $[1]$, for each integer k. *(Distinguished) triangles* in $\mathcal{D}$ will be denoted by $X \longrightarrow Y \longrightarrow Z \overset{+}{\longrightarrow}$. A *triangulated functor* $F : \mathcal{D}_1 \longrightarrow \mathcal{D}_2$ between triangulated categories is an additive functor together with a natural isomorphism $F \circ [1] \simeq [1] \circ F$, which sends triangles to triangles. For more details on triangulated categories see [40].

Let $\mathcal{D}$ be a triangulated category and let $\mathcal{S}$ be a class of objects in $\mathcal{D}$. We are going to use the following subcategories of $\mathcal{D}$:

$$\mathcal{S}^{\perp} = \{X \in \mathcal{D} \mid \mathrm{Hom}_{\mathcal{D}}(S, X) = 0 \text{ for any } S \in \mathcal{S}\}$$

$$^{\perp}\mathcal{S} = \{X \in \mathcal{D} \mid \mathrm{Hom}_{\mathcal{D}}(X, S) = 0 \text{ for any } S \in \mathcal{S}\}$$

for an integer n and $*$ standing for $\leqslant n, \geqslant n, > n, < n$ or $k \in \mathbb{Z}$

$$\mathcal{S}^{\perp *} = \{X \in \mathcal{D} \mid \mathrm{Hom}_{\mathcal{D}}(S, X[k]) = 0 \text{ for any } S \in \mathcal{S} \text{ and } k \in \mathbb{Z} \text{ satisfying } *\}$$

$$^{\perp *}\mathcal{S} = \{X \in \mathcal{D} \mid \mathrm{Hom}_{\mathcal{D}}(X, S[k]) = 0 \text{ for any } S \in \mathcal{S} \text{ and } k \in \mathbb{Z} \text{ satisfying } *\}.$$

Given two subcategories $\mathcal{X}$ and $\mathcal{Y}$ of a triangulated category $\mathcal{D}$, we will denote by $\mathcal{X} \star \mathcal{Y}$ the subcategory of $\mathcal{D}$ consisting of the objects M which fit into a triangle $X \longrightarrow M \longrightarrow Y \overset{+}{\longrightarrow}$, where $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. Due to the octahedral axiom, the operation $\star$ is associative, so for a family of subcategories $(\mathcal{X}_i)_{1 \leqslant i \leqslant n}$ the subcategory $\mathcal{X}_1 \star \mathcal{X}_2 \star \cdots \star \mathcal{X}_n$ is well-defined (see [10]). A subcategory $\mathcal{X}$ is *closed under extensions* when $\mathcal{X} \star \mathcal{X} \subseteq \mathcal{X}$.

Given a triangulated category $\mathcal{D}$, a subcategory $\mathcal{E}$ will be called a *suspended* (resp. *strongly suspended*) *subcategory* if $\mathcal{E}[1] \subseteq \mathcal{E}$ and $\mathcal{E}$ is closed under extensions (resp. extensions and direct summands). If $\mathcal{E}$ is strongly suspended and $\mathcal{E} = \mathcal{E}[1]$, we will say that $\mathcal{E}$ is a *thick subcategory*. When $\mathcal{D}$ has coproducts, a triangulated subcategory closed under taking arbitrary coproducts is called a *localizing subcategory*. Note that such a subcategory is always thick by [40, proposition 1.6.8]. Clearly, there are dual concepts of a *(strongly) cosuspended subcategory* and a *colocalizing subcategory*, while that of a thick subcategory is self-dual. Given a class $\mathcal{S}$ of objects of $\mathcal{D}$, we will denote by $\mathrm{susp}_{\mathcal{D}}(\mathcal{S})$ (resp. $\mathrm{thick}_{\mathcal{D}}(\mathcal{S})$) the smallest strongly suspended (resp. thick) subcategory of $\mathcal{D}$ containing $\mathcal{S}$. When $\mathcal{D}$ has coproducts, we will let $\mathrm{Susp}_{\mathcal{D}}(\mathcal{S})$ and $\mathrm{Loc}_{\mathcal{D}}(\mathcal{S})$ be the smallest (strongly) suspended subcategory

closed under taking coproducts and the smallest localizing subcategory containing $\mathcal{S}$, respectively.

2.1. Torsion pairs, t-structures and co-t-structures

A pair of subcategories $(\mathcal{X}, \mathcal{Y})$ in $\mathcal{D}$ is a *torsion pair* if

- $\mathcal{X}$ and $\mathcal{Y}$ are closed under direct summands;

- $\mathrm{Hom}_{\mathcal{D}}(X, Y) = 0$, for all $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$;

- $\mathcal{D} = \mathcal{X} \star \mathcal{Y}$.

A *t-structure* in $\mathcal{D}$ is a pair $(\mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant 0})$ such that $(\mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant 0}[-1])$ is a torsion pair and $\mathcal{D}^{\leqslant 0}[1] \subseteq \mathcal{D}^{\leqslant 0}$. A *co-t-structure* or a *weight structure* is a pair $(\mathcal{D}_{\geqslant 0}, \mathcal{D}_{\leqslant 0})$ such that $(\mathcal{D}_{\geqslant 0}[-1], \mathcal{D}_{\leqslant 0})$ is a torsion pair and $\mathcal{D}_{\geqslant 0}[-1] \subseteq \mathcal{D}_{\geqslant 0}$. Adopting the terminology used for t-structures, given a torsion pair $(\mathcal{X}, \mathcal{Y})$, we will call $\mathcal{X}$ and $\mathcal{Y}$ the *aisle* and the *co-aisle* of the torsion pair. Note that the aisle of a torsion pair $(\mathcal{X}, \mathcal{Y})$ is suspended (resp. cosuspended) if and only if $(\mathcal{X}, \mathcal{Y}[1])$ (resp. $(\mathcal{X}[1], \mathcal{Y})$) is a t-structure (resp. co-t-structure).

For a t-structure $(\mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant 0})$, the objects U and V in a triangle $U \longrightarrow M \longrightarrow V \overset{+}{\longrightarrow}$, with $U \in \mathcal{D}^{\leqslant 0}$ and $V \in \mathcal{D}^{>0} := \mathcal{D}^{\geqslant 0}[-1]$, are uniquely determined by $M \in \mathcal{D}$ up to isomorphism. The assignments $M \rightsquigarrow U$ and $M \rightsquigarrow V$ coincide on objects with the action of the functors $\tau^{\leqslant 0} : \mathcal{D} \longrightarrow \mathcal{D}^{\leqslant 0}$ and $\tau^{>0} : \mathcal{D} \longrightarrow \mathcal{D}^{>0}$, which are right and left adjoint to the inclusion functors. The functors $\tau^{\leqslant 0}$ and $\tau^{>0}$ are called the *left and right truncation functors* with respect to the t-structure. When $\tau = (\mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant 0})$ (resp. $(\mathcal{D}_{\geqslant 0}, \mathcal{D}_{\leqslant 0})$) is a t-structure (resp. a co-t-structure), the intersection $\mathcal{H} := \mathcal{D}^{\leqslant 0} \cap \mathcal{D}^{\geqslant 0}$ (resp. $\mathcal{C} := \mathcal{D}_{\geqslant 0} \cap \mathcal{D}_{\leqslant 0}$) is called the *heart (resp. co-heart) of the t-structure (resp. co-t-structure)*. Recall that $\mathcal{H}$ is an abelian category in which the short exact sequences are induced by the triangles with all the three terms in $\mathcal{H}$ (see [**10**]). Sometimes we shall use the term *co-heart of the t-structure* τ, meaning the intersection $\mathcal{C}_{\tau} = {}^{\perp}(\mathcal{D}^{\leqslant 0})[1] \cap \mathcal{D}^{\leqslant 0}$. A *semi-orthogonal decomposition* of $\mathcal{D}$ is a torsion pair $(\mathcal{X}, \mathcal{Y})$ such that $\mathcal{X} = \mathcal{X}[1]$ (or, equivalently, $\mathcal{Y} = \mathcal{Y}[1]$). Note that such a pair is both a t-structure and a co-t-structure in $\mathcal{D}$, and the corresponding truncation functors are triangulated. The notions of torsion pair, t-structure, co-t-structure and semi-orthogonal decomposition are self-dual.

If $\mathcal{D}'$ is a thick subcategory of $\mathcal{D}$, we say that a torsion pair $\tau = (\mathcal{X}, \mathcal{Y})$ in $\mathcal{D}$ *restricts to* $\mathcal{D}'$ when $\tau' := (\mathcal{X} \cap \mathcal{D}', \mathcal{Y} \cap \mathcal{D}')$ is a torsion pair in $\mathcal{D}'$. In this case τ' is called the *restriction* of τ to $\mathcal{D}'$. Conversely, when $\tau' = (\mathcal{X}', \mathcal{Y}')$ is a torsion pair in $\mathcal{D}'$, we say that it *lifts to* $\mathcal{D}$ if there is a torsion pair $\tau = (\mathcal{X}, \mathcal{Y})$ in $\mathcal{D}$ which restricts to τ'. Then τ is called a *lifting of* τ' *to* $\mathcal{D}$.

Given two torsion pairs $\tau = (\mathcal{X}, \mathcal{Y})$ and $\tau' = (\mathcal{Y}', \mathcal{Z})$ in $\mathcal{D}$, we shall say that τ *is left adjacent to* τ' or that τ' *is right adjacent to* τ or that τ and τ' (in this order) are *adjacent torsion pairs* when $\mathcal{Y} = \mathcal{Y}'$. Note that the torsion pairs associated to the co-t-structure $(\mathcal{D}_{\geqslant 0}, \mathcal{D}_{\leqslant 0})$ and the t-structure $(\mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant 0})$ are adjacent if and only if $\mathcal{D}_{\leqslant 0} = \mathcal{D}^{\leqslant 0}$. In this case their co-hearts coincide. A triple of subcategories $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ of $\mathcal{D}$ is called a *torsion torsion-free triple (TTF triple, for short)* when $(\mathcal{X}, \mathcal{Y})$ and $(\mathcal{Y}, \mathcal{Z})$ are adjacent t-structures, which is equivalent to saying that they

are adjacent semi-orthogonal decompositions. We shall say that such a TTF triple is *extendable to the right* when $(\mathcal{Y}, \mathcal{Z}, \mathcal{Z}^\perp)$ is also a TTF triple. By [**45**, proposition 3.4], we know that this is always the case when $\mathcal{D}$ and the torsion pair $(\mathcal{Y}, \mathcal{Z})$ are compactly generated (see §2.3 for the definition of compact generation). As before, one can consider lifting and restriction of TTF triples (see [**46**] for details).

2.2. Recollements

Let $\mathcal{D}$, $\mathcal{X}$ and $\mathcal{Y}$ be triangulated categories. $\mathcal{D}$ is said to be a *recollement* of $\mathcal{X}$ and $\mathcal{Y}$ if there are six triangulated functors as in the following diagram

$$
\mathcal{Y} \xrightarrow[\xleftarrow{\quad i_* \quad}]{\substack{\xleftarrow{\ i^* \ } \\ \xrightarrow{\ i_* \ } \\ \xrightarrow{\ i^! \ }}} \mathcal{D} \xrightarrow[\xleftarrow{\quad j_* \quad}]{\substack{\xleftarrow{\ j_! \ } \\ \xrightarrow{\ j^* \ } \\ \xrightarrow{\ j_* \ }}} \mathcal{X}
\tag{2.1}
$$

such that

(1) (i^*, i_*), $(i_*, i^!)$, $(j_!, j^*)$, (j^*, j_*) are adjoint pairs,

(2) i_*, j_*, $j_!$ are full embeddings,

(3) $i^! j_* = 0$ (and, hence $j^* i_* = 0$ and $i^* j_! = 0$),

(4) for any $Z \in \mathcal{D}$ the units and the counits of the adjunctions give triangles:

$$
i_* i^! Z \longrightarrow Z \longrightarrow j_* j^* Z \xrightarrow{\ +\ },
$$
$$
j_! j^* Z \longrightarrow Z \longrightarrow i_* i^* Z \xrightarrow{\ +\ }.
$$

To any recollement one canonically associates the TTF triple $(\mathrm{Im}(j_!), \mathrm{Im}(i_*),$ $\mathrm{Im}(j_*))$ in $\mathcal{D}$. Conversely, if $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ is a TTF triple in $\mathcal{D}$, then one obtains a recollement as above, where $j_! : \mathcal{X} \hookrightarrow \mathcal{D}$ and $i_* : \mathcal{Y} \hookrightarrow \mathcal{D}$ are the inclusion functors. Two recollements $(\mathcal{Y} \equiv \mathcal{D} \equiv \mathcal{X})$ and $(\tilde{\mathcal{Y}} \equiv \mathcal{D} \equiv \tilde{\mathcal{X}})$ are said to be *equivalent* when the associated TTF triples coincide. It is easy to see that this is equivalent to the existence of triangulated equivalences $F : \mathcal{X} \xrightarrow{\cong} \tilde{\mathcal{X}}$ and $G : \mathcal{Y} \xrightarrow{\cong} \tilde{\mathcal{Y}}$ such that the sextuple of functors associated to the second recollement is pointwise naturally isomorphic to $(G \circ i^*, i_* \circ G^{-1}, G \circ i^!, j_! \circ F^{-1}, F \circ j^*, j_* \circ F^{-1})$, for any choice of quasi-inverses F^{-1} and G^{-1} of F and G.

Given thick subcategories $\mathcal{Y}' \subseteq \mathcal{Y}$, $\mathcal{D}' \subseteq \mathcal{D}$ and $\mathcal{X}' \subseteq \mathcal{X}$ and a recollement

$$
\mathcal{Y}' \xrightarrow[\xleftarrow{\quad \tilde{i}_* \quad}]{\substack{\xleftarrow{\ \tilde{i}^* \ } \\ \xrightarrow{\ \tilde{i}_* \ } \\ \xrightarrow{\ \tilde{i}^! \ }}} \mathcal{D}' \xrightarrow[\xleftarrow{\quad \tilde{j}_* \quad}]{\substack{\xleftarrow{\ \tilde{j}_! \ } \\ \xrightarrow{\ \tilde{j}^* \ } \\ \xrightarrow{\ \tilde{j}_* \ }}} \mathcal{X}' \ ,
\tag{2.2}
$$

we say that the recollement (2.1) *restricts* to the recollement (2.2) or that the recollement (2.2) *lifts* to the recollement (2.1), when the functors in the recollement (2.2) are naturally isomorphic to the restrictions of the functors in the recollement (2.1). We shall say that the recollement (2.1) *restricts, up to equivalence,* to the recollement (2.2), or that the recollement (2.2) *lifts, up to equivalence,* to the

recollement (2.1) when the TTF triple $(\mathrm{Im}(j_!), \mathrm{Im}(i_*), \mathrm{Im}(j_*))$ restricts to $\mathcal{D}'$ and the restriction coincides with $(\mathrm{Im}(\tilde{j}_!), \mathrm{Im}(\tilde{i}_*), \mathrm{Im}(\tilde{j}_*))$. Obviously 'restricts' implies 'restricts up to equivalence'.

REMARK 2.1. The typical situation throughout this paper is that of a recollement (2.2), where we know that $\mathcal{Y}'$, $\mathcal{D}'$ and $\mathcal{X}'$ are thick subcategories of $\mathcal{Y}$, $\mathcal{D}$ and $\mathcal{X}$, respectively. The condition that the TTF triple $(\mathrm{Im}(\tilde{j}_!), \mathrm{Im}(\tilde{i}_*), \mathrm{Im}(\tilde{j}_*))$ in $\mathcal{D}'$ lifts to a TTF triple in $\mathcal{D}$ does not mean that it lifts to a TTF triple coming from a recollement (2.1), i.e. it might happen that we cannot find functors $i^*, i_*, i^!, j_!, j^*$ and j_*, which restrict to the functors $\tilde{i}^*, \tilde{i}_*, \tilde{i}^!, \tilde{j}_!, \tilde{j}^*$ and $\tilde{j}_*$. Therefore if the recollement (2.2) lifts up to equivalence to a recollement (2.1), then the TTF triple $(\mathrm{Im}(\tilde{j}_!), \mathrm{Im}(\tilde{i}_*), \mathrm{Im}(\tilde{j}_*))$ lifts to a TTF triple in $\mathcal{D}$, but the converse need not be true. Similarly, it might happen that a TTF triple coming from the recollement (2.1) restricts to the subcategory $\mathcal{D}'$ but the functors from (2.1) do not restrict to $\mathcal{Y}', \mathcal{D}'$ and $\mathcal{X}'$.

Given torsion pairs $(\mathcal{X}', \mathcal{X}'')$ and $(\mathcal{Y}', \mathcal{Y}'')$ in $\mathcal{X}$ and $\mathcal{Y}$, respectively, *the torsion pair glued with respect to the recollement (2.1)* is the pair $(\mathcal{D}', \mathcal{D}'')$ in $\mathcal{D}$, where

$$\mathcal{D}' = \{Z \in \mathcal{D} | j^*Z \in \mathcal{X}', i^*Z \in \mathcal{Y}'\},$$

$$\mathcal{D}'' = \{Z \in \mathcal{D} | j^*Z \in \mathcal{X}'', i^!Z \in \mathcal{Y}''\}.$$

Moreover, when the original torsion pairs are associated to t-structures (resp. co-t-structures or semi-orthogonal decompositions), the resulting torsion pair is associated to a t-structure (resp. co-t-structure or semi-orthogonal decomposition) in $\mathcal{D}$ (see [**10**, théorème 1.4.10] for t-structures and [**13**, theorem 8.2.3] for co-t-structures).

A *ladder of recollements* $\mathcal{L}$ is a finite or infinite diagram of triangulated categories and triangulated functors

$$
\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
\mathcal{C}' & \begin{array}{c}\longrightarrow\\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow\end{array} & \mathcal{C} & \begin{array}{c}\longrightarrow\\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow\end{array} & \mathcal{C}'' \\
\vdots & & \vdots & & \vdots
\end{array}
$$

such that any three consecutive rows form a recollement (see [**5, 11**]). The *height* of a ladder is the number of recollements contained in it (counted with multiplicities).

2.3. Generation and Milnor colimits

For a class of objects $\mathcal{S}$ in $\mathcal{D}$ one can consider the pair of subcategories $(\mathcal{X}, \mathcal{Y}) = (^\perp(\mathcal{S}^\perp), \mathcal{S}^\perp)$. Then, $\mathcal{X}$ and $\mathcal{Y}$ are closed under direct summands and $\mathrm{Hom}_{\mathcal{D}}(X, Y) = 0$ for all $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. However, the inclusion $\mathcal{X} \star \mathcal{Y} \subseteq \mathcal{D}$

might be strict, so that $(\mathcal{X}, \mathcal{Y})$ is not necessarily a torsion pair. We shall say that $\mathcal{S}$ *generates a torsion pair in* $\mathcal{D}$ or that *the torsion pair generated by* $\mathcal{S}$ *in* $\mathcal{D}$ *exists* if $(^{\perp}(\mathcal{S}^{\perp}), \mathcal{S}^{\perp})$ is a torsion pair. Slightly abusing common terminology, we will say that $\mathcal{S}$ *generates a t-structure (resp. co-t-structure or semiorthogonal decomposition)* or that *the t-structure (resp. co-t-structure or semiorthogonal decomposition) generated by* $\mathcal{S}$ *in* $\mathcal{D}$ *exists* when the torsion pair generated by $\bigcup_{k \geq 0} \mathcal{S}[k]$ (resp. $\bigcup_{k \leq 0} \mathcal{S}[k]$ or $\bigcup_{k \in \mathbb{Z}} \mathcal{S}[k]$) exists. In all those cases $\mathcal{S}$ is contained in the aisle of the corresponding t-structure (resp. co-t-structure, resp. semi-orthogonal decomposition). That is, if $(\mathcal{X}, \mathcal{Y})$ is the torsion pair constructed above, then $(\mathcal{X}, \mathcal{Y}[1]) = (^{\perp}(\mathcal{S}^{\perp \leq 0}), \mathcal{S}^{\perp < 0})$ and $(\mathcal{X}, \mathcal{Y}[-1]) = (^{\perp}(\mathcal{S}^{\perp \geq 0}), \mathcal{S}^{\perp > 0})$ are the t-structure and co-t-structure generated by $\mathcal{S}$. The semi-orthogonal decomposition generated by $\mathcal{S}$ is $(^{\perp}(\mathcal{S}^{\perp k \in \mathbb{Z}}), \mathcal{S}^{\perp k \in \mathbb{Z}})$ and not $(^{\perp}(\mathcal{S}^{\perp k \in \mathbb{Z}}), \mathcal{S}^{\perp k \in \mathbb{Z}})$. The definition of the dual notions is left to the reader. We just point out that, keeping the dual philosophy of forcing $\mathcal{S}$ to be contained in the co-aisle, the *t-structure (resp. co-t-structure) cogenerated by* $\mathcal{S}$, when it exists, is the pair $(^{\perp < 0}\mathcal{S}, (^{\perp \leq 0}\mathcal{S})^{\perp})$ (resp. $(^{\perp > 0}\mathcal{S}, (^{\perp \geq 0}\mathcal{S})^{\perp})$). A class (resp. set) $\mathcal{S} \subset \mathrm{Ob}(\mathcal{D})$ is a *generating class (resp. set)* of $\mathcal{D}$ if $\mathcal{S}^{\perp k \in \mathbb{Z}} = 0$, in this case we will also say, that $\mathcal{S}$ *generates* $\mathcal{D}$. We say that $\mathcal{D}$ satisfies the *property of infinite dévissage with respect to* $\mathcal{S}$ when $\mathcal{D} = \mathrm{Loc}_{\mathcal{D}}(\mathcal{S})$, a fact that implies that $\mathcal{S}$ generates $\mathcal{D}$. When $\mathcal{D}$ has coproducts, a *compact object* is an object X such that the canonical map $\coprod_{i \in I} \mathrm{Hom}_{\mathcal{D}}(X, M_i) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(X, \coprod_{i \in I} M_i)$ is bijective, for each family of objects $(M_i)_{i \in I}$. A torsion pair is called *compactly generated* when there exists a <u>set</u> of compact objects which generates the torsion pair. We say that $\mathcal{D}$ *is a compactly generated triangulated category* when it has a generating <u>set</u> of compact objects. It is well-known that in this case the subcategory $\mathcal{D}^c$ of the compact objects of $\mathcal{D}$ is skeletally small (see, e.g. [**40**, lemma 4.5.13]). A triangulated category is called *algebraic* when it is triangulated equivalent to the stable category of a Frobenius exact category (see [**26**] and [**49**]).

Assuming that $\mathcal{D}$ has coproducts, for a sequence of morphisms

$$0 = X_{-1} \xrightarrow{x_0} X_0 \xrightarrow{x_1} X_1 \xrightarrow{x_2} \cdots \xrightarrow{x_n} X_n \xrightarrow{x_{n+1}} \cdots \qquad (2.3)$$

let us denote by $\sigma : \coprod_{n \in \mathbb{N}} X_n \longrightarrow \coprod_{n \in \mathbb{N}} X_n$ the unique morphism such that $\sigma \circ \iota_k = \iota_{k+1} \circ x_{k+1}$, where $\iota_k : X_k \longrightarrow \coprod_{n \in \mathbb{N}} X_n$ is the canonical inclusion, for all $k \in \mathbb{N}$. The *Milnor colimit* (or homotopy colimit [**40**]) of the sequence is the object $X = \mathrm{Mcolim}(X_n)$ which appears in the triangle

$$\coprod_{n \in \mathbb{N}} X_n \xrightarrow{1 - \sigma} \coprod_{n \in \mathbb{N}} X_n \longrightarrow X \xrightarrow{+} .$$

We will frequently use the fact that, when $\mathcal{D}$ has coproducts and $\mathcal{S}$ is a set of compact objects, the pair $(^{\perp}(\mathcal{S}^{\perp}), \mathcal{S}^{\perp})$ is a torsion pair in $\mathcal{D}$ (see [**1**, theorem 4.3] or [**2**, **42**]). In the case of the t-structure (resp. semi-orthogonal decomposition) generated by $\mathcal{S}$, one has $^{\perp}(\mathcal{S}^{\perp \leq 0}) = \mathrm{Susp}_{\mathcal{D}}(\mathcal{S})$ (resp. $^{\perp}(\mathcal{S}^{\perp i \in \mathbb{Z}}) = \mathrm{Loc}_{\mathcal{D}}(\mathcal{S})$) (see [**28**, theorem 12.1], [**45**, lemma 2.3]). Furthermore, the objects of $\mathrm{Loc}_{\mathcal{D}}(\mathcal{S})$ (resp. $\mathrm{Susp}_{\mathcal{D}}(\mathcal{S})$, for a non-positive $\mathcal{S}$) are precisely the Milnor colimits of sequences of the form (2.3), where the cone of each x_n, denoted by $\mathrm{cone}(x_n)$, is a coproduct of objects from $\bigcup_{k \in \mathbb{Z}} \mathcal{S}[k]$ (resp. of objects from $\mathcal{S}[n]$), for each $n \in \mathbb{N}$ (see the proof

of [**40**, theorem 8.3.3] for $\mathrm{Loc}_{\mathcal{D}}(\mathcal{S})$, and [**28**, theorem 12.2] and [**47**, theorem 2] for $\mathrm{Susp}_{\mathcal{D}}(\mathcal{S})$) (see also [**57**, theorem 3.7]).

A class of objects $\mathcal{T}$ is called *non-positive* if $\mathrm{Hom}_{\mathcal{D}}(T, T'[i]) = 0$ for any $T, T' \in \mathcal{T}, i > 0$. Two non-positive sets $\mathcal{T}$ and $\mathcal{T}'$ are said to be *add- (resp. Add-) equivalent* when $\mathrm{add}(\mathcal{T}) = \mathrm{add}(\mathcal{T}')$ (resp. $\mathrm{Add}(\mathcal{T}) = \mathrm{Add}(\mathcal{T}')$).

2.4. DG categories and algebras

A *differential graded (=dg) category* is a category $\mathcal{A}$ such that, for each pair (A, B) of its objects, the K-module of morphisms, denoted by $\mathcal{A}(A, B)$, has a structure of a differential graded K-module (or equivalently a structure of a complex of K-modules) so that the composition map $\mathcal{A}(B, C) \otimes \mathcal{A}(A, B) \longrightarrow \mathcal{A}(A, C)$ $(g \otimes f \rightsquigarrow g \circ f)$ is a morphism of degree zero of the underlying graded K-modules which commutes with the differentials. This means that $d(g \circ f) = d(g) \circ f + (-1)^{|g|} g \circ d(f)$ whenever $g \in \mathcal{A}(B, C)$ and $f \in \mathcal{A}(A, B)$ are homogeneous morphisms and $|g|$ is the degree of g. The reader is referred to [**24**] and [**26**] for details on dg categories. The most important concept for us is the *derived category of a small dg category*, denoted by $\mathcal{D}(\mathcal{A})$. It is the localization, in the sense of Gabriel–Zisman [**18**] of $\mathcal{C}(\mathcal{A})$ with respect to the class of quasi-isomorphisms. Here $\mathcal{C}(\mathcal{A})$ denotes the category whose objects are the (right) dg $\mathcal{A}$-modules (i.e. the dg functors $M : \mathcal{A}^{op} \longrightarrow \mathcal{C}_{dg}K$, where $\mathcal{C}_{dg}K$ is the category of dg K-modules) and the morphisms $f : M \longrightarrow N$ are the morphisms of degree zero in the underlying graded category which commute with the differentials. The category $\mathcal{D}(\mathcal{A})$ is triangulated and it turns out that, up to triangulated equivalence, the derived categories $\mathcal{D}(\mathcal{A})$ are precisely the compactly generated algebraic triangulated categories (see [**24**, theorem 4.3]). The canonical set of compact generators of $\mathcal{D}(\mathcal{A})$ is the set of *representable dg $\mathcal{A}$-modules* $\{A^\wedge : A \in \mathcal{A}\}$, where $A^\wedge : \mathcal{A}^{op} \longrightarrow \mathcal{C}_{dg}K$ takes A' to $\mathcal{A}(A', A)$, for each $A' \in \mathcal{A}$. We will frequently use the fact that there is a natural isomorphism of K-modules $\mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(A^\wedge, M[k]) \cong H^k(M(A))$, for $A \in \mathcal{A}$ and $M \in \mathcal{D}(\mathcal{A})$.

Two particular cases of small dg categories $\mathcal{A}$ will be of special interest to us. Any small K-category can be considered as a dg category concentrated in degree zero. A dg algebra A, i.e. an associative unital graded algebra A with a differential $d : A \longrightarrow A$ which satisfies the graded Leibniz rule, is a dg category with just one object. The intersection of both cases is the case of an associative unital algebra, called *ordinary algebra* throughout the paper, which is then considered as a dg category with just one object concentrated in degree zero. Such an algebra will be called *a finite length algebra* when it has finite length as a K-module (note that we are neither requiring K to be a field nor an Artin ring). Note also that an *Artin algebra* is just an ordinary algebra which is finite length over its centre, which, in turn, is a commutative Artin ring. Conversely, an ordinary K-algebra A is finite length if, and only if, it is an Artin algebra whose centre is a (commutative) finite length K-algebra. For any ordinary algebra A, we will denote by Mod-A (resp. mod-A, fl-A, Proj-A, proj-A) the category of all (resp. finitely presented, finite length, projective, finitely generated projective) right A-modules. We refer the reader to [**3**, **8**, **9**, **61**] for the classical terminology concerning ordinary rings, algebras and their modules.

