Math. Struct. in Comp. Science (2012), *vol.* 22, *pp.* 880–900.
doi:10.1017/S0960129511000569

Incomputability in Physics and Biology†

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Received 1 October 2010; revised 1 July 2011

Computability has its origins in Logic within the framework formed along the original path laid down by the founding fathers of the modern foundational analysis for Mathematics (Frege and Hilbert). This theoretical itinerary, which was largely focused on Logic and Arithmetic, departed in principle from the renewed relations between Geometry and Physics occurring at the time. In particular, the key issue of physical measurement, as our only access to 'reality', played no part in its theoretical framework. This is in stark contrast to the position in Physics, where the role of measurement has been a core theoretical and epistemological issue since Poincaré, Planck and Einstein. Furthermore, measurement is intimately related to unpredictability, (in-)determinism and the relationship with physical space–time. Computability, despite having exact access to its own discrete data type, provides a unique tool for the investigation of 'unpredictability' in both Physics and Biology through its fine-grained analysis of undecidability – note that unpredictability coincides with physical randomness in both classical and quantum frames. Moreover, it now turns out that an understanding of randomness in Physics and Biology is a key component of the intelligibility of Nature. In this paper, we will discuss a few results following along this line of thought.

1. The issue of physical measurement

Before the crisis in the foundations in Mathematics, in other words, before the invention of non-euclidean geometries, Mathematics was secured by a direct link between Euclid's Geometry and physical space. There was certainty in the relation between our human experience of space and Euclid's ruler and compass constructions, in particular, once Descartes had embedded them in abstract spaces and Newton had made them absolute. So a theorem proved at the scale or our everyday experiences could be transferred equally well to the stars or to Democritus' atoms: for example, the sum of internal angles of a triangle is always 180^0. But Riemann claimed that the relevant space manifolds, where 'the cohesive forces among physical bodies could be related to the metrics' (Riemann 1854), were not closed under homotheties (to be precise, this was Klein's version). Hence, the scale invariance (under homotheties – dilations) that allowed us to extend our ruler and compass proofs to the astronomical and microscopic realms was lost. Then, as a result of Einstein's work, Euclid's spaces finally turned out to be an irrelevant singularity, corresponding to zero curvature, or, at best, a local approximation within the new geometrisation of

† This paper is an extended and revised version of Longo (2010a), which was an Invited Lecture presented at the Computability in Europe conference held in the Azores in June 2010.

Physics. In short, the physically crucial meaning of curved spaces demolished the assumed absolute correspondence between our intuitions, which were grounded on action in human-scale space, and physical space–time (see Boi (1995) for a survey of the epistemological challenges thrown up). This was a true 'epistemological break', in Bachelard's terms (Bachelard 1940), or a 'revolution' (Khun 1962), as it stimulated a new and foundational approach.

The response by Frege and Hilbert to this major crisis, though different in their approach to meaning and existence, agreed in giving a central role to arithmetic, with its 'absolute laws', and in turning away from the loss of certainty arising from the non-Euclidean developments with regard to our intuitive relationship with space. Hence, logic or formal systems should allow Mathematics to be reconstructed categorically (if we put it in modern terms for Frege) or completely (for Hilbert).

This 'royal way out' in the foundations of Mathematics led it to break away from its relationship with physical space and, thus also with Physics. In particular, it proposed a foundational culture in Mathematics, thereby programmatically disregarding its constitutive interaction with Physics, with its core of Riemannanian Geometry, and thus ignoring our forms of 'access to the world' through measurement in space and time. The foundations of Mathematics had to be divorced from our forms of life, action and space, and built on pure logic: in other words, formal computations over meaningless strings of signs[†].

As a matter of fact, our only 'access' to physical processes is through measurement: ranging from the cognitive/perceptive measurements arising from our relations with the environment to the very refined tools of quantum measurement. By ignoring this, an entire arithmetising community by-passed a key aspect of the revolution arising in Physics at the turn of the twentieth century: *viz.* the novel role played by physical measurement in our understanding of Nature. Now, when turning back to natural phenomena, some people project on them these arithmetic/computational views, with their discrete structures of determination, as I will explain below.

Poincaré first understood, through his Three Body Theorem of 1890, that the intrinsically approximate measurement of the initial conditions, in combination with the non-linearity of the mathematical description (and gravitational 'resonances'), led to an unpredictable, though deterministic, continuous dynamics. And thus the Geometry of Dynamical Systems was born, and classical randomness has since then become understood as deterministic unpredictability (see below). Einstein's relativity required reference systems and their measure-invariant properties to be given explicitly, with what was at the time a revolutionary correlation between space and time measurements. Quantum Mechanics, in a rather different way, also brought the process of measurement into the limelight

[†] The foundations of Mathematics had to be independent of the 'wildest visions of delirium' that were being proposed by the interpretations of non-Euclidean theories (in Frege's words – Frege (1884, page 20)) and instead directed towards axiomatic systems for Geometry that must contain no references to intuition or meaning in space (see Hilbert (1899) and Poincaré's critique of it in Poincaré (1903), as well as his critique of Frege's and Russell's approach (Goldfarb 1988)). Hermann Weyl similarly interpreted and criticised formalism and logicism, but from a geometric perspective Weyl (1918; 1927; 1985).

by introducing an intrinsic randomness at the point that Schrödinger's deterministic dynamics of the state function (a probability density in Hilbert spaces) is made accessible (by being measured in our space and time). And Planck's *h* gave a lower bound for the simultaneous measurement of conjugate (complementary) variables.

Shortly after, the arithmetising culture in foundations, which was explicitly born as a reaction to Riemann and Poincaré's physicalisation of Geometry (and geometrisation of Physics), produced some remarkable formal-arithmetic machines. Computability was invented within Logic, in the 1930s, for purely foundational purposes by Herbrand, Gödel, Church, Kleene, and others. And Turing's Logical Computing Machine introduced a further – metaphysical – separation from the world: the perfect (Cartesian) dualism provided by the distinction between hardware and software, based on a remarkable theoretical (and later practical) split between the 'logic' of a process and its physical realisation. Turing's idea was the description of a human in the 'least act of computing', or actually of thought. But wasn't Frege's and even Hilbert's project also an analysis of general human reasoning? Weren't neural spikes, which were just the most visible traces of complex critical transitions of electrostatic potentials in the brain, quickly regarded as some sort of binary values?

