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POWER PARTITIONS AND SEMI-*m*-FIBONACCI PARTITIONS

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Abstract

Andrews [‘Binary and semi-Fibonacci partitions’, *J. Ramanujan Soc. Math. Math. Sci.* **7**(1) (2019), 1–6] recently proved a new identity between the cardinalities of the set of semi-Fibonacci partitions and the set of partitions into powers of 2 with all parts appearing an odd number of times. We extend the identity to the set of semi-m-Fibonacci partitions of n and the set of partitions of n into powers of m in which all parts appear with multiplicity not divisible by m. We also give a new characterisation of semi-m-Fibonacci partitions and some congruences satisfied by the associated number sequence.

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1. Introduction

A partition λ of an integer $n > 0$ is a finite nonincreasing integer sequence whose sum is n. The terms of the sequence are called *parts* of λ. As in [2], a partition with k parts will generally be expressed as

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k), \quad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0,$$

or

$$\lambda = (\lambda_1^{v_1}, \lambda_2^{v_2}, \ldots, \lambda_t^{v_t}), \quad \lambda_1 > \lambda_2 > \cdots > \lambda_t > 0,\ t \leq k,$$

where $\lambda_i^{v_i}$ indicates that λ_i occurs with multiplicity v_i for each i, and $v_1 + \cdots + v_t = k$.

Andrews [1] describes the set $SF(n)$ of semi-Fibonacci partitions as follows: $SF(1) = \{(1)\}$, $SF(2) = \{(2)\}$; if $n > 2$ and n is even, then

$$SF(n) = \{\lambda \mid \lambda \text{ is a partition of } n/2 \text{ with each part doubled}\};$$

if n is odd, then a member of $SF(n)$ is obtained by inserting 1 into each partition in $SF(n-1)$ or by adding 2 to the single odd part in a partition in $SF(n-2)$.

TABLE 1. The map $SF(n) \to OB(n)$ for $n = 9$.

$SF(9)$	$\longrightarrow$	$OB(9)$
(8,1)	$\mapsto$	(8,1)
(4,3,2)	$\mapsto$	(4,2,1,1,1)
(6,3)	$\mapsto$	(2,2,2,1,1,1)
(5,4)	$\mapsto$	(4,1,1,1,1,1)
(7,2)	$\mapsto$	(2,1,1,1,1,1,1,1)
(9)	$\mapsto$	(1,1,1,1,1,1,1,1,1)

The cardinality $sf(n) = |SF(n)|$ satisfies the recurrence relation

$$sf(n) = \begin{cases} sf(n/2) & \text{if } n \text{ is even,} \\ sf(n-1) + sf(n-2) & \text{if } n \text{ is odd,} \end{cases}$$

for all $n > 0$ (with $sf(-1) = 0, sf(0) = 1$).

The semi-Fibonacci sequence $\{sf(n)\}_{n>0}$ occurs as sequence number A030067 in Sloane's database [5]. Beck [3] has previously considered the properties of a set of polynomials related to the semi-Fibonacci partitions.

Andrews stated the following relation between the number of semi-Fibonacci partitions of n and the number $ob(n)$ of binary partitions of n in which every part occurs an odd number of times.

THEOREM 1.1 [1, Theorem 1]. *For each $n \geq 0$,*

$$sf(n) = ob(n).$$

Andrews gave a generating function proof and asked for a bijective proof.

The proof turns out to be remarkably simple. It goes as follows. Each part t of $\lambda \in SF(n)$ can be expressed as $t = 2^i \cdot h$, $i \geq 0$, where h is odd. Now transform t as

$$t = 2^i \cdot h \longmapsto 2^i, 2^i, \ldots, 2^i \quad (h \text{ times}).$$

This gives a partition of n into powers of 2 in which every part has odd multiplicity. Conversely, consider $\beta \in OB(n)$, the set of binary partitions of n in which every part occurs an odd number of times. Since every part (a power of 2) has odd multiplicity, we simply write β in the exponent notation $\beta = (\beta_1^{u_1}, \ldots, \beta_s^{u_s})$, with $\beta_1 > \cdots > \beta_s$ and the u_i odd and positive. Since each $\beta_i^{u_i}$ has the form $(2^{j_i})^{u_i}$, $j_i \geq 0$, we apply the transformation

$$\beta_i^{u_i} = (2^{j_i})^{u_i} \longmapsto 2^{j_i} u_i.$$

This gives a unique partition in $SF(n)$. Indeed, the image may contain at most one odd part which occurs precisely when $j_i = 0$. We illustrate the map for $n = 9$ in Table 1.