3. On lifting recollements and TTF triples

In this section for thick subcategories of compactly generated algebraic triangulated categories we investigate the relation between the preservation of compactness by the functors of the recollement and lifting TTF triples and recollements.

The following lemma was proved in [**45**, lemma 2.4] and [**45**, lemma 2.3].

LEMMA 3.1. *Let $\mathcal{T}$ be a triangulated category with coproducts. If $(\mathcal{E}, \mathcal{F})$ is a compactly generated semi-orthogonal decomposition in $\mathcal{T}$, then $\mathcal{E}$ is compactly generated as a triangulated category and $\mathcal{E}^c = \mathcal{E} \cap \mathcal{T}^c$. Moreover, $\mathcal{F}$ has coproducts, calculated as in $\mathcal{T}$, and the left adjoint $\tau : \mathcal{T} \longrightarrow \mathcal{F}$ of the inclusion functor preserves compact objects. When, in addition, $\mathcal{T}$ is compactly generated, $\mathcal{F}$ is compactly generated by $\tau(\mathcal{T}^c)$.*

LEMMA 3.2. *Let $\hat{\mathcal{D}}$ be a compactly generated triangulated category and let $(\mathcal{U}_0, \mathcal{V}_0)$ be a semi-orthogonal decomposition of $\hat{\mathcal{D}}^c$. Then $(\mathcal{U}, \mathcal{V}, \mathcal{W}) := (Loc_{\hat{\mathcal{D}}}(\mathcal{U}_0), Loc_{\hat{\mathcal{D}}}(\mathcal{V}_0), Loc_{\hat{\mathcal{D}}}(\mathcal{V}_0)^\perp)$ is a TTF triple in $\hat{\mathcal{D}}$ such that $(\mathcal{U}, \mathcal{V})$ and $(\mathcal{V}, \mathcal{W})$ are compactly generated semiorthogonal decompositions. Moreover, $(\mathcal{U} \cap \hat{\mathcal{D}}^c, \mathcal{V} \cap \hat{\mathcal{D}}^c) = (\mathcal{U}_0, \mathcal{V}_0) = (\mathcal{U}^c, \mathcal{V}^c)$.*

Proof. The pair $(\mathcal{V}, \mathcal{W})$ is clearly a torsion pair. We need to prove that $(\mathcal{U}, \mathcal{V})$ is a torsion pair in $\hat{\mathcal{D}}$. The argument is standard and can be found in the literature (see [**46**]). We sketch it, leaving some details to the reader. Since objects in $\mathcal{U}_0$ are compact and objects in $\mathcal{V}$ are Milnor colimits of sequences of morphisms with cones in $\text{Add}(\mathcal{V}_0)$, we see that $\mathcal{V} \subseteq \mathcal{U}^\perp$. For $M \in \mathcal{U}^\perp = (\mathcal{U}_0)^\perp$ let us consider the truncation triangle $V \longrightarrow M \longrightarrow W \overset{+}{\longrightarrow}$ with respect to $(\mathcal{V}, \mathcal{W})$. We get $W \in \mathcal{U}^\perp$ and $W \in \mathcal{V}^\perp$. Therefore, $W \in (\mathcal{U}_0 \cup \mathcal{V}_0)^\perp$. But for each $D \in \hat{\mathcal{D}}^c$ there is a triangle $U_0 \longrightarrow D \longrightarrow V_0 \overset{+}{\longrightarrow}$, whose outer terms are in $\mathcal{U}_0$ and $\mathcal{V}_0$, respectively. It follows that $\text{Hom}_{\hat{\mathcal{D}}}(D, W) = 0$, for all $D \in \hat{\mathcal{D}}^c$. This implies that $W = 0$, so $V \cong M$ belongs to $\mathcal{V}$. Then the pair $(\mathcal{U}, \mathcal{V})$ is of the form $(\text{Loc}_{\hat{\mathcal{D}}}(\mathcal{U}_0), \text{Loc}_{\hat{\mathcal{D}}}(\mathcal{U}_0)^\perp)$, and hence is a torsion pair. The torsion pairs $(\mathcal{U}, \mathcal{V})$ and $(\mathcal{V}, \mathcal{W})$ are compactly generated by sets $\mathcal{U}_0$ and $\mathcal{V}_0$, respectively.

By lemma 3.1, we know that $(\mathcal{U} \cap \hat{\mathcal{D}}^c, \mathcal{V} \cap \hat{\mathcal{D}}^c) = (\mathcal{U}^c, \mathcal{V}^c)$. Since $\mathcal{U}_0$ is a thick subcategory of a compactly generated triangulated category $\mathcal{U}$ which generates $\mathcal{U}$ and consists of compact objects of $\mathcal{U}$, we get $\mathcal{U}_0 = \mathcal{U}^c$. Similarly, $\mathcal{V}_0 = \mathcal{V}^c$. $\square$

The key result of the section is the following.

THEOREM 3.3. *Let*

$$\mathcal{Y} \underset{\underset{i^!}{\overset{i_*}{\longleftarrow}}}{\overset{\overset{i^*}{\longleftarrow}}{\longrightarrow}} \mathcal{D} \underset{\underset{j_*}{\overset{j^*}{\longleftarrow}}}{\overset{\overset{j_!}{\longleftarrow}}{\longrightarrow}} \mathcal{X}$$

be a recollement, where $\mathcal{Y}$, $\mathcal{D}$ and $\mathcal{X}$ are thick subcategories of compactly generated triangulated categories $\hat{\mathcal{Y}}$, $\hat{\mathcal{D}}$ and $\hat{\mathcal{X}}$ which contain the respective subcategories of compact objects. Consider the following assertions:

(1) *The given recollement lifts to a recollement*

$$\hat{\mathcal{Y}} \xrightleftharpoons[\hat{i}^!]{\overset{\hat{i}^*}{\underset{\hat{i}_*}{\leftarrow}}} \hat{\mathcal{D}} \xrightleftharpoons[\hat{j}_*]{\overset{\hat{j}_!}{\underset{\hat{j}^*}{\leftarrow}}} \hat{\mathcal{X}}$$

which is the upper part of a ladder of recollements of height two.

(2) *The TTF triple $(Im(j_!), Im(i_*), Im(j_*))$ in $\mathcal{D}$ lifts to a TTF triple $(\mathcal{U}, \mathcal{V}, \mathcal{W})$ in $\hat{\mathcal{D}}$ such that:*

   (a) *The torsion pairs $(\mathcal{U}, \mathcal{V})$ and $(\mathcal{V}, \mathcal{W})$ are compactly generated (whence $(\mathcal{U}, \mathcal{V}, \mathcal{W})$ is extendable to the right);*

   (b) $j_!(\hat{\mathcal{X}}^c) = \mathcal{U} \cap \hat{\mathcal{D}}^c$ and $i_*(\hat{\mathcal{Y}}^c) = \mathcal{V} \cap \hat{\mathcal{D}}^c$.

(3) *The functors $j_!$, j^*, i^* and i_* preserve compact objects, i.e. $j_!(\hat{\mathcal{X}}^c) \subseteq \hat{\mathcal{D}}^c$ and similarly for j^*, i^* and i_*.*

The implications $(1) \implies (2) \implies (3)$ hold. Moreover, when $Im(i_*)$ cogenerates $Loc_{\hat{\mathcal{D}}}(i_*(\hat{\mathcal{Y}}^c))$ or $\mathcal{D}$ cogenerates $\hat{\mathcal{D}}$, the implication $(3) \implies (2)$ also holds.

Proof. $(1) \implies (2)$ The functors $\hat{j}_!$, $\hat{j}^*$, $\hat{i}^*$ and $\hat{i}_*$ preserve compact objects, since they have right adjoints which preserve coproducts, because they also have right adjoints. Let us consider the TTF triple $(\mathcal{U}, \mathcal{V}, \mathcal{W}) := (Im(\hat{j}_!), Im(\hat{i}_*), Im(\hat{j}_*))$ associated with the recollement from assertion 1. The torsion pair $(\mathcal{U}, \mathcal{V})$ is generated by $\hat{j}_!(\hat{\mathcal{X}}^c) = j_!(\hat{\mathcal{X}}^c)$. Indeed, the inclusion $(\hat{j}_!(\hat{\mathcal{X}}^c))^\perp \supseteq (Im(\hat{j}_!))^\perp$ is obvious, the inverse inclusion follows from the fact that, by infinite dévissage, $\hat{\mathcal{X}} = Loc_{\hat{\mathcal{X}}}(\hat{\mathcal{X}}^c)$ and $\hat{j}_!$ commutes with coproducts. Since $j_!(\hat{\mathcal{X}}^c)$ consists of compact objects and is skeletally small there is a set of compact objects generating $(\mathcal{U}, \mathcal{V})$. Similarly, the torsion pair $(\mathcal{V}, \mathcal{W})$ is generated by $\hat{i}_*(\hat{\mathcal{Y}}^c) = i_*(\hat{\mathcal{Y}}^c)$, and hence by a set of compact objects. Thus condition 2.a holds and, moreover, we have inclusions $j_!(\hat{\mathcal{X}}^c) \subseteq \mathcal{U} \cap \hat{\mathcal{D}}^c$ and $i_*(\hat{\mathcal{Y}}^c) \subseteq \mathcal{V} \cap \hat{\mathcal{D}}^c$. On the other hand, if $U \in \mathcal{U} \cap \hat{\mathcal{D}}^c$, then $\hat{j}^*U \in \hat{\mathcal{X}}^c$. Choosing now $X \in \hat{\mathcal{X}}$ such that $U = \hat{j}_!X$, we have $X \cong \hat{j}^*\hat{j}_!X \cong \hat{j}^*U \in \hat{\mathcal{X}}^c$ and, hence, $U \cong j_!X \in j_!(\hat{\mathcal{X}}^c)$. Similarly, $\mathcal{V} \cap \hat{\mathcal{D}}^c \subseteq i_*(\hat{\mathcal{Y}}^c)$.

$(2) \implies (3)$ By condition 2.b we know that $j_!$ and i_* preserve compact objects. Moreover, $j_!(\hat{\mathcal{X}}^c) = Im(j) \cap \hat{\mathcal{D}}^c$ and $i_*(\hat{\mathcal{Y}}^c) = Im(i_*) \cap \hat{\mathcal{D}}^c$, since $\mathcal{U} \cap \mathcal{D} = Im(j_!)$ and $\mathcal{V} \cap \mathcal{D} = Im(i_*)$. This implies that $j_!$ and i_* also reflect compact objects, i.e. $j_!(X) \in \hat{\mathcal{D}}^c$ (resp. $i_*(Y) \in \hat{\mathcal{D}}^c$) if and only if $X \in \hat{\mathcal{X}}^c$ (resp. $Y \in \hat{\mathcal{Y}}^c$). For any $D \in \hat{\mathcal{D}}^c$ let us consider the associated triangle $j_!j^*D \longrightarrow D \longrightarrow i_*i^*D \overset{+}{\longrightarrow}$, we get that j^* preserves compact objects if and only if so does i^*.

Let us prove that i^* preserves compact objects. Since the semi-orthogonal decomposition $(Im(j_!), Im(i_*))$ is the restriction to $\mathcal{D}$ of $(\mathcal{U}, \mathcal{V})$, the associated truncation functor $\tau : \hat{\mathcal{D}} \longrightarrow \mathcal{V}$ has the property that $\tau(\mathcal{D}) = \mathcal{V} \cap \mathcal{D} = Im(i_*)$. By lemma 3.1, we know that $\tau(\hat{\mathcal{D}}^c) = \mathcal{V}^c = \mathcal{V} \cap \hat{\mathcal{D}}^c = Im(i_*) \cap \hat{\mathcal{D}}^c$. We next decompose i_* as $\mathcal{Y} \overset{\cong}{\longrightarrow} Im(i_*) = \mathcal{D} \cap \mathcal{V} \hookrightarrow \mathcal{D}$, where the first arrow i_* is an equivalence

of categories. Then the left adjoint i^* is naturally isomorphic to the composition $\mathcal{D} \xrightarrow{\tau} \mathrm{Im}(i_*) = \mathcal{D} \cap \mathcal{V} \xrightarrow{i_*^{-1}} \mathcal{Y}$. But $i_*^{-1}(\mathcal{V} \cap \hat{\mathcal{D}}^c) = \hat{\mathcal{Y}}^c$, since $i_*(\hat{\mathcal{Y}}^c) = \mathcal{V} \cap \hat{\mathcal{D}}^c$. Therefore we get that $i^*(\hat{\mathcal{D}}^c) = \hat{\mathcal{Y}}^c$.

3) $\Longrightarrow$ 2) (assuming any of the extra hypotheses). Setting $\mathcal{U}_0 := j_!(\hat{\mathcal{X}}^c)$ and $\mathcal{V}_0 := i_*(\hat{\mathcal{Y}}^c)$, by lemma 3.2, $(\mathcal{U}, \mathcal{V}, \mathcal{W}) := (\mathrm{Loc}_{\hat{\mathcal{D}}}(j_!(\hat{\mathcal{X}}^c)), \mathrm{Loc}_{\hat{\mathcal{D}}}(i_*(\hat{\mathcal{Y}}^c)), \mathrm{Loc}_{\hat{\mathcal{D}}}(i_*(\hat{\mathcal{Y}}^c))^\perp)$ is a TTF-triple with $(\mathcal{U}, \mathcal{V})$ and $(\mathcal{V}, \mathcal{W})$ compactly generated by $j_!(\hat{\mathcal{X}}^c)$ and $i_*(\hat{\mathcal{Y}}^c)$.

We need to check that $\mathcal{U} \cap \mathcal{D} = \mathrm{Im}(j_!)$ and $\mathcal{V} \cap \mathcal{D} = \mathrm{Im}(i_*)$. The equality $\mathrm{Im}(j_*) = \mathcal{W} \cap \mathcal{D}$ will then follow automatically. Indeed, the inclusion $\mathrm{Im}(j_*) \subseteq \mathcal{W} \cap \mathcal{D}$ is obvious, the other inclusion follows from orthogonality. By properties of recollements (see [10]), $\mathrm{Im}(i_*) = \mathrm{Ker}(j^*)$. Since $\hat{\mathcal{X}}$ is compactly generated, an object D of $\mathcal{D}$ belongs to $\mathrm{Ker}(j^*)$ if and only if $0 = \mathrm{Hom}_{\mathcal{X}}(X, j^*D) \cong \mathrm{Hom}_{\mathcal{D}}(j_!X, D)$, for all $X \in \hat{\mathcal{X}}^c$. This happens exactly when $D \in \mathrm{Loc}_{\hat{\mathcal{D}}}(j_!(\hat{\mathcal{X}}^c))^\perp \cap \mathcal{D} = \mathcal{V} \cap \mathcal{D}$, and thus $\mathcal{V} \cap \mathcal{D} = \mathrm{Im}(i_*)$.

Let us check that $\mathcal{U} \cap \mathcal{D} = \mathrm{Im}(j_!)$. Since each object of $\mathcal{U}$ is the Milnor colimit of a sequence of morphisms in $\hat{\mathcal{D}}$ with successive cones in $\mathrm{Add}(j_!(\hat{\mathcal{X}})^c)$ and since $\mathrm{Hom}_{\hat{\mathcal{D}}}(j_!X, -)$ vanishes on $\mathrm{Im}(i_*)$, for each $X \in \mathcal{X}$, we get that $\mathcal{U} \cap \mathcal{D} \subseteq \mathrm{Im}(j_!)$. Indeed, $(\mathrm{Im}(j_!), \mathrm{Im}(i_*))$ is a torsion pair in $\mathcal{D}$ and hence $\mathrm{Im}(j_!) = {}^\perp\mathrm{Im}(i_*) \cap \mathcal{D}$. Conversely, for $D \in \mathrm{Im}(j_!)$ let us consider the truncation triangle $U \longrightarrow D \longrightarrow V \xrightarrow{+}$ in $\hat{\mathcal{D}}$ with respect to $(\mathcal{U}, \mathcal{V})$. As before, $\mathrm{Hom}_{\hat{\mathcal{D}}}(U, -)$ vanishes on $\mathrm{Im}(i_*)$, and hence $\mathrm{Hom}_{\hat{\mathcal{D}}}(V, -)$ vanishes on $\mathrm{Im}(i_*)$. In the assumption that $\mathrm{Im}(i_*)$ cogenerates $\mathcal{V} = \mathrm{Loc}_{\hat{\mathcal{D}}}(i_*(\hat{\mathcal{Y}}^c))$, we immediately get $V = 0$. In the other case, we also have that $\mathrm{Hom}_{\hat{\mathcal{D}}}(V, -)$ vanishes on $\mathcal{W}$ and, hence, it also vanishes on $\mathrm{Im}(j_*)$. It follows that $\mathrm{Hom}_{\hat{\mathcal{D}}}(V, -)$ vanishes both on $\mathrm{Im}(j_*)$ and $\mathrm{Im}(i_*)$. This implies that $\mathrm{Hom}_{\hat{\mathcal{D}}}(V, -)$ vanishes on $\mathcal{D}$, and hence that $V = 0$ since, by hypothesis, $\mathcal{D}$ cogenerates $\hat{\mathcal{D}}$. Under both extra hypotheses, we then get that $U \cong D \in \mathcal{U} \cap \mathcal{D}$.

Let us prove the inclusions $\mathcal{U} \cap \hat{\mathcal{D}}^c \subseteq j_!(\hat{\mathcal{X}}^c)$ and $\mathcal{V} \cap \hat{\mathcal{D}}^c \subseteq i_*(\hat{\mathcal{Y}}^c)$, the inverse inclusions are obvious. For $U \in \mathcal{U} \cap \hat{\mathcal{D}}^c \subseteq \mathrm{Im}(j_!)$ the adjunction map $j_!j^*(U) \longrightarrow U$ is an isomorphism. It follows that $U \in j_!(\hat{\mathcal{X}}^c)$, since $j^*(U) \in \hat{\mathcal{X}}^c$. The second inclusion is analogous. $\qquad\square$

COROLLARY 3.4. *If in theorem 3.3 we assume that* $\mathcal{D} = \hat{\mathcal{D}}^c$, *then assertion 2 of the theorem holds if and only if* $\mathcal{Y} = \hat{\mathcal{Y}}^c$ *and* $\mathcal{X} = \hat{\mathcal{X}}^c$.

Proof. Let us suppose that $\mathcal{D} = \hat{\mathcal{D}}^c$ in the recollement of theorem 3.3. If assertion 2 of the theorem holds so does assertion 3, and hence the functors $j_!$, j^*, i^* and i_* preserve compact objects. Hence, $\mathrm{Im}(j^*) \subseteq \hat{\mathcal{X}}^c$ and $\mathrm{Im}(i^*) \subseteq \hat{\mathcal{Y}}^c$. It follows that $\mathcal{Y} = \hat{\mathcal{Y}}^c$ and $\mathcal{X} = \hat{\mathcal{X}}^c$, since the functors i^* and j^* are dense, for any recollement. Conversely, if $\mathcal{Y} = \hat{\mathcal{Y}}^c$ and $\mathcal{X} = \hat{\mathcal{X}}^c$, then $(\mathrm{Im}(j_!), \mathrm{Im}(i_*)) = (j_!(\hat{\mathcal{X}}^c), i_*(\hat{\mathcal{Y}}^c))$ is a semi-orthogonal decomposition of $\mathcal{D} = \hat{\mathcal{D}}^c$ and by lemma 3.2 there is a TTF triple $(\mathcal{U}, \mathcal{V}, \mathcal{W}) = (\mathrm{Loc}_{\hat{\mathcal{D}}}(\mathrm{Im}(j_!)), \mathrm{Loc}_{\hat{\mathcal{D}}}(\mathrm{Im}(i_*)), \mathrm{Loc}_{\hat{\mathcal{D}}}(\mathrm{Im}(i_*))^\perp)$ in $\hat{\mathcal{D}}$, with compactly generated constituent torsion pairs, such that $(\mathcal{U} \cap \mathcal{D}, \mathcal{V} \cap \mathcal{D}) = (\mathrm{Im}(j_!), \mathrm{Im}(i_*))$. Since $\mathcal{W} \cap \mathcal{D} = \mathcal{V}^\perp \cap \mathcal{D} = \mathrm{Im}(i_*)^\perp \cap \mathcal{D} = \mathrm{Im}(j_*)$, the TTF triple $(\mathcal{U}, \mathcal{V}, \mathcal{W})$ satisfies all the conditions of assertion 2 in theorem 3.3. $\qquad\square$

We immediately get:

COROLLARY 3.5. *Let $\hat{\mathcal{Y}}$, $\hat{\mathcal{D}}$ and $\hat{\mathcal{X}}$ be compactly generated triangulated categories and suppose that we have a recollement*

$$\hat{\mathcal{Y}}^c \underset{\underset{i^!}{\longrightarrow}}{\overset{\overset{i^*}{\longleftarrow}}{\underset{\overset{i_*}{\longrightarrow}}{}}} \hat{\mathcal{D}}^c \underset{\underset{j_*}{\longleftarrow}}{\overset{\overset{j_!}{\longleftarrow}}{\underset{\overset{j^*}{\longrightarrow}}{}}} \hat{\mathcal{X}}^c$$

Then the TTF triple $(Im(j_!), Im(i_*), Im(j_*))$ in $\hat{\mathcal{D}}^c$ lifts to a TTF triple $(\mathcal{U}, \mathcal{V}, \mathcal{W})$ in $\hat{\mathcal{D}}$, where the torsion pairs $(\mathcal{U}, \mathcal{V})$ and $(\mathcal{V}, \mathcal{W})$ are compactly generated.

Here is a list of examples where the last corollary applies:

EXAMPLE 3.6. Let us consider one of the following situations:

(1) Let A, B and C be finite dimensional algebras over a field. Let the triple $\mathcal{Y} \equiv \mathcal{D} \equiv \mathcal{X}$ be $\underline{mod}\text{-}\hat{B} \equiv \underline{mod}\text{-}\hat{A} \equiv \underline{mod}\text{-}\hat{C}$, where $\underline{mod}\text{-}\hat{A}$ is the stable category of the repetitive algebra $\hat{A}$ of A (see [**21**, § 2.2]) and let $\hat{\mathcal{D}}$ be $\underline{Mod}\text{-}\hat{A}$.

(2) Let A, B and C be self-injective finite length algebras. Let the triple $\mathcal{Y} \equiv \mathcal{D} \equiv \mathcal{X}$ be $\underline{mod}\text{-}B \equiv \underline{mod}\text{-}A \equiv \underline{mod}\text{-}C$, where $\underline{mod}\text{-}A$ is the stable category of A and let $\hat{\mathcal{D}}$ be $\underline{Mod}\text{-}A$.

(3) Let $\mathbf{U}$, $\mathbf{X}$ and $\mathbf{Z}$ be separated Noetherian schemes. Let the triple $\mathcal{Y} \equiv \mathcal{D} \equiv \mathcal{X}$ be $\mathcal{D}^b(\mathrm{coh}(\mathbf{U})) \equiv \mathcal{D}^b(\mathrm{coh}(\mathbf{X})) \equiv \mathcal{D}^b(\mathrm{coh}(\mathbf{Z}))$ and let $\hat{\mathcal{D}}$ be $\mathcal{K}(\mathrm{Inj}\text{-}\mathbf{X})$ – the homotopy category of injective objects of $\mathrm{Qcoh}(\mathbf{X})$.

(4) Let A, B and C be right Noetherian rings. Let the triple $\mathcal{Y} \equiv \mathcal{D} \equiv \mathcal{X}$ be $D^b(\mathrm{mod}\text{-}B) \equiv D^b(\mathrm{mod}\text{-}A) \equiv D^b(\mathrm{mod}\text{-}C)$ and let $\hat{\mathcal{D}}$ be the homotopy category of injectives $\mathcal{K}(\mathrm{Inj}\text{-}A)$.

Suppose there is a recollement

$$\mathcal{Y} \underset{\underset{i^!}{\longrightarrow}}{\overset{\overset{i^*}{\longleftarrow}}{\underset{\overset{i_*}{\longrightarrow}}{}}} \mathcal{D} \underset{\underset{j_*}{\longleftarrow}}{\overset{\overset{j_!}{\longleftarrow}}{\underset{\overset{j^*}{\longrightarrow}}{}}} \mathcal{X}$$

and consider $\mathcal{D}$ as a full triangulated subcategory of $\hat{\mathcal{D}}$, then there exists a TTF triple $(\mathcal{U}, \mathcal{V}, \mathcal{W})$ in $\hat{\mathcal{D}}$, with compactly generated constituent torsion pairs $(\mathcal{U}, \mathcal{V})$ and $(\mathcal{V}, \mathcal{W})$, which restricts to the TTF triple $(\mathrm{Im}(j_!), \mathrm{Im}(i_*), \mathrm{Im}(j_*))$ in $\mathcal{D}$.

Proof. In all the cases, it turns out that the three categories of the recollements are the subcategories of compact objects in the appropriate compactly generated triangulated categories.

(1) Let $\hat{\Lambda}$ denotes the repetitive algebra of Λ. It is easy to see that the category Mod-$\hat{\Lambda}$ of unitary $\hat{\Lambda}$-modules (i.e. modules M such that $M\hat{\Lambda} = M$) is a Frobenius category, so that its stable category $\underline{Mod}\text{-}\hat{\Lambda}$ is triangulated and compactly generated. Its subcategory of compact objects is precisely $\underline{mod}\text{-}\hat{\Lambda}$,

where mod-$\hat{\Lambda}$ is the subcategory of Mod-$\hat{\Lambda}$ consisting of the finitely generated $\hat{\Lambda}$-modules, which coincide with the $\hat{\Lambda}$-modules of finite length.

(2) It is well-known that if Λ is a self-injective Artin algebra, in particular a self-injective finite length algebra, then its module category Mod-Λ is Frobenius and its associated triangulated stable category $\underline{Mod}$-Λ has $\underline{mod}$-Λ as its subcategory of compact objects.

(3) and (4) By [**33**, theorem 1.1], we can identify $\mathcal{D}^b(\mathrm{coh}(\mathbf{Y}))$ with the subcategory of compact objects of $\mathcal{K}(\mathrm{Inj}\text{-}\mathbf{Y})$, for any separated Noetherian scheme $\mathbf{Y}$, and by [**33**, proposition 2.3], we can identify $D^b(\mathrm{mod}\text{-}R)$ with the subcategory of compact objects of $\mathcal{K}(\mathrm{Inj}\text{-}R)$, for any right Noetherian ring R (here mod-R is the category of finitely generated R-modules, which coincides with that of Noetherian modules).

With all these considerations in mind, the result is now a direct consequence of the previous corollary. $\qquad\square$

In order to provide some examples where condition 3 of the last theorem implies condition 2, we introduce the following terminology.

NOTATION AND TERMINOLOGY 3.7: *Given any triangulated category $\mathcal{D}$ and any class $\mathcal{X}$ of its objects, we denote by $\mathcal{D}_{\mathcal{X}}^-$ (resp. $\mathcal{D}_{\mathcal{X}}^+$ or $\mathcal{D}_{\mathcal{X}}^b$) the (thick) subcategory of $\mathcal{D}$ consisting of objects M such that, for each $X \in \mathcal{X}$, one has $\mathrm{Hom}_{\mathcal{D}}(X, M[k]) = 0$ for $k \gg 0$ (resp. $k \ll 0$ or $|k| \gg 0$). We denote by $\mathcal{D}_{\mathcal{X}, fl}$ the (thick) subcategory of $\mathcal{D}$ consisting of objects M such that, for each $X \in \mathcal{X}$ and each $k \in \mathbb{Z}$, the K-module $\mathrm{Hom}_{\mathcal{D}}(X, M[k])$ is of finite length. We finally put $\mathcal{D}_{\mathcal{X}, \dagger}^\star = \mathcal{D}_{\mathcal{X}}^\star \cap \mathcal{D}_{\mathcal{X}, \dagger}$, for $\star \in \{\emptyset, +, -, b\}$ and $\dagger \in \{\emptyset, fl\}$. In the particular case when $\mathcal{D}$ is compactly generated and $\mathcal{X} = \mathcal{D}^c$, we will simply write $\mathcal{D}_\dagger^\star$ instead of $\mathcal{D}_{\mathcal{X}, \dagger}^\star$. Note that, in order to define $\mathcal{D}_\dagger^\star$ in the latter case, one can replace $\mathcal{D}^c$ by any set $\mathcal{X}$ of compact generators of $\mathcal{D}$, since $\mathcal{D}^c = thick_{\mathcal{D}}(\mathcal{X})$. Let us denote by $\mathbf{P}_\dagger^\star$ the property that defines the full subcategory $\mathcal{D}_\dagger^*$ of $\mathcal{D}$. For instance, if $\star = -$ and $\dagger = fl$, then, for a given $M \in \mathcal{D}$, we will say that M satisfies property $\mathbf{P}_\dagger^\star$, for some $X \in \mathcal{D}^c$, when $\mathrm{Hom}_{\mathcal{D}}(X, M[k])$ is zero, for $k \gg 0$, and is a K-module of finite length, for all $k \in \mathbb{Z}$.*

EXAMPLE 3.8. If $\mathcal{A}$ is a small dg category, then $\mathcal{D}^-(\mathcal{A}) := \mathcal{D}(\mathcal{A})^-$ (resp. $\mathcal{D}^+(\mathcal{A}) := \mathcal{D}(\mathcal{A})^+$ or $\mathcal{D}^b(\mathcal{A}) := \mathcal{D}(\mathcal{A})^b$) is the subcategory of $\mathcal{D}(\mathcal{A})$ consisting of dg $\mathcal{A}$-modules M such that, for each $A \in \mathcal{A}$, one has $H^k M(A) = 0$ for $k \gg 0$ (resp. $k \ll 0$ or $|k| \gg 0$). Similarly, for $\star \in \{\emptyset, +, -, b\}$, one has $\mathcal{D}_{fl}^\star(\mathcal{A})$ consists of dg $\mathcal{A}$-modules $M \in \mathcal{D}^\star(\mathcal{A})$ such that $H^k M(A)$ is a K-module of finite length, for each $A \in \mathcal{A}$ and each $k \in \mathbb{Z}$.