The physical realisation of the Machine preserved Turing's fundamental separation: its soft 'soul' could act on discrete data types, without any access problem, and independently from the hardware. There was no need for physically approximate measurements or random information since its databases are given exactly. Digital processes are exact, and evolve separately in their own world, over a very artificial device, which was indeed a most remarkable human alphanumeric invention: a discrete state physical process, which is capable of repeating identically whatever program it is given and, as a result, it is reliable. Nothing like this had existed before in the World, since Nature very rarely repeats anything exactly (there may be a few chemical processes, *in vitro*, which do, but we will return to this later).

I would like to stress that the mathematical origins of the Machine are reflected in its exactness over discrete data types and, thus, in the reliable and identically repeatable interaction between hardware and software. This is given by the arithmetical certainty postulated by Frege and Hilbert within Logic, which is distinct from the imprecise measurements of classical/relativistic dynamics, and the randomness of Quantum Mechanics. However, networks of concurrent computers, which are distributed in physical space–time, are now challenging this original view.

2. Preliminaries: from equational determination to incompleteness.

I suggested in a short note in 2001 that Poincaré's Three Body Theorem may be considered to be an epistemological predecessor of Gödel's undecidability result† if we understand Hilbert's completeness conjecture as a meta-mathematical revival of Laplace's idea of the predictability of formally (equationally) determined systems. For Laplace, once the

† This is discussed more extensively in Longo (2009; 2010b)

equations are given, we can completely derive the future states of affairs (with some, preserved, approximation). Or, more precisely, in 'Le système du monde', he claimed that the mathematical mechanics of moving particles, one by one, two by two, three by three, and so on, compositionally and *completely* 'covers', or makes understandable, the entire Universe. Specifically in relation to celestial bodies, he said that through this progressive mathematical integration 'We should be able to deduce all facts of astronomy'.

The challenge, if we are to make a closer comparison, is that Hilbert was speaking about the completeness of formal systems as deducibility or *decidability* of purely mathematical *'yes or no'* questions, while unpredictability shows up in the relations between a physical system and a mathematical set of equations (or evolution function).

In order to relate unpredictability to undecidability consistently, we need to make the dynamical spaces and measure theory effective (typically, using a computable Lebesgue measure), since they are the loci for dynamic randomness. This allows us to have a sound and purely mathematical treatment of the epistemological issue (and obtain a convincing correspondence between unpredictability and undecidability – see the next section).

Remark 2.1 (On proof methods). The theorems produced by Poincaré and Gödel are also methodologically related: they both (and independently, of course) destroy the conjecture of predictability (Laplace) and decidability (Hilbert) *from within*. Poincaré does not need to refer concretely to an unpredictable physical process by measuring it 'before and after'. He shows, by a pure analysis of the equations, that the resulting bifurcations and 'homoclinic intersections' (intersection points of stable and unstable manifolds or trajectories) lead to deterministic unpredictability (of course, the equations were derived with reference to three bodies interacting through their gravitational fields, in much the same way as Peano's Axioms were invented with reference to the ordered structure of numbers). Similarly, in the *statements* and *proofs* of his 1931 paper, Gödel formally constructed an undecidable sentence by playing a purely syntactic game, with no reference whatsoever to 'semantics', 'truth' or suchlike, in other words, with no reference to the underlying mathematical structure.

By contrast, modern 'concrete incompleteness' theorems (that is, Girard's normalisation, Paris–Harrington or Friedman–Kruskal Theorems – see Longo (2002; 2010b) for references and a discussion) resemble Laskar's results of the 1990s (Laskar 1994), where 'concrete unpredictability' was shown for the *Solar system*, as the physical system of interest, with reference to the *best possible astronomical measurements*. Similarly, concrete incompleteness was given by proving (unprovability and) truth over the (*standard*) *model*, thus comparing formal syntax and the intended mathematical structure.

Philosophically, the incompleteness of our formal (and equational) approaches to knowledge is a general and fundamental epistemological issue. It motivates our permanent need for new science: by inventing new *contentual* – that is, content based or meaningful – principles and conceptual constructions, we can change direction, propose new understandings and get a meaningful grasp or organisation of new fragments of the world. There is no such thing as 'the final solution to the foundational problem' in Mathematics (as Hilbert dreamed – a true nightmare), or indeed in other sciences.

As a preliminary hint in the field of Biology, note that the 'incompleteness' of molecular theories for understanding life-related phenomena is a comparable issue. There is no way to understand/derive embryogenesis nor phylogenesis (Evolution) completely by just looking at the four letters of the bases of DNA (the formal language of Molecular Biology) – despite the claims of too many biologists (see Monod (1973) and Maynard Smith (1989) for two classic examples, and Fox Keller (2000) for a critical survey). The massive control and feedback response of the cell and organism on DNA expression and subsequent molecular cascades is being increasingly acknowledged in Molecular Biology. Thus, the analysis of the global structure of the cell (and the organism) must go in parallel with the absolutely crucial molecular analyses. The difficult philosophical point to explain to our colleagues in Molecular Biology is that 'incomplete' does not mean useless, but that we also need a (still missing) autonomous theory of the organism and to further develop the Darwinian theory of Evolution.

Incidentally, randomness plays a key role in both Evolution and embryogenesis. But what kind of randomness? Classical and Quantum Physics proposes two distinct notions of randomness. Can logical undecidability (incomputability) help us in understanding this? We will focus on these questions at the end of this paper.

3. Randomness versus undecidability

As I mentioned earlier, classical (physical) randomness is the unpredictability of deterministic systems in finite time. For example, dice trajectories are theoretically determined: they follow the Hamiltonian, and thus a unique geodetics; yet they are so sensitive to the initial and boundary conditions that it is not worth our writing the equations of motion. On the other hand, algorithmic randomness, that is Martin-Löf's (and Chaitin's) number-theoretic randomness, is defined for infinite sequences (Calude 2002), so how can this yield a connection between Poincaré's unpredictability and Gödel's undecidability?

Classical physical randomness is deterministic unpredictability, so it manifests itself at the 'equations/process' interface and shows up in finite time (that is, after a finite number of iterations of the dynamics, if this is represented, as usual, using the technique of Poincaré sections). However, this physical randomness can also be expressed as a limit or asymptotic notion, and in this way can be soundly turned into a purely mathematical problem: this is Birkhoff's ergodicity (for any observable, time averages coincide with space averages, and this is an equality of two infinite sums or integrals (Petersen 1983)). And in this sense, it applies in (weakly chaotic) dynamical systems within the framework of Poincaré's Geometry of Dynamical Systems.