We also consider the following congruence, which Andrews proved with generating functions.

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THEOREM 1.2 [1, Theorem 2]. *For each $n \geq 0$, $sf(n)$ is even if $3 \mid n$ and odd otherwise.*

PROOF. We give a combinatorial proof based on mathematical induction. The result holds for $n = 1, 2, 3$, since $sf(1) = 1 = sf(2)$ and $sf(3) = |\{(1, 2), (3)\}| = 2$. Now let $n > 3$ and assume that the result holds for all integers less than n.

If $n \equiv 1 \pmod 3$, then $sf(n)$ is the sum of $sf(n-1)$ and $sf(n-2)$ which have opposite parities since, by the inductive hypothesis, $sf(n-1)$ is even (since $3 \mid (n-1)$) and $sf(n-2)$ is odd.

If $n \equiv 2 \pmod 3$, then $sf(n)$ is the sum of $sf(n-1)$ which is odd (since $3 \nmid (n-1)$) and $sf(n-2)$ which is even. Thus, $sf(n)$ is odd.

If $3 \mid n$ and n is even, then $sf(n) = sf(n/2)$. Since $3 \mid n/2$, it follows that $sf(n/2)$ is even by the inductive hypothesis.

Lastly, if $3 \mid n$ and n is odd, then $sf(n) = sf(n-1) + sf(n-2)$ which is even since $3 \nmid (n-1)$ and $3 \nmid (n-2)$. This completes the proof. □

The following result is easily deduced from the definition of the sets counted by $sf(n)$.

COROLLARY 1.3. *Given a nonnegative integer v,*

$$sf(2^v) = 1.$$

In Section 2 we define the semi-m-Fibonacci partitions by extending the previous construction using a fixed integer modulus $m > 1$. A generalised identity is then stated between the set of semi-m-Fibonacci partitions and the set of partitions into powers of m with multiplicities not divisible by m (Theorem 2.2). Then in Section 3 we give an independent characterisation of the semi-m-Fibonacci partitions. Lastly, in Section 4 we discuss some arithmetic properties satisfied by the semi-m-Fibonacci sequence.

2. Generalisation

We generalise the set of semi-Fibonacci partitions to the set $SF(n, m)$ of semi-m-Fibonacci partitions as follows: $SF(n, m) = \{(n)\}$ for $n = 1, 2, \ldots, m$; if $n > m$ and n is a multiple of m, then

$$SF(n, m) = \{\lambda \mid \lambda \text{ is a partition of } n/m \text{ with each part multiplied by } m\};$$

if n is not a multiple of m, that is, $n \equiv r \pmod m$, $1 \leq r \leq m-1$, then $SF(n, m)$ arises from two sources: first, partitions obtained by inserting r into each partition in $SF(n-r, m)$ and, second, partitions obtained by adding m to the single part of each partition $\lambda \in SF(n-m, m)$ which is congruent to $r \pmod m$ (since λ contains exactly one part which is congruent to r modulo m, as shown in Lemma 2.1 below).