REMARK 3.9. In [**46**] $\mathcal{D}^-(\mathcal{A})$ was defined as the union $\bigcup_{k \geqslant 0} \mathcal{U}[k]$, where $\mathcal{U} = \mathcal{D}^{\leqslant 0}(\mathcal{A})$. Here $\mathcal{D}^{\leqslant 0}(\mathcal{A}) = \mathrm{Susp}_{\mathcal{D}(\mathcal{A})}(A^\wedge : A \in \mathcal{A})$, which is the aisle of a t-structure in $\mathcal{D}(\mathcal{A})$. That definition does not agree in general with the one given here, although they coincide when $\mathcal{A} = A$ is a dg algebra.

DEFINITION 3.10. Let $\mathcal{D}$ be a compactly generated triangulated category. We will say that $\mathcal{D}$ is *homologically locally bounded* when $\mathcal{D}^c \subseteq \mathcal{D}^b$ and $\mathcal{D}$ is *homologically locally finite length* when $\mathcal{D}^c \subseteq \mathcal{D}^b_{fl}$.

EXAMPLE 3.11. If $\mathcal{A}$ is a small dg category and $\mathcal{D} = \mathcal{D}(\mathcal{A})$ is its derived category, then $\mathcal{D}$ is homologically locally bounded if and only if the set $\{k \in \mathbb{Z}: H^k \mathcal{A}(A, A') \neq 0\}$ is finite, for all $A, A' \in \mathcal{A}$. Moreover, $\mathcal{D}$ is homologically locally finite length if, in addition, $H^k \mathcal{A}(A, A')$ is a K-module of finite length, for all $k \in \mathbb{Z}$ and all $A, A' \in \mathcal{A}$. Slightly abusing the terminology, we will say in those cases that $\mathcal{A}$ is a *homologically locally bounded* or a *homologically locally finite length dg category*, respectively. When $\mathcal{A} = A$ is a dg algebra, we will simply say that A is *homologically bounded* if $H^k(A) = 0$, for almost all $k \in \mathbb{Z}$, or that A is *homologically finite length* if $H^*(A) := \oplus_{k \in \mathbb{Z}} H^k(A)$ is a K-module of finite length.

We are ready to give examples where condition 3 of theorem 3.3 implies condition 2.

COROLLARY 3.12. *Let $\hat{\mathcal{Y}}$, $\hat{\mathcal{D}}$ and $\hat{\mathcal{X}}$ be compactly generated triangulated categories. For $\star \in \{\emptyset, +, -, b\}$ and $\dagger \in \{\emptyset, fl\}$ let*

$$\hat{\mathcal{Y}}^\star_\dagger \underset{\substack{\xleftarrow{\quad i^* \quad} \\ \xrightarrow{\quad i_* \quad} \\ \xleftarrow{\quad i^! \quad}}}{} \hat{\mathcal{D}}^\star_\dagger \underset{\substack{\xleftarrow{\quad j_! \quad} \\ \xrightarrow{\quad j^* \quad} \\ \xleftarrow{\quad j_* \quad}}}{} \hat{\mathcal{X}}^\star_\dagger$$

be a recollement, such that the subcategories involved contain the respective subcategories of compact objects and such that the functors $j_!, j^*, i^*, i_*$ preserve compact objects. If $\hat{\mathcal{D}}$ is homologically locally bounded (resp. homologically locally finite length), then the subcategory $\hat{\mathcal{D}}^\star$ (resp. $\mathcal{D}^\star_{fl}$) cogenerates $\hat{\mathcal{D}}$, and hence assertion 2 of theorem 3.3 holds for $\dagger = \emptyset$ (resp. $\dagger = fl$).

Proof. Let us check that $\hat{\mathcal{D}}^b$ (resp. $\hat{\mathcal{D}}^b_{fl}$) cogenerates $\hat{\mathcal{D}}$. For this take a minimal injective cogenerator E of Mod-K and use the notion of Brown–Comenetz dual. Since compactly generated (or even well-generated) triangulated categories satisfy Brown representability theorem (see [**40**, proposition 8.4.2]), for each $X \in \hat{\mathcal{D}}^c$, the functor $\mathrm{Hom}_K(\mathrm{Hom}_{\hat{\mathcal{D}}}(X, -), E): \hat{\mathcal{D}}^{op} \longrightarrow$ Mod-K is naturally isomorphic to the representable functor $\mathrm{Hom}_{\hat{\mathcal{D}}}(-, D(X))$, for an object $D(X)$, uniquely determined up to isomorphism, called the *Brown–Comenetz dual* of X. It immediately follows that $\{D(X): X \in \hat{\mathcal{D}}^c\}$ is a skeletally small cogenerating class of $\hat{\mathcal{D}}$. Our task reduces to check that $D(X) \in \hat{\mathcal{D}}^b$ (resp. $D(X) \in \hat{\mathcal{D}}^b_{fl}$), when $\hat{\mathcal{D}}$ is homologically locally bounded (resp. homologically locally finite length). But this is clear since, given any $Y, X \in \hat{\mathcal{D}}^c$, we have that $\mathrm{Hom}_{\hat{\mathcal{D}}}(Y, D(X)[k]) \cong \mathrm{Hom}_K(\mathrm{Hom}_{\hat{\mathcal{D}}}(Y, X[-k]), E)$ and the homologically locally bounded condition on $\hat{\mathcal{D}}$ implies that $\mathrm{Hom}_{\hat{\mathcal{D}}}(Y, X[-k]) = 0$, for all but finitely many $k \in \mathbb{Z}$. When $\hat{\mathcal{D}}$ is homologically locally finite length, we have in addition that each $\mathrm{Hom}_{\hat{\mathcal{D}}}(Y, X[-k])$ is of finite length as K-module, which implies that the same is true for $\mathrm{Hom}_{\hat{\mathcal{D}}}(Y, D(X)[k])$. $\square$

Our next result, inspired by the analogous results for derived categories of ordinary algebras [**5**], says that 'restriction up to equivalence' and 'restriction' are equivalent concepts for recollements in some interesting cases.

PROPOSITION 3.13. *Let*

$$\hat{\mathcal{Y}} \underset{\xrightarrow{\hat{i}^!}}{\overset{\xleftarrow{\hat{i}^*}}{\underset{\xrightarrow{\hat{i}_*}}{\longleftarrow}}} \hat{\mathcal{D}} \underset{\xrightarrow{\hat{j}_*}}{\overset{\xleftarrow{\hat{j}_!}}{\underset{\xrightarrow{\hat{j}^*}}{\longleftarrow}}} \hat{\mathcal{X}}$$

be a recollement of compactly generated triangulated categories that is the upper part of a ladder of recollements of height two, let $\star \in \{\emptyset, b, -, +\}$ *and* $\dagger \in \{\emptyset, fl\}$, *and suppose that* $\mathcal{E}^c \subseteq \mathcal{E}_\dagger^\star$, *for* $\mathcal{E} = \hat{\mathcal{X}}, \hat{\mathcal{D}}, \hat{\mathcal{Y}}$. *The following assertions are equivalent:*

(1) *The recollement restricts to* $(-)_\dagger^\star$-*level.*

(2) *The recollement restricts, up to equivalence, to* $(-)_\dagger^\star$-*level, i.e. the associated TTF triple* $(Im(\hat{j}_!), (Im(\hat{i}_*), (Im(\hat{j}_*)))$ *restricts to* $\hat{\mathcal{D}}_\dagger^\star$

(3) $\hat{i}^*(\hat{\mathcal{D}}_\dagger^\star) \subseteq \hat{\mathcal{Y}}_\dagger^\star$.

(4) $\hat{j}_!(\hat{\mathcal{X}}_\dagger^\star) \subseteq \hat{\mathcal{D}}_\dagger^\star$

Proof. The fact that the recollement is the upper part of a ladder of recollements of height two implies that the functors $\hat{i}^*, \hat{i}_*, \hat{j}_!, \hat{j}^*$ preserve compact objects, since their right adjoints preserve coproducts. Note that if in an adjoint pair of triangulated functors $(F : \mathcal{D} \longrightarrow \mathcal{E}, G : \mathcal{E} \longrightarrow \mathcal{D})$ between compactly generated triangulated categories the functor F preserves compact objects, then $G(\mathcal{E}_\dagger^\star) \subseteq \mathcal{D}_\dagger^\star$. Indeed, let $E \in \mathcal{E}_\dagger^\star$ be any object. Then $G(E) \in \mathcal{D}_\dagger^\star$ if and only if $G(E)$ satisfies property $\mathbf{P}_\dagger^*$ for all $X \in \mathcal{D}^c$. Due to the natural isomorphism $\mathrm{Hom}_\mathcal{E}(F(X), E[n]) \cong \mathrm{Hom}_\mathcal{D}(X, G(E)[n])$, for all $X \in \mathcal{D}^c$ and $n \in \mathbb{Z}$, and since $F(X) \in \mathcal{E}^c$ we get that $G(E)$ satisfies property $\mathbf{P}_\dagger^*$ for all $X \in \mathcal{D}^c$, since E satisfies property $\mathbf{P}_\dagger^*$, for all $Z \in \mathcal{E}^c$. This implies that all the functors $\hat{i}_*, \hat{i}^!, \hat{j}^*$ and $\hat{j}_*$ restrict to the $(-)_\dagger^\star$-level. In particular, the triangle $\hat{i}_*\hat{i}^!D \longrightarrow D \longrightarrow \hat{j}_*\hat{j}^*D \overset{+}{\longrightarrow}$, which is the truncation triangle with respect to $(\mathcal{V}, \mathcal{W}) := (\mathrm{Im}(\hat{i}_*), \mathrm{Im}(\hat{j}_*))$, belongs to $\hat{\mathcal{D}}_\dagger^\star$ for each $D \in \hat{\mathcal{D}}_\dagger^\star$. Therefore the semi-orthogonal decomposition $(\mathcal{V}, \mathcal{W})$ restricts to $\hat{\mathcal{D}}_\dagger^\star$. As a consequence, assertion 2 holds if and only if the triangle $\hat{j}_!\hat{j}^*D \longrightarrow D \longrightarrow \hat{i}_*\hat{i}^*D \overset{+}{\longrightarrow}$ (I), which is the truncation triangle with respect to $(\mathcal{U}, \mathcal{V}) := (\mathrm{Im}(\hat{j}_!), \mathrm{Im}(\hat{i}_*))$, belongs to $\hat{\mathcal{D}}_\dagger^\star$ for each $D \in \hat{\mathcal{D}}_\dagger^\star$.

Since $\hat{i}_*$ and $\hat{j}_*$ are fully faithful, the counits of the adjoint pairs $(\hat{i}^*, \hat{i}_*)$ and $(\hat{j}^*, \hat{j}_*)$ are natural isomorphisms, and, hence $\hat{\mathcal{Y}}_\dagger^\star \subseteq \hat{i}^*(\hat{\mathcal{D}}_\dagger^\star)$ and $\hat{j}^*(\hat{\mathcal{D}}_\dagger^\star) = \hat{\mathcal{X}}_\dagger^\star$. Using this, we get that the triangle (I) belongs to $\hat{\mathcal{D}}_\dagger^\star$, for each $D \in \hat{\mathcal{D}}_\dagger^\star$ if and only if $\hat{j}_!(\hat{\mathcal{X}}_\dagger^\star) \subseteq \hat{\mathcal{D}}_\dagger^\star$. Thus, assertions 2 and 4 are equivalent.

On the other hand, we have $\hat{i}^*(\hat{\mathcal{D}}^c) = \hat{\mathcal{Y}}^c$. Indeed, we only need to check the inclusion $\supseteq$, since $\hat{i}^*$ preserves compact objects. If $Y \in \hat{\mathcal{Y}}^c$ then $Y \cong \hat{i}^*\hat{i}_*(Y)$, and $\hat{i}_*Y \in \hat{\mathcal{D}}^c$. The equality $\hat{i}^*(\hat{\mathcal{D}}^c) = \hat{\mathcal{Y}}^c$ implies, that for $Y \in \hat{\mathcal{Y}}$, one has that $\hat{i}_*Y \in \hat{\mathcal{D}}_\dagger^\star$

if and only if $Y \in \hat{\mathcal{Y}}_{\dagger}^{\star}$. Indeed $\hat{i}_* Y \in \hat{\mathcal{D}}_{\dagger}^{\star}$ if and only if $\hat{i}_* Y$ satisfies property $\mathbf{P}_{\dagger}^{\star}$, for all $D_0 \in \hat{\mathcal{D}}^c$, which, by adjunction, is equivalent to saying that Y satisfies property $\mathbf{P}_{\dagger}^{\star}$ for all $\hat{i}^*(D_0)$, with $D_0 \in \hat{\mathcal{D}}^c$. That is, if and only if Y satisfies property $\mathbf{P}_{\dagger}^{\star}$, for all $Y_0 \in \hat{\mathcal{Y}}^c$, if and only if $Y \in \hat{\mathcal{Y}}_{\dagger}^{\star}$. Thus, triangle (I) belongs to $\hat{\mathcal{D}}_{\dagger}^{\star}$, for each $D \in \hat{\mathcal{D}}_{\dagger}^{\star}$, if and only if $\hat{i}_* \hat{i}^* D \in \hat{\mathcal{D}}_{\dagger}^{\star}$ if and only if $\hat{i}^* D \in \hat{\mathcal{Y}}_{\dagger}^{\star}$, if and only if $\hat{i}^*(\hat{\mathcal{D}}_{\dagger}^{\star}) \subseteq \hat{\mathcal{Y}}_{\dagger}^{\star}$. Therefore assertions 2 and 3 are also equivalent. Finally, taking into account the first paragraph of this proof, it is clear that if the equivalent assertions 3 and 4 hold, then assertion 1 holds. $\square$

THEOREM 3.14. *Let $\hat{\mathcal{D}}$ and $\hat{\mathcal{X}}$ be compactly generated algebraic triangulated categories and let $\mathcal{B}$ be a small K-linear category. For $\star \in \{\emptyset, +, -, b\}$ and $\dagger \in \{\emptyset, fl\}$ let*

$$\mathcal{D}_{\dagger}^{\star}(\mathcal{B}) \; \underset{i^!}{\overset{\underset{i_*}{\overset{i^*}{\longleftarrow}}}{\longrightarrow}} \; \hat{\mathcal{D}}_{\dagger}^{\star} \; \underset{j_*}{\overset{\underset{j^*}{\overset{j_!}{\longleftarrow}}}{\longrightarrow}} \; \hat{\mathcal{X}}_{\dagger}^{\star}$$

be a recollement, such that the categories involved contain the respective subcategories of compact objects and such that the functors $j_!, j^*, i^*, i_*$ preserve compact objects (that is $j_!(\hat{\mathcal{X}}^c) \subseteq \hat{\mathcal{D}}^c$ and similarly for j^*, i^* and i_*). Then $\mathrm{Im}(i_*)$ cogenerates $\mathcal{V} = \mathrm{Loc}_{\hat{\mathcal{D}}}(i_*(\mathcal{D}^c(\mathcal{B})))$ and assertion 2 of theorem 3.3 holds.

If in addition $\hat{\mathcal{X}} = \mathcal{D}(\mathcal{C})$, for some small K-linear category $\mathcal{C}$, then the given recollement lifts up to equivalence to a recollement

$$\mathcal{D}(\mathcal{B}) \; \underset{\hat{i}^!}{\overset{\underset{\hat{i}_*}{\overset{\hat{i}^*}{\longleftarrow}}}{\longrightarrow}} \; \hat{\mathcal{D}} \; \underset{\hat{j}_*}{\overset{\underset{\hat{j}^*}{\overset{\hat{j}_!}{\longleftarrow}}}{\longrightarrow}} \; \mathcal{D}(\mathcal{C}) \;,$$

which is the upper part of a ladder of recollements of height two.

Proof. We are going to use the notation from the proof of theorem 3.3. To prove the first assertion, we are going to check that $\mathcal{D}(\mathcal{B}) \overset{\cong}{\longrightarrow} \mathcal{V}$ and that this equivalence restricts to $\mathcal{D}_{\dagger}^{\star}(\mathcal{B}) \overset{\cong}{\longrightarrow} \mathrm{Im}(i_*)$. Note that $\mathcal{B}$ is homologically locally bounded considered as a dg category. When $\dagger = fl$, due to the inclusion $\mathcal{D}^c(\mathcal{B}) \subseteq \mathcal{D}_{\dagger}^{\star}(\mathcal{B})$, the dg category $\mathcal{B}$ is also homologically locally finite length. By lemma 3.2, $(\mathcal{U}, \mathcal{V}, \mathcal{W}) = (\mathrm{Loc}_{\hat{\mathcal{D}}}(j_!(\hat{\mathcal{X}}^c)), \mathrm{Loc}_{\hat{\mathcal{D}}}(i_*(\mathcal{D}^c(\mathcal{B}))), \mathrm{Loc}_{\hat{\mathcal{D}}}(i_*(\mathcal{D}^c(\mathcal{B})))^{\perp})$ is a TTF triple in $\hat{\mathcal{D}}$ and $\mathcal{V}^c = \mathcal{V} \cap \hat{\mathcal{D}}^c$. By the proof of the implication $(3) \Longrightarrow (2)$ of theorem 3.3, $\mathrm{Im}(i_*) = \mathcal{V} \cap \hat{\mathcal{D}}_{\dagger}^{\star}$. Indeed, the additional condition on $\mathrm{Im}(i_*)$ was used only to check that $\mathrm{Im}(j_!) = \mathcal{U} \cap \hat{\mathcal{D}}_{\dagger}^{\star}$.

By hypothesis, we have $i_*(\mathcal{D}^c(\mathcal{B})) \subseteq \mathcal{V} \cap \hat{\mathcal{D}}^c$. For $V \in \mathcal{V} \cap \hat{\mathcal{D}}^c \subseteq \mathrm{Im}(i_*)$ take W such that $i_* W \simeq V$, since i^* preserves compact objects $W \simeq i^* i_* W$ is compact and $V \in i_*(\mathcal{D}^c(\mathcal{B}))$. So $\mathcal{V}^c = \mathcal{V} \cap \hat{\mathcal{D}}^c = i_*(\mathcal{D}^c(\mathcal{B}))$. Hence, also $i_* i^* \hat{\mathcal{D}}^c = i_* \mathcal{D}^c(\mathcal{B})$.

Note that $\mathcal{V}$ is a quotient of an algebraic compactly generated triangulated category by a localizing subcategory generated by a set of compact objects. Then $\mathcal{V}$ is compactly generated by [**39**, theorem 2.1] (using the description of compact objects and the right adjoint to the localization functor), and it is also algebraic

as a triangulated subcategory of an algebraic triangulated category. Furthermore, by [**24**, theorem 9.2] there is a triangulated equivalence $F : \mathcal{D}(\mathcal{B}) \xrightarrow{\cong} \mathcal{V}$ such that $F(B^\wedge) \cong i_*(B^\wedge)$, for each $B \in \mathcal{B}$. Since $\mathcal{D}^c(\mathcal{B}) = \mathrm{thick}_{\mathcal{D}(\mathcal{B})}(B^\wedge \colon B \in \mathcal{B})$, we get $F(\mathcal{D}^c(\mathcal{B})) = i_*(\mathcal{D}^c(\mathcal{B}))$.

Consider the canonical triangle $j_! j^* Z \longrightarrow Z \longrightarrow i_* i^* Z \xrightarrow{+}$, for each $Z \in \hat{\mathcal{D}}^c$. Since $j_! j^*(Z)$ is compact $\mathrm{Hom}_{\hat{\mathcal{D}}}(j_! j^*(Z), -)$ vanishes on $\mathrm{Im}(F) = \mathcal{V}$. For $Y \in \mathcal{D}(\mathcal{B})$ we get that $F(Y)$ belongs to $\hat{\mathcal{D}}^*_\dagger$ iff it satisfies property $\mathbf{P}^\star_\dagger$, for all $Z \in \hat{\mathcal{D}}^c$, iff it satisfies property $\mathbf{P}^\star_\dagger$, for all $Z' \in \hat{\mathcal{D}}^c$ such that $Z' \cong i_* i^*(Z)$. Since $i_* i^* \hat{\mathcal{D}}^c = i_* \mathcal{D}^c(\mathcal{B})$, we get that $F(Y)$ is in $\hat{\mathcal{D}}^*_\dagger$ iff it satisfies property $\mathbf{P}^\star_\dagger$, for all $Z = i_*(M) \cong F(M')$, with $M, M' \in \mathcal{D}^c(\mathcal{B})$. The fact that F is an equivalence implies that $F(Y) \in \hat{\mathcal{D}}^*_\dagger$ iff Y satisfies property $\mathbf{P}^\star_\dagger$ in $\mathcal{D}(\mathcal{B})$ for all $M' \in \mathcal{D}^c(\mathcal{B})$. Thus, $F(Y) \in \hat{\mathcal{D}}^*_\dagger$ iff $Y \in \mathcal{D}^*_\dagger \mathcal{B}$. This means that $F(\mathcal{D}^*_\dagger(\mathcal{B})) = \hat{\mathcal{D}}^*_\dagger \cap \mathrm{Im}(F) = \hat{\mathcal{D}}^*_\dagger \cap \mathcal{V} = \mathrm{Im}(i_*)$. Therefore F induces an equivalence of triangulated categories $F : \mathcal{D}^*_\dagger(\mathcal{B}) \xrightarrow{\cong} \mathrm{Im}(i_*)$. Note that, this equivalence need not be naturally isomorphic to the one induced by i_*. By the proof of corollary 3.12, $\mathcal{D}^*_\dagger$ cogenerates $\mathcal{D}(\mathcal{B})$, and hence $\mathrm{Im}(i_*)$ cogenerates $\mathcal{V}$.

Let us prove the second assertion of the proposition. From the TTF triple constructed above we get a recollement:

$$
\mathcal{D}(\mathcal{B}) \; \underset{\xrightarrow{\hat{i}^!}}{\overset{\xleftarrow{\hat{i}^*}}{\xrightarrow[\xleftarrow{\hat{i}_*}]{}}} \; \hat{\mathcal{D}} \; \underset{\xrightarrow{\hat{j}_*}}{\overset{\xleftarrow{\hat{j}_!}}{\xrightarrow[\xleftarrow{\hat{j}^*}]{}}} \; \mathcal{W} \;, \tag{3.1}
$$

where $\hat{j}_* : \mathcal{W} \hookrightarrow \hat{\mathcal{D}}$ is the inclusion, such that the associated TTF triple $(\mathrm{Im}(\hat{j}_!), \mathrm{Im}(\hat{i}_*), \mathrm{Im}(\hat{j}_*))$ in $\hat{\mathcal{D}}$ restricts to the TTF triple $(\mathrm{Im}(j_!), \mathrm{Im}(i_*), \mathrm{Im}(j_*))$ in $\hat{\mathcal{D}}^*_\dagger$. In particular, there is an equivalence of categories $\mathcal{U} \xrightarrow{\cong} \mathcal{W}$ which restricts to the canonical equivalence $\mathrm{Im}(j_!) \xrightarrow{\cong} \mathrm{Im}(j_*)$ given by $j_* j_!^{-1}$. As a quotient of an algebraic triangulated category $\mathcal{W}$ is algebraic. Since $\mathcal{W} \xrightarrow{\cong} \mathcal{U} = \mathrm{Loc}_{\hat{\mathcal{D}}}(j_!(\mathcal{D}^c(\mathcal{C}))) = \mathrm{Loc}_{\hat{\mathcal{D}}}(j_!(C^\wedge) \colon C \in \mathcal{C})$, we conclude that $\mathcal{W}$ is compactly generated by $\{j_*(C^\wedge) \colon C \in \mathcal{C}\}$.

As before, by [**24**, theorem 9.2], there is a triangulated equivalence $G : \mathcal{D}(\mathcal{C}) \xrightarrow{\cong} \mathcal{W}$ such that $G(C^\wedge) \cong j_*(C^\wedge)$, for each $C \in \mathcal{C}$. Let $X \in \mathcal{D}(\mathcal{C})$ be any object. We claim that $G(X) \in \mathrm{Im}(j_*) = \mathcal{W} \cap \hat{\mathcal{D}}^*_\dagger$ iff $X \in \mathcal{D}^*_\dagger(\mathcal{C})$. Indeed, $G(X) \in \hat{\mathcal{D}}^*_\dagger$ iff $\mathrm{Hom}_{\hat{\mathcal{D}}}(Z, G(X)[k])$ satisfies property $\mathbf{P}^\star_\dagger$, for all $Z \in \hat{\mathcal{D}}^c$. Using the triangle $i_* i^! Z \longrightarrow Z \longrightarrow j_* j^* Z \xrightarrow{+}$ and the fact that $\mathrm{Hom}_{\hat{\mathcal{D}}}(i_* i^! Z, -)$ vanishes on $\mathcal{W} = \mathrm{Im}(G)$, we get that $G(X)$ is in $\hat{\mathcal{D}}^*_\dagger$ iff $\mathrm{Hom}_{\hat{\mathcal{D}}}(j_* j^* Z, G(X)[k])$ satisfies $\mathbf{P}^\star_\dagger$, for all $Z \in \hat{\mathcal{D}}^c$. By hypothesis, we have an inclusion $j^*(\hat{\mathcal{D}}^c) \subseteq \mathcal{D}^c(\mathcal{C})$. Conversely, if $X' \in \mathcal{D}^c(\mathcal{C})$ then $j_!(X') \in \hat{\mathcal{D}}^c$, so that $X' \cong j^* j_!(X') \in j^*(\hat{\mathcal{D}}^c)$. Thus, when Z runs through the objects of $\hat{\mathcal{D}}^c$, the object $j^* Z$ runs through the objects of $\mathcal{D}^c(\mathcal{C})$. Since $\mathcal{D}^c(\mathcal{C}) = \mathrm{thick}_{\mathcal{D}(\mathcal{C})}(C^\wedge \colon C \in \mathcal{C})$, we easily conclude that $G(X) \in \hat{\mathcal{D}}^*_\dagger$ iff $\mathrm{Hom}_{\hat{\mathcal{D}}}(j_*(C^\wedge), G(X)[k])$ satisfies $\mathbf{P}^\star_\dagger$, for all $C \in \mathcal{C}$. Since G is an equivalence of categories and $j_*(C^\wedge) \cong G(C^\wedge)$, we have $G(X) \in \hat{\mathcal{D}}^*_\dagger$ iff $\mathrm{Hom}_{\mathcal{D}(\mathcal{C})}(C^\wedge, X[k])$ satisfies $\mathbf{P}^\star_\dagger$, for all $C \in \mathcal{C}$. That is, iff $X \in \mathcal{D}^*_\dagger(\mathcal{C})$.

The previous paragraph yields an equivalence of categories $G : \mathcal{D}(\mathcal{C}) \xrightarrow{\cong} \mathcal{W}$ which induces by restriction another equivalence $\mathcal{D}^*_\dagger(\mathcal{C}) \xrightarrow{\cong} \mathrm{Im}(j_*)$. This implies that we can replace $\mathcal{W}$ by $\mathcal{D}(\mathcal{C})$ in the recollement (3.1), thus obtaining a recollement as in the final assertion of the proposition, which in turn restricts to a recollement whose associated TTF triple is $(\mathrm{Im}(j_!), \mathrm{Im}(i_*), \mathrm{Im}(j_*))$. This last recollement is then equivalent to the original one.

It remains to prove that the obtained recollement

$$
\mathcal{D}(\mathcal{B}) \underset{\substack{\longleftarrow \\ \xrightarrow{\hat{i}^!} \\ \xleftarrow{\hat{i}_*} \\ \xleftarrow{\hat{i}^*}}}{} \hat{\mathcal{D}} \underset{\substack{\longleftarrow \\ \xrightarrow{\hat{j}_*} \\ \xleftarrow{\hat{j}^*} \\ \xleftarrow{\hat{j}_!}}}{} \mathcal{D}(\mathcal{C})
$$

is the upper part of a ladder of recollements of height two. This is a direct consequence of [19, proposition 3.4] since $\hat{i}_*$ preserves compact objects.   □

REMARK 3.15. When $\mathcal{B}$ is a K-linear category, the assumption $\mathcal{D}^c(\mathcal{B}) \subseteq \mathcal{D}^*_\dagger(\mathcal{B})$ always holds when $\dagger = \emptyset$. When $\dagger = fl$ the assumption holds iff $\mathcal{B}(B, B')$ is a K-module of finite length, for all $B, B' \in \mathcal{B}$. In particular when $\mathcal{B} = B$ is an ordinary algebra, the inclusion $\mathcal{D}^c(\mathcal{B}) \subseteq \mathcal{D}^*_{fl}(\mathcal{B})$ holds iff B is finite length.

For our next result, we shall use the following concept.