As for algorithmic randomness, Martin-Löf randomness is a 'Gödelian' notion of randomness since it is based on recursion theory and yields a strong form of undecidability for infinite 0-1 sequences (in short, a sequence is random if it passes all *effective statistical tests*; as a consequence, it contains no infinite recursively enumerable subsequences). Recently, under Galatolo's and my supervision, M. Hoyrup and C. Rojas proved that dynamic randomness (*à la* Poincaré, and thus in the ergodic sense, but at the purely mathematical limit) in suitable *effectively given* measurable dynamical systems is equivalent to (a generalisation of) Martin-Löf randomness (specifically, Schnorr randomness). This

https://doi.org/10.1017/S0960129511000569 Published online by Cambridge University Press

is a non-obvious result, which was also based on a collaboration with P. Gacs (Gacs *et al.* 2011), and developed in the two parallel doctoral dissertations mentioned above (which were defended in June 2008)[†].

When it comes to quantum randomness, note that because of entanglement, it differs mathematically from classical randomness: if two classical dice interact and then separate, the probabilistic analysis of their values are independent; but when two quanta interact and form a 'system', they can no longer be separated, since measurements on them give correlated probabilities of the results. Mathematically, they violate Bell's inequalities – see Bailly and Longo (2007) for a comparative introduction.

Algorithmic randomness provides a close analysis of classical randomness, but how can this mathematics of discrete structures tell us more about randomness in general?

4. Discrete versus continua

One of the ideas extensively developed in the book Bailly and Longo (2011) and in several papers written with two physicists, Francis Bailly[‡] and Thierry Paul (see our downloadable papers) is that the mathematical structures constructed to develop an understanding of physical phenomena may, depending on their continuous (mostly in Physics) or discrete (generally in computing) nature, suggest different views of Nature.

In other words, the 'causal relations' in Physics, as mathematical determinations and intelligibility structures (through which we 'understand Nature'), are usually given in terms of (differential) equations or evolution functions. Their physical meaning is dependent on the use of the continuum or the discrete, and may differ deeply according to the choice of one of these mathematical frameworks. For example, in most non-linear systems, discrete approximations soon diverge from continuous evolutions and do not provide actual 'models' of the intended physical processes. In a few cases, such as hyperbolic systems, 'shadowing theorems' can, at most, guarantee that continuous evolutions approximate discrete ones (but not the converse!) – see Pilyugin (1999). In modern terms, the continuous or discrete underlying mathematical structures induce different symmetries and symmetry-breaking (Bailly and Longo 2011; Longo and Montévil 2011, Chapters 4 and 5).

But what discrete (mathematical) structures are we referring to? A clear mathematical definition of 'discrete' is that a structure is *discrete* when the discrete topology on it is 'natural'. Of course, this is not a formal definition, but in Mathematics we all know what 'natural with respect to the intended purpose means. For example, we can endow Cantor's real line with the discrete topology, but it is not 'natural' (you cannot do much with it), while, the integer numbers or a digital data base are naturally endowed with the discrete

[†] Undecidability over dynamical systems had already been investigated and proved to yield a form of unpredictability: the undecidability, say, of a point crossing or not a given region of phase space – see Moore (1990) and da Costa and Doria (1991) among others. Here, however, we are comparing an independent notion of physical unpredictability, as classical randomness *à la* Birkhoff, with algorithmic randomness, as a strong form of Gödelian undecidability.

[‡] The enlightening collaboration with Francis, a physicist who was also interested in Biology, has been fundamental for me. Francis recently passed away: a recorded Colloquium in his memory may be accessed from my web page, `http://www.di.ens.fr/users/longo`.

topology. Even though we may have good reason to use a different structure for some purposes, the interval topology on the reals is 'natural' for Mathematical Physics in being derived from the *interval* (approximate) nature of measurement. And this induces the discrete topology on the subset of integers or on any finite set of approximating numbers.

Church's thesis, which was introduced in the 1930s following the functional equivalence proofs of various formal systems for computability, is only concerned with computability over integers or discrete data types. As such, it is an extremely robust thesis: it ensures that any sufficiently expressive *finitistic formal system* over integers (a Hilbertian-type logico-formal system) computes exactly the recursive functions, as defined by Herbrand, Gödel, Kleene, Church, Turing, and so on. This thesis therefore emerged within the context of Mathematical Logic. It is a thesis grounded on *languages* and *formal systems* for arithmetic computations on discrete data types, which were programmatically invented by the founding fathers, without any connections with Physics and its space–time.

So the first question we can ask is what happens if we extend *just* the formal framework? If we want to refer to continuous (differentiable) mathematical structures, the extension to consider is to the computable real numbers (Pour-El and Richards 1989). And we then need to ask whether the various formalisms for computability over real numbers are equivalent? An affirmative answer could suggest an extension of Church's thesis to computability on 'continua'. Of course, there are only countably many computable reals, but they are dense in the 'natural' (interval) topology over Cantor's reals, and, as we shall see below, this yields a crucial difference.

By posing this question, we get closer to current Physics, since we assume spatial and often temporal continuity when we represent dynamical systems, that is, in most mathematical models for (classical) Physics. This does not imply that the World is continuous, but only that since Newton and Leibniz we have a better understanding of large parts of Physics, such as space–time and movement within it, using continuous tools, as was later very well specified by Cantor (though his continuum is not the only possible one: for example, Lawvere and Bell proposed a topos-theoretic continuum without points (Bell 1998)).

Now, when we pass to computability over real numbers, very little remains of this equivalence of formalisms, which lies at the heart of Church's thesis: the theories proposed are demonstrably different in terms of computational expressiveness (the classes of defined functions). The various systems (such as: recursive analysis, which was first developed by Lacombe and Grezgorzcyk in 1955–57; the Blum, Shub and Smale, BSS, system; the Moore-type recursive real functions; and different forms of 'analogue' systems, such as threshold neurones, the GPAC) yield different classes of 'continuous' computable functions. Some recent work has established links, or reductions, between the various systems (more precisely, pairwise relations between subsystems and/or extensions), but the full equivalence of the discrete case is lost. Moreover, these systems have no 'universal function' in Turing's sense. In the discrete case, this function is constructed by a computable isomorphism between spaces of different dimension, that is $N^2 = N$. However, there is no continuous and computable isomorphism between the computable reals R_c and R_c^2 that would allow us to transfer the notion of a Universal Machine to computability over continua – see Costa *et al.* (2009) for more.