LEMMA 2.1. *Let $\lambda \in SF(n, m)$.*

(1) *If $m \mid n$, then every part of λ is a multiple of m.*
(2) *If $n \equiv r \pmod m$, $1 \leq r < m$, then λ contains exactly one part $\equiv r \pmod m$.*

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TABLE 2. Semi-3-partitions for $n = 1, 2, \dots, 10$.

n	$SF(n,3)$	$sf(n,3)$
1	$\{(1)\}$	1
2	$\{(2)\}$	1
3	$\{(3)\}$	1
4	$\{(4),(3,1)\}$	2
5	$\{(5),(3,2)\}$	2
6	$\{(6)\}$	1
7	$\{(7),(4,3),(6,1)\}$	3
8	$\{(8),(5,3),(6,2)\}$	3
9	$\{(9)\}$	1
10	$\{(10),(6,4),(7,3),(9,1)\}$	4

PROOF. If $m \mid n$, the parts of a partition in $SF(n,n)$ are clearly divisible by m by construction.

For induction, note that $SF(r,m) = \{(r)\}$, $r = 1, \dots, m-1$, so the assertion holds trivially. Assume that the assertion holds for the partitions of all integers $< n$ and consider $\lambda \in SF(n,m)$ with $1 \le r < m$. Then λ may be obtained by inserting r into a partition $\alpha \in SF(n-r,m)$. Since α consists of multiples of m (as $m \mid (n-r)$), λ contains exactly one part $\equiv r \pmod{m}$. Alternatively, λ is obtained by adding m to the single part of a partition $\beta \in SF(n-m,m)$ which is $\equiv r \pmod{m}$. Indeed, β contains exactly one such part by the inductive hypothesis. Hence, the assertion is proved. □

Define $sf(n,m) = |SF(n,m)|$. Table 2 illustrates the semi-3-Fibonacci partitions for small n.

For $m > 1$, we see that $sf(n,m) = 0$ if $n < 0$, $sf(0,m) = 1$ and, for $n > 0$,

$$sf(n,m) = \begin{cases} sf(n/m, m) & \text{if } n \equiv 0 \pmod{m}, \\ sf(n-r,m) + sf(n-m,m) & \text{if } n \equiv r \pmod{m}, \quad 0 < r < m. \end{cases} \tag{2.1}$$

The case $m = 2$ gives the function considered by Andrews: $sf(n,2) = sf(n)$.

Power partitions are partitions into powers of a positive integer m, also known as m-power partitions [4]. Let $nd(n,m)$ be the number of m-power partitions of n in which the multiplicity of each part is not divisible by m. For example, $nd(10,3) = 4$, the enumerated partitions being

$$(9,1), (3,3,1,1,1,1), (3,1,1,1,1,1,1,1), (1,1,1,1,1,1,1,1,1,1).$$

THEOREM 2.2. *For integers $n \ge 0, m > 1$,*

$$sf(n,m) = nd(n,m).$$

We give two proofs, the first analytic and the second combinatorial.

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First Proof. Let $|q| < 1$ and define

$$G_m(q) = \sum_{n\geq 0} sf(n,m)q^n. \tag{2.2}$$

Then

$$\begin{aligned}
G_m(q) &= \sum_{n\geq 0} sf(mn,m)q^{mn} + \sum_{n\geq 0} sf(mn+1,m)q^{mn+1} + \cdots \\
&\quad + \sum_{n\geq 0} sf(mn+m-1,m)q^{mn+m-1} \\
&= \sum_{n\geq 0} sf(mn,m)q^{mn} + \sum_{r=1}^{m-1}\sum_{n\geq 0} sf(mn+r,m)q^{mn+r} \qquad (2.3) \\
&= \sum_{n\geq 0} sf(n,m)q^{mn} + \sum_{r=1}^{m-1}\sum_{n\geq 0}(sf(mn,m) + sf(mn+r-m,m))q^{mn+r} \quad \text{(by (2.1))} \\
&= \sum_{n\geq 0} sf(n,m)q^{mn} + \sum_{r=1}^{m-1}\sum_{n\geq 0}(sf(n,m)q^{mn+r} + \sum_{r=1}^{m-1}\sum_{n\geq 0} sf(mn+r-m,m)q^{mn+r} \\
&= \Big(1 + \sum_{r=1}^{m-1} q^r\Big)\sum_{n\geq 0} sf(n,m)q^{mn} + \sum_{r=1}^{m-1}\sum_{n\geq 0} sf(m(n-1)+r,m)q^{mn+r} \\
&= G_m(q^m)\sum_{r=0}^{m-1} q^r + \sum_{r=1}^{m-1}\sum_{n\geq 0} sf(mn+r,m)q^{mn+m+r} \\
&= G_m(q^m)\sum_{r=0}^{m-1} q^r + q^m\Big(\sum_{n\geq 0} sf(n,m)q^n - \sum_{n\geq 0} sf(mn,m)q^{mn}\Big) \quad \text{(by (2.3))} \\
&= G_m(q^m)\sum_{r=0}^{m-1} q^r + q^m(G_m(q) - G_m(q^m)) \\
&= \Big(-q^m + \sum_{r=0}^{m-1} q^r\Big)G_m(q^m) + q^m G_m(q).
\end{aligned}$$