DEFINITION 3.16. A compactly generated triangulated category $\mathcal{E}$ will be called *compact-detectable in finite length* when $\mathcal{E}^c$ consists of the objects $X \in \mathcal{E}^b_{fl}$ such that $\mathrm{Hom}_\mathcal{E}(X, E[k]) = 0$ for $E \in \mathcal{E}^b_{fl}$ and $k \gg 0$. Note that such a category is homologically locally finite length.

EXAMPLE 3.17. The following triangulated K-categories $\hat{\mathcal{D}}$ are compact-detectable in finite length and have the property that $\hat{D}^b_{fl}$ is Hom-finite (i.e. $\mathrm{Hom}_{\hat{\mathcal{D}}}(M, N)$ is a K-module of finite length for any $M, N \in \hat{D}^b_{fl}$):

(1) $\hat{\mathcal{D}} = \mathcal{D}(\mathrm{Qcoh}(\mathbb{X}))$, for a projective scheme $\mathbb{X}$ over a perfect field K [54].

(2) $\hat{\mathcal{D}} = \mathcal{D}(A)$, where A is a homologically non-positive homologically finite length dg K-algebra, where K is any commutative ring.

Proof. (1) By [54, lemma 7.46], for an arbitrary projective scheme $\mathbb{X}$ over K, we have $\mathcal{D}^b(\mathrm{coh}(\mathbb{X})) = \mathcal{D}(\mathrm{Qcoh}(\mathbb{X}))^b_{fl}$, which is well-known to be Hom-finite over K. By [54, lemma 7.49], $\mathcal{D}(\mathrm{Qcoh}(\mathbb{X}))^c$ consists of $X \in \mathcal{D}(\mathrm{Qcoh}(\mathbb{X}))$ such that, for each $M \in \mathcal{D}^b(\mathrm{coh}(\mathbb{X}))$, the K-vector space $\oplus_{k \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{D}(\mathrm{Coh}(\mathbb{X}))}(X, M[k])$ is finite dimensional. But if $X, M \in \mathcal{D}^b(\mathrm{coh}(\mathbb{X}))$, then $\mathrm{Hom}_{\mathcal{D}(\mathrm{Qcoh}(\mathbb{X}))}(X, M[k]) = 0$ for $k \ll 0$. We also know that $\mathcal{D}^b(\mathrm{coh}(\mathbb{X}))$ is Hom-finite. Thus, the subcategory consisting of $X \in \mathcal{D}^b(\mathrm{coh}(\mathbb{X}))$ such that, for each $M \in \mathcal{D}^b(\mathrm{coh}(\mathbb{X}))$, one has $\mathrm{Hom}_{\mathcal{D}(\mathrm{Qcoh}(\mathbb{X}))}(X, M[k]) = 0$ for $k \gg 0$, coincides with the subcategory of $X \in \mathcal{D}^b(\mathrm{coh}(\mathbb{X}))$ such that $\oplus_{k \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{D}(\mathrm{Qcoh}(\mathbb{X}))}(X, M[k])$ is finite dimensional. This subcategory is precisely $\mathcal{D}(\mathrm{Qcoh}(\mathbb{X}))^c \cap \mathcal{D}^b(\mathrm{coh}(\mathbb{X})) = \mathcal{D}(\mathrm{Qcoh}(\mathbb{X}))^c$.

(2) By [**46**, § 3.7] or [**28**, example 6.1], there is a *canonical t-structure* $(\mathcal{D}^{\leqslant 0}A, \mathcal{D}^{\geqslant 0}A)$ in $\mathcal{D}(A)$, where $\mathcal{D}^{\leqslant 0}(A)$ (resp. $\mathcal{D}^{\geqslant 0}(A)$) consists of the dg A-modules M such that $H^k(M) = 0$, for all $k > 0$ (resp. $k < 0$). Moreover, $\mathcal{D}^{\leqslant 0}(A) = \text{Susp}_{\mathcal{D}(A)}(A)$. By corollary 4.8 below for $\mathcal{T} = \{A\}$, this t-structure restricts to $\mathcal{D}^b_{fl}(A)$.

Note that there is a dg subalgebra $\tilde{A}$ of A, given by $\tilde{A}^n = A^n$, for $n < 0$, $\tilde{A}^0 = Z^0(A) = \{0\text{-cycles of } A\}$, and $\tilde{A}^n = 0$, for $n > 0$. The inclusion $\lambda : \tilde{A} \hookrightarrow A$ is a quasi-isomorphism, and the associated restriction of scalars $\lambda_* : \mathcal{D}(A) \longrightarrow \mathcal{D}(\tilde{A})$ is a triangulated equivalence which takes A to $\tilde{A}$. As a consequence, this equivalence preserves the canonical t-structure, and hence it induces an equivalence between the corresponding hearts. The heart of $(\mathcal{D}^{\leqslant 0}(\tilde{A}), \mathcal{D}^{\geqslant 0}(\tilde{A}))$ is known to be equivalent to the category of modules over $H^0(\tilde{A}) \cong H^0(A)$ (see [**28**, example 6.1]). This equivalence is given by $H^0 : \mathcal{H} \xrightarrow{\cong} \text{Mod-}H^0(A)$ ($M \rightsquigarrow H^0(M)$). Putting $\mathcal{H}_{fl} := \mathcal{H} \cap \mathcal{D}^b_{fl}(A)$, which is the heart of the restricted t-structure $(\mathcal{D}^{\leqslant 0}(A) \cap \mathcal{D}^b_{fl}(A), \mathcal{D}^{\geqslant 0}(A) \cap \mathcal{D}^b_{fl}(A))$ in $\mathcal{D}^b_{fl}(A)$, we deduce an equivalence of categories $H^0 : \mathcal{H}_{fl} \xrightarrow{\cong} \text{mod-}H^0(A)$, bearing in mind that $H^0(A)$ is a finite length K-algebra.

Let now $X \in \mathcal{D}^b_{fl}(A)$ be any object. By the proof of [**47**, theorem 2] and the fact that $\text{Hom}_{\mathcal{D}(A)}(A[k], M) \cong H^{-k}M$ is a K-module of finite length, for each $M \in \mathcal{D}^b_{fl}(A)$ and each $k \in \mathbb{Z}$, we know that X is the Milnor colimit of a sequence $0 = X_{-1} \xrightarrow{f_0} X_0 \xrightarrow{f_1} X_1 \longrightarrow \cdots \longrightarrow X_n \xrightarrow{f_n} \cdots$ such that $\text{cone}(f_n) \in \text{add}(A)[n]$, for each $n \in \mathbb{N}$. Let $r > 0$ be arbitrary and, for each $n > r$, put $u_n := f_n \circ \cdots \circ f_{r+1} : X_r \longrightarrow X_n$ and $C_n = \text{cone}(u_n)$. By Verdier's 3×3 lemma (see [**37**, lemma 1.7]), we have a commutative diagram, where all rows and columns are triangles

$$
\begin{array}{ccccccc}
\coprod_{n>r} X_r & \xrightarrow{\coprod_{n>r} u_n} & \coprod_{n>r} X_n & \longrightarrow & \coprod_{n>r} C_n & \longrightarrow & \\
\downarrow{\scriptstyle 1-\sigma} & & \downarrow{\scriptstyle 1-\sigma} & & \downarrow & & \\
\coprod_{n>r} X_r & \xrightarrow{\coprod_{n>r} u_n} & \coprod_{n>r} X_n & \longrightarrow & \coprod_{n>r} C_n & \longrightarrow & \\
\downarrow & & \downarrow & & \downarrow & & \\
X_r & \longrightarrow & X & \longrightarrow & C & \longrightarrow & .
\end{array}
$$

Since each C_n is a finite iterated extension of objects in $\text{add}(A)[n]$, with $n > r$, we get $C_n \in \mathcal{D}^{<-r}(A)$, for each $n > r$. It follows that $C \in \mathcal{D}^{\leqslant -r}(A)$. But $C \in \mathcal{D}^b_{fl}(A)$, since the left two terms of the triangle in the bottom row of the diagram are in $\mathcal{D}^b_{fl}(A)$. We then get a triangle $X_r \longrightarrow X \longrightarrow C \xrightarrow{+}$ in $\mathcal{D}^b_{fl}(A)$ such that $X_r \in \mathcal{D}^c(A)$ and $C \in \mathcal{D}^{\leqslant -r}(A)$. If now $Y \in \mathcal{D}^b_{fl}(A)$ is any object, then we know that $Y \in \mathcal{D}^{>-r}(A)$, for some integer $r > 0$. For this integer, we then get a monomorphism $\text{Hom}_{\mathcal{D}(A)}(X, Y) \longrightarrow \text{Hom}_{\mathcal{D}(A)}(X_r, Y)$ whose target is a K-module of finite length. This proves that $\mathcal{D}^b_{fl}(A)$ is Hom-finite.

On the other hand, by [**15**, lemma 3.7], we know that

$$\mathcal{D}^{\leqslant 0}(A) \cap \mathcal{D}^b_{fl}(A) = \bigcup_{n \in \mathbb{N}} \mathrm{add}(\mathcal{H}_{fl}[n] \star \mathcal{H}_{fl}[n-1] \star \cdots \star \mathcal{H}_{fl}[0]). \qquad (3.2)$$

Let $X \in \mathcal{D}^b_{fl}(A)$ be an object such that, for each $M \in \mathcal{D}^b_{fl}(A)$, one has $\mathrm{Hom}_{\mathcal{D}(A)}(X, M[k]) = 0$ for $k \gg 0$ (and, hence, also for $|k| \gg 0$). Let us choose a (necessarily finite) set $\mathcal{S}$ of representatives of the isoclasses of simple objects of $\mathcal{H}_{fl}$. If m is the Loewy length of $H^0(A)$, then $\mathcal{H}_{fl} \subset \mathrm{add}(\mathcal{S}) \star \overset{m}{\cdots} \star \mathrm{add}(\mathcal{S})$. Fixing $r \in \mathbb{N}$ such that $\mathrm{Hom}_{\mathcal{D}(A)}(X, \mathcal{S}[k]) = 0$, for $k \geqslant r$, we get that $\mathrm{Hom}_{\mathcal{D}(A)}(X, -[k])$ vanishes on $\mathcal{H}_{fl}$, for all $k \geqslant r$. By (3.2) above, $\mathrm{Hom}_{\mathcal{D}(A)}(X, -)$ vanishes on $\mathcal{D}^{\leqslant -r}(A) \cap \mathcal{D}^b_{fl}(A)$. If for this r we consider the triangle $X_r \longrightarrow X \longrightarrow C \overset{+}{\longrightarrow}$ constructed above, then the arrow $X \longrightarrow C$ is the zero map, and hence X is isomorphic to a direct summand of X_r and $X \in \mathcal{D}^c(A)$. $\qquad\square$

REMARK 3.18. While preparing the manuscript we have learnt that Neeman has introduced the powerful tool of approximable triangulated categories. Using it, one can derive the compact-detectability in finite length for the categories from the last example, using the fact that they are approximable, with the equivalence class of the canonical t-structure as the preferred one (see [**43**, examples 3.3 and 3.6]). Although nontrivial, the only thing left to prove would be the fact that what is $\mathcal{T}_c^-$, in Neeman's terminology, coincides with $\hat{\mathcal{D}}^b_{fl}$ in our case. Once this is proved the compact-detectability in finite length follows from [**41**, theorem 0.3].

PROPOSITION 3.19. *Let $\hat{\mathcal{Y}}$, $\hat{\mathcal{D}}$ and $\hat{\mathcal{X}}$ be compactly generated triangulated categories which are compact-detectable in finite length and let*

$$\hat{\mathcal{Y}}^b_{fl} \xrightarrow[\overset{\textstyle\xleftarrow{\;i^*\;}}{\xrightarrow{\;i_*\;}}]{\xleftarrow{\;i^!\;}} \hat{\mathcal{D}}^b_{fl} \xrightarrow[\overset{\textstyle\xleftarrow{\;j_!\;}}{\xrightarrow{\;j^*\;}}]{\xleftarrow{\;j_*\;}} \hat{\mathcal{X}}^b_{fl}$$

be a recollement. Then the functors $j_!$, j^*, i^* and i_* preserve compact objects. In particular, the associated TTF triple $(Im(j_!), Im(i_*), Im(j_*))$ in $\hat{\mathcal{D}}^b_{fl}$ lifts to a TTF triple $(\mathcal{U}, \mathcal{V}, \mathcal{W})$ in $\hat{\mathcal{D}}$ such that the torsion pairs $(\mathcal{U}, \mathcal{V})$ and $(\mathcal{V}, \mathcal{W})$ are compactly generated and $j_!(\hat{\mathcal{X}}^c) = \mathcal{U} \cap \hat{\mathcal{D}}^c$ and $i_*(\hat{\mathcal{Y}}^c) = \mathcal{V} \cap \hat{\mathcal{D}}^c$.

In the particular case when $\hat{\mathcal{Y}} = \mathcal{D}(B)$ and $\hat{\mathcal{X}} = \mathcal{D}(C)$, for ordinary finite length K-algebras B and C (and hence $\hat{\mathcal{Y}}^b_{fl} \cong \mathcal{D}^b(mod\text{-}B)$ and $\hat{\mathcal{X}}^b_{fl} \cong \mathcal{D}^b(mod\text{-}C)$), and $\hat{\mathcal{D}}$ is algebraic, the given recollement lifts, up to equivalence, to a recollement

$$\mathcal{D}(B) \xrightarrow[\overset{\textstyle\xleftarrow{\;\hat{i}^*\;}}{\xrightarrow{\;\hat{i}_*\;}}]{\xleftarrow{\;\hat{i}^!\;}} \hat{\mathcal{D}} \xrightarrow[\overset{\textstyle\xleftarrow{\;\hat{j}_!\;}}{\xrightarrow{\;\hat{j}^*\;}}]{\xleftarrow{\;\hat{j}_*\;}} \mathcal{D}(C),$$

which is the upper part of a ladder of recollements of height two.

Proof. It is clear that if $\mathcal{D}$ and $\mathcal{E}$ are triangulated categories which are compact-detectable in finite length and $F : \mathcal{D}^b_{fl} \longrightarrow \mathcal{E}^b_{fl}$ is a functor that has a right adjoint,

then F preserves compact objects. Therefore $j_!$, j^*, i^* and i_* preserve compact objects. Now corollary 3.12 says that assertion 2 of theorem 3.3 holds. The last assertion of the proposition is a direct consequence of the last assertion of theorem 3.14. □

REMARK 3.20. Last proposition applies to any recollement

$$
D^b(\text{mod-}B) \xrightarrow[\substack{\longleftarrow \\ i_* \\ \longrightarrow \\ i^!}]{\substack{i^* \\ \longleftarrow \\ \longrightarrow}} D^b(\text{mod-}A) \xrightarrow[\substack{\longleftarrow \\ j^* \\ \longrightarrow \\ j_*}]{\substack{j_! \\ \longleftarrow \\ \longrightarrow}} D^b(\text{mod-}C),
$$

where A, B and C are finite length algebras.

4. Partial silting sets

Recall that a silting set in a triangulated category $\mathcal{D}$ is a non-positive set $\mathcal{T}$ such that $\text{thick}_{\mathcal{D}}(\mathcal{T}) = \mathcal{D}$ (see [1]). In this paper, we will call a silting set with this property a *classical silting set*. In [47, 50] the authors introduced the notion of a silting set in any triangulated category with coproducts. We take the following definition, given in [47] for triangulated categories with coproducts, and consider it in an arbitrary triangulated category $\mathcal{D}$.

DEFINITION 4.1. Let $\mathcal{D}$ be a triangulated category. A set of objects $\mathcal{T}$ in $\mathcal{D}$ will be called *partial silting* when the following conditions hold:

(1) The t-structure generated by $\mathcal{T}$ exists in $\mathcal{D}$;

(2) $\text{Hom}_{\mathcal{D}}(T, -[1])$ vanishes on the aisle of that t-structure, for all $T \in \mathcal{T}$.

Such a set will be called a *silting set* when it generates $\mathcal{D}$.

A t-structure $(\mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant 0})$ in $\mathcal{D}$ is called a *(partial) silting t-structure* when it is generated by a (partial) silting set.

REMARK 4.2. Note that this definition of a silting set from [47] coincides with the one from [50], i.e. a set of objects $\mathcal{T}$ such that $(\mathcal{T}^{\perp > 0}, \mathcal{T}^{\perp < 0})$ is a t-structure. Furthermore, 'silting set in $\mathcal{D}$ consisting of compact objects' and 'classical silting set in $\mathcal{D}^c$' are the same.

EXAMPLE 4.3.

(a) If $\mathcal{D}$ has coproducts, then any non-positive set of compact objects is partial silting (see [47, example 2(1)]).

(b) Let A be an ordinary algebra and let $\mathcal{K}^b(\text{Proj-}A)$ denote the bounded homotopy category of complexes of projective modules. A complex $P^\bullet \in \mathcal{K}^b(\text{Proj-}A)$ is called a semi-tilting complex in [59] if $\text{Hom}_{\mathcal{D}(A)}(P^\bullet, P^{\bullet(I)}[k]) = 0$, for all sets I and all integers $k > 0$, and $\text{thick}_{\mathcal{D}(A)}(\text{Add}(P^\bullet)) = \mathcal{K}^b(\text{Proj-}A)$. In such a case $\mathcal{T} = \{P^\bullet\}$ is a silting set in $\mathcal{D}(A)$ (see [47, example 2(2)]).

The following gives a good source of examples of partial silting sets.

PROPOSITION 4.4. *Let $\mathcal{D}$ be a thick subcategory of a triangulated category $\mathcal{E}$ and let $\mathcal{T} \subset \mathcal{D}$ be a set of objects. If $\mathcal{T}$ is partial silting in $\mathcal{E}$ and the associated t-structure τ in $\mathcal{E}$ restricts to $\mathcal{D}$, then $\mathcal{T}$ is partial silting in $\mathcal{D}$. Moreover, when $\mathcal{E}$ has coproducts and $\mathcal{T}$ is a silting set in $\mathcal{E}$ consisting of compact objects that is partial silting in $\mathcal{D}$, then τ restricts to the t-structure generated by $\mathcal{T}$ in $\mathcal{D}$.*

Proof. Assume $\mathcal{T}$ is partial silting in $\mathcal{E}$ and the associated t-structure τ in $\mathcal{E}$ restricts to $\mathcal{D}$. The restricted t-structure in $\mathcal{D}$ is $({}^{\perp}(\mathcal{T}^{\perp \leqslant 0}) \cap \mathcal{D}, \mathcal{T}^{\perp < 0} \cap \mathcal{D})$, where the orthogonals are taken in $\mathcal{E}$. Since $\operatorname{Hom}_{\mathcal{E}}(T, -)$ vanishes on $({}^{\perp}\mathcal{T}^{\perp \leqslant 0})[1]$, it vanishes on $({}^{\perp}(\mathcal{T}^{\perp \leqslant 0}) \cap \mathcal{D})[1]$. It remains to see that the restricted t-structure is generated by $\mathcal{T}$ in $\mathcal{D}$. That is, that $({}^{\perp}(\mathcal{T}^{\perp \leqslant 0}) \cap \mathcal{D}, \mathcal{T}^{\perp < 0} \cap \mathcal{D}) = ({}^{\perp}(\mathcal{T}^{\perp \leqslant 0} \cap \mathcal{D}) \cap \mathcal{D}, \mathcal{T}^{\perp < 0} \cap \mathcal{D})$. Right parts of these pairs coincide and we clearly have the inclusion $\subseteq$ on the left parts. If $X \in {}^{\perp}(\mathcal{T}^{\perp \leqslant 0} \cap \mathcal{D}) \cap \mathcal{D}$ and $U \xrightarrow{f} X \xrightarrow{g} V \xrightarrow{+}$ is the truncation triangle with respect to the restricted t-structure, then $g = 0$ and hence X is a direct summand of $U \in {}^{\perp}(\mathcal{T}^{\perp \leqslant 0}) \cap \mathcal{D}$. This implies that X belongs to ${}^{\perp}(\mathcal{T}^{\perp \leqslant 0}) \cap \mathcal{D}$ since this class is closed under direct summands.

For the second part of the statement note that by [**47**, theorem 1], the set $\mathcal{T}$ is partial silting in $\mathcal{E}$ and ${}^{\perp}(\mathcal{T}^{\perp \leqslant 0}) = \mathcal{T}^{\perp > 0}$. On the other hand, the t-structure in $\mathcal{D}$ generated by $\mathcal{T}$ is $\tau' = ({}^{\perp}(\mathcal{T}^{\perp \leqslant 0} \cap \mathcal{D}) \cap \mathcal{D}, \mathcal{T}^{\perp < 0} \cap \mathcal{D})$, and the partial silting condition of $\mathcal{T}$ in $\mathcal{D}$ gives ${}^{\perp}(\mathcal{T}^{\perp \leqslant 0} \cap \mathcal{D}) \cap \mathcal{D} \subseteq \mathcal{T}^{\perp > 0}$, so ${}^{\perp}(\mathcal{T}^{\perp \leqslant 0} \cap \mathcal{D}) \cap \mathcal{D} \subseteq \mathcal{T}^{\perp > 0} \cap \mathcal{D}$. Thus, we get a chain of inclusions ${}^{\perp}(\mathcal{T}^{\perp \leqslant 0}) \cap \mathcal{D} \subseteq {}^{\perp}(\mathcal{T}^{\perp \leqslant 0} \cap \mathcal{D}) \cap \mathcal{D} \subseteq \mathcal{T}^{\perp > 0} \cap \mathcal{D}$, all of which must be equalities. Hence, $\tau' = (\mathcal{T}^{\perp > 0} \cap \mathcal{D}, \mathcal{T}^{\perp < 0} \cap \mathcal{D})$ is the restriction of τ to $\mathcal{D}$. $\square$

We now address the question on the uniqueness of the partial silting set which generates a given partial silting t-structure. The following is a consequence of the results in [**47**, § 4].

PROPOSITION 4.5. *Let $\mathcal{D}$ be a triangulated category with coproducts. If $(\mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant 0})$ is a partial silting t-structure in $\mathcal{D}$, then the partial silting set which generates the t-structure is uniquely determined up to Add-equivalence.*

When $\mathcal{D}$ is a subcategory of a category with coproducts and the t-structure is generated by a partial silting set of compact objects, we still have a certain kind of uniqueness, as the following result shows.

PROPOSITION 4.6. *Let $\mathcal{D}$ be a thick subcategory of a triangulated category with coproducts $\hat{\mathcal{D}}$ such that $\hat{\mathcal{D}}^c \subseteq \mathcal{D}$ and $\hat{\mathcal{D}}^c$ is skeletally small. Suppose that $(\mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant 0})$ is a t-structure in $\mathcal{D}$ generated by a partial silting set which consists of compact objects in $\hat{\mathcal{D}}$. There is a non-positive set $\mathcal{T} \subseteq \hat{\mathcal{D}}^c$, uniquely determined up to add-equivalence, such that the following two conditions hold:*

(a) *$\mathcal{T}$ is partial silting in $\mathcal{D}$ and it generates $(\mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant 0})$.*

(b) *If $\mathcal{T}' \subset \hat{\mathcal{D}}^c$ is any partial silting set in $\mathcal{D}$ which generates $(\mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant 0})$, then $add(\mathcal{T}') \subseteq add(\mathcal{T})$.*

If $(\mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant 0})$ is the restriction of a t-structure $(\hat{\mathcal{D}}^{\leqslant 0}, \hat{\mathcal{D}}^{\geqslant 0})$ in $\hat{\mathcal{D}}$ generated by some non-positive set $\mathcal{T}_0 \subset \hat{\mathcal{D}}^c$, then $add(\mathcal{T}) = add(\mathcal{T}_0)$.

Proof. Let $\mathcal{C} := {}^{\perp}(\mathcal{D}^{\leqslant 0}[1]) \cap \mathcal{D}^{\leqslant 0}$ be the co-heart of $(\mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant 0})$ and let $\mathcal{T}' \subset \mathcal{D}^c$ be any partial silting set in $\mathcal{D}$ which generates this t-structure. Since $\mathrm{Hom}_{\mathcal{D}}(T', -)$ vanishes on $\mathcal{D}^{\leqslant 0}[1]$, for all $T' \in \mathcal{T}'$, we have $T' \subset \mathcal{C}$, and hence $\mathrm{add}(\mathcal{T}') \subseteq \mathcal{C} \cap \hat{\mathcal{D}}^c$. Let $\mathcal{T}$ be a set of representatives of isomorphism classes of objects of $\mathcal{C} \cap \hat{\mathcal{D}}^c$. Let us check that $\mathcal{T}$ generates $(\mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant 0})$.

Without loss of generality, we can assume that $\mathcal{T}' \subseteq \mathcal{T}$. This implies that $\mathcal{T}^{\perp_{\leqslant 0}} \cap \mathcal{D} \subseteq \mathcal{T}'^{\perp_{\leqslant 0}} \cap \mathcal{D} = \mathcal{D}^{>0} := \mathcal{D}^{\geqslant 0}[-1]$. Since $\mathcal{T} \subset \mathcal{D}^{\leqslant 0}$, we have that $\mathrm{Hom}_{\mathcal{D}}(T, Y) = 0$, for all $T \in \mathcal{T}$ and $Y \in \mathcal{D}^{>0}$. Hence, the inclusion $\mathcal{D}^{>0} = \mathcal{T}'^{\perp_{\leqslant 0}} \cap \mathcal{D} \subseteq \mathcal{T}^{\perp_{\leqslant 0}} \cap \mathcal{D}$ also holds and $\mathcal{T}$ generates the t-structure $(\mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant 0})$.

Let us prove the last assertion of the proposition. Suppose that $(\mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant 0}) = ({}^{\perp}(\mathcal{T}_0^{\perp_{\leqslant 0}}) \cap \mathcal{D}, \mathcal{T}_0^{\perp_{<0}} \cap \mathcal{D})$, for some non-positive set $\mathcal{T}_0 \subset \hat{\mathcal{D}}^c$. Recall that ${}^{\perp}(\mathcal{T}_0^{\perp_{\leqslant 0}}) = \mathrm{Susp}_{\hat{\mathcal{D}}}(\mathcal{T}_0)$ (see [**47**, theorem 2]). If $C \in \mathcal{C} \cap \hat{\mathcal{D}}^c$, then $\mathrm{Hom}_{\hat{\mathcal{D}}}(C, -)$ vanishes on $\mathrm{Susp}_{\hat{\mathcal{D}}}(\mathcal{T}_0)[1] = {}^{\perp}(\mathcal{T}_0^{\perp_{\leqslant 0}})[1]$. Indeed, $\bigcup_{k>0} \mathcal{T}_0[k] \subset \mathcal{D}^{\leqslant 0}[1]$, $\mathrm{Hom}_{\hat{\mathcal{D}}}(C, -)$ vanishes on $\mathcal{D}^{\leqslant 0}[1]$ and C is compact. Hence, $\mathcal{C} \cap \hat{\mathcal{D}}^c$ belongs to the co-heart $\hat{\mathcal{C}} := {}^{\perp}\mathrm{Susp}_{\hat{\mathcal{D}}}(\mathcal{T}_0)[1] \cap \mathrm{Susp}_{\hat{\mathcal{D}}}(\mathcal{T}_0)$ of the t-structure $({}^{\perp}(\mathcal{T}_0^{\perp_{\leqslant 0}}), \mathcal{T}_0^{\perp_{<0}})$ in $\hat{\mathcal{D}}$. By [**47**, lemma 6], we conclude that $\mathcal{C} \cap \hat{\mathcal{D}}^c \subseteq \mathrm{Add}(\mathcal{T}_0)$ and, since $\mathcal{C} \cap \hat{\mathcal{D}}^c$ consists of compact objects, $\mathcal{C} \cap \hat{\mathcal{D}}^c \subseteq \mathrm{add}(\mathcal{T}_0)$. On the other hand, by proposition 4.4, $\mathcal{T}_0$ is a partial silting set in $\mathcal{D}$ which generates $(\mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant 0})$. By the first paragraph of the proof, $\mathrm{add}(\mathcal{T}_0) \subseteq \mathcal{C} \cap \hat{\mathcal{D}}^c$, and hence $\mathrm{add}(\mathcal{T}_0) = \mathrm{add}(\mathcal{T})$. □

Recall the notation and terminology of 3.7.