As a consequence of the $N^2 = N$ computable isomorphism, when we consider computability on the discrete, the work spaces may be of any finite dimension: they are all effectively isomorphic, or, in other words, the 'Cartesian dimension' does not matter! This is a highly unsuitable property for Physics. First, dimensional analysis is a fundamental tool (one cannot confuse energy with force, or with the square of energy, and so on). Second, dimension is a topological invariant in all space manifolds for classical and relativistic Physics. Specifically, if we take physical measurement, an interval, as a 'natural' starting point for the metric (which gives us the interval or 'real' topology), then you can *prove* that if two such spaces have isomorphic open subsets, they have the same dimension. The topological invariance of dimension on physically meaningful topologies is a very simple, but beautiful, correspondence between Mathematics and Physics.

These facts weaken computational approaches to the analysis of *physical* invariants over continua since two fundamental *computational* invariants are lost: equivalence (which is the basis of the Church Thesis) and universality.

In summary, in discrete computability, a cloud of isolated points has no dimension *per se*, and, for all theoretical purposes we may encode them on a line. When we put dimension back in to consideration, we lose the universal function and the equivalence of systems in computability over continua, where the trace of the interval topology maintains good physical properties. There is thus a huge gap between the theoretical world of discrete computability and physico-mathematical continua. While I believe that we should be able to do better than Cantor for continua, I would not give a penny for a physical theory whose dynamics only takes place on discrete spaces, thereby divorcing itself from physical measurement, dimensional analysis and the general relevance of dimensions in Physics (again, space dimension is crucial in problems ranging from heat propagation, through mean field theory, to relativity theory). The analysis over 'computable continua' provides a more interesting framework for Physics by adding relevant information, though it loses two key invariants of computations over the discrete (typically, the Universal Function). We will discuss the peculiar 'discrete' nature of Quantum Mechanics later in the paper.

Remark 4.1 (On physical constants). As an aside, we might ask whether the main physical constants (G, c, h) are *computable* (real numbers). Of course, it depends on our choice of reference system and the metrics. So, suppose we fix $h = 1$. We then have to renormalise all metrics and re-calculate, using equations, dimensional analyses *and* physical measurement, G and c. But physical measurement will always give an interval, as we have already said, or, in the quantum case, the probability of a value. If we interpret the classical measurement interval as a Cantorian continuum (which is, so far, the best way we have for grasping fluctuations), then, for believers in the absolute existence of the real numbers R, where are G and c? The computable reals form a dense subset of Lebesgues measure 0, with plenty of gaps but no jumps. Why should (the mathematical understanding of) fluctuations avoid falling into the gaps, and jump from computable real to computable real? Cristian Calude and I conjecture that random (and thus highly incomputable) reals form a better structure for understanding non-observable events.

However, the most striking mistake many 'computationalists' make is to say that if 'Physics is not fully computable', then some physical processes would super-compute (that is, 'compute' non-computable functions)! But this is not the point. Most physical processes simply do not *define* a mathematical function. And the challenge again arises from the fact that our only form of access to the (physical) 'world' is *measurement*. In order to force a classical process to define a function, you have to fix a time for the input, associate a (rational) number with the interval of measurement, and then let the process go. Then you wait until the output time and measure again. In order for the process to define the mathematical function $f(x) = y$ at the rational input x, it must *always* produce the same rational output y every time the process is repeated on input x. But if, say, we try this with a physical double pendulum (a simple deterministic and chaotic process), and restart it on x, that is restart it within the interval of measurement corresponding to x, then a minor (typically, thermal) fluctuation *below the interval defined by* x will yield a different observable result y', even if we fix a very short 'computation' time. Of course, for processes that are modelled by non-linear dynamics, taking intervals as input and output does not solve the physical problem. Non-linear maps (or the solutions of non-linear equations, if any) may enlarge the interval exponentially (following Lyapounov exponents (Cencini 2010)) and are *mixing*, that is, the extremes and the maxima/minima of the intervals are shuffled around. As numerical analysts know very well, this even makes the rarely applicable 'shadowing theorems' fail (and they would at least guarantee that the discrete dymanics is approximated by the continuous one, but not conversely, as mentioned earlier (Pilyugin 1999)).

So a better question would be to consider a physical process that does define *a function, and ask is this function computable?*

The idea then is that the process should be sufficiently insensitive to initial conditions (which is sometimes called robust) as to actually define a function. But the question then changes radically (and becomes trivial). Typically, we should be able to partition the World into little cubes of the smallest size, according to the best measurement of the insensitivity (the scale at which fluctuations have no effect on the dynamics). *If* the Accessible World is considered finite†, we can make a finite list of the input–output relation established by the given process. *This is a 'program'*. But then a good programming question would be *is this program compressible?*

Remark 4.2 (On quanta and discrete space–time). When it comes to the relevance of the discrete case, Quantum Mechanics started precisely with the discovery of a key (and unexpected) discretisation of atomic energy absorption and emission spectra. A few people dared to propose a discrete lower bound to measurement of *action*, that is, of the quantity energy × time. It is this physical dimension that reveals a discrete structure. Clearly, we can then compute, by assuming the relativistic maximum for the speed of light, a Planck

† But according to which *measure* would it be finite? What about Riemann's sphere as a model for the Relativistic Universe, and the little human moving with his/her metre rule towards the poles being squeezed to 0? For the God, who holds it in His hand, this Universe is finite, but it is infinite for our little human's measurements.

length and time. But this does not in any way mean that space and time are discretised in small 'quantum boxes or cubes'. And this is the most striking and crucial feature of Quantum Mechanics: the 'systemic' or entanglement effects that yield the non-separability of observables. In this context, no discrete space topology is natural as it would yield a separability of all (measurable) observables. That is, these quantum effects in space–time are the opposite of a discrete, and thus well-separated, organisation of space, and this fact lies at the core of its scientific originality. In particular, they motivate quantum computing (as well as our analysis of quantum randomness above). In fact, Thierry Paul and I claim that the belief in an absolutely separable topology of space continua was Einstein's mistake in EPR (Einstein *et al.* 1935), where entanglement was first examined and considered to be impossible (this is ongoing work – see Longo (2010b) for some further analysis).