Hence,

$$G_m(q) = \frac{1 + q + q^2 + q^3 + \cdots + q^{m-1} - q^m}{1 - q^m} G_m(q^m). \tag{2.4}$$

Using (2.4) iteratively gives

$$G_m(q) = \prod_{n=0}^{N}\Big(\frac{1 + q^{m^n} + q^{2m^n} + \cdots + q^{(m-1)m^n} - q^{m^{n+1}}}{1 - q^{m^{n+1}}}\Big)G_m(q^{m^{N+1}}).$$

TABLE 3. The map $SF(n,m) \to ND(n,m)$ for $n = 11$, $m = 3$.

$SF(11,3)$	$\longrightarrow$	$ND(11,3)$
(11)	$\mapsto$	(1,1,1,1,1,1,1,1,1,1,1)
(8,3)	$\mapsto$	(3,1,1,1,1,1,1,1,1)
(6,5)	$\mapsto$	(3,3,1,1,1,1,1)
(9,2)	$\mapsto$	(9,1,1)

Taking the limit as $N \to \infty$, we have $G_m(q^{m^{N+1}}) \to G_m(0) = 1$ (since $|q| < 1$), so that

$$\begin{aligned} G_m(q) &= \prod_{n=0}^{\infty}\Big(\frac{1 + q^{m^n} + q^{2m^n} + \cdots + q^{(m-1)m^n} - q^{m^{n+1}}}{1 - q^{m^{n+1}}}\Big) \\ &= \prod_{n=0}^{\infty}\Big(1 + \frac{q^{m^n} + q^{2m^n} + \cdots + q^{(m-1)m^n}}{1 - q^{m^{n+1}}}\Big) \\ &= \prod_{n=0}^{\infty}\Big(1 + (q^{m^n} + q^{2m^n} + \cdots + q^{(m-1)m^n})\sum_{j=0}^{\infty} q^{j(m^{n+1})}\Big) \\ &= \prod_{n=0}^{\infty}\Big(1 + \sum_{j=0}^{\infty} q^{m^n(jm+1)} + \sum_{j=0}^{\infty} q^{m^n(jm+2)} + \sum_{j=0}^{\infty} q^{m^n(jm+3)} + \cdots + \sum_{j=0}^{\infty} q^{m^n(jm+m-1)}\Big) \\ &= \sum_{n\geq 0} nd(n,m)q^n. \qquad (2.5) \end{aligned}$$

The assertion follows by comparing coefficients in (2.2) and (2.5). □

SECOND PROOF. Each part t of $\lambda \in SF(n,m)$ can be expressed as $t = m^i \cdot h$, $i \geq 0$, where m does not divide h. Now transform t as

$$t = m^i \cdot h \longmapsto m^i, m^i, \ldots, m^i \quad (h \text{ times}).$$

This gives a partition of n into powers of m in which every part has multiplicity not divisible by m. Conversely, consider $\beta \in ND(n,m)$. Since every part (a power of m) has a nonmultiple of m as multiplicity, we simply write β in the exponent notation $\beta = (\beta_1^{u_1}, \ldots, \beta_s^{u_s})$, with $\beta_1 > \cdots > \beta_s$ and the $u_i \not\equiv 0 \pmod{m}$. Since each $\beta_i^{u_i}$ has the form $(m^{j_i})^{u_i}$, we can apply the transformation

$$\beta_i^{u_i} = (m^{j_i})^{u_i} \longmapsto m^{j_i} u_i.$$

This gives a unique partition in $SF(n,m)$. If $m \mid n$, this image contains only multiples of m. If $n \equiv r \pmod{m}$, $1 \leq r < m$, the image consists of multiples of m and exactly one part $\equiv r \pmod{m}$ which occurs when $j_i = 0$. We illustrate the map for $n = 11, m = 3$ in Table 3. □