COROLLARY 4.7. *Let $\mathcal{D}$ be a triangulated category and let $\mathcal{T}$ be a partial silting set in $\mathcal{D}$. For any $\star \in \{\emptyset, +, -, b\}$ and $\dagger \in \{\emptyset, fl\}$ the t-structure $\tau_{\mathcal{T}} = ({}^{\perp}(\mathcal{T}^{\perp_{\leqslant 0}}), \mathcal{T}^{\perp_{<0}})$ restricts to $\mathcal{D}_{\mathcal{T}, \dagger}^{\star}$. In particular, if $\mathcal{T}$ is contained in $\mathcal{D}_{\mathcal{T}, \dagger}^{\star}$, then $\mathcal{T}$ is a partial silting set in $\mathcal{D}_{\mathcal{T}, \dagger}^{\star}$.*

Proof. For $M \in \mathcal{D}$, let us consider the truncation triangle with respect to $\tau_{\mathcal{T}}$

$$U \longrightarrow M \longrightarrow V \stackrel{+}{\longrightarrow} .$$

Then $V \in \mathcal{T}^{\perp_{\leqslant 0}}$ and, since $\mathrm{Hom}_{\mathcal{D}}(\mathcal{T}, -)$ vanishes on ${}^{\perp}(\mathcal{T}^{\perp_{\leqslant 0}})[1]$, we get $U \in \mathcal{T}^{\perp_{>0}}$. This gives induced isomorphisms $\mathrm{Hom}_{\mathcal{D}}(T, U[k]) \cong \mathrm{Hom}_{\mathcal{D}}(T, M[k])$, for $k \leqslant 0$, and $\mathrm{Hom}_{\mathcal{D}}(T, M[k]) \cong \mathrm{Hom}_{\mathcal{D}}(T, V[k])$, for $k > 0$. It immediately follows that $\tau_{\mathcal{T}}$ restricts to $\mathcal{D}_{\mathcal{T}, \dagger}^{*}$, for any choices $\star \in \{\emptyset, +, -, b\}$ and $\dagger \in \{\emptyset, fl\}$. □

Using that for any generating set $\mathcal{X}$ of $\mathcal{D}$, consisting of compact objects, $\mathcal{D}_{\mathcal{X}, \dagger}^{*} = \mathcal{D}_{\dagger}^{*}$, we get:

COROLLARY 4.8. *Let $\mathcal{D}$ be a compactly generated triangulated category and let $\mathcal{T}$ be a classical silting set in $\mathcal{D}^c$. For any $\star \in \{\emptyset, +, -, b\}$ and $\dagger \in \{\emptyset, fl\}$ the t-structure $\tau_{\mathcal{T}} = ({}^{\perp}(\mathcal{T}^{\perp_{\leqslant 0}}), \mathcal{T}^{\perp_{<0}}) = (\mathcal{T}^{\perp_{>0}}, \mathcal{T}^{\perp_{<0}})$ restricts to $\mathcal{D}_{\dagger}^{\star}$. And if $\mathcal{T} \subset \mathcal{D}_{\dagger}^{\star}$ (equivalently, if $\mathcal{D}^c \subset \mathcal{D}_{\dagger}^{\star}$), then $\mathcal{T}$ is a silting set of $\mathcal{D}_{\dagger}^{\star}$.*

5. (Pre)envelopes and their constructions

Recall that in any category $\mathcal{C}$, a morphism $f : C \longrightarrow C'$ is *left (resp. right) minimal* when any endomorphism $g \in \mathrm{End}_{\mathcal{C}}(C')$ (resp. $g \in \mathrm{End}_{\mathcal{C}}(C)$) such that

$g \circ f = f$ (resp. $f \circ g = f$) is an isomorphism. When $\mathcal{X}$ is a subcategory, a morphism $f : C \longrightarrow X_C$, with $X_C \in \mathcal{X}$, is called an $\mathcal{X}$-*preenvelope* or *left $\mathcal{X}$-approximation* of C if each morphism $g : C \longrightarrow X$, with $X \in \mathcal{X}$, factors through f. The dual concept is that of $\mathcal{X}$-*precover* or *right $\mathcal{X}$-approximation*. An $\mathcal{X}$-*envelope* (resp. $\mathcal{X}$-*cover*) or *minimal left $\mathcal{X}$-approximation* (resp. *minimal right $\mathcal{X}$-approximation*) is an $\mathcal{X}$-preenvelope (resp. $\mathcal{X}$-precover) which is a left (resp. right) minimal morphism. The subcategory $\mathcal{X}$ is called *(pre)enveloping* (resp. *(pre)covering*) when each object of $\mathcal{C}$ has an $\mathcal{X}$-(pre)envelope (resp. $\mathcal{X}$-(pre)cover). In this section we show some relationship between (pre)enveloping subcategories and t- and co-t-structures in a triangulated category $\mathcal{D}$.

The following result is folklore and follows from [**34**, corollary 1.4].

LEMMA 5.1. *Let $\mathcal{V}$ be a full subcategory of $\mathcal{D}$ such that $\mathcal{V}$ is Krull–Schmidt. If an object M of $\mathcal{D}$ has a $\mathcal{V}$-preenvelope (resp. $\mathcal{V}$-precover), then it has a $\mathcal{V}$-envelope (resp. $\mathcal{V}$-cover).*

LEMMA 5.2. *Let $\mathcal{V}$ be a full subcategory of $\mathcal{D}$ closed under extensions, let $f : M \longrightarrow V$ be a morphism with $V \in \mathcal{V}$. Consider the following assertions:*

(1) *f is a $\mathcal{V}$-envelope*

(2) *the object U in the triangle $U \longrightarrow M \xrightarrow{f} V \xrightarrow{+}$ belongs to $^{\perp}\mathcal{V}$*

(3) *f is a $\mathcal{V}$-preenvelope.*

Then $(1) \Longrightarrow (2) \Longrightarrow (3)$ holds.

Proof. $(1) \Longrightarrow (2)$ Adapt the proof of [**16**, lemma 1.3].

$(2) \Longrightarrow (3)$ Applying the functor $\operatorname{Hom}_{\mathcal{D}}(-, V')$ to the triangle from assertion 2, we get that $\operatorname{Hom}_{\mathcal{D}}(f, V') : \operatorname{Hom}_{\mathcal{D}}(V, V') \longrightarrow \operatorname{Hom}_{\mathcal{D}}(M, V')$ is an epimorphism for any $V' \in \mathcal{V}$, thus f is a $\mathcal{V}$-preenvelope. $\square$

LEMMA 5.3. *Let $\mathcal{E}$ and $\mathcal{F}$ be full subcategories of $\mathcal{D}$. Consider the following homotopy pushout diagram, where the rows are triangles.*

$$
\begin{array}{ccccc}
C & \xrightarrow{u} & M & \xrightarrow{h} & F \\
\downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} & & \| \\
E & \xrightarrow{u'} & X & \xrightarrow{h'} & F
\end{array}
$$

(1) *If h is an $\mathcal{F}$-preenvelope and g is an $\mathcal{E}$-preenvelope, then f is an $\mathcal{E} \star \mathcal{F}$-preenvelope.*

(2) *Suppose that $\mathcal{E}$ and $\mathcal{F}$ are closed under extensions, and that the inclusion $\mathcal{F} \subseteq \mathcal{E}[1]$ holds. If g is an $\mathcal{E}$-envelope and h is an $\mathcal{F}$-envelope, then f is an $\mathcal{E} \star \mathcal{F}$-envelope (and hence an add$(\mathcal{E} \star \mathcal{F})$-envelope), provided that one of the following conditions hold:*

(a) $\mathcal{D}$ *is Krull–Schmidt.*

(b) $Hom_{\mathcal{D}}(E, F) = 0$, *for all $E \in \mathcal{E}$ and $F \in \mathcal{F}$.*

Proof. (1) Let $f' : M \longrightarrow X'$ be any morphism, where $X' \in \mathcal{E} \star \mathcal{F}$ and fix a triangle $E' \xrightarrow{\gamma} X' \xrightarrow{\delta} F' \xrightarrow{+}$, with $E' \in \mathcal{E}$ and $F' \in \mathcal{F}$. The $\mathcal{F}$-preenveloping condition on h gives a morphism $\rho : F \longrightarrow F'$ such that $\rho \circ h = \delta \circ f'$. We then get a morphism $g' : C \longrightarrow E'$ making commutative the following diagram:

$$
\begin{array}{ccccc}
C & \xrightarrow{\ u\ } & M & \xrightarrow{\ h\ } & F \\
\downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle \rho} \\
E' & \xrightarrow{\ \gamma\ } & X' & \xrightarrow{\ \delta\ } & F'
\end{array}
$$

The $\mathcal{E}$-preenveloping condition of g gives a morphism $\lambda : E \longrightarrow E'$ such that $g' = \lambda \circ g$. Thus $\gamma \circ \lambda \circ g = \gamma \circ g' = f' \circ u$ and there exists $\mu : X \longrightarrow X'$ such that $\mu \circ u' = \gamma \circ \lambda$ and $\mu \circ f = f'$, since the diagram we started from is a homotopy pushout. In particular, f' factors through f so that f is an $\mathcal{E} \star \mathcal{F}$-preenvelope.

(2) Since any $\mathcal{E} \star \mathcal{F}$-envelope is an $\mathrm{add}(\mathcal{E} \star \mathcal{F})$-envelope, we only need to check the left minimality of f.

(2.a) When $\mathcal{D}$ is Krull–Schmidt, there is a decomposition $f = \begin{pmatrix} f' & 0 \end{pmatrix}^t : M \longrightarrow X_1 \oplus X_2 = X$, where $f' : M \longrightarrow X_1$ is left minimal. Thus we can assume that the triangle $\mathrm{C}(f) \longrightarrow M \xrightarrow{f} X \xrightarrow{+}$, coincides with the triangle

$$
\mathrm{C}(f') \oplus X_2[-1] \xrightarrow{\begin{pmatrix} \gamma & 0 \end{pmatrix}} M \xrightarrow{\begin{pmatrix} f' & 0 \end{pmatrix}^t} X_1 \oplus X_2 \xrightarrow{+}.
$$

Since homotopy pushout squares are also homotopy pullback we get a triangle $(*)$

$$
C(g) = \mathrm{C}(f') \oplus X_2[-1] \xrightarrow{\begin{pmatrix} \alpha & \beta \end{pmatrix}} C \xrightarrow{g} E \xrightarrow{+},
$$

so $u \circ \begin{pmatrix} \alpha & \beta \end{pmatrix} = \begin{pmatrix} \gamma & 0 \end{pmatrix}$ and $u \circ \beta = 0$. Thus β admits a factorization $\beta : X_2[-1] \longrightarrow F[-1] \longrightarrow C$. But $F[-1] \in \mathcal{F}[-1] \subseteq \mathcal{E}$ and $X_2[-1]$ is a direct summand of $\mathrm{C}(g)$. By lemma 5.2, $\mathrm{C}(g) \in {}^{\perp}\mathcal{E}$, which implies $\beta = 0$. Hence, the triangle $(*)$ is isomorphic to $\mathrm{C}(f') \oplus X_2[-1] \xrightarrow{\begin{pmatrix} \alpha & 0 \end{pmatrix}} C \xrightarrow{\begin{pmatrix} g' & 0 \end{pmatrix}^t} E' \oplus X_2 \xrightarrow{+}$, where $E \simeq E' \oplus X_2$. The left minimality of g implies $X_2 = 0$ and, hence, that f is left minimal.

(2.b) Assume now that $\mathrm{Hom}_{\mathcal{D}}(\mathcal{E}, \mathcal{F}) = 0$. Let $\alpha \in \mathrm{End}_{\mathcal{D}}(X)$ be such that $\alpha \circ f = f$. Since $h' \circ \alpha \circ u' \in \mathrm{Hom}_{\mathcal{D}}(E, F) = 0$, there are $\alpha_1 : E \longrightarrow E$ and $\alpha_2 : F \longrightarrow F$ making the following diagram commutative:

$$
\begin{array}{ccccccc}
F[-1] & \xrightarrow{\ \lambda\ } & E & \xrightarrow{\ u'\ } & X & \xrightarrow{\ h'\ } & F \\
\downarrow & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha_2} \\
F[-1] & \xrightarrow{\ \lambda\ } & E & \xrightarrow{\ u'\ } & X & \xrightarrow{\ h'\ } & F
\end{array}
$$

Then $u' \circ \alpha_1 \circ g = \alpha \circ u' \circ g = \alpha \circ f \circ u = f \circ u = u' \circ g$, which implies $u' \circ (g - \alpha_1 \circ g) = 0$ and, hence, $g - \alpha_1 \circ g$ factors in the form $C \xrightarrow{t} F[-1] \xrightarrow{\lambda} E$. But $F[-1] \in \mathcal{F}[-1] \subseteq \mathcal{E}$ and since g is an $\mathcal{E}$-envelope, there is a morphism $\pi : E \longrightarrow F[-1]$ such that $t = \pi \circ g$. It follows that $g - \alpha_1 \circ g = \lambda \circ \pi \circ g$ and $g = (\alpha_1 + \lambda \circ \pi) \circ g$. The left minimality of g implies that $\alpha_1 + \lambda \circ \pi$ is an isomorphism. But $u' \circ (\alpha_1 + \lambda \circ \pi) = u' \circ \alpha_1$ since $u' \circ \lambda = 0$. This means that we can replace α_1 by $\alpha_1 + \lambda \circ \pi$ (and α_2 by some new α_2) and assume that α_1 is an isomorphism.

Note now that $\alpha_2 \circ h = \alpha_2 \circ h' \circ f = h' \circ \alpha \circ f = h' \circ f = h$. Then the left minimality of h implies that α_2 is an isomorphism and, as a consequence, α is an isomorphism. $\qquad\square$

COROLLARY 5.4. *Let $\mathcal{E}$ and $\mathcal{F}$ be enveloping subcategories of the triangulated category $\mathcal{D}$ closed under extensions and such that $\mathcal{F} \subseteq \mathcal{E}[1]$. If either $\mathcal{D}$ is Krull–Schmidt or $Hom_\mathcal{D}(\mathcal{E}, \mathcal{F}) = 0$, then $\mathcal{E} \star \mathcal{F}$ is an enveloping subcategory of $\mathcal{D}$ and, in particular, it is closed under direct summands. If, moreover, $Hom_\mathcal{D}(\mathcal{E}, \mathcal{F}[1]) = 0$ then $\mathcal{E} \star \mathcal{F}$ is also closed under extensions in $\mathcal{D}$.*

Proof. The enveloping condition on $\mathcal{E} \star \mathcal{F}$ is a direct consequence of lemma 5.3, and it is well-known that any enveloping subcategory is closed under direct summands. The final statement follows from [**47**, lemma 8]. $\qquad\square$

By [**1**, lemma 2.15] we have the following:

LEMMA 5.5. *Let $\mathcal{T}$ be a non-positive set of objects of $\mathcal{D}$. Then*

(1) $thick_\mathcal{D}(\mathcal{T}) = \bigcup_{r \leqslant s} add(add(\mathcal{T})[r] \star add(\mathcal{T})[r+1] \star \cdots \star add(\mathcal{T})[s])$. *Moreover, if $add(\mathcal{T})$ is an enveloping subcategory of $\mathcal{D}$, then*

$$thick_\mathcal{D}(\mathcal{T}) = \bigcup_{r \leqslant s}(add(\mathcal{T})[r] \star add(\mathcal{T})[r+1] \star \cdots \star add(\mathcal{T})[s])$$

(2) $susp_\mathcal{D}(\mathcal{T}) = \bigcup_{r \geqslant 0} add(add(\mathcal{T}) \star add(\mathcal{T})[1] \star \cdots \star add(\mathcal{T})[r])$. *If $add(\mathcal{T})$ is an enveloping subcategory of $\mathcal{D}$, then $susp_\mathcal{D}(\mathcal{T}) = \bigcup_{r \geqslant 0}(add(\mathcal{T}) \star add(\mathcal{T})[1] \star \cdots \star add(\mathcal{T})[r])$.*

Proof. First equality in assertion (1) is [**1**, lemma 2.15], first equality in assertion (2) is proved analogously. When $add(\mathcal{T})$ is enveloping, the assertions follow by an iterative application of corollary 5.4. $\qquad\square$

For an object M and a subcategory $\mathcal{T}$ in $\mathcal{D}$, we shall use the notation

$$s(M, \mathcal{T}) := Sup\{k \in \mathbb{N} \mid \operatorname{Hom}_\mathcal{D}(M, -[k])_{|\mathcal{T}} \neq 0\} \in \mathbb{N} \cup \{\infty\},$$

when this subset of natural numbers is nonempty. When this subset is empty, by convention, we put $add(\mathcal{T}) \star add(\mathcal{T})[1] \star \cdots \star add(\mathcal{T})[s(M, \mathcal{T})] := 0$.

LEMMA 5.6. *Let $\mathcal{T}$ be a nonpositive set of objects of $\mathcal{D}$ and $\mathcal{U} := susp_{\mathcal{D}}(\mathcal{T})$. The following assertions are equivalent for an object $M \in \mathcal{D}$:*

(1) *M has a $\mathcal{U}$-(pre)envelope.*

(2) *$Hom_{\mathcal{D}}(M, -[k])_{|\mathcal{T}} = 0$ for $k \gg 0$, and M has an $add(add(\mathcal{T}) \star add(\mathcal{T})[1] \star \cdots \star add(\mathcal{T})[s])$-(pre)envelope, where $s = s(M, \mathcal{T})$.*

Proof. $(1) \Longrightarrow (2)$ Let us check that $Hom_{\mathcal{D}}(M, -[k])_{|\mathcal{T}} = 0$ for $k \gg 0$. Let $f : M \longrightarrow U$ be a $\mathcal{U}$-(pre)envelope. By lemma 5.5 there exists an $r \in \mathbb{N}$ such that $U \in add(add(\mathcal{T}) \star add(\mathcal{T})[1] \star \cdots \star add(\mathcal{T})[r])$. If $k > r$ and $g : M \longrightarrow T[k]$ is a morphism, with $T \in \mathcal{T}$, then g factors in the form $g : M \xrightarrow{f} U \xrightarrow{h} T[k]$, where the second arrow is zero since $Hom_{\mathcal{D}}(-, T[k])$ vanishes on $add(\mathcal{T})[j]$, for $j = 0, 1, \ldots, r$.

There is a triangle $U' \xrightarrow{(v_1 \ \ v_2)^t} U \oplus Z \xrightarrow{(p_1 \ \ p_2)} U'' \xrightarrow{+}$, where $U' \in add((\mathcal{T}) \star add(\mathcal{T})[1] \star \cdots \star add(\mathcal{T})[s]$ and $U'' \in add(\mathcal{T})[s+1] \star \cdots \star add(\mathcal{T})[r]$. By definition of $s = s(M, \mathcal{T})$, we have that $Hom_{\mathcal{C}}(M, U'') = 0$, and so $0 = p_1 \circ f = (p_1 \ \ p_2) \circ (f \ \ 0)^t$. This implies that $(f \ \ 0)^t : M \longrightarrow U \oplus Z$ admits a factorization $(f \ \ 0)^t :$ $M \xrightarrow{f'} U' \xrightarrow{(v_1 \ \ v_2)^t} U \oplus Z$, and so $f = v_1 \circ f'$. Then f' is clearly the desired preenvelope. If f was an envelope, then U is a summand of U' and f is the desired envelope.

$(2) \Longrightarrow (1)$ Let $0 \neq f : M \longrightarrow X$ be any morphism with $X \in \mathcal{U}$, then $X \in add(add(\mathcal{T}) \star add(\mathcal{T})[1] \star \cdots \star add(\mathcal{T})[r])$, for some $r \in \mathbb{N}$. Without loss of generality, we assume that $X \in add(\mathcal{T}) \star add(\mathcal{T})[1] \star \cdots \star add(\mathcal{T})[r]$. There is a triangle $X' \xrightarrow{v} X \xrightarrow{p} X'' \xrightarrow{+}$, where $X' \in add((\mathcal{T}) \star add(\mathcal{T})[1] \star \cdots \star add(\mathcal{T})[s]$ and $X'' \in add(\mathcal{T})[s+1] \star \cdots \star add(\mathcal{T})[r]$. As before, $Hom_{\mathcal{C}}(M, X'') = 0$, and so $p \circ f = 0$. This implies that f admits a factorization $f : M \xrightarrow{f'} X' \xrightarrow{v} X$. Thus any morphism from M to $\mathcal{U}$ factors through $add(add(\mathcal{T}) \star add(\mathcal{T})[1] \star \cdots \star add(\mathcal{T})[s])$ and we are done. $\qquad \square$

PROPOSITION 5.7. *Let $\mathcal{D}$ be a triangulated category and $\mathcal{T}$ be a non-positive set of objects in $\mathcal{D}$. Consider the following assertions:*

1 *$Hom_{\mathcal{D}}(M, -[k])_{|\mathcal{T}} = 0$ for any $M \in \mathcal{D}$, $k \gg 0$ and M has an $add(\mathcal{T})[s(M, \mathcal{T})]$-envelope.*

1′ *$Hom_{\mathcal{D}}(M, -[k])_{|\mathcal{T}} = 0$ for any $M \in \mathcal{D}$, $k \gg 0$ and M has an $add(\mathcal{T})[s(M, \mathcal{T})]$-preenvelope.*

2 *$Hom_{\mathcal{D}}(M, -[k])_{|\mathcal{T}} = 0$ for any $M \in \mathcal{D}$, $k \gg 0$ and M has an $add(add(\mathcal{T}) \star add(\mathcal{T})[1] \star \cdots \star add(\mathcal{T})[s(M, \mathcal{T})])$-envelope.*

2′ *$Hom_{\mathcal{D}}(M, -[k])_{|\mathcal{T}} = 0$ for any $M \in \mathcal{D}$, $k \gg 0$ and M has an $add(add(\mathcal{T}) \star add(\mathcal{T})[1] \star \cdots \star add(\mathcal{T})[s(M, \mathcal{T})])$-preenvelope.*

3 *$susp_{\mathcal{D}}(\mathcal{T})$ is an enveloping class in $\mathcal{D}$.*

3′ susp$_\mathcal{D}$($\mathcal{T}$) is a preenveloping class in $\mathcal{D}$.

4 ($^\perp susp_\mathcal{D}(\mathcal{T})[1], susp_\mathcal{D}(\mathcal{T})$) is a co-t-structure in $\mathcal{D}$.

Then implications

$$(1') \Longleftrightarrow (2')$$
$$(1) \qquad\qquad (3') \Longleftrightarrow (4)$$
$$(2) \Longleftrightarrow (3)$$
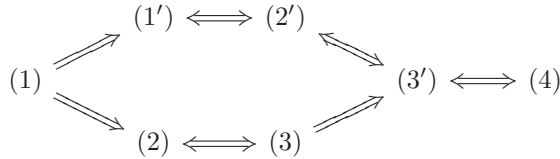
hold and if $\mathcal{D}$ is Krull–Schmidt, then all the assertions are equivalent. Moreover, when assertion 1 holds, the envelope $M \longrightarrow U$ from assertion 2, which is also a susp$_\mathcal{D}$($\mathcal{T}$)-envelope, can be constructed inductively.

Proof. The implications $(2) \Longleftrightarrow (3) \Longrightarrow (4) \Longrightarrow (3')$ follow from lemma 5.6 and lemma 5.2. The equivalence $(2)' \Longleftrightarrow (3')$ also follows from lemma 5.6, and the implications $(1) \Longrightarrow (1')$ and $(3) \Longrightarrow (3')$ are clear. Apart from the statement about inductive construction, it is enough to prove implications $(1) \Longrightarrow (2)$ and $(2') \Longrightarrow (1') \Longrightarrow (4)$, then the equivalence of all assertions when $\mathcal{D}$ is Krull–Schmidt will follow from lemma 5.1.

$(1) \Longrightarrow (2)$ Without loss of generality, we only consider M such that $\mathrm{Hom}_\mathcal{C}(M, -[k])_{|\mathcal{T}} \neq 0$, for some $k \in \mathbb{N}$. Let us prove by induction on $r \geqslant 0$ that if M is an object such that $0 \leqslant s := s(M, \mathcal{T}) \leqslant r$, then M has an $\mathrm{add}(\mathcal{T}) \star \mathrm{add}(\mathcal{T})[1] \star \cdots \star \mathrm{add}(\mathcal{T})[r]$-envelope. Note that if $s := s(M, \mathcal{T}) < r$, then by the induction hypothesis, there is an $\mathrm{add}(\mathcal{T}) \star \mathrm{add}(\mathcal{T})[1] \star \cdots \star \mathrm{add}(\mathcal{T})[s]$-envelope, which is easily seen to be an $\mathrm{add}(\mathcal{T}) \star \mathrm{add}(\mathcal{T})[1] \star \cdots \star \mathrm{add}(\mathcal{T})[r]$-envelope. Assume $r = s$, and fix an $\mathrm{add}(\mathcal{T})[s]$-envelope $h : M \longrightarrow T_M[s]$, which we complete to a triangle $C \xrightarrow{u} M \xrightarrow{h} T_M[s] \xrightarrow{+} (*)$. Since $\mathrm{add}(\mathcal{T})[s]$ is closed under extensions, by lemma 5.2, $\mathrm{Hom}_\mathcal{C}(C, -[s])_{|\mathcal{T}} = 0$. Applying $\mathrm{Hom}_\mathcal{C}(-, T[k])$ to the triangle $(*)$, we see that $\mathrm{Hom}_\mathcal{C}(C, -[k])_{|\mathcal{T}} = 0$, for $k \geqslant s$. Then $s(C, \mathcal{T}) < s$. By the induction hypothesis there is an $\mathrm{add}(\mathcal{T}) \star \mathrm{add}(\mathcal{T})[1] \star \cdots \star \mathrm{add}(\mathcal{T})[r-1]$-envelope $g : C \longrightarrow E$. Then, for $\mathcal{E} = \mathrm{add}(\mathcal{T}) \star \mathrm{add}(\mathcal{T})[1] \star \cdots \star \mathrm{add}(\mathcal{T})[r-1]$, $\mathcal{F} = \mathrm{add}(\mathcal{T})[r]$ and $F = T_M[r]$, lemma 5.3 implies that M has an $\mathcal{E} \star \mathcal{F} = \mathrm{add}(\mathcal{T}) \star \mathrm{add}(\mathcal{T})[1] \star \cdots \star \mathrm{add}(\mathcal{T})[r]$-envelope.

$(2') \Longrightarrow (1')$ Let $f : M \longrightarrow U$ be an $\mathrm{add}(\mathcal{T}) \star \mathrm{add}(\mathcal{T})[1] \star \cdots \star \mathrm{add}(\mathcal{T})[s]$-preenvelope, where $s = s(M, \mathcal{T}) \geqslant 0$. There is a triangle $X' \longrightarrow U \xrightarrow{g} T[s] \xrightarrow{+}$, where $X' \in \mathrm{add}(\mathcal{T}) \star \mathrm{add}(\mathcal{T})[1] \star \cdots \star \mathrm{add}(\mathcal{T})[s-1]$ and $T \in \mathrm{add}(\mathcal{T})$. Let $h : M \longrightarrow T'[s]$ be any morphism, where $T' \in \mathcal{T}$. Then there is a morphism $\eta : U \longrightarrow T'[s]$ such that $\eta \circ f = h$. Since $\mathrm{Hom}_\mathcal{C}(X', T'[s]) = 0$, there is a morphism $\mu : T[s] \longrightarrow T'[s]$ such that $\mu \circ g = \eta$. Thus, $h = \eta \circ f = \mu \circ g \circ f$ and $g \circ f$ is an $\mathrm{add}(\mathcal{T})[s]$-preenvelope of M.

$(1') \Longrightarrow (4)$ Put $\mathcal{U} := \mathrm{susp}_\mathcal{D}(\mathcal{T})$. Let us prove that any object M fits into a triangle $V_M \longrightarrow M \longrightarrow U_M \xrightarrow{+}$, where $U_M \in \mathcal{U}$ and $V_M \in {}^\perp\mathcal{U}$. If $M \in {}^\perp\mathcal{U}$ there is nothing to prove. We then assume that $M \notin {}^\perp\mathcal{U}$, so that $s(M, \mathcal{T}) \geqslant 0$. Let us prove

the statement by induction on $s(M, \mathcal{T})$. Assume $s(M, \mathcal{T}) = 0$ and consider the triangle $V_M \longrightarrow M \xrightarrow{f} T_0 \xrightarrow{+}$, where f is an $\mathrm{add}(\mathcal{T})$-preenvelope, which exists by the hypothesis. It follows that the map $f^* : \mathrm{Hom}_{\mathcal{D}}(T_0, T) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(M, T)$ is an epimorphism and $\mathrm{Hom}_{\mathcal{D}}(V_M, T[k]) = 0$, for all $T \in \mathcal{T}$ and all integers $k \geqslant 0$. Given the description of $\mathcal{U}$ from lemma 5.5, we conclude that $V_M \in {}^{\perp}\mathcal{U}$.

Suppose $s := s(M, \mathcal{T}) > 0$ and that all $N \in \mathcal{D}$ such that $s(N, \mathcal{T}) < s$ admit the desired triangle. Consider a triangle $X \longrightarrow M \xrightarrow{g} T_s[s] \xrightarrow{+}$, where g is an $\mathrm{add}(T)[s]$-preenvelope. Applying the functor $\mathrm{Hom}_{\mathcal{D}}(-, T[k])$, for $T \in \mathcal{T}$, to this triangle we see that $\mathrm{Hom}_{\mathcal{D}}(X, T[k]) = 0$, for all $T \in \mathcal{T}$ and $k \geqslant s$. It follows that $s(X, \mathcal{T}) < s$. By the induction hypothesis $X \in {}^{\perp}\mathcal{U} \star \mathcal{U}$ and $T_s[s] \in {}^{\perp}\mathcal{U} \star \mathcal{U}$. It follows that $M \in {}^{\perp}\mathcal{U} \star \mathcal{U}$ since ${}^{\perp}\mathcal{U} \star \mathcal{U}$ is closed under extensions (see [**47**, lemma 8]).

Finally, the proof of implication (1) $\implies$ (2) shows how to construct $\mathrm{susp}_{\mathcal{D}}(\mathcal{T})$-envelopes inductively. □

DEFINITION 5.8. We shall say that a non-positive set $\mathcal{T}$ in $\mathcal{D}$ is *weakly preenveloping* when it satisfies condition (1′) of proposition 5.7. The notion of a *weakly precovering* nonpositive set of objects is defined dually.