Note, finally, that loop gravitation and string theory do in fact assert that our world might be composed of (a very large number of) finite objects with discrete relations. However, and crucially, these objects and their dynamical relations are handled in abstract mathematical spaces, such as Hilbert or Fock spaces, with possibly infinite dimensions. In those spaces, which are very remote from 'ordinary' space–time, processes may even be represented by a linear, and thus fully computable, mathematical dynamics, such as given by the Schrödinger equation. The problem then, as usual, is to 'return' these dynamics to our own space–time, through measurement, since that is where the intrinsic uncertainty pops out.

In summary, continua, Cantorian or topos theoretic, though not absolute, take rather good care of the approximate nature of (classical) physical measurement, which is represented as an interval: with unknowable fluctuations lying within the interval. In Quantum Physics, the peculiar correlation of conjugated variabes and intrinsic uncertainty gives measurement an even more important role. And (physical) measurement is our only form to access 'reality'. The arithmetising foundations of Mathematics went along another (and very fruitful) direction, but based on perfectly accessible data types.

5. The originality of the Discrete State Machine

As I have already mentioned, the Discrete State Alphanumeric Machine is a remarkable and very original human invention, with a long history. As hinted in Longo (2008), this story began with the invention of the alphabet, which is probably the oldest example of discretisation. Instead of trying to capture concepts and ideas in designs (by recalling 'meaning', like in ideograms), the continuous aural stream of speech is discretised by annotating phonetic pitches, which is an amazing idea (due to the people of Altham, in Mesopotamia, 3300 B.C.). Meaning is reconstructed by the sound, which acts as a compiler, either aloud or in silence (it was only after the fourth century A.D. that we learned to read 'within the head'!).

The other key passage towards alphanumeric discretisation was the invention of a discrete coding structure. This originated with Gödel numbering, which seems an obvious thing to do now, but was another idea that was as remarkable as it was artificial. Turing's work followed: the Logical Computing Machine (LCM), as he first called it, is at the core of computing (right/left, 0, 1 ...). Of course, between the invention of the alphabet and Turing, there was Descartes' 'discretisation' of thought (stepwise reasoning, along a discrete chain of intuitive certitudes), and much more.

When, in the late 1940s, Turing worked again in Physics, he changed the name of his LCM: in Turing (1950; 1952), he refers to it as a *Discrete State* Machine (this is what matters for its physical behaviour). And twice in his 1950 paper (the 'imitation game'), he calls it 'Laplacian'. Its evolution is theoretically predictable, even if there may be practical unpredictability (programs that are too long to be grasped, he says).

So, by using ideas from formal Logic, we invented and later physically realised an incredibly stable processor, which, by working on discrete data types, does what it is expected to do. And it produces repeatable results, very faithfully, which I stress is its key feature. Primitive recursion and the portability of software are forms of repeatability: iteration and the update of a register do what they are supposed to do, even when in slightly different contexts, and they do it over and over again. For example, suppose we program the evolution function of the most chaotic strange attractor we know, but if we push 'restart', the digital evolution, by starting on the same initial digits, will follow exactly the same trajectory. This makes no physical sense, but it is very useful (for example, in meteorology, you can restart your turbulent flow, exactly, and try to understand better how it evolves). Of course, one may imitate unpredictability using some pseudo-random generator or by incorporating some true physical randomness, added *ad hoc*. But this is cheating the observer, in the same way as Turing's *imitation* of a woman's brain is meant to cheat the observer, rather than 'model' the brain. He says this explicitly, while working in his 1952 paper on a *model* of morphogenesis as (non-)linear dynamics. He speculates that brain activity may depend on fluctuations below the level of measurement, and not his DSM – see Longo (2008) for a closer analysis and, of course, Turing's two papers, which should always be read together. In contrast to imitation, a mathematical model tries to propose a 'structure of determination' (for example, the equations for action, reaction and diffusion in the 1952 paper). Finally, we may observe that our colleagues working in networks and concurrency are so good that programming in concurrent networks is also reliable: programs do what they are supposed to do, they run again and give you the web page you want, identically every time, one thousands times, one million times. And this is hard, as physical space–time, which we understand better through continua and continuous approximations, steps in, but still using discrete data types, and this allows repeatability. Of course, identical repetition is the opposite of randomness (many define a process to be random when, repeated with the 'same' initial conditions – as in the case of physical measurement – it follows a different evolution).

Those who claim that the Universe is a big digital computer miss the originality of this machine of ours: in particular, its history, from the alphabet through Hilbert's formal systems and on to the current work in concurrency in networks, where reliability is a key objective. The construction of computers has been a remarkable achievement in producing a reliable (and thus programmable), physical (but artificial) device, divorced from the natural world, and repeating as we wish and any time we wish, even in networks. One should not ignore the principles that guided this invention, as well as the principles through which we have understood physical dynamics, since Poincaré. The very rich relations between computing and the (physical) world, and with those dynamics in particular, is a non-trivial issue, and a long way from the flat transfer of techniques or the identification of models.

6. The relevance of negative results

Even if we do not consider the Universe to be a big computer, some still claim that the 'Laws of Physics' are computable. However, this is hard to (dis-)prove, as I have never seen the 'Complete Table of the Laws of Physics', or their enumeration algorithm.

What should be analysed is the effectiveness of our *mathematical description* of physical invariants. Of course, equations, evolution functions, and so on, are given in terms of sums, products, exponents, derivations, integrations, and so on, which are all computable operations. And no one would be so crazy as to put an incomputable real as a coefficient or exponent in an equation (even if h might be one). This allows us to make some remarkable approximations and, frequently, also gives us qualitative information: Poincaré's Geometry of Dynamical Systems or Hadamard's analysis of the geodetic flow on hyperbolic surfaces do not give predictions, but do provide some very relevant global information (through attractors, for example, or regularities in flows, which we can observe beautifully today, as never before, through fantastic approximations, which are 'shadowed' on our computers' screens). On the other hand, we can write two equations that 'model' in the best way we know the non-linear dynamics of a double pendulum, and even compute a computable solution – it is just too bad that that solution does not follow the actual physical dynamics: no matter how well we measure its initial conditions, a fluctuation below our best measurement lets the pendulum go along a completely different trajectory from the one we rushed to compute.