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3. A characterisation of semi-m-Fibonacci partitions

Define the max *m-power* of an integer N as the largest power of m that divides N (not just the exponent of the power). Thus, using the notation $x_m(N)$, we find that $N = u \cdot m^s$, $s \geq 0$, where $m \nmid u$ and $x_m(N) = m^s$. So, $x_m(N) > 0$ for all N.

For example, $x_2(50) = 2$, $x_2(40) = 8$, $x_3(216) = 27$ and $x_5(216) = 1$.

Note that by unique factorisation, if the parts of a partition λ have distinct max m-powers, then the parts are distinct.

We define three (reversible) operations on a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ with an integer $m > 1$.

(i) If the last part of λ is less than m, delete it: $\tau_1(\lambda) = (\lambda_1, \ldots, \lambda_{k-1})$.
(ii) If $m \nmid \lambda_t > m$, then $\tau_2(\lambda) = (\lambda_1, \ldots, \lambda_{t-1}, \lambda_t - m, \lambda_{t+1}, \ldots, \lambda_k)$.
(iii) If λ consists of multiples of m, divide every part by m: $\tau_3(\lambda) = (\lambda_1/m, \ldots, \lambda_k/m)$.

These operations are consistent with the recursive construction of the set $SF(n, m)$, where τ_3^{-1}, τ_1^{-1} and τ_2^{-1} correspond, respectively, to the three quantities in the recurrence (2.1).

Lemma 3.1. *Let $B(n, m)$ denote the set of partitions of n in which the parts have distinct max m-powers and at most one nonmultiple of m. If $\lambda \in B(n, m)$ and $\tau_i(\lambda) \neq \emptyset$, then, for each $i = 1, 2$ or 3, there is an N such that $\tau_i(\lambda) \in B(N, m)$.*

Proof. Let $\lambda = (\lambda_1, \ldots, \lambda_k) \in B(n, m)$. If λ contains one part less than m, the part is λ_k. So, $\tau_1(\lambda) \in B(n - \lambda_k, m)$ since the max m-powers remain distinct. It is obvious that the parity of λ is inherited by $\tau_2(\lambda) = (\lambda_1, \ldots, \lambda_{t-1}, \lambda_t - m, \lambda_{t+1}, \ldots, \lambda_k) \in B(n - m, m)$. Lastly, since the parts of λ have distinct max m-powers, $\tau_3(\lambda) = (\lambda_1/m, \ldots, \lambda_k/m)$ may contain at most one nonmultiple of m as a part. Hence, $\tau_3(\lambda) \in B(n/m, m)$. □

We state an independent characterisation of the semi-m-Fibonacci partitions.

Theorem 3.2. *A partition of n is a semi-m-Fibonacci partition if and only if the parts have distinct max m-powers and at most one nonmultiple of m.*