Recall that an object G of a triangulated category $\mathcal{D}$ is called a *classical generator* when $\mathrm{thick}_{\mathcal{D}}(G) = \mathcal{D}$. Recall also that if a pair $(\mathcal{X}, \mathcal{Y})$ is a t-structure or a co-t-structure in $\mathcal{D}$, it is called *left (resp. right) bounded* when $\mathcal{D} = \bigcup_{k \in \mathbb{Z}} \mathcal{X}[k]$ (resp. $\mathcal{D} = \bigcup_{k \in \mathbb{Z}} \mathcal{Y}[k]$). The pair is called *bounded* when it is left and right bounded.

The following proposition together with its dual generalizes [**23**, proposition 3.2].

PROPOSITION 5.9. *Let $\mathcal{D}$ be a skeletally small triangulated category with split idempotents. The assignment $\mathcal{T} \rightsquigarrow ({}^{\perp}\mathrm{susp}_{\mathcal{D}}(\mathcal{T})[1], \mathrm{susp}_{\mathcal{D}}(\mathcal{T}))$ gives a one-to-one correspondence between (add-)equivalence classes of weakly preenveloping non-positive sets and left bounded co-t-structures in $\mathcal{D}$. Its inverse associates to such a co-t-structure a set of representatives of the isomorphism classes of the objects of its co-heart.*

This correspondence restricts to a bijection between equivalence classes of classical silting sets and bounded co-t-structures in $\mathcal{D}$. When $\mathcal{D}$ has a classical generator, this induces a bijection between equivalence classes of silting objects and bounded co-t-structures in $\mathcal{D}$.

Proof. By proposition 5.7, $\tau(\mathcal{T}) := ({}^{\perp}\mathcal{U}[1], \mathcal{U}) := ({}^{\perp}\mathrm{susp}_{\mathcal{D}}(T)[1], \mathrm{susp}_{\mathcal{D}}(\mathcal{T}))$ is a co-t-structure in $\mathcal{D}$. Moreover, for each M in $\mathcal{D}$, there exists $r \in \mathbb{N}$ such that $\mathrm{Hom}_{\mathcal{D}}(M, -[k])_{|\mathcal{T}} = 0$, for $k \geqslant r$. It follows that $M \in {}^{\perp}\mathcal{U}[r]$ and $\tau(\mathcal{T})$ is left bounded.

By [**13**], the co-heart of any co-t-structure τ is a non-positive class of objects. In our case it is skeletally small, so we can chose a set $\mathcal{T}(\tau)$ of representatives of isomorphism classes of its objects. We claim that $\mathcal{T}$ and $\mathcal{T}(\tau(\mathcal{T}))$ are equivalent non-positive sets. The inclusion $\mathcal{T} \subset \mathcal{C}$, where $\mathcal{C}$ is the co-heart of $\tau(\mathcal{T})$, clearly holds, so we need to prove that $\mathcal{C} \subseteq \mathrm{add}(\mathcal{T})$. For $0 \neq C \in \mathcal{C}$ we get $s(C, \mathcal{T}) = 0$, since $\mathrm{Hom}_{\mathcal{D}}(C, -)$ vanishes on $\mathcal{U}[1]$. Since $\mathcal{T}$ is weakly preenveloping, there is an $\mathrm{add}(\mathcal{T})$-preenvelope $f : C \longrightarrow T_C$, let us consider a triangle $V_C \xrightarrow{g} C \xrightarrow{f} T_C \xrightarrow{+}$.

As before (see the proof of implication $(1') \implies (4)$ in proposition 5.7), $V_C \in {}^\perp \mathcal{U}$ and, hence, $g = 0$. It follows that f is a section and $C \in \text{add}(\mathcal{T})$.

Let $\tau = ({}^\perp \mathcal{U}_\tau[1], \mathcal{U}_\tau)$ be any left bounded co-t-structure in $\mathcal{D}$, let $\mathcal{C} := {}^\perp \mathcal{U}_\tau[1] \cap \mathcal{U}_\tau$ be its co-heart and let $\mathcal{T}$ be a set of representatives of its isomorphism classes. The left boundedness of τ implies that $\text{Hom}_{\mathcal{D}}(M, -[k])_{|\mathcal{T}} = 0$ for any $M \in \mathcal{D}$ for $k \gg 0$. Clearly $s := s(M, \mathcal{U}) \geqslant s(M, \mathcal{T})$. We claim that the inverse inequality also holds, provided $s(M, \mathcal{U}) \geqslant 0$. Let us consider the triangle coming from the co-t-structure τ: $V \longrightarrow M[-s] \xrightarrow{f} U \xrightarrow{+}$. For an arbitrary $U' \in \mathcal{U}_\tau$ applying $\text{Hom}_{\mathcal{D}}(-, U')$ to this triangle gives $\text{Hom}_{\mathcal{D}}(U, U'[k]) = 0$, for all $k > 0$. Thus, $U \in {}^\perp \mathcal{U}_\tau[1] \cap \mathcal{U}_\tau = \mathcal{C}$ and $U \in \text{add}(\mathcal{T})$. Clearly, the map $f : M[-s] \longrightarrow \mathcal{U}$ is an $\text{add}(\mathcal{T})$-preenvelope. This in turn implies that $f[s] : M \longrightarrow U[s]$ is an $\text{add}(\mathcal{T})[s]$-preenvelope. Note that f is a nonzero map, since, otherwise $M \in {}^\perp \mathcal{U}_\tau$, contradicting the hypothesis. This implies that $s(M, \mathcal{T}) = s$ and that M has an $\text{add}(\mathcal{T})[s(M, \mathcal{T})]$-preenvelope. Hence, $\mathcal{T}$ is weakly preenveloping and the map from the set of left bounded co-t-structures to weakly preenveloping non-positive sets is well-defined.

The last paragraph shows that if $M \notin {}^\perp \mathcal{U}_\tau$, then there exists a nonzero morphism $f : M \longrightarrow T[s]$, for some $T \in \mathcal{T}$, where $s = s(M, \mathcal{U}) = s(M, \mathcal{T})$. It follows that ${}^\perp \mathcal{U}_\tau = {}^\perp (\bigcup_{k \geqslant 0} \mathcal{T}[k])$ and, hence, that ${}^\perp \mathcal{U}_\tau = {}^\perp \text{susp}_{\mathcal{D}}(\mathcal{T})$. Due to the weak preenveloping condition on $\mathcal{T}$, proposition 5.7 provides a co-t-structure $\tau' := ({}^\perp \text{susp}_{\mathcal{D}}(\mathcal{T})[1], \text{susp}_{\mathcal{D}}(\mathcal{T}))$ in $\mathcal{D}$. Clearly, $\tau' = \tau$. Since $\tau = \tau(\mathcal{T}(\tau))$, the assignments $\mathcal{T} \rightsquigarrow \tau(\mathcal{T})$ and $\tau \rightsquigarrow \mathcal{T}(\tau)$ define mutually inverse maps.

As for the last statement, note that the dual version of the result above gives the bijection $\mathcal{T} \rightsquigarrow (\text{cosusp}_{\mathcal{D}}(\mathcal{T}), \text{cosusp}_{\mathcal{D}}(\mathcal{T})^\perp[-1])$ between the equivalence classes of weakly precovering non-positive sets and right bounded co-t-structures in $\mathcal{D}$, the inverse of this map takes any such co-t-structure τ to a set of representatives of isomorphism classes of objects of the co-heart of τ. If τ is a bounded co-t-structure in $\mathcal{D}$ and $\mathcal{T}_\tau$ is a set of representatives of isomorphism classes of objects of its co-heart, then we deduce from the bijections and from the construction of the triangle with respect to τ that $\tau = (\text{cosusp}_{\mathcal{D}}(\mathcal{T}), \text{susp}_{\mathcal{D}}(\mathcal{T}))$. In particular, any object $M \in \mathcal{D}$ fits into a triangle $V \longrightarrow M \longrightarrow U \xrightarrow{+}$, where $V \in \text{cosusp}_{\mathcal{D}}(\mathcal{T})[-1] \subset \text{thick}_{\mathcal{D}}(\mathcal{T})$ and $U \in \text{susp}_{\mathcal{D}}(\mathcal{T}) \subset \text{thick}_{\mathcal{D}}(\mathcal{T})$. It follows that $\mathcal{D} = \text{thick}_{\mathcal{D}}(\mathcal{T})$, so that $\mathcal{T}$ is a classical silting set. The fact that if $\mathcal{T}$ is a classical silting set in $\mathcal{D}$, then $(\text{cosusp}_{\mathcal{D}}(\mathcal{T}), \text{susp}_{\mathcal{D}}(\mathcal{T}))$ is a bounded co-t-structure is well-known (see [**13**, theorem 4.3.2(II.1)]).

Finally, if $\mathcal{D}$ has a classical generator G and $\mathcal{T}$ is a silting set in $\mathcal{D}$, then $G \in \text{thick}_{\mathcal{D}}(\mathcal{T})$, which implies the existence of a finite subset $\mathcal{T}_0 \subseteq \mathcal{T}$ such that $G \in \text{thick}_{\mathcal{D}}(\mathcal{T}_0)$, so that $\mathcal{D} = \text{thick}_{\mathcal{D}}(\mathcal{T}_0)$, and hence $\mathcal{T}_0$ is a classical silting set. By [**1**, theorem 2.18], we conclude that $\mathcal{T}_0 = \mathcal{T}$ and $\hat{T} := \coprod_{T \in \mathcal{T}} T$ is a classical silting object. $\qquad \square$

We point out the following consequence of the proof of last proposition.

COROLLARY 5.10. *Let $\mathcal{D}$ be a skeletally small triangulated category with split idempotents and let $\mathcal{T}$ be a classical silting set. Then it is weakly precovering and weakly preenveloping in $\mathcal{D}$.*

PROPOSITION 5.11. *Let $\mathcal{D}$ be a triangulated category with coproducts, let $\mathcal{T}$ be a nonpositive set of compact objects and let $(\mathcal{U}_\mathcal{T}, \mathcal{U}_\mathcal{T}^\perp[1]) = ({}^\perp(\mathcal{T}^{\perp \leqslant 0}), \mathcal{T}^{\perp < 0})$ be the associated t-structure in $\mathcal{D}$. The following assertions are equivalent:*

(1) *$\mathcal{T}$ is a weakly preenveloping set in $\mathcal{D}^c$.*

(2) *$({}^\perp \text{susp}_\mathcal{D}(\mathcal{T})[-1] \cap \mathcal{D}^c, \text{susp}_\mathcal{D}(\mathcal{T}))$ is a co-t-structure in $\mathcal{D}^c$.*

(3) *For each $M \in \mathcal{D}^c$, there is a triangle $V_M \longrightarrow M \longrightarrow U_M \xrightarrow{+}$, where $U_M \in \mathcal{U}_\mathcal{T}$ and $V_M \in {}^\perp \mathcal{U}_\mathcal{T}$.*

Proof. $(1) \Longleftrightarrow (2)$ is just the equivalence $(1') \Longleftrightarrow (4)$ of proposition 5.7 applied to $\mathcal{D}^c$.

$(2) \Longrightarrow (3)$ Let M be compact and fix a triangle $V_M \longrightarrow M \longrightarrow U_M \xrightarrow{+}$, where $U_M \in \text{susp}_\mathcal{D}(\mathcal{T})$ and $V_M \in {}^\perp \text{susp}_\mathcal{D}(\mathcal{T})$. We clearly have that $U_M \in \mathcal{U}_\mathcal{T}$. It remains to prove that $\text{Hom}_\mathcal{D}(V_M, -)$ vanishes on $\mathcal{U}_\mathcal{T}[1]$. But, by the proof of [**28**, theorem 12.2] (see also [**47**, theorem 2]), we know that if $U \in \mathcal{U}_\mathcal{T}$ then it is the Milnor colimit $U = \text{Mcolim} U_n$ of a sequence

$$U_0 \xrightarrow{h_1} U_1 \xrightarrow{h_2} \cdots \xrightarrow{h_n} U_n \xrightarrow{h_{n+1}},$$

where $U_0 \in \text{Sum}(\mathcal{T})$ and $\text{cone}(h_n) \in \text{Sum}(\mathcal{T})[n]$, for all $n > 0$. The compactness of V_M gives $\varinjlim \text{Hom}_\mathcal{D}(V_M, U_n) \cong \text{Hom}_\mathcal{D}(V_M, U) = 0$.

$(3) \Longrightarrow (1)$ Let $M \in \mathcal{D}^c$ be arbitrary and let $V_M \longrightarrow M \xrightarrow{f} U_M \xrightarrow{+}$ be the triangle given by assertion 3. As mentioned above, we have a sequence of morphisms

$$U_0 \xrightarrow{h_1} U_1 \xrightarrow{h_2} \cdots \xrightarrow{h_n} U_n \xrightarrow{h_{n+1}},$$

where $U_0 \in \text{Sum}(\mathcal{T})$ and $\text{cone}(h_n) \in \text{Sum}(\mathcal{T})[n]$ for all $n > 0$, such that $U_M \cong \text{Mcolim} U_n$. Due to compactness of M, the canonical morphism $\varinjlim \text{Hom}_\mathcal{D}(M, U_n) \longrightarrow \text{Hom}_\mathcal{D}(M, U_M)$ is an isomorphism. Thus, there exists $g : M \longrightarrow U_t$, for some $t \in \mathbb{N}$, such that f factors in the form $f : M \xrightarrow{g} U_t \xrightarrow{u_t} U_M$, where u_t is the canonical morphism into the Milnor colimit. It immediately follows that g is a $\mathcal{U}_\mathcal{T}$-preenvelope since so is f. But $\text{Hom}_\mathcal{D}(U_t, -)$ vanishes on $\text{Sum}(\mathcal{T}[k])$, for all $k > t$, hence, $\text{Hom}_\mathcal{D}(M, -[k])_{|\mathcal{T}} = 0$ for $k > t$.

Let us consider the sequences of morphisms $U_s \xrightarrow{1} U_s \xrightarrow{1} \cdots \longrightarrow U_s \xrightarrow{1} \cdots$ and $U_s \xrightarrow{h_{s+1}} U_{s+1} \xrightarrow{h_{s+2}} \cdots \xrightarrow{h_n} U_n \xrightarrow{h_{n+1}} \cdots$. There is a morphism of sequences $(U_s, 1) \longrightarrow (U_n, h_n)$ that for $n \geqslant s$ is the map $h'_n := h_n \circ \cdots \circ h_{s+1} : U_s \longrightarrow U_n$ and for $n = s$ is $h'_s = 1_{U_s}$. Thus, there is a triangle $U_s \xrightarrow{h'_n} U_n \longrightarrow U'_n \xrightarrow{+}$, for each $n \geqslant s$, where $U'_n \in \text{Add}(\mathcal{T}[s+1]) \star \text{Add}(\mathcal{T}[s+2]) \star \cdots \star \text{Add}(\mathcal{T}[n])$, for each $n \geqslant s$. Using Verdier's 3×3 lemma and [**40**, lemmas 1.6.6 and 7.1.1], we get a triangle $U_s = \text{Mcolim}(U_s, 1) \xrightarrow{f'} U_M = \text{Mcolim}(U_n, h_n) \longrightarrow U_{>s} \xrightarrow{+}$, where $U_{>s}$ fits into a triangle $\coprod_{n \geqslant s} U'_n \longrightarrow \coprod_{n \geqslant s} U'_n \longrightarrow U_{>s} \xrightarrow{+}$. In particular, $\text{Hom}_\mathcal{D}(M, U_{>s}) = 0$, and hence $f'_* : \text{Hom}_\mathcal{D}(M, U_s) \longrightarrow \text{Hom}_\mathcal{D}(M, U_M)$ is surjective, since M is compact and $\text{Hom}_\mathcal{D}(M, U'_n[k]) = 0$ for each $k \geqslant 0$ and $n \geqslant s$. It follows that there exists a

factorization $f : M \xrightarrow{f'} U_s \longrightarrow U_M$ of the map f of the previous paragraph, Then f' is a $\mathcal{U}_{\mathcal{T}}$-preenvelope since so is f.

Let us consider finally the triangle $U_{s-1} \xrightarrow{h_s} U_s \xrightarrow{p} \coprod_{i \in I} T_i[s] \xrightarrow{+}$. An argument similar to that of the proof of implication $(2') \Longrightarrow (1')$ in proposition 5.7 shows that the composition $p \circ f' : M \longrightarrow \coprod_{i \in I} T_i[s]$ is an $\mathrm{Add}(\mathcal{T})[s]$-preenvelope. The compactness of M gives a factorization $p \circ f' : M \xrightarrow{\alpha} \coprod_{i \in F} T_i[s] \xrightarrow{\iota_F} \coprod_{i \in I} T_i[s]$, for some finite subset $F \subseteq I$, where ι_F is the canonical section. It follows that $\alpha : M \longrightarrow \coprod_{i \in F} T_i[s]$ is also an $\mathrm{Add}(\mathcal{T})[s]$-preenvelope and, hence, an $\mathrm{add}(\mathcal{T})[s]$-preenvelope. $\qquad\square$

We can now deduce the following consequence.

COROLLARY 5.12. *Let $\mathcal{D}$ be a triangulated category with coproducts, and let $\mathcal{T}$ be a silting set in $\mathcal{D}$ consisting of compact objects, i.e. a classical silting set in $\mathcal{D}^c$. The following assertions hold:*

(1) *$\mathcal{T}$ is a weakly preenveloping and weakly precovering set in $\mathcal{D}^c$.*

(2) *The associated t-structure in $\mathcal{D}$ is $\tau_{\mathcal{T}} = (\mathcal{T}^{\perp_{>0}}, \mathcal{T}^{\perp_{<0}}) = (\mathrm{Susp}_{\mathcal{D}}(\mathcal{T}), \mathcal{T}^{\perp_{<0}})$, and it has a left adjacent co-t-structure $(^{\perp}\mathrm{Susp}_{\mathcal{D}}(\mathcal{T})[1], \mathrm{Susp}_{\mathcal{D}}(\mathcal{T}))$ which restricts to $\mathcal{D}^c$.*

(3) *$\mathrm{Susp}_{\mathcal{D}}(\mathcal{T}) \cap \mathcal{D}^c = \mathcal{T}^{\perp_{>0}} \cap \mathcal{D}^c = \mathrm{susp}_{\mathcal{D}}(\mathcal{T})$ and $^{\perp}\mathrm{Susp}_{\mathcal{D}}(\mathcal{T})[1] \cap \mathcal{D}^c = \mathrm{cosusp}_{\mathcal{D}}(\mathcal{T})$.*

Proof. Assertion 1 is a particular case of corollary 5.10, and assertion 2 follows from [**48**, corollary 7]. We just need to prove assertion 3. Indeed the pair $(^{\perp}\mathrm{Susp}_{\mathcal{D}}(\mathcal{T}) \cap \mathcal{D}^c, \mathrm{Susp}_{\mathcal{D}}(\mathcal{T}) \cap \mathcal{D}^c)$ is a pair of orthogonal subcategories of $\mathcal{D}^c$. But, by proposition 5.9 and its proof we know that $(\mathrm{cosusp}_{\mathcal{D}}(\mathcal{T})[-1], \mathrm{susp}_{\mathcal{D}}(\mathcal{T}))$ is a torsion pair in $\mathcal{D}^c$. Since we have inclusions $\mathrm{cosusp}_{\mathcal{D}}(\mathcal{T})[-1] \subseteq {}^{\perp}\mathrm{Susp}_{\mathcal{D}}(\mathcal{T}) \cap \mathcal{D}^c$ and $\mathrm{susp}_{\mathcal{D}}(\mathcal{T}) \subseteq \mathrm{Susp}_{\mathcal{D}}(\mathcal{T}) \cap \mathcal{D}^c$, these inclusions are necessarily equalities. $\qquad\square$

6. Gluing partial silting sets

In this section, we will give criteria for the gluing of partial silting t-structures to be a partial silting t-structure.

6.1. Sufficient condition

LEMMA 6.1. *Let*

$$\mathcal{Y} \underset{\underset{i^!}{\overset{i_*}{\longrightarrow}}}{\overset{i^*}{\longleftarrow}} \mathcal{D} \underset{\underset{j_*}{\overset{j^*}{\longrightarrow}}}{\overset{j_!}{\longleftarrow}} \mathcal{X} \tag{6.1}$$

be a recollement of triangulated categories, let $(\mathcal{X}', \mathcal{X}'')$ and $(\mathcal{Y}', \mathcal{Y}'')$ be torsion pairs in $\mathcal{X}$ and $\mathcal{Y}$, respectively, generated by classes of objects $\mathcal{S}_X \subseteq \mathcal{X}'$ and $\mathcal{S}_Y \subseteq \mathcal{Y}'$. The glued torsion pair $(\mathcal{D}', \mathcal{D}'')$ is generated by $j_!(\mathcal{S}_X) \cup i_*(\mathcal{S}_Y)$.

Proof. An object $Z \in \mathcal{D}$ belongs to the class $(j_!(\mathcal{S}_X) \cup i_*(\mathcal{S}_Y))^\perp$ iff $\mathrm{Hom}_\mathcal{D}(j_!(S), Z) = 0 = \mathrm{Hom}_\mathcal{D}(i_*(S'), Z)$, for all objects $S \in \mathcal{S}_X$ and $S' \in \mathcal{S}_Y$ iff $\mathrm{Hom}_\mathcal{X}(S, j^*(Z)) = 0 = \mathrm{Hom}_\mathcal{Y}(S', i^!(Z))$ for all $S \in \mathcal{S}_X$ and $S' \in \mathcal{S}_Y$ iff $j^*(Z) \in \mathcal{X}''$ and $i^!(Z) \in \mathcal{Y}''$ iff $Z \in \mathcal{D}''$. $\qquad\square$

REMARK 6.2. In the recollement (6.1) if $(\mathcal{X}^{\leqslant 0}, \mathcal{X}^{\geqslant 0})$ is a t-structure in $\mathcal{X}$, then $j_!(\mathcal{X}^{\leqslant 0})$ is the aisle of a t-structure in $\mathcal{D}$. Indeed $j_!(\mathcal{X}^{\leqslant 0}) = \{D \in \mathcal{D} \mid j^*D \in \mathcal{X}^{\leqslant 0}, i^*D = 0\}$ and so it is the aisle of the t-structure in $\mathcal{D}$ glued from $(\mathcal{X}^{\leqslant 0}, \mathcal{X}^{\geqslant 0})$ and the trivial t-structure $(0, \mathcal{Y})$ in $\mathcal{Y}$.

We are ready to prove the technical criteria on which all further results of this section are based. The main applications will be given in corollaries 6.11 and 6.12.

THEOREM 6.3. *Let*

$$\mathcal{Y} \xrightarrow[\underset{i^!}{\overset{i_*}{\longrightarrow}}]{\overset{i^*}{\longleftarrow}} \mathcal{D} \xrightarrow[\underset{j_*}{\overset{j^*}{\longrightarrow}}]{\overset{j_!}{\longleftarrow}} \mathcal{X}$$

be a recollement of triangulated categories, let $\mathcal{T}_X$ and $\mathcal{T}_Y$ be (partial) silting sets in $\mathcal{X}$ and $\mathcal{Y}$, let $(\mathcal{X}^{\leqslant 0}, \mathcal{X}^{\geqslant 0})$, $(\mathcal{Y}^{\leqslant 0}, \mathcal{Y}^{\geqslant 0})$ be the associated t-structures in $\mathcal{X}$ and $\mathcal{Y}$ and let $(\mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant 0})$ be the glued t-structure. Suppose that the following condition holds:

($\star$) For each object $T_Y \in \mathcal{T}_Y$, there is a triangle $\tilde{T}_Y \longrightarrow i_*T_Y \xrightarrow{f_{T_Y}} U_{T_Y}[1] \xrightarrow{+}$ such that $U_{T_Y} \in j_!(\mathcal{X}^{\leqslant 0})$ and $\tilde{T}_Y \in {}^\perp j_!(\mathcal{X}^{\leqslant 0})[1]$.

Then for $\tilde{\mathcal{T}}_Y := \{\tilde{T}_Y : T_Y \in \mathcal{T}_Y\}$ the set $j_!(\mathcal{T}_X) \cup \tilde{\mathcal{T}}_Y$ is a (partial) silting set in $\mathcal{D}$ which generates $(\mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant 0})$.

Proof. Let us prove that $\mathcal{T} := j_!(\mathcal{T}_X) \cup \tilde{\mathcal{T}}_Y$ generates $(\mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant 0})$. By lemma 6.1, the class $j_!\mathcal{X}^{\leqslant 0} \cup i_*(\mathcal{T}_Y)$ generates $(\mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant 0})$, so $\mathcal{D}^{> 0} = (j_!\mathcal{X}^{\leqslant 0} \cup i_*(\mathcal{T}_Y))^{\perp_{\leqslant 0}}$. Condition ($\star$) implies that $(j_!\mathcal{X}^{\leqslant 0} \cup i_*(\mathcal{T}_Y))^{\perp_{\leqslant 0}} = (j_!\mathcal{X}^{\leqslant 0} \cup \tilde{\mathcal{T}}_Y)^{\perp_{\leqslant 0}}$. Note that $(j_!\mathcal{X}^{\leqslant 0})^\perp \subseteq (j_!(\mathcal{T}_X))^{\perp_{\leqslant 0}}$. Conversely, if $Z \in (j_!(\mathcal{T}_X))^{\perp_{\leqslant 0}}$ then $\mathrm{Hom}_\mathcal{X}(T_X[k], j^*Z) \cong \mathrm{Hom}_\mathcal{D}(j_!T_X[k], Z) = 0$, for all $T_X \in \mathcal{T}_X$ and all integers $k \geqslant 0$. Then $j^*Z \in \mathcal{X}^{> 0}$, and so $\mathrm{Hom}_\mathcal{D}(j_!X, Z) \cong \mathrm{Hom}_\mathcal{X}(X, j^*Z) = 0$, for all $X \in \mathcal{X}^{\leqslant 0}$ since $\mathcal{T}_X$ generates $(\mathcal{X}^{\leqslant 0}, \mathcal{X}^{\geqslant 0})$. Thus, there is an equality $(j_!\mathcal{X}^{\leqslant 0})^\perp = (j_!(\mathcal{T}_X))^{\perp_{\leqslant 0}}$. It follows that $\mathcal{D}^{> 0} = (j_!\mathcal{X}^{\leqslant 0} \cup \tilde{\mathcal{T}}_Y)^{\perp_{\leqslant 0}} = (j_!\mathcal{X}^{\leqslant 0})^{\perp_{\leqslant 0}} \cap \tilde{\mathcal{T}}_Y^{\perp_{\leqslant 0}} = (j_!\mathcal{X}^{\leqslant 0})^\perp \cap \tilde{\mathcal{T}}_Y^{\perp_{\leqslant 0}} = (j_!(\mathcal{T}_X))^{\perp_{\leqslant 0}} \cap \tilde{\mathcal{T}}_Y^{\perp_{\leqslant 0}} = (j_!(\mathcal{T}_X) \cup \tilde{\mathcal{T}}_Y)^{\perp_{\leqslant 0}}$. Therefore $\mathcal{T} := j_!(\mathcal{T}_X) \cup \tilde{\mathcal{T}}_Y$ generates $(\mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant 0})$.

In order to prove that $\mathcal{T}$ is partial silting, we need to check that the functors $\mathrm{Hom}_\mathcal{D}(j_!T_X, -)$ and $\mathrm{Hom}_\mathcal{D}(\tilde{T}_Y, -)$ vanish on $\mathcal{D}^{< 0} := \mathcal{D}^{\leqslant 0}[1]$, for all $T_X \in \mathcal{T}_X$ and $T_Y \in \mathcal{T}_Y$. Consider $D \in \mathcal{D}^{\leqslant 0}$. Due to adjunction, $\mathrm{Hom}_\mathcal{D}(j_!T_X, D[1]) \cong \mathrm{Hom}_\mathcal{X}(T_X, j^*D[1])$, for all $T_X \in \mathcal{T}_X$. But $j^*D \in \mathcal{X}^{\leqslant 0}$, so the partial silting condition on $\mathcal{T}_X$ gives $\mathrm{Hom}_\mathcal{X}(T_X, j^*D[1]) = 0$, for all $T_X \in \mathcal{T}_X$. On the other hand, $j^*D \in \mathcal{X}^{\leqslant 0}$ and, by definition of $\tilde{T}_Y$, we have $\tilde{T}_Y \in {}^\perp(j_!\mathcal{X}^{\leqslant 0}[1])$, for each $T_Y \in \mathcal{T}_Y$. Then $\mathrm{Hom}_\mathcal{D}(\tilde{T}_Y, D[1]) \cong \mathrm{Hom}_\mathcal{D}(\tilde{T}_Y, i_*i^*D[1]) \cong \mathrm{Hom}_\mathcal{Y}(i^*\tilde{T}_Y, i^*D[1])$. The equality $i^* \circ j_! = 0$ implies that i^* vanishes on $j_!\mathcal{X}^{\leqslant 0}[k]$, for all $k \in \mathbb{Z}$, and hence $i^*\tilde{T}_Y \cong i^*i_*T_Y \cong T_Y$. Thus, $\mathrm{Hom}_\mathcal{D}(\tilde{T}_Y, D[1]) \cong \mathrm{Hom}_\mathcal{Y}(T_Y, i^*D[1])$, for each $T_Y \in$

$\mathcal{T}_Y$. Since, by definition of $\mathcal{D}^{\leqslant 0}$, we know that $i^*D \in \mathcal{Y}^{\leqslant 0}$, the partial silting condition on $\mathcal{T}_Y$ implies that $\operatorname{Hom}_{\mathcal{D}}(\tilde{T}_Y, D[1]) \cong \operatorname{Hom}_{\mathcal{Y}}(T_Y, i^*D[1]) = 0$, for all $T_Y \in \mathcal{T}_Y$. Therefore $\mathcal{T} := j_!(\mathcal{T}_X) \cup \tilde{T}_Y$ is partial silting in $\mathcal{D}$.