Moreover, Pour-El, Richards, Baverman, Yampolsky, and so on, were able to find unique (or just) non-computable solutions of effective equations (Pour-El and Richards 1989; Braverman and Yampolski 2006; Weihrauch and Zhong 2002). In the end, these results reduce the question to the halting problem, that is, to the *diagonal writing* of sequences of digits. Turing said that computing is 'a man provided with paper, pencil and rubber'. Computing does not exist in the real World, but is an extraordinary alpha-numeric invention of ours, based on written language, and is a form of re-writing made possible by the very abstract and dualistic nature of the alphabet (the signs have no meaning in themselves – classical Chinese computers are geometric devices). The alphabet and computing are very rough approximations of the expressivity of the continuous stream of spoken language and of the physical world, respectively.

As I said earlier, isolating the Absolute Laws of Nature is too hard for us: our human insight is provided by the constructive theorising on the phenomenal veil, at the interface between us and the World. These active constructions are effective (we understand a lot and can transform the World, though not always for the best) and mostly computable (we use the alphabet, computable operations and codings).

Yet, *there are very few predictable processes in Nature*: you can compute the date of a few forthcoming Eclipses, on a human time scale, but the Solar System is chaotic in astronomical times, as Poincaré proved and Laskar later quantified by computing an upper bound to its predictability.

Note that the unpredictable processes are the mathematical and computational challenge, as a computable physical process is, by definition, deterministic and predictable. In order to predict (pre-dicere, 'to say in advance' in Latin), we just 'say' or write the corresponding program and compute in advance. Thus, the results mentioned above, by showing the equivalence of unpredictability and (strong) undecidability (ML-randomness), prove this equivalence by logical duality. Unpredictability may pop-up in computer networks because of *physical* space–time, as observed earlier, but we then make them computable and predictable/reliable by enforcing semaphores, handling interleaving, and so on. In Nature, many (fortunately, most) processes evade prediction, and thus our computations. Fortunately, because otherwise there would be no change, in particular, there would be no life: randomness is crucial. And when we compute unpredictable evolutions, we just approximate their initial paths, or provide some qualitative information, though both may be very relevant tasks. Thus the detailed analysis of unpredictability and randomness is an essential component of scientific knowledge. Moreover, by 'saying no' to strong programs (Laplace, Hilbert), unpredictability and undecidability led to the discovery of new science: modern dynamics and Mathematical Logic.

Computability is as artificial (and, thus, as useful) as the alphabet. By formalising what we can effectively say (compute/predict), we get *detached* symbolic images of the World and, in particular, the best way (the only one?) we have to describe what we *cannot* say (in advance). As mentioned above, computability was invented by proving incomputability. This is why I emphasise that a peculiar and relevant role must be given to incomputability (and randomness) in the relation between Physics (and Biology, see the next section) and Mathematics as a form of meaningful 'writing' about the World.

As a matter of fact, the only mathematical way I know to define randomness in classical dynamics is Birkhoff's ergodicity. But it is very specific (to certain dynamics). Otherwise, randomness is given in terms of probability measure. But this is unsatisfactory, as probability just gives a *measure* of randomness, not a definition. It is the theory of algorithms that, asymptotically, gave us a fully general and mathematical notion of randomness as a strong form of incomputability, and independently of probability theory. Again, physical (classical) randomness is deterministic unpredictability, and the bridging results mentioned above, and others in the literature, bring the role of computational randomness further into the limelight. In particular, it provides a very flexible theory of randomness: you can adjust the class of effective randomness tests (Martin-Löf, Schnorr, and many more). A wild conjecture is that this may help us get a better grasp of, for example, the mathematical difference between classical and quantum randomness.

7. Randomness, entropy and anti-entropy in Biology

7.1. *Embryogenesis and ontogenesis*

In Biology, randomness is even more relevant than in Physics. About 50% of conceptions in mammals fail (do not produce a birth), which is a very bad performance for DNA

https://doi.org/10.1017/S0960129511000569 Published online by Cambridge University Press

as a program. This is because DNA is not a 'program' *in any reasonable sense*. While repeatability, as reliability, is at the core of software (and hardware) design, the key principle for understanding life – at the phenotypic level – is *variability*, a form of non-iterability, combined with 'structural stability' against unexpected change, which is a very different matter. In Evolution, as in ontogenesis, a cell is never identical to its mother cell, and this is crucial to life. So the principles of intelligibility are exactly opposite: variability and failure correspond to the crucial *possibility* that a random mutant is a better fit in a changing environment.

Of course, some molecular processes do iterate, in particular, *in vitro*, but there is an increasing tendency to analyse molecular cascades in terms of stochastic phenomena, in particular, in eukaryotic cells (Kaern *et al.* 2005; Raj and van Oudenaarden 2008), and this is where good computational approaches may also help our understanding through stochastic network interactions – see, for example, Krivine *et al.* (2008). It has even been said that 'the DNA is a random generator of proteins, regulated by the cell, the organism and the environment' (Kupiec 2009). This is an extreme, but empirically motivated, reaction to the over-long domination of the 'one gene–one protein' hypothesis and Crick's 'central dogma' of 1958 (on the unidirectional, linear determinism from DNA to RNA to proteins and, then, to the phenotype). These imposed, until early this decade, a Laplacian framework for Molecular Biology (in other words, the DNA is a program paradigm).

I believe that the complexity of life processes is also a blend of conceptually opposite aspects. The central dogma is almost always false, but a few molecular cascades may actually follow Crick's dogma: the colours in plants, apparently, are predictably and uniquely determined by a fragment of DNA, a gene in the classical sense, but this is a very rare example. And some Laplacian molecular cascades can be reproduced *in vitro* or observed in bacteria (but very rarely in eukaryote cells). However, large parts of DNA or RNA interact statistically in a non-linear way and in a turbulent context, in the presence of quasi-chaotic enthalpic oscillations of huge molecules – in particular, in the cytoplasm of eukaryotes.

Whatever the validity or level of the blend I suggest, these new views on randomness open the way to an increasing role for epigenetics, and thus to the relevance of downward regulating effects from the cell and the organism to DNA expression.

Random effects persist during life. In particular, the recent Darwinian perspective in cancerogenesis proposes *growth* as the 'default state' of all cells (Sonnenschein and Soto 1999). This largely random proliferation is usually controlled by the various regulating activities of the organism on the cells: cancer should then be mostly viewed as the failure of this control and/or of the exchanges between cells in a tissue, generally in the presence of external carcinogenic factors.