Proof. We show first that $SF(n, m) = B(n, m)$. Let $\lambda = (\lambda_1, \ldots, \lambda_k) \in SF(n, m)$ be such that $\lambda \notin B(n, m)$. Assume that there are $\lambda_i > \lambda_j$ satisfying $x_m(\lambda_i) = x_m(\lambda_j)$ and let $\lambda_i = u_i m^s$, $\lambda_j = u_j m^s$ with $m \nmid u_i, u_j$. Observe that τ_1 deletes a part less than m if it exists. So, we can use repeated applications of τ_2 to reduce a nonmultiple modulo m, followed by τ_1. This is tantamount to simply deleting the nonmultiple of m, say λ_t, to obtain a member of $B(n - \lambda_t, m)$ from Lemma 3.1. By successively deleting nonmultiples and applying τ_3^c, $c > 0$, we obtain a partition $\beta = (\beta_1, \beta_2, \ldots)$ with $\beta_i = v_i m^w > \beta_j = v_j m^w$, where $m \nmid v_i, v_j$ and $w \leq s$. Then apply τ_3^w to obtain a partition γ with two nonmultiples of m. Then, by Lemma 2.1, $\gamma \notin SF(n, m)$. Therefore, $SF(n, m) \subseteq B(n, m)$.

Conversely, let $\lambda = (\lambda_1, \ldots, \lambda_k) \in B(n, m)$. If $\lambda = (t)$, $1 \leq t \leq m$, then $\lambda \in SF(t, m)$. If $m \mid \lambda_i$ for all i, then $\tau_3(\lambda) = (\lambda_1/m, \ldots, \lambda_k/m) \in B(n/m, m)$ contains at most one part $\not\equiv 0 \pmod{m}$, so $\lambda \in SF(n, m)$. Lastly, assume that $n \equiv r \not\equiv 0 \pmod{m}$. Then $r \in \lambda$ or $\lambda_t \equiv r$

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(mod m) for exactly one index t. Thus, $\tau_1(\lambda) = (\lambda_1, \ldots, \lambda_{k-1})$ consists of multiples of m while $\tau_2(\lambda) = (\lambda_1, \ldots, \lambda_{t-1}, \lambda_t - m, \lambda_{t+1}, \ldots, \lambda_k)$ still contains one part $\not\equiv 0$ (mod m). In either case $\lambda \in SF(n, m)$. Hence, $B(n, m) \subseteq SF(n, m)$. Thus, the two sets are identical. □

Remark 3.3. Notice that Theorem 3.2 certifies the second (bijective) proof of Theorem 2.2. If $\lambda = (\lambda_1, \ldots, \lambda_k) \in SF(n, m)$ but $\lambda \notin B(n, m)$ on account of having two parts λ_i, λ_j such that $\lambda_i = u_i m^s > \lambda_j = u_j m^s$ with $m \nmid u_i, u_j$, then it cannot have an inverse image. Assume that λ maps to $\beta \in ND(n, m)$, which then includes the parts $m^{u_i+u_j}$ ($u_i + u_j$ copies of m). Then $u_i + u_j$ may be a multiple of m (for example, when $u_i = 1, u_j = m - 1$), which implies that $\beta \notin ND(n, m)$, which is a contradiction. Alternatively, the pre-image of β would include the part $m(u_i + u_j)$ and so cannot be λ.

4. Arithmetic properties

We prove several congruence properties of the numbers $sf(n, m)$.

Theorem 4.1. *Let n, m be integers with $n \geq 0$, $m > 1$. Then*

$$sf(nm + 1, m) = sf(nm + 2, m) = \cdots = sf(nm + m - 1, m) = \sum_{j=0}^{n} sf(j, m).$$

Proof. Let $J_{r,m}(q) = \sum_{n\geq 0} sf(nm + r, m)q^n$, where $r = 1, 2, 3, \ldots, m - 1$. Then

$$\begin{aligned} J_{r,m}(q) &= \sum_{n\geq 0} sf(nm, m)q^n + \sum_{n\geq 0} sf(mn + r - m, m)q^n \quad \text{(by (2.1))} \\ &= \sum_{n\geq 0} sf(n, m)q^n + \sum_{n\geq 0} sf(mn + r, m)q^{n+1} \\ &= G_m(q) + q\sum_{n\geq 0} sf(mn + r, m)q^n \\ &= G_m(q) + qJ_{r,m}(q), \end{aligned}$$

so that

$$J_{r,m}(q) = \frac{G_m(q)}{1 - q}. \tag{4.1}$$

Since the right side of (4.1) is independent of r, we must have $J_{1,m}(q) = J_{2,m}(q) = \cdots = J_{m-1,m}(q)$, so that $sf(nm + 1, m) = sf(nm + 2, m) = \cdots = sf(nm + m - 1, m)$. Furthermore, from (4.1),