Assume now that $\mathcal{T}_X$ and $\mathcal{T}_Y$ generate $\mathcal{X}$ and $\mathcal{Y}$, respectively. Let us prove that $\mathcal{T} = j_!(\mathcal{T}_X) \cup \tilde{T}_Y$ generates $\mathcal{D}$. Let $Z \in \mathcal{D}$ be an object such that $\operatorname{Hom}_{\mathcal{D}}(T[k], Z) = 0$, for all $T \in \mathcal{T}$, $k \in \mathbb{Z}$. Then $\operatorname{Hom}_{\mathcal{X}}(T_X[k], j^*Z) \cong \operatorname{Hom}_{\mathcal{D}}(j_! T_X[k], Z) = 0$, for all $k \in \mathbb{Z}$ and $T_X \in \mathcal{T}_X$. Since $\mathcal{T}_X$ generates $\mathcal{X}$, we get $j^*Z = 0$. In particular, $\operatorname{Hom}_{\mathcal{D}}(j_! X[k], Z) = 0$, for all $X \in \mathcal{X}^{\leqslant 0}$. Condition $(\star)$ implies $\operatorname{Hom}_{\mathcal{Y}}(T_Y[k], i^!Z) \cong \operatorname{Hom}_{\mathcal{D}}(i_* T_Y[k], Z) = 0$, for all $k \in \mathbb{Z}$ and all $T_Y \in \mathcal{T}_Y$. The generating condition on $\mathcal{T}_Y$ gives $i^!Z = 0$. From the canonical triangle $i_* i^!Z \longrightarrow Z \longrightarrow j_* j^*Z \overset{+}{\longrightarrow}$, we conclude that $Z = 0$. Therefore $j_!(\mathcal{T}_X) \cup \tilde{T}_Y$ generates $\mathcal{D}$, and hence is a silting set in $\mathcal{D}$. $\qquad\square$

It is natural to ask when condition $(\star)$ from theorem 6.3 holds. The following result is useful.

LEMMA 6.4. *Let $\mathcal{Y}, \mathcal{D}, \mathcal{X}$ be triangulated categories and let $\mathcal{L}$ be a ladder of recollements of height two*

$$\mathcal{Y} \; \overset{\overset{\overset{i_\sharp}{\longrightarrow}}{\overset{i^*}{\longleftarrow}}}{\underset{\overset{i_*}{\longleftarrow}}{\underset{i^!}{\longrightarrow}}} \; \mathcal{D} \; \overset{\overset{\overset{j^\sharp}{\longrightarrow}}{\overset{j_!}{\longleftarrow}}}{\underset{\overset{j^*}{\longrightarrow}}{\underset{j_*}{\longleftarrow}}} \; \mathcal{X} \qquad (6.2)$$

Let $(\mathcal{X}_1, \mathcal{X}_1')$ and $(\mathcal{X}_2, \mathcal{X}_2')$ be adjacent torsion pairs in $\mathcal{X}$ and let $(\mathcal{Y}_1, \mathcal{Y}_1')$ and $(\mathcal{Y}_2, \mathcal{Y}_2')$ be adjacent torsion pairs in $\mathcal{Y}$. Consider the torsion pairs $(\mathcal{D}_i, \mathcal{D}_i')$ $(i = 1, 2)$ defined as follows:

(1) *$(\mathcal{D}_1, \mathcal{D}_1')$ is the torsion pair in $\mathcal{D}$ glued from $(\mathcal{X}_1, \mathcal{X}_1')$ and $(\mathcal{Y}_1, \mathcal{Y}_1')$ with respect to the upper recollement of the ladder.*

(2) *$(\mathcal{D}_2, \mathcal{D}_2')$ is the torsion pair in $\mathcal{D}$ glued from $(\mathcal{X}_2, \mathcal{X}_2')$ and $(\mathcal{Y}_2, \mathcal{Y}_2')$ with respect to the lower recollement of the ladder.*

Then $(\mathcal{D}_1, \mathcal{D}_1')$ and $(\mathcal{D}_2, \mathcal{D}_2')$ are adjacent torsion pairs in $\mathcal{D}$.

Proof. By the gluing procedure we get: $\mathcal{D}_2 = \{Z \in \mathcal{D} \mid j^*Z \in \mathcal{X}_2, i^*Z \in \mathcal{Y}_2\} = \{Z \in \mathcal{D} \mid i^*Z \in \mathcal{Y}_1', j^*Z \in \mathcal{X}_1'\} = \mathcal{D}_1'$. $\qquad\square$

COROLLARY 6.5. *Let $\mathcal{L}$ be a ladder of recollements as in lemma 6.4, and let $\mathcal{T}_X$ and $\mathcal{T}_Y$ be partial silting sets which generate t-structures $\tau_X = (\mathcal{X}^{\leqslant 0}, \mathcal{X}^{\geqslant 0})$ and $\tau_Y = (\mathcal{Y}^{\leqslant 0}, \mathcal{Y}^{\geqslant 0})$ in $\mathcal{X}$ and $\mathcal{Y}$, respectively. If τ_X has a left adjacent co-t-structure in $\mathcal{X}$, then condition $(\star)$ of theorem 6.3 holds, so $j_!(\mathcal{T}_X) \cup \tilde{T}_Y$ is a partial silting set which generates the t-structure $(\mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant 0})$ in $\mathcal{D}$ glued with respect to the lower recollement of the ladder.*

Proof. Gluing the co-t-structures $(^{\perp}(\mathcal{X}^{\leqslant 0})[-1], \mathcal{X}^{\leqslant 0})$ and $(\mathcal{Y}, 0)$ with respect to the upper recollement of the ladder, we obtain a co-t-structure in $\mathcal{D}$ whose right component is $j_!(\mathcal{X}^{\leqslant 0})$. Then condition ($\star$) of theorem 6.3 holds. $\square$

EXAMPLE 6.6. In the situation of the last corollary, let $\mathcal{X}$ have coproducts, let $\mathcal{T}_X$ be a silting set in $\mathcal{X}$ with the associated t-structure $\tau_X := (\mathcal{X}^{\leqslant 0}, \mathcal{X}^{\geqslant 0})$. If either of the following conditions holds, then τ_X has a left adjacent co-t-structure:

(1) $\mathcal{X}$ is the stable category of an efficient Frobenius exact category with coproducts in the terminology of [**55**];

(2) $\mathcal{T}_X$ consists of compact objects.

(See [**14**, theorem 4.3.1] for a more general condition than 2 where the argument below also works.)

Proof. By [**47**, theorem 1] (see also [**50**]), we have $(\mathcal{X}^{\leqslant 0}, \mathcal{X}^{\geqslant 0}) = (\mathcal{T}_X^{\perp >0}, \mathcal{T}_X^{\perp <0})$. Then the proof reduces to check that $(^{\perp}(\mathcal{T}_X^{\perp >0}), \mathcal{T}_X^{\perp >0})$ is a torsion pair in $\mathcal{X}$. Under condition 1, this follows from [**55**, corollary 3.5]. Under condition 2 it follows from [**1**, theorem 4.3] or from corollary 5.12. $\square$

REMARK 6.7. The following diagram is a recollement of triangulated categories

$$\mathcal{Y} \xrightarrow[i_*]{\begin{smallmatrix} i^* \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ i^! \end{smallmatrix}} \mathcal{D} \xrightarrow[j_*]{\begin{smallmatrix} j_! \\ \longleftarrow \\ j^* \\ \longrightarrow \\ \longleftarrow \end{smallmatrix}} \mathcal{X} \quad \text{if and only if so is} \quad \mathcal{Y}^{op} \xrightarrow[i^*]{\begin{smallmatrix} i^! \\ \longleftarrow \\ i_* \\ \longrightarrow \\ \longleftarrow \end{smallmatrix}} \mathcal{D}^{op} \xrightarrow[j_!]{\begin{smallmatrix} j_* \\ \longleftarrow \\ j^* \\ \longrightarrow \\ \longleftarrow \end{smallmatrix}} \mathcal{X}^{op}. \quad \text{As a con-}$$

sequence, after defining *(partial) cosilting set* as the dual of (partial) silting set, many results in this section admit a dualization. We leave their statement to the reader.

6.2. Gluing partial silting sets of compact objects

When some of the functors in a recollement preserve compact objects, we can approach condition ($\star$) of theorem 6.3 on the compact level.

SETUP 6.8. *In this subsection we consider:*

(1) *A recollement*

$$\mathcal{Y} \xrightarrow[i^!]{\begin{smallmatrix} i^* \\ \longleftarrow \\ i_* \\ \longrightarrow \\ \longleftarrow \end{smallmatrix}} \mathcal{D} \xrightarrow[j_*]{\begin{smallmatrix} j_! \\ \longleftarrow \\ j^* \\ \longrightarrow \\ \longleftarrow \end{smallmatrix}} \mathcal{X} \ , \tag{6.3}$$

where $\mathcal{Y}$, $\mathcal{D}$ and $\mathcal{X}$ are thick subcategories of triangulated categories with coproducts $\hat{\mathcal{Y}}$, $\hat{\mathcal{D}}$ and $\hat{\mathcal{X}}$ which contain the corresponding subcategories of compact objects.

(2) *Partial silting sets $\mathcal{T}_X$ and $\mathcal{T}_Y$ in $\mathcal{X}$ and $\mathcal{Y}$, respectively, consisting of compact objects, and the t-structures $(\mathcal{X}^{\leqslant 0}, \mathcal{X}^{\geqslant 0})$ and $(\mathcal{Y}^{\leqslant 0}, \mathcal{Y}^{\geqslant 0})$ in $\mathcal{X}$ and $\mathcal{Y}$, generated by $\mathcal{T}_X$ and $\mathcal{T}_Y$.*

(3) $j_!(\mathcal{T}_X) \cup i_*(\mathcal{T}_Y) \subset \hat{\mathcal{D}}^c$ and $j_!(\mathcal{T}_X)$ weakly preenveloping in $\hat{\mathcal{D}}^c$.

The following are examples of weakly preenveloping sets of compact objects.

EXAMPLE 6.9. Let $\hat{\mathcal{D}}$ be a compactly generated triangulated category. Under either of the following conditions, the set $\mathcal{T}$ is weakly preenveloping in $\hat{\mathcal{D}}^c$:

(1) $\hat{\mathcal{D}}$ is homologically locally bounded, $\mathrm{Hom}_{\hat{\mathcal{D}}}(M, N)$ is a finitely generated K-module, for all $M, N \in \hat{\mathcal{D}}^c$, and $\mathcal{T}$ is a finite non-positive set in $\hat{\mathcal{D}}^c$.

(2) $\mathcal{T}$ is classical silting in $\hat{\mathcal{D}}^c$.

Proof. Example 2 follows from corollary 5.10. As for example 1, using the finiteness of $\mathcal{T}$, we can assume that $\mathcal{T} = \{T\}$. The homological local boundedness of $\hat{\mathcal{D}}$ implies that, for each $M \in \hat{\mathcal{D}}^c$, one has $\mathrm{Hom}_{\hat{\mathcal{D}}}(M, T[k]) = 0$, for $k \gg 0$. Moreover, if $s = s(M, T) = \mathrm{Sup}\{k \in \mathbb{N} \colon \mathrm{Hom}_{\hat{\mathcal{D}}}(M, T[k]) \neq 0\}$, then M has an $\mathrm{add}(T)[s]$-preenvelope because $\mathrm{Hom}_{\hat{\mathcal{D}}}(M, T[s])$ is finitely generated as a K-module. $\qquad\square$

In setup 6.8, there exists a triangle $(\dagger) \colon \tilde{T}_Y \longrightarrow i_* T_Y \longrightarrow U_{T_Y}[1] \overset{+}{\longrightarrow}$ in $\hat{\mathcal{D}}^c$, where $U_{T_Y} \in \mathrm{susp}_{\hat{\mathcal{D}}}(j_!(\mathcal{T}_X))$ and $\tilde{T}_Y \in {}^{\perp}\mathrm{Susp}_{\hat{\mathcal{D}}}(j_!(\mathcal{T}_X))[1]$, for each $T_Y \in \mathcal{T}_Y$ (see proposition 5.11). The natural question is: does condition $(\star)$ of theorem 6.3 hold? Our next result gives a partial answer.

THEOREM 6.10. *In setup* 6.8, *if* $j_!(\mathcal{X}^{\leqslant 0}) \subseteq \mathrm{Susp}_{\hat{\mathcal{D}}}(j_!(\mathcal{T}_X))$, *then condition* $(\star)$ *of theorem* 6.3 *holds and, for* $\tilde{\mathcal{T}}_Y = \{\tilde{T}_Y \colon T_Y \in \mathcal{T}_Y\}$, *the set* $\mathcal{T} := j_!(\mathcal{T}_X) \cup \tilde{\mathcal{T}}_Y \subseteq \hat{\mathcal{D}}^c$ *is partial silting in* $\mathcal{D}$ *and generates the glued t-structure* $(\mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant 0})$.
The inclusion $j_!(\mathcal{X}^{\leqslant 0}) \subseteq \mathrm{Susp}_{\hat{\mathcal{D}}}(j_!(\mathcal{T}_X))$ *holds if, in addition to setup* 6.8, *either one of the following conditions hold:*

(1) *Recollement* (6.3) *is the restriction of a recollement*

$$
\hat{\mathcal{Y}} \;\; \overset{\overset{\hat{i}^*}{\overset{\longleftarrow}{\underset{\hat{i}_*}{\longleftarrow}}}}{\underset{\hat{i}^!}{\longrightarrow}} \;\; \hat{\mathcal{D}} \;\; \overset{\overset{\hat{j}_!}{\overset{\longleftarrow}{\underset{\hat{j}^*}{\longleftarrow}}}}{\underset{\hat{j}_*}{\longrightarrow}} \;\; \hat{\mathcal{X}} \;\; ,
$$

and $(\mathcal{X}^{\leqslant 0}, \mathcal{X}^{\geqslant 0})$ *is the restriction of the t-structure in* $\hat{\mathcal{X}}$ *generated by* $\mathcal{T}_X$.

(2) *The triangulated categories* $\hat{\mathcal{Y}}$, $\hat{\mathcal{D}}$ *and* $\hat{\mathcal{X}}$ *are compactly generated,* $\mathcal{T}_X$ *is a classical silting set in* $\hat{\mathcal{X}}^c$, *the functors* $j_!$, j^*, i_* *and* i^* *preserve compact objects and either* $\mathrm{Im}(i_*)$ *cogenerates* $\mathrm{Loc}_{\hat{\mathcal{D}}}(i_*(\hat{\mathcal{Y}}^c))$ *or* $\mathcal{D}$ *cogenerates* $\hat{\mathcal{D}}$.

Proof. Let us consider the triangle $(\dagger) \colon \tilde{T}_Y \longrightarrow i_* T_Y \longrightarrow U_{T_Y}[1] \overset{+}{\longrightarrow}$. The inclusion $j_!(\mathcal{X}^{\leqslant 0}) \subseteq \mathrm{Susp}_{\hat{\mathcal{D}}}(j_!(\mathcal{T}_X))$ implies that $\tilde{T}_Y \in {}^{\perp}j_!(\mathcal{X}^{\leqslant 0})[1]$, and hence condition $(\star)$ of theorem 6.3 holds.
Let us check that $j_!(\mathcal{X}^{\leqslant 0}) \subseteq \mathrm{Susp}_{\hat{\mathcal{D}}}(j_!(\mathcal{T}_X))$ under conditions 1 or 2.

(1) Since $\hat{j}_! \colon \hat{\mathcal{X}} \longrightarrow \hat{\mathcal{D}}$ has a right adjoint, it preserves Milnor colimits. Moreover, since $(\mathcal{X}^{\leqslant 0}, \mathcal{X}^{\geqslant 0}) = (\hat{\mathcal{X}}^{\leqslant 0} \cap \mathcal{X}, \hat{\mathcal{X}}^{\geqslant 0} \cap \mathcal{X})$, where $(\hat{\mathcal{X}}^{\leqslant 0}, \hat{\mathcal{X}}^{\geqslant 0}) =$

$({}^{\perp}(\mathcal{T}_X^{\perp \leqslant 0}), \mathcal{T}_X^{\perp <0})$, each object X of $\mathcal{X}^{\leqslant 0}$ is the Milnor colimit of a sequence

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} X_n \xrightarrow{f_{n+1}} \cdots,$$

where $X_0 \in \mathrm{Add}(\mathcal{T}_X)$ and $\mathrm{cone}(f_n) \in \mathrm{Add}(\mathcal{T}_X)[n]$, for each $n > 0$ (see [**47**, theorem 2]). Thus, $j_!(X) \in \mathrm{Susp}_{\hat{\mathcal{D}}}(j_!(\mathcal{T}_X))$, for each $X \in \mathcal{X}^{\leqslant 0}$.

(2) By theorem 3.3, $(\mathrm{Loc}_{\hat{\mathcal{D}}}(j_!(\mathcal{X}^c)), \mathrm{Loc}_{\hat{\mathcal{D}}}(i_*(\mathcal{Y})^c), \mathrm{Loc}_{\hat{\mathcal{D}}}(i_*(\mathcal{Y}^c))^{\perp}) =: (\mathcal{U}, \mathcal{V}, \mathcal{W})$ is a TTF triple in $\hat{\mathcal{D}}$ such that $(\mathcal{U} \cap \mathcal{D}, \mathcal{V} \cap \mathcal{D}, \mathcal{W} \cap \mathcal{D}) = (\mathrm{Im}(j_!), \mathrm{Im}(i_*), \mathrm{Im}(j_!))$ and it satisfies conditions 2.a and 2.b of that theorem. We then get an associated recollement

$$\mathcal{V} \underset{\xrightarrow{\hat{i}^!}}{\overset{\xleftarrow{\hat{i}^*}}{\underset{\xrightarrow{\hat{i}_*}}{\longrightarrow}}} \hat{\mathcal{D}} \underset{\xrightarrow{\hat{j}_*}}{\overset{\xleftarrow{\hat{j}_!}}{\underset{\xrightarrow{\hat{j}^*}}{\longrightarrow}}} \mathcal{U} \ ,$$

which restricts up to equivalence to the recollement

$$\mathrm{Im}(i_*) \underset{\xrightarrow{i^!}}{\overset{\xleftarrow{i^*}}{\underset{\xrightarrow{i_*}}{\longrightarrow}}} \mathcal{D} \underset{\xrightarrow{j_*}}{\overset{\xleftarrow{j_!}}{\underset{\xrightarrow{j^*}}{\longrightarrow}}} \mathrm{Im}(j_!) \ . \tag{6.4}$$

Recollement (6.4) is equivalent to the one in setup 6.8 via the equivalences of triangulated categories $i_* : \mathcal{Y} \xrightarrow{\cong} \mathrm{Im}(i_*)$ and $j_! : \mathcal{X} \xrightarrow{\cong} \mathrm{Im}(j_!)$.

By proposition 4.4, we know that the t-structure $(\mathcal{T}_X^{\perp >0}, \mathcal{T}_X^{\perp <0})$ in $\hat{\mathcal{X}}$ restricts to $\mathcal{X}$, and so $(\mathcal{X}^{\leqslant 0}, \mathcal{X}^{\geqslant 0}) = (\mathcal{T}_X^{\perp >0} \cap \mathcal{X}, \mathcal{T}_X^{\perp <0} \cap \mathcal{X}) := (\mathcal{T}_X^{\perp >0(\mathcal{X})}, \mathcal{T}_X^{\perp <0(\mathcal{X})})$. Using the equivalence $j_! : \mathcal{X} \xrightarrow{\cong} \mathrm{Im}(j_!) = \mathcal{U} \cap \mathcal{D}$, we get that $(j_!(\mathcal{X}^{\leqslant 0}), j_!(\mathcal{X}^{\geqslant 0})) = (j_!(\mathcal{T}_X)^{\perp >0(\mathcal{U} \cap \mathcal{D})}, j_!(\mathcal{T}_X)^{\perp <0(\mathcal{U} \cap \mathcal{D})})$, which is the restriction to $\mathcal{U} \cap \mathcal{D}$ of the pair $(j_!(\mathcal{T}_X)^{\perp >0(\mathcal{U})}, j_!(\mathcal{T}_X)^{\perp <0(\mathcal{U})})$. The equality $\hat{\mathcal{X}}^c = \mathrm{thick}_{\mathcal{D}}(\mathcal{T}_X)$ gives the equality $\mathcal{U}^c = \mathcal{U} \cap \hat{\mathcal{D}}^c = j_!(\hat{\mathcal{X}}^c) = \mathrm{thick}_{\mathcal{U} \cap \mathcal{D}}(j_!(\mathcal{T}_X)) = \mathrm{thick}_{\mathcal{U}}(j_!(\mathcal{T}_X))$, so $j_!(\mathcal{T}_X)$ is a silting set of compact objects in $\mathcal{U}$ and $(j_!(\mathcal{T}_X)^{\perp >0}, j_!(\mathcal{T}_X)^{\perp <0})$ is a t-structure in $\mathcal{U}$ which restricts to $(j_!(\mathcal{X}^{\leqslant 0}), j_!(\mathcal{X}^{\geqslant 0}))$.

We have checked that condition (1) holds for the recollement (6.4), and hence the inclusion $j_!(\mathcal{X}^{\leqslant 0}) \subseteq \mathrm{Susp}_{\hat{\mathcal{D}}}(j_!(\mathcal{T}_X))$ holds by the first part of the proof, since the corresponding functor in (6.4) is the inclusion. $\qquad \square$

Several consequences of the last theorem (condition 2) and the results in § 3 can be obtained. For the sake of brevity, we provide two of them.

COROLLARY 6.11. *Let $\hat{\mathcal{Y}}, \hat{\mathcal{D}}$ and $\hat{\mathcal{X}}$ be compactly generated triangulated categories, where $\hat{\mathcal{D}}$ is homologically locally bounded and such that $\mathrm{Hom}_{\hat{\mathcal{D}}}(M, N)$ is finitely*

generated (resp. of finite length) as a K-module, for all $M, N \in \hat{\mathcal{D}}^c$. Let

$$\hat{\mathcal{Y}}_\dagger^\star \underset{\substack{\longrightarrow \\ i^! \\ \xleftarrow{\hspace{1em}} \\ \xrightarrow{i_*} \\ \xleftarrow{i^*}}}{} \hat{\mathcal{D}}_\dagger^\star \underset{\substack{\xleftarrow{j_!} \\ \xrightarrow{j^*} \\ \xleftarrow{j_*} \\ \xrightarrow{j_*}}}{} \hat{\mathcal{X}}_\dagger^\star$$

be a recollement, such that the subcategories involved contain the respective subcategories of compact objects and the functors $j_!$, j^*, i^* and i_* preserve compact objects. Let $\mathcal{T}_X$ and $\mathcal{T}_Y$ be partial silting sets in $\hat{\mathcal{X}}_\dagger^\star$ and $\hat{\mathcal{Y}}_\dagger^\star$ consisting of compact objects, with $\mathcal{T}_X$ finite and classical silting in $\hat{\mathcal{X}}^c$, and let $(\mathcal{X}^{\leqslant 0}, \mathcal{X}^{\geqslant 0})$ and $(\mathcal{Y}^{\leqslant 0}, \mathcal{Y}^{\geqslant 0})$ be the associated t-structures in $\hat{\mathcal{X}}_\dagger^\star$ and $\hat{\mathcal{Y}}_\dagger^\star$, respectively. Then condition $(\star)$ of theorem 6.3 holds for $\star \in \{\emptyset, +, -, b\}$ and $\dagger = \emptyset$ (resp. $\dagger = fl$). Therefore $j_!(\mathcal{T}_X) \cup \tilde{\mathcal{T}}_Y$ is a partial silting set in $\hat{\mathcal{D}}_\dagger^\star$, consisting of compact objects, which generates the glued t-structure.

Proof. Let us check that the conditions from setup 6.8 hold. For this we only need to check that $j_!(\mathcal{T}_X)$ is weakly preenveloping in $\hat{\mathcal{D}}^c$. This follows from example 6.9. Thus, the result is a direct consequence of theorem 6.10 (condition 2) and corollary 3.12. □

COROLLARY 6.12. *Let $\hat{\mathcal{Y}}$, $\hat{\mathcal{D}}$ and $\hat{\mathcal{X}}$ be compactly generated algebraic triangulated categories and let*

$$\hat{\mathcal{Y}}_{fl}^b \underset{\substack{\longrightarrow \\ i^! \\ \xleftarrow{\hspace{1em}} \\ \xrightarrow{i_*} \\ \xleftarrow{i^*}}}{} \hat{\mathcal{D}}_{fl}^b \underset{\substack{\xleftarrow{j_!} \\ \xrightarrow{j^*} \\ \xleftarrow{j_*} \\ \xrightarrow{j_*}}}{} \hat{\mathcal{X}}_{fl}^b \tag{6.5}$$

be a recollement, such that the subcategories involved contain the respective subcategories of compact objects and $\hat{\mathcal{D}}$ is compact-detectable in finite length. Let $T_X \in \hat{\mathcal{X}}^c$ and $T_Y \in \hat{\mathcal{Y}}^c$ be classical silting in $\hat{\mathcal{X}}^c$ and $\hat{\mathcal{Y}}^c$, respectively. Let $(\mathcal{X}^{\leqslant 0}, \mathcal{X}^{\geqslant 0})$ and $(\mathcal{Y}^{\leqslant 0}, \mathcal{Y}^{\geqslant 0})$ be the corresponding t-structures in $\hat{\mathcal{X}}_{fl}^b$ and $\hat{\mathcal{Y}}_{fl}^b$. There is a triangle $\tilde{T}_Y \longrightarrow i_*(T_Y) \longrightarrow U_{T_Y}[1] \overset{+}{\longrightarrow}$ in $\hat{\mathcal{D}}^c$ such that $U_{T_Y} \in susp_{\hat{\mathcal{D}}}(j_!(T_X))$ and $\tilde{T}_Y \in {}^\perp j_!(\mathcal{X}^{\leqslant 0})[1]$. In particular $T := j_!(T_X) \oplus \tilde{T}_Y$ is a silting object in $\hat{\mathcal{D}}^c$, uniquely determined up to add-equivalence, which generates the glued t-structure in $\hat{\mathcal{D}}_{fl}^b$.

Proof. The object T_X generates $\hat{\mathcal{X}}$. By [**24**, theorem 4.3], there is a dg algebra C and a triangulated equivalence $F : \mathcal{D}(C) \overset{\cong}{\longrightarrow} \hat{\mathcal{X}}$ which takes C to T_X. In particular, there is an isomorphism $H^k C \cong \mathrm{Hom}_{\mathcal{D}(C)}(C, C[k]) \overset{\cong}{\longrightarrow} \mathrm{Hom}_{\hat{\mathcal{X}}}(T_X, T_X[k])$, for each $k \in \mathbb{Z}$. Hence, C is homologically non-positive and homologically finite length. In addition, F restricts to an equivalence $\mathcal{D}_{fl}^b(C) \overset{\cong}{\longrightarrow} \hat{\mathcal{X}}_{fl}^b$, since $\mathcal{D}_{fl}^b(C) = \mathcal{D}(C)_{fl}^b$. Similarly, there is a homologically non-positive homologically finite length dg algebra B and a triangulated equivalence $G : \mathcal{D}(B) \overset{\cong}{\longrightarrow} \hat{\mathcal{Y}}$ which takes B to T_Y and restricts to a triangulated equivalence $\mathcal{D}_{fl}^b(B) \overset{\cong}{\longrightarrow} \hat{\mathcal{Y}}_{fl}^b$.

Thus, we can assume, that the recollement (6.5) is of the form

$$
\mathcal{D}_{fl}^b(B) \;
\underset{\xrightarrow{\;\;i^!\;\;}}{\overset{\xleftarrow{\;\;i^*\;\;}}{\underset{\xleftarrow{\;\;i_*\;\;}}{\xrightarrow{\hspace{1.2cm}}}}}
\; \hat{\mathcal{D}}_{fl}^b \;
\underset{\xrightarrow{\;\;j_*\;\;}}{\overset{\xleftarrow{\;\;j_!\;\;}}{\underset{\xleftarrow{\;\;j^*\;\;}}{\xrightarrow{\hspace{1.2cm}}}}}
\; \mathcal{D}_{fl}^b(C) \;\;,
$$

where B and C are homologically non-positive homologically finite length dg algebras. Moreover, by the proof of corollary 6.11, we know that we are in the situation of setup 6.8. Then, by proposition 3.19 (see example 3.17(2)) and theorem 6.10 (condition 2) with $\mathcal{T}_X = \{T_X\}$ and $\mathcal{T}_Y = \{T_Y\}$, the result follows, except for the uniqueness of T. This uniqueness is a consequence of propositions 4.4 and 4.6. $\qquad\square$

6.3. Gluing with respect to t-structures versus gluing with respect to co-t-structures over finite length algebras

If A is a finite length K-algebra there is a triangulated equivalence $\mathcal{D}_{fl}^b(A) \cong \mathcal{D}^b(\mathrm{mod}\text{-}A)$ and $\mathcal{D}^c(A)$ may be identified with $\mathcal{K}^b(\mathrm{proj}\text{-}A)$, the homotopy category of finitely generated projective A-modules. The following result is a direct consequence of proposition 3.19 (see remark 3.20) and corollary 6.12, except for its last sentence which follows from proposition 5.7.