7.2. *Evolution*

Through a remarkable analysis spanning many articles and two books (Gould 1989; Gould 1998), S. J. Gould stressed the role of randomness in Evolution. In particular, we – primates – are a random complexification in a bacterial world, appearing along a

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contingent diffusive path. The expansion of life, which has been 'punctuated' by sudden explosions and massive extinctions of species, has preserved a few invariants, while constantly changing organisms and their ecosystems. In order to set these remarks and the associated paleontological evidence on mathematical grounds, we have proposed the notion of anti-entropy, as formalised biological complexity (a qualitative evaluation of cellular, functional and phenotypical differentiation) – see Bailly and Longo (2009). I will briefly survey some aspects of that technical paper as an application of the role of randomness in Evolution.

Anti-entropy formally opposes entropy as a new observable (it has the opposite sign), in the same way that anti-matter opposes matter, as required by new (observable) particles in Quantum Mechanics. Anti-entropy is a form of (increasing) information in embryogenesis and Evolution. Organisms become more 'complex', from bacteria to eukaryotes to multicellular organisms, and this is the result of an asymmetric diffusion of biomass over anti-entropy, following along random paths. Anti-entropy locally opposes the (increasing) entropy inherent in all irreversible processes, by the (increasing) structuring of organisms, both in embryogenesis and Evolution.

In order to give a brief comparison between anti-entropy and information, observe that traditionally Shannon's information is considered as negentropy (Brillouin). Then, by definition, the sum of a quantity of information (negentropy) and an equal quantity of entropy gives 0. Information (as defined by Shannon, but also Kolmogorov) is 'insensitive to coding' – it is also an analysis of coding, since one can 'encrypt' and 'decrypt' as much as one wishes and, at least in principle, the information content will not be increased or decreased.

Note that in doing this we have now passed from Turing's *elaboration* of information (Computability Theory), which is also insensitive to coding and dimension, to the analysis of information (Shannon, Brillouin, and so on). However, this notion, though hugely important for the development of machines and the transmission of data, is not sufficient for an investigation of the living state of matter. DNA, which is usually considered as digital information, is the most important component of the cell, but it is also necessary to analyse the *organisation* of the living objects, as an observable specific to biological theory. Without a proper theory of the organism and its genesis comparable to the remarkable one we have for Evolution, we may get stuck in the current situation, where there is no general theoretical framework relating genotype to phenotype (only very long lists of mosty differential correlations, and only a few direct, positive ones relating the wild gene to the phenotype (Fox Keller 2000; Longo and Tendero 2007)). In short, modulo a few exceptions, we have no idea how the discrete chemical structures of DNA and other active macromolecules contribute to the construction of biological 'forms' in general. It is certain that randomness and constraints (including deformations, torsions, relative geometric positions, and so on), regulation and integration, as well as timing, and so on, play a radically different role from the one, if any, they may have in programs that generate forms from digits in machines.

In our work, anti-entropy, as biological complexity, may be understood as 'information specific to the form', including the intertwining and enwrapping of levels of organisation, which lie at the core of the autonomy of life. Anti-entropy yields a *strict* extension, in a logical sense, of the thermodynamics of entropy in that it extends some balance equations (Bailly and Longo 2009). It is compatible with information as negentropy, but it differs from it. First, the production of entropy and anti-entropy are summed in an 'extended critical singularity' (Bailly and Longo 2008; Longo and Montévil 2011) in the form of an organism, and never to 0, which is in direct contrast to Brillouin's and others' negentropy. Second, as it is linked to spatial forms, anti-entropy is '*sensitive to coding*', unlike digital information, since it depends on the dimensions of embedding manifolds, on folds, on fractal structures, on singularities, and so on.

Its use in metabolic balance equations has produced a number of results. We have, in particular, following Schrödinger's 'operational method' in Quantum Mechanics, proposed a diffusion equation of biomass over anti-entropy. This has enabled us to operate a mathematical reconstruction of this diffusion, which fits Gould's curve describing phenotypic complexity along the evolution of species. As mentioned earlier, we could then mathematically describe the random complexification during Evolution, as evidenced by Gould, by an asymmetric diffusion equation. The original asymmetry (Gould's '*left* wall' of least complexity, which is the formation of bacteria, and which was a critical transition from the inert to the living state of matter) propagates as *right* average bias along *random* evolving paths. Then, through purely local effects, as in any asymmetric random diffusion, biological complexity, qualitatively described by anti-entropy, propogates Evolution with no goal-directedness or program whatsoever.

7.3. *Computability in bio-chemical cycles*

Rosen (1991) hinted at the possibility of a stimulating investigation of incomputability at the molecular level. Through an abstract analysis of metabolic cycles and a refined distinction between mechanical simulation and modelling, the claim is made that some auto-referential steps would lead to incomputability (or non-mechanisability). Unfortunately, the technical sections on this topic are flawed by notational ambiguities and (crucial) misprints. By proposing a *possible* interpretation of Rosen's equations, we gave in Mossio *et al.* (2009) a λ-calculus, and thus computable, fixed-point solution to these equations. Other interpretations and, of course, other, possibly incomputable, solutions can be given. However, a computable (and possibly optimal in Scott domains) version may be obtained. Unfortunately, incorrect ideas have for too long plagued a lively 'Rosenian' debate. Some people (see Mossio *et al.* (2009) for references) have claimed that 'life is not computable' *because* Rosen's equations:

(1) lead to divergence (while computable functions should always be total);
(2) are circular (self-referential);
(3) are impredicative;
(4) are set-theoretically non-well-founded.

However, in the computablity community, we know very well that:

(1) This branch of Mathematics was initiated through proofs of the existence of '(relevant) *partial* computable functions that cannot be extended to total ones' (they are

https://doi.org/10.1017/S0960129511000569 Published online by Cambridge University Press

intrinsically diverging: *cf.* Gödel's work on incompleteness and Turing's work on halting).

(2) *Circularity* lies at the core of recursion; type-free λ-calculus, in particular, is based on self-reference (such as $f(f) = f$ equations, which, for some Rosenians, represents an incomputable miracle with a one-line computable solution in λ-calculus). Reflexive domain equations and λ-calculus fixed-point constructions, which are both rich forms of circularity, may give interesting (and useful) non-normalising terms. And this may correspond to the correct intuition that formal (computable) metabolic 'cycles' are not supposed to stop.