$$\sum_{n\geq 0} sf(mn + r, m)q^n = \sum_{n\geq 0} q^n \sum_{n\geq 0} sf(n, m)q^n = \sum_{n\geq 0}\sum_{j=0}^{n} sf(j, m)q^n,$$

which implies that $sf(mn + r, m) = \sum_{j=0}^{n} sf(j, m)$. □

Corollary 4.2. *Given integers $m \geq 2$, for any $j \geq 0$ and a fixed $v \in \{0, 1, \ldots, m\}$,*

$$sf(m^j(mv + r), m) = v + 1 \quad \text{for } 1 \leq r \leq m - 1.$$

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PROOF. By applying (2.1) several times (the case when $m \mid n$), it is clear that for any $j \geq 0$,

$$sf(m^j(mv+r),m) = sf(m^{j-1}(mv+r),m) = sf(m^{j-2}(mv+r),m) = \cdots = sf(mv+r,m).$$

By the last equality in Theorem 4.1,

$$sf(mv+r,m) = \sum_{i=0}^{v} sf(i,m) = 1 + \sum_{i=1}^{v} sf(i,m), \quad v \geq 0, 1 \leq r < m.$$

If $1 \leq v < m$, then $\sum_{i=1}^{v} sf(i,m) = \sum_{i=1}^{v}(sf(i-i,m) + sf(i-m,m))$ (by (2.1)). Since $0 < i \leq v < m$, this gives $sf(mv+r,m) = 1 + \sum_{i=1}^{v}(1+0) = 1+v$. If $v = m$, then $\sum_{i=1}^{v} sf(i,m) = \sum_{i=1}^{m-1} sf(i,m) + sf(m,m) = m-1+sf(1,m) = m-1+1 = m$; thus, $sf(mv+r) = v+1$ is true in this case. Finally, if $v = 0$, it is not difficult to see that $sf(r,m) = 1$. □

We note a few interesting special cases of Corollary 4.2.

COROLLARY 4.3. *For any integer $m \geq 2$:*

(i) $sf(m^i,m) = 1$ *for* $i \geq 0$;
(ii) $sf(m^i h,m) = 1$ *for* $1 \leq h \leq m-1$, $i \geq 0$;
(iii) *given an integer $n \geq 0$, then, for each $n \in \{0,1,\ldots,m\}$,*

$$sf(nm+1,m) = sf(nm+2,m) = \cdots = sf((n+1)m-1,m) = v+1.$$

PROOF. Part (i) is the case $h = 1$ of part (ii). Parts (ii) and (iii) are obtained by setting $v = 0$ and $j = 0$, respectively, in Corollary 4.2. □

Note that part (i) of Corollary 4.3 implies Corollary 1.3. Also, when $m = 2$, part (iii) gives just the three values $sf(1) = 1$, $sf(3) = 2$ and $sf(5) = 3$, the parities of which are consistent with Theorem 1.2. Part (iii) is a stronger version of Theorem 4.1 since the restriction of n to the set $\{0,1,\ldots,m\}$ specifies a common value.

THEOREM 4.4. *For any $j \geq 0$,*

$$\sum_{r=0}^{2j+1} sf(r,3) \equiv 0 \pmod 2.$$

Consequently,

$$sf(3j+4,3) = sf(3j+5,3) \equiv 0 \pmod 2, \quad \textit{where } j \equiv 0 \pmod 2, \tag{4.2}$$
$$sf(3^r j+4,3) = sf(3^r j+5,3) \equiv 0 \pmod 2 \quad \textit{for all } j \geq 0,\ r \geq 2. \tag{4.3}$$