Corollary 6.13.

Let A, B and C be finite length algebras, let

$$
D^b(\mathrm{mod}\text{-}B) \;
\underset{\xrightarrow{\;\;i^!\;\;}}{\overset{\xleftarrow{\;\;i^*\;\;}}{\underset{\xleftarrow{\;\;i_*\;\;}}{\xrightarrow{\hspace{1.2cm}}}}}
\; D^b(\mathrm{mod}\text{-}A) \;
\underset{\xrightarrow{\;\;j_*\;\;}}{\overset{\xleftarrow{\;\;j_!\;\;}}{\underset{\xleftarrow{\;\;j^*\;\;}}{\xrightarrow{\hspace{1.2cm}}}}}
\; D^b(\mathrm{mod}\text{-}C) \qquad (6.6)
$$

be a recollement, and let T_C and T_B be silting complexes in $\mathcal{K}^b(\mathrm{proj}\text{-}C)$ and $\mathcal{K}^b(\mathrm{proj}\text{-}B)$ which generate t-structures $(\mathcal{X}^{\leqslant 0}, \mathcal{X}^{\geqslant 0})$ in $D^b(\mathrm{mod}\text{-}C)$ and $(\mathcal{Y}^{\leqslant 0}, \mathcal{Y}^{\geqslant 0})$ in $D^b(\mathrm{mod}\text{-}B)$. There is a triangle $\tilde{T}_B \longrightarrow i_*(T_B) \overset{f}{\longrightarrow} U_{T_B}[1] \overset{+}{\longrightarrow}$ in $\mathcal{K}^b(\mathrm{proj}\text{-}A)$ such that $U_{T_B} \in \mathrm{susp}_{\mathcal{D}(A)}(j_!(T_C))$ and $\tilde{T}_B \in {}^{\perp}j_!(\mathcal{X}^{\leqslant 0})[1]$. The object $T := j_!(T_C) \oplus \tilde{T}_B$ is a silting complex in $\mathcal{K}^b(\mathrm{proj}\text{-}A)$, uniquely determined up to add-equivalence, which generates the glued t-structure $(\mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant 0})$ in $D^b(\mathrm{mod}\text{-}A)$.

Moreover, the map f can be taken to be a $\mathrm{susp}(j_!T_C)[1]$-envelope, which can be calculated inductively.

In this section we compare the gluing of silting objects via the co-t-structures [35] and the one given by the last corollary. Conceptually, t-structures and co-t-structures corresponding to silting objects are adjacent, hence by lemma 6.4 gluing with respect to two recollements in a ladder of recollements should give the same result. We provide the details below.

PROPOSITION 6.14. *Let A, B, C be finite length algebras and suppose that there are recollements*

$$D^b(mod\text{-}B) \xrightarrow[\substack{\xleftarrow{i^*} \\ \xrightarrow{i_*} \\ \xleftarrow{i^!}}]{} D^b(mod\text{-}A) \xrightarrow[\substack{\xleftarrow{j_!} \\ \xrightarrow{j^*} \\ \xleftarrow{j_*}}]{} D^b(mod\text{-}C) \qquad (6.7)$$

and

$$\mathcal{K}^b(proj\text{-}C) \xrightarrow[\substack{\xleftarrow{j^\sharp} \\ \xrightarrow{j_!} \\ \xleftarrow{j^*}}]{} \mathcal{K}^b(proj\text{-}A) \xrightarrow[\substack{\xleftarrow{i^\sharp} \\ \xrightarrow{i^*} \\ \xleftarrow{i_*}}]{} \mathcal{K}^b(proj\text{-}B) \qquad (6.8)$$

where the functors $j_!$, j^*, i^* and i_* of the second recollement are the restrictions of the corresponding functors in the first one. Let T_C and T_B be silting objects in $\mathcal{K}^b(proj\text{-}C)$ and $\mathcal{K}^b(proj\text{-}B)$, let $(\mathcal{X}^{\leqslant 0}, \mathcal{X}^{\geqslant 0})$ and $(\mathcal{Y}^{\leqslant 0}, \mathcal{Y}^{\geqslant 0})$ be the associated t-structures in $\mathcal{D}^b(mod\text{-}C)$ and $\mathcal{D}^b(mod\text{-}B)$ and $(\mathcal{X}_{\geqslant 0}, \mathcal{X}_{\leqslant 0})$ and $(\mathcal{Y}_{\geqslant 0}, \mathcal{Y}_{\leqslant 0})$ the associated co-t-structures in $\mathcal{K}^b(proj\text{-}C)$ and $\mathcal{K}^b(proj\text{-}B)$, respectively.

If $(\mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant 0})$ is the glued t-structure in $\mathcal{D}^b(mod\text{-}A)$ with respect to the recollement (6.7) and $(\mathcal{D}_{\geqslant 0}, \mathcal{D}_{\leqslant 0})$ is the glued co-t-structure in $\mathcal{K}^b(proj\text{-}A)$ with respect to the recollement (6.8), then there is a silting complex $T \in \mathcal{K}^b(proj\text{-}A)$, uniquely determined up to add-equivalence, such that $(\mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant 0})$ is the t-structure in $\mathcal{D}^b(mod\text{-}A)$ associated with T and $(\mathcal{D}_{\geqslant 0}, \mathcal{D}_{\leqslant 0})$ is the co-t-structure in $\mathcal{K}^b(proj\text{-}A)$ associated with T.

Proof. By corollary 6.13, there is a triangle $\tilde{T}_B \longrightarrow i_*(T_B) \xrightarrow{f} U_{T_B}[1] \xrightarrow{+} (\star)$ in $\mathcal{K}^b(\text{proj-}A)$ such that $U_{T_B} \in \mathrm{susp}_{\mathcal{D}(A)}(j_!(T_C))$ and $\tilde{T}_B \in {}^\perp j_!(\mathcal{X}^{\leqslant 0})[1]$ and $(\mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant 0})$ is the t-structure in $\mathcal{D}^b(\text{mod} - A)$ associated with $T = j_! T_C \oplus \tilde{T}_B$. Note that, by construction, this is a triangle associated with the co-t-structure $({}^\perp\mathrm{susp}_{\mathcal{D}(A)}(j_! T_C))[1] \cap \mathcal{K}^b(\text{proj-}A), \mathrm{susp}_{\mathcal{D}(A)}(j_! T_C))[1])$ in $\mathcal{K}^b(\text{proj-}A)$ (see the paragraph immediately before theorem 6.10).

On the other hand, $(\mathcal{X}_{\geqslant 0}, \mathcal{X}_{\leqslant 0}) = (\mathrm{cosusp}_{\mathcal{D}(C)}(T_C), \mathrm{susp}_{\mathcal{D}(C)}(T_C))$ and $(\mathcal{Y}_{\geqslant 0}, \mathcal{Y}_{\leqslant 0}) = (\mathrm{cosusp}_{\mathcal{D}(B)}(T_B), \mathrm{susp}_{\mathcal{D}(B)}(T_B))$ are the co-t-structures in $\mathcal{K}^b(\text{proj-}C)$ and $\mathcal{K}^b(\text{proj-}B)$ cogenerated by T_C and T_B, respectively. By the dual of lemma 6.1 (see remark 6.7), the glued co-t-structure $(\mathcal{D}_{\geqslant 0}, \mathcal{D}_{\leqslant 0})$ with respect to recollement (6.8) is precisely the one cogenerated by $S = j_! T_C \oplus i_* T_B$. That is, $(\mathcal{D}_{\geqslant 0}, \mathcal{D}_{\leqslant 0}) = ({}^{\perp_{>0}}S \cap \mathcal{K}^b(\text{proj-}A), ({}^{\perp_{\geqslant 0}}S \cap \mathcal{K}^b(\text{proj-}A))^\perp \cap \mathcal{K}^b(\text{proj-}A))$. By the triangle $(\star)$ above, we clearly have that ${}^{\perp_{>0}}S \subseteq {}^{\perp_{>0}}\tilde{T}_B \cap {}^{\perp_{>0}} j_! T = {}^{\perp_{>0}}(\tilde{T}_B \oplus j_! T_C) = {}^{\perp_{>0}}T$. The same triangle also shows that any object $M \in {}^{\perp_{>0}}T$ is also in ${}^{\perp_{>0}}i_* T_B$, which in turn implies that $M \in {}^{\perp_{>0}}(j_! T_C \oplus i_* T_B) = {}^{\perp_{>0}}S$. Therefore $(\mathcal{D}_{\geqslant 0}, \mathcal{D}_{\leqslant 0})$ is the co-t-structure in $\mathcal{K}^b(\text{proj-}A)$ cogenerated by T, i.e. $(\mathcal{D}_{\geqslant 0}, \mathcal{D}_{\leqslant 0}) = (\mathrm{cosusp}_{\mathcal{D}(A)}(T), \mathrm{susp}_{\mathcal{D}(A)}(T))$.

The uniqueness of T up to *add*-equivalence follows from corollary 6.13 for $(\mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant 0})$ and from proposition 5.9 for $(\mathcal{D}_{\geqslant 0}, \mathcal{D}_{\leqslant 0})$. Alternatively, it follows from the Koenig–Yang bijection for the case of a finite-dimensional algebra over a field (see [**32**]). $\qquad \square$

A natural subsequent question is when the 'overlap' of recollements like in the last corollary exists. Our next result gives two situations for which it is the case:

PROPOSITION 6.15. *Let A, B, C be finite length K-algebras and let*

$$D^b(\text{mod-}B) \underset{\substack{\xleftarrow{i^*} \\ \xrightarrow{i_*} \\ \xleftarrow{i^!}}}{} D^b(\text{mod-}A) \underset{\substack{\xleftarrow{j_!} \\ \xrightarrow{j^*} \\ \xleftarrow{j_*}}}{} D^b(\text{mod-}C) \qquad (6.9)$$

be a recollement. The following assertions are equivalent:

(1) *There is a recollement*

$$\mathcal{K}^b(\text{proj-}C) \underset{\substack{\xleftarrow{j^\sharp} \\ \xrightarrow{j_!} \\ \xleftarrow{j^*}}}{} \mathcal{K}^b(\text{proj-}A) \underset{\substack{\xleftarrow{i^\sharp} \\ \xrightarrow{i^*} \\ \xleftarrow{i_*}}}{} \mathcal{K}^b(\text{proj-}B) \qquad (6.10)$$

where the functors $j_!$, j^*, i^* and i_* are the restrictions of the corresponding functors in the recollement (6.9) *(see proposition* 3.19).

(2) *The induced functor $j_! : \mathcal{K}^b(\text{proj-}C) \longrightarrow \mathcal{K}^b(\text{proj-}A)$ has a left adjoint.*

Under either one of the following two conditions, the assertions above hold:

(a) *K is a field (and hence A, B and C are finite dimensional algebras);*

(b) *The algebra C has finite global dimension.*

Proof. (1) $\Longrightarrow$ (2) is clear.

(2) $\Longrightarrow$ (1) For each $D \in \mathcal{K}^b(\text{proj-}A)$, the triangle $j_! j^* D \longrightarrow D \longrightarrow i_* i^* D \xrightarrow{+}$ given by the recollement (6.9) belongs to $\mathcal{K}^b(\text{proj-}A)$ by proposition 3.19, implying that $(j_!(\mathcal{K}^b(\text{proj-}C)), i_*(\mathcal{K}^b(\text{proj-}B)))$ is a semi-orthogonal decomposition in $\mathcal{K}^b(\text{proj-}A)$. Let $j^\sharp : \mathcal{K}^b(\text{proj-}A) \longrightarrow \mathcal{K}^b(\text{proj-}C)$ be the left adjoint to $j_!$. Since $j_!$ is fully faithful, the counit $\delta : j^\sharp \circ j_! \longrightarrow 1_{\mathcal{K}^b(\text{proj-}C)}$ is an isomorphism. Let us consider the triangle $K_D \longrightarrow D \xrightarrow{\eta_D} j_! j^\sharp D \xrightarrow{+}$, for any $D \in \mathcal{K}^b(\text{proj-}A)$, where η_D is the unit of the adjunction. For each $X \in \mathcal{K}^b(\text{proj-}C)$, there is a composition of morphisms $\text{Hom}_{\mathcal{K}^b(\text{proj-}C)}(j^\sharp D, X) \xrightarrow{j_!} \text{Hom}_{\mathcal{K}^b(\text{proj-}A)}(j_! j^\sharp D, j_! X) \longrightarrow \text{Hom}_{\mathcal{K}^b(\text{proj-}A)}(D, j_! X)$, which is easily identified with the adjunction isomorphism. The first arrow of this composition is an isomorphism, since $j_!$ is fully faithful. Hence, $\eta_D^* = \text{Hom}_{\mathcal{K}^b(\text{proj-}A)}(\eta_D, j_! X)$ is an isomorphism, for all $X \in \mathcal{K}^b(\text{proj-}C)$, yielding $K_D \in {}^\perp j_!(\mathcal{K}^b(\text{proj-}C))$. So $({}^\perp j_!(\mathcal{K}^b(\text{proj-}C)), j_!(\mathcal{K}^b(\text{proj-}C)))$ is a semi-orthogonal decomposition of $\mathcal{K}^b(\text{proj-}A)$, and $(\mathcal{S}_0, \mathcal{U}_0, \mathcal{V}_0) := ({}^\perp j_!(\mathcal{K}^b(\text{proj-}C)), j_!(\mathcal{K}^b(\text{proj-}C)), i_*(\mathcal{K}^b(\text{proj-}B)))$ is a TTF triple in $\mathcal{K}^b(\text{proj-}A)$. One then obtains the recollement of assertion (1) by standard methods. In particular, the functor $i^\sharp : \mathcal{K}^b(\text{proj-}B) \longrightarrow \mathcal{K}^b(\text{proj-}A)$ is the composition $\mathcal{K}^b(\text{proj-}B) \xrightarrow{i_*} \mathcal{V}_0 \xrightarrow{\cong} \mathcal{S}_0 \xhookrightarrow{incl} \mathcal{K}^b(\text{proj-}A)$, where the central arrow is the canonical equivalence induced by the TTF triple.

We next prove that, under either one of conditions (a) or (b), assertion 2 holds. By theorem 3.14 and proposition 3.13, there is a recollement

$$\mathcal{D}(B) \xrightarrow[\overset{\hat{i}_*}{\underset{\hat{i}^!}{\longleftarrow}}]{\overset{\hat{i}^*}{\longleftarrow}} \mathcal{D}(A) \xrightarrow[\overset{\hat{j}^*}{\underset{\hat{j}_*}{\longleftarrow}}]{\overset{\hat{j}_!}{\longleftarrow}} \mathcal{D}(C) \ . \tag{6.11}$$

that restricts to the $\mathcal{D}^b(\mathrm{mod})$-level and whose restriction is equivalent to the recollement 6.9. When K is a field, using [5, proposition 3.2(b)], we get that $\hat{j}_!$ has a left adjoint $\hat{j}^\sharp$. The functor $\hat{j}^\sharp$ preserves compact objects since its right adjoint preserves coproducts. The induced functors $(\hat{j}^\sharp : \mathcal{K}^b(\mathrm{proj}\text{-}A) \longrightarrow \mathcal{K}^b(\mathrm{proj}\text{-}C), \hat{j}_! : \mathcal{K}^b(\mathrm{proj}\text{-}C) \longrightarrow \mathcal{K}^b(\mathrm{proj}\text{-}A))$ form an adjoint pair. Bearing in mind that $j_!(\mathcal{K}^b(\mathrm{proj}\text{-}C)) = \mathcal{K}^b(\mathrm{proj}\text{-}A) \cap \mathrm{Im}(j_!) = \mathcal{K}^b(\mathrm{proj}\text{-}A) \cap \mathrm{Im}(\hat{j}_!) = \hat{j}_!(\mathcal{K}^b(\mathrm{proj}\text{-}C))$, we get a triangulated autoequivalence $\varphi : \mathcal{K}^b(\mathrm{proj}\text{-}C) \xrightarrow{\cong} \mathcal{K}^b(\mathrm{proj}\text{-}C)$ such that $(\hat{j}_!)|_{\mathcal{K}^b(\mathrm{proj}\text{-}C)} \circ \varphi \cong (j_!)|_{\mathcal{K}^b(\mathrm{proj}\text{-}C)}$. Therefore assertion 2 holds.

When C has finite global dimension, we consider, for any $M \in \mathcal{K}^b(\mathrm{proj}\text{-}A)$, the homological functor $H := \mathrm{Hom}_{\mathcal{K}^b(\mathrm{proj}\text{-}A)}(M, j_!(-))$. Note that, when R is a finite length K-algebra, all Hom spaces in $\mathcal{K}^b(\mathrm{proj}\text{-}R)$ are modules over K/I_R, where $I_R = \{\lambda \in K : \lambda R = 0\}$ is the annihilator of R in K. Since K/I_R is a K-submodule of R, it is of finite length. Putting $I := I_A \cap I_B \cap I_C$ in our case, one gets a commutative ring K/I that has finite length as a K-module, and hence is an Artinian (whence Noetherian) ring. Moreover it is clear that all the functors in the recollement are K/I-linear. Therefore, replacing K by K/I if necessary, we can assume that K is an Artinian (whence Noetherian) commutative ring. Then the dual of [54, corollary 4.18], obtained first in [12, theorem 1.3] when K is a field, applies to our case. Indeed, in the terminology of [54], $\mathcal{K}^b(\mathrm{proj}\text{-}C)^{op}$ is Ext-finite and strongly finitely generated (see [54, proposition 7.25]) and the cohomological functor $H : (\mathcal{K}^b(\mathrm{proj}\text{-}C)^{op})^{op} \longrightarrow \mathrm{Mod} - K$ is locally finite. Then H is representable, so that we get an object $X_M \in \mathcal{K}^b(\mathrm{proj}\text{-}C)^{op}$ such that $H \cong \mathrm{Hom}_{\mathcal{K}^b(\mathrm{proj}\text{-}C)^{op}}(-, X_M) \cong \mathrm{Hom}_{\mathcal{K}^b(\mathrm{proj}\text{-}C)}(X_M, -)$. It is routine to check that the assignment $M \rightsquigarrow X_M$ is the definition on objects of a functor $j^\sharp : \mathcal{K}^b(\mathrm{proj}\text{-}A) \longrightarrow \mathcal{K}^b(\mathrm{proj}\text{-}C)$ which is left adjoint to $j_! : \mathcal{K}^b(\mathrm{proj}\text{-}C) \longrightarrow \mathcal{K}^b(\mathrm{proj}\text{-}A)$. Therefore assertion 2 holds. □

We do not know of any recollement of derived categories of finite length algebras where the equivalent assertions 1 and 2 of last proposition do not hold.

We would like to finish this section with examples of gluing computed applying the algorithm from corollary 6.13 and proposition 5.7. Let A be a finite dimensional algebra over a field. Recall the following example of a recollement of the unbounded derived category $\mathcal{D}(A)$. Let $e \in A$ be an idempotent element such that $eA(1 - e) = 0$. It induces a homological ring epimorphism $i : A \longrightarrow A/AeA$ (see [45, example 4.1]). Moreover, since $eA = eAe$, the left eAe-module eA is projective; $A/AeA \cong A(1 - e)$ is projective as a left A-module as well. We then get a recollement

$$\mathcal{D}(A/AeA) \xrightarrow[\overset{i_*}{\underset{i^!}{\longleftarrow}}]{\overset{i^*}{\longleftarrow}} \mathcal{D}(A) \xrightarrow[\overset{j^*}{\underset{j_*}{\longleftarrow}}]{\overset{j_!}{\longleftarrow}} \mathcal{D}(eAe) \ , \tag{6.12}$$

where $i_* = - \otimes^{\mathbb{L}}_{A/AeA}(A/AeA) = - \otimes_{A/AeA}(A/AeA)$, $i^* = - \otimes^{\mathbb{L}}_A (A/AeA) = - \otimes_A (A/AeA)$, $i^! = \mathbb{R}\mathrm{Hom}_A(A/AeA, -))$, $j^* = - \otimes^{\mathbb{L}}_A (Ae) = - \otimes_A Ae$, $j_! = - \otimes^{\mathbb{L}}_{eAe} eA = - \otimes_{eAe} eA$ and $j_* = \mathbb{R}\mathrm{Hom}_{eAe}(Ae, -)$. It is clear that the functors $i^*, i_*, j_!, j^*$ restrict to the $\mathcal{D}^b(\mathrm{mod})$-level. So by [**5**, theorem 4.6] the recollement restricts to the $\mathcal{D}^b(\mathrm{mod})$-level if and only if $i_*(A/AeA) \in K^b(\mathrm{proj}\text{-}A)$. That is, if and only if A/AeA (equivalently AeA) has finite projective dimension as a right A-module. So if the idempotent e satisfies the following two conditions

(a) $eA(1 - e) = 0$;

(b) the projective dimension of AeA as a right A-module is finite;

we get an induced recollement

$$D^b(\mathrm{mod}\text{-}A/AeA) \; \underset{\underset{i^!}{\xleftarrow{\hspace{0.7cm}}}}{\overset{\overset{i^*}{\xleftarrow{\hspace{0.7cm}}}}{\xrightarrow[i_*]{\hspace{0.7cm}}}} \; D^b(\mathrm{mod}\text{-}A) \; \underset{\underset{j_*}{\xleftarrow{\hspace{0.7cm}}}}{\overset{\overset{j_!}{\xleftarrow{\hspace{0.7cm}}}}{\xrightarrow[j^*]{\hspace{0.7cm}}}} \; D^b(\mathrm{mod}\text{-}eAe) \; . \tag{6.13}$$

EXAMPLE 6.16. Let $A = KA_3 = K(1 \xrightarrow{a} 2 \xrightarrow{b} 3)$ be the path algebra of the linear orientation of A_3. We will denote by P_i, I_i, S_i the indecomposable projective, injective and simple modules corresponding to the vertex i. Let $e = e_3$ be the idempotent corresponding to the vertex 3. It is clear that conditions (a) and (b) are satisfied. Then $A/AeA \simeq K(1 \xrightarrow{a} 2)$ and $eAe \simeq K$. Let us consider the following silting complexes for these algebras. For A/AeA we consider $T_Y := \bar{P}_1[1] \oplus \bar{P}_2$, where $\bar{P}_i$ is the indecomposable projective A/AeA-module corresponding to the vertex i. For eAe we take $T_X := eAe$. Then in recollement 6.13 we have: $j_! eAe = eAe \otimes_{eAe} eA \simeq e_3 A = P_3$, $i_* T_Y = i_*(\bar{P}_1[1] \oplus \bar{P}_2) = (\bar{P}_1[1] \oplus \bar{P}_2) \otimes_{A/AeA} (A/AeA) \simeq I_2[1] \oplus S_2$. Note that $I_2 \simeq (P_3 \xrightarrow{ab} P_1)$ and $S_2 \simeq (P_3 \xrightarrow{b} P_2)$ in $D^b(\mathrm{mod}\text{-}A)$. We need to construct $susp(P_3)[1]$-envelope of $I_2[1] \oplus S_2$. The maximal s such that $Hom(I_2[1] \oplus S_2, P_3[1][s]) \neq 0$ is 1, the $add(P_3[2])$-envelope of $I_2[1] \oplus S_2$ is given by the chain map h depicted by the following diagram:

$$
\begin{array}{ccccc}
P_3 & \xrightarrow{(0,ab)^t} & P_3 \oplus P_1 & \xrightarrow{(b,0)} & P_2 \\
\downarrow{\scriptstyle id} & & & & \\
P_3 & & & &
\end{array} \; .
$$

The cocone of this map is the complex $P_3 \oplus P_1 \xrightarrow{(b,0)} P_2$, which is isomorphic to $P_1[1] \oplus S_2$ in $D^b(\mathrm{mod}\text{-}A)$. We denote by u the map representing the morphism $P_1[1] \oplus S_2 \longrightarrow I_2[1] \oplus S_2$ from the cocone. The $add(P_3[1])$-envelope of $P_1[1] \oplus S_2$ is

the chain map g given by the following diagram:

$$P_3 \oplus P_1 \xrightarrow{(b,0)} P_2 \ ,$$

$$\downarrow {\scriptstyle (id,0)}$$

$$P_3$$

the cocone of which is the complex $P_1 \xrightarrow{0} P_2$ isomorphic to $P_1[1] \oplus P_2$. Denote by v the map representing the morphism $P_1[1] \oplus P_2 \longrightarrow P_1[1] \oplus S_2$ from the cocone.

By lemma 5.3 the $susp(P_3)[1]$-envelope f of $I_2[1] \oplus S_2$ is the map from $I_2[1] \oplus S_2$ to the cone of $u \circ v$. It easily follows that the desired triangle $\tilde{T}_Y \longrightarrow i_*T_Y \xrightarrow{f} U[1] \xrightarrow{+}$ is, up to isomorphism in $D^b(\text{mod-}A)$, of the form $P_1[1] \oplus P_2 \longrightarrow I_2[1] \oplus S_2 \longrightarrow P_3[2] \oplus P_3[1] \xrightarrow{+}$. The glued silting complex is then $j_!T_X \oplus \tilde{T}_Y \cong P_1[1] \oplus P_2 \oplus P_3$.

EXAMPLE 6.17. Let A be any finite dimensional algebra and $e \in A$ be an idempotent satisfying conditions (a) and (b) above. Let $(\mathcal{X}^{\leqslant 0}, \mathcal{X}^{\geqslant 0})$ be the canonical t-structure in $D^b(\text{mod-}eAe)$, induced by the tilting object eAe, and let $(\mathcal{Y}^{\leqslant 0}, \mathcal{Y}^{\geqslant 0})$ be a partial silting t-structure in $D^b(\text{mod-}A/AeA)$ induced by a compact partial silting object T_Y in $D^b(\text{mod-}A/AeA)$ such that $T_Y \in \mathcal{D}^{\leqslant 0}(A/AeA)$. Fix a quasi-isomorphism $s : P^\bullet \longrightarrow i_*(T_Y)$, where $P^\bullet \in \mathcal{K}^{\leqslant 0}(\text{proj-}A)$ is assumed to be minimal, i.e. such that the image of the differential $d^k : P^k \longrightarrow P^{k+1}$ is contained in $P^{k+1}\text{rad}(A)$, for each $i \in \mathbb{Z}$. The exact sequence of complexes $0 \to P^\bullet(1-e)A \longrightarrow P^\bullet \longrightarrow P^\bullet/P^\bullet(1-e)A \to 0$ splits in each degree and induces a triangle $\tilde{T}_Y \longrightarrow i_*T_Y \xrightarrow{h} U[1] \xrightarrow{+}$ in $D^b(\text{mod-}A)$, where h is a $j_!(\mathcal{X}^{\leqslant 0})[1]$-envelope. In particular $T = eA \oplus P^\bullet(1-e) \cong j_!(eAe) \oplus \tilde{T}_Y$ is a partial silting object of $\mathcal{D}^b(\text{mod}-A)$ which generates the glued t-structure by corollary 6.11.

Proof. Each projective right A-module P decomposes as $_eP \oplus_{1-e} P$, with $_eP \in \text{add}(eA)$ and $_{1-e}P \in \text{add}((1-e)A)$. Since $eA(1-e) = 0$, $_eP(1-e)A = 0$ and $P(1-e)A =_{1-e} P$. It follows that $_{1-e}P^\bullet \cong P^\bullet(1-e)A$ is a subcomplex of $P^\bullet$ and the three terms in the exact sequence $0 \to P^\bullet(1-e)A \longrightarrow P^\bullet \longrightarrow P^\bullet/P^\bullet (1-e)A \to 0$ are complexes of projectives. The component P^0 of the complex $P^\bullet$ is a projective cover of $H^0(i_*T_Y)$. Since $H^0(i_*T_Y)$ is a quotient of a module in $\text{add}(A/AeA)$, we get $P^0 \in \text{add}((1-e)A)$. Thus, $P^\bullet/P^\bullet(1-e)A \cong \ _eP^\bullet$ belongs to $\mathcal{K}^{<0}(\text{add}(eA))$.

Note that $_ePe \otimes_{eAe} eA \cong \ _eP$. Moreover, $j_!(_eP^\bullet e) \cong \ _eP^\bullet e \otimes^{\mathbb{L}}_{eAe} eA \cong \ _eP^\bullet e \otimes_{eAe} eA \cong \ _eP^\bullet \in j_!(D^{<0}(\text{mod-}eAe) \cap D^b(\text{mod-}eAe)) = j_!(\mathcal{X}^{\leqslant 0})[1]$. On the other hand, $\text{Im}(j_!) = \mathcal{K}^-(\text{add}(eA)) \cap D^b(\text{mod-}A)$, when we view $\mathcal{K}^-(\text{add}(eA))$ as a full subcategory of $D(A)$. Hence $\text{Hom}_{D(A)}(P^\bullet(1-e)A, -)$ vanishes on $\text{Im}(j_!)$, and so $\pi : P^\bullet \longrightarrow P^\bullet/P^\bullet(1-e)A$ is a $j_!(\mathcal{X}^{\leqslant 0})[1]$-preenvelope. It is easy to see that π is a left minimal morphism in $D^b(\text{mod-}A)$, and therefore a $j_!(\mathcal{X}^{\leqslant 0})[1]$-envelope of $P^\bullet \cong i_*T_Y$. $\square$

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