(3) *Impredicativity* is an integral part of Girard's Type Theory, which computes exactly the recursive functions that are provably total in II order Arithmetic (see Asperti and Longo (1991) for models of λ-calculus and Impredicative Type Theory).

(4) Models of *anti-foundation* axioms (non-well-founded sets) can be (relatively) interpreted in constructive models of type-free λ-calculus (and conversely – see Mossio *et al.* (2009) for a discussion and references).

(In-)computability in Biology is a delicate issue, which is also related to computer simulation and Artificial Life. Equational descriptions and the solutions are generally computable, as discussed above, but with very hard ('construed') counterexamples. These counterexamples are always proved by reducing the problem to Turing's halting problem, which is a pure 'diagonal' game of signs. Once again, computability and its opposite are a (very relevant) alpha-numeric linguistic construction – they are not in the World. As for incomputability, I would say, by way of a metaphor, that it applies to Natural Sciences in the same way as it relates to Cantor's real numbers: these are 'almost all' non-computable (in fact, they are all ML-random, apart from a set of measure 0), yet it is extremely hard, *using our mathematical language* to describe one: Turing's example and Chaitin's random number Ω are the only examples that have been given so far, though, with infinitely many variants, of course. In Nature, we can *informally* point out many incomputable (or unpredictable) processes, but it has been very difficult to single them out formally. Poincaré had to give meaning to the *absence* of analytic solutions of certain equations – an unusual step in Mathematical Physics. The computational difficulty is that not only is our writing effective, but also, I stress, that we invented computability and its machines as alphanumeric *(re-)writing systems*, of which λ-calculus is a paradigm. That is, when we write (and re-write), we obtain computable structures. And in order to depart from them and formally provide an example of an incomputable object, we can only, at least to date, use diagonalisation or a reduction to some diagonalisation process. The first examples were invented by Gödel and Turing, and the reduction to the second of these has been applied by a few (from Chaitin to Pour-El, Braverman and collaborators).

Yet, as a form of unpredictability, incomputability in Biology should be analysed well beyond classical randomness, that is, in addition to the strong incomputability of deterministic unpredictability or quantum randomness. An understanding of unpredictability and incomputability may certainly depend on non-linear and network interactions, but also on an understanding of organisms as *specific* (contingent) autonomous systems in a changing ecosystem, as proposed by many, including Rosen. Equationally determined objects, by contrast, are *generic* (mathematically invariant), which is a crucial difference

– see Bailly and Longo (2011). Moreover, 'resonance' effects may take place between different levels of organisation, the analysis of which has not yet formed a part of the mathematical tools inherited from Physics (since Poincaré, we have understood the importance of gravitational resonance, which is a critical non-linear interaction, as a source of deterministic unpredictability at a single level of organisation: a planetary system, say). An organism contains networks of cells in tissues as parts of organs subject to morphogenesis, which are also integrated and regulated within an organism by hormones, the neural and immune systems, and so on. Thus, proper biological randomness is a further mathematical challenge, as yet unexplored, and one component of the many difficult issues in the 'mathematisation' of Biology (Buiatti and Longo 2012).

8. Conclusion

In this paper, we have compared physical (dynamic) randomness synthetically with algorithmic randomness (which lies at the heart of *algorithmic theories of information*). This has been a way to discuss incomputability in Physics. With reference to randomness in Evolution, we mentioned the concept of anti-entropy (which forms a '*geometrical* extension' of the notion of information) as an observable that we have added to thermodynamic balance equations. This has allowed us to mathematise the globally random complexification of life (where any diffusion is based on random paths). A further, ongoing, analysis of some aspects of the stable/unstable, far from equilibrium, dissipative state of living matter is based on a theory of 'extended criticality', which is a mathematical extension of point-wise critical transitions in Physics.

The scientific outcome of this work, whose conceptual framework is presented in Bailly and Longo (2011), may also entail some epistemological consequences.

First, through science, we grasp, at most and approximately, some relevant, but changing fragments of the World; relevant for us, that is, from our perspective of randomly complexified bacteria, up to the neotenic monkeys we are, with free hands and relatively too big brains, in constant historical evolution. The understanding that physical and biological processes do not coincide with formal computations, and that our symbolic writings cannot even represent them faithfully and completely, constitutes a constant need to search for new forms of knowledge. Our (mathematical) descriptions are not absolute, and they are 'reasonably effective' exactly because they are derived from a concrete friction between our evolutive (and historical) being and the World.

The close analysis of incomputability or unpredictability, through the negative results it produces, and its contribution to a qualitative analysis leading to a better understanding of limits and, thus, to a search for new perspectives, is part of this quest. If, one day, Molecular Biology is able to *prove from the inside* (in the same way as Poincaré's and Gödel's analyses did in Mathematics and Logic) that we are unable to predict the shape of the ear by analysing the DNA alone, I would say that something theoretically very original and relevant has happened. Instead, there are still such claims as the identification of the gene for marital fidelity (Young *et al.* 1999).

Second, as part of the many existing interactions with Physics and Biology, this view should also make a contribution to the epistemological debate on the notion of information, and, in particular, to the updating of its theoretical principles. Both the concept and theory of information in the current sense (of negentropy, say) seem largely inadequate as a way of expressing biological dynamics. And a possible outcome of these interactions for the notion of information could be that we are led to start thinking about the 'next machine', but along a different path from the one explored by quantum computing. I am prepared to bet that this nice DSM of ours will not be the 'final machine' invented by Mankind, despite what computationalists seem to claim when they consider the World to be identical to it, or a faithful image of it, or its logic.

Biology presents us with the need for a radical departure from this narrow view, particularly when we are faced with issues like 'structural stability' as a non-identical iteration of a morphogenetic process, as well as the role of contingency in phylogenesis and ontogenesis. And randomness, which lies at the core of life's contingency, also seems to depend on quantum effects, which are increasingly showing up in cells, and on non-linear interactions, which are amplified by (metabolic) circularities, and also on 'resonance' effects between different levels of organisation, both within the cytoplasm and the organism's integration–regulation system. But this is ongoing work – see Buiatti and Longo (2012).

Acknowledgements

I would like to thank the Editors of this special issue for inviting me to publish in such an excellent journal(!), as well as the anonymous referees for their very helpful comments and critiques.

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