PROOF. Note the identity

$$\frac{1}{1-q} = \prod_{n=0}^{\infty}(1 + q^{3^n} + q^{2\cdot 3^n}) \tag{4.4}$$

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and recall that

$$\begin{aligned}\sum_{n\geq 0} sf(n,3)q^n &= \prod_{n=0}^{\infty}\left(\frac{1+q^{3^n}+q^{2\cdot 3^n}-q^{3\cdot 3^n}}{1-q^{3\cdot 3^n}}\right)\\ &\equiv \prod_{n=0}^{\infty}\left(\frac{1+q^{3^n}+q^{2\cdot 3^n}+q^{3\cdot 3^n}}{1+q^{3\cdot 3^n}}\right) \pmod 2\\ &= \prod_{n=0}^{\infty}\frac{(1+q^{3^n})(1+q^{2\cdot 3^n})}{1+q^{3\cdot 3^n}}\\ &= \prod_{n=0}^{\infty}\left(\frac{1+q^{2\cdot 3^n}}{1+q^{3^n}+q^{2\cdot 3^n}}\right)\\ &= (1-q)\prod_{n=0}^{\infty}(1+q^{2\cdot 3^n}) \quad (\text{by } (4.4)).\end{aligned}$$

Thus,

$$\sum_{n\geq 0}\sum_{r=0}^{n} sf(r,3)q^n = \frac{1}{1-q}\sum_{n\geq 0} sf(n,3)q^n \equiv \prod_{n=0}^{\infty}(1+q^{2\cdot 3^n}) \pmod 2.$$

Since the series expansion of the right-hand side of the preceding equation has even exponents, the result follows.

To prove (4.2), we observe that

$$\begin{aligned}sf(3j+4,3) = sf(3(j+1)+1,3) &= sf(3(j+1)+2,3) \quad (\text{by Theorem } 4.1)\\ &= \sum_{r=0}^{j+1} sf(r,3) \quad (\text{by Theorem } 4.1)\\ &\equiv 0 \pmod 2 \quad (\text{since } j+1 \text{ is odd}).\end{aligned}$$

Furthermore, for (4.3), observe that

$$3^{r-1}j+1 \equiv \begin{cases} 0 & \text{if } j\equiv 1 \pmod 2,\\ 1 & \text{otherwise.}\end{cases}$$

Now, if j is odd, then

$$\begin{aligned}sf(3^r j+4,3) = sf(3(3^{r-1}j+1)+1,3) &= sf(3(3^{r-1}j+1)+2,3)\\ &= \sum_{r=0}^{3^{r-1}j+1} sf(r,3) \quad (\text{by Theorem } 4.1)\\ &= sf(3^{r-1}j+1,3) + \sum_{r=0}^{3^{r-1}j} sf(r,3)\end{aligned}$$

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$$\equiv sf(3^{r-1}j+1,3) \pmod 2 \quad (\text{since } 3^{r-1}j \text{ is odd})$$
$$= \sum_{r=0}^{3^{r-2}j} sf(r,3)$$
$$\equiv 0 \pmod 2 \quad (\text{since } 3^{r-2}j \text{ is odd}).$$

On the other hand, if j is even, we reach the result using (4.2). □

Theorem 4.5. *Let $k \equiv m + r \pmod{2m}$ and $k \le m^2 + r$ for $1 \le r \le m-1$. If $n \ge 0$, $m \ge 2$ and $n = m^i k$ for $i \ge 0$, then $sf(n,m)$ is even.*

Proof. Note that $k \equiv m + r \pmod{2m}$ and $k \le m^2 + r$ for $1 \le r \le m-1$ imply that $k = m(2t+1) + r \le m^2 + r$, for some positive integer t, and so $2t+1 \le m$. Then, from Corollary 4.2,

$$\begin{aligned} sf(m^i k, m) &= sf(m^i(m(2t+1)+r), m) = sf(m(2t+1)+r, m) \quad (\text{by } (2.1)) \\ &= 2t+1+1 \quad (\text{by Corollary 4.2 and since } 2t+1 \le m) \\ &= 2t+2. \end{aligned}$$
□

Remark 4.6. When $m = 3$, Theorem 4.5 reduces to Theorem 4.4 without the restriction $k \le m^2 + r$.

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