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A REMARK ON A THEOREM OF M. HALL

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Let $A=(a_{ij})$ be an $m\times n$ (0, 1)-matrix, $m\le n$. The *permanent* of A is defined by

$$\mathrm{Per}(A) = \sum_{\sigma}\prod_{i=1} a_{i\sigma(i)},$$

where the summation is over all one-one functions $\sigma:\{1, 2, \ldots, m\}\to\{1, 2, \ldots, n\}$. If A is regarded as the incidence matrix of a configuration of m subsets of an n-set, then Per(A) is the number of systems of distinct representatives in the configuration. A well-known lower bound for the number of systems of distinct representatives, due to M. Hall [1], can be expressed in terms of permanents as follows.

THEOREM 1. (Hall). *Let A be an $m\times n$ (0, 1)-matrix, $m\le n$. If* Per$(A)>0$ *and all row sums of A are greater than or equal to t, then*

$$\mathrm{Per}(A) \ge t! \qquad \text{if} \quad m \ge t, \tag{1}$$

$$\mathrm{Per}(A) \ge \frac{t!}{(t-m)!} \quad \text{if} \quad m \le t. \tag{2}$$

Inequality (2), which is an immediate consequence of (1), seems to have been stated explicitly for the first time by Mann and Ryser [2].

The purpose of this note is to show that, unlike (1), the inequality (2) does not require the additional hypothesis that Per$(A)>0$. In fact, we prove the following result.

THEOREM 2. *If A is an $m\times n$ (0, 1)-matrix, $m\le n$, and if each row sum of A is greater than or equal to m, then*

$$\mathrm{Per}(A) > 0.$$

We shall require a preliminary result which is a corollary to the following classical theorem.

THEOREM 3. (Frobenius-König [3]). *Let $A=(a_{ij})$ be an n-square (0, 1)-matrix. Then every diagonal $(a_{1\sigma(1)}, \ldots, a_{n\sigma(n)})$ of A contains a zero if and only if there exists an $s\times t$ zero submatrix of A with $s+t=n+1$.*

COROLLARY *If A is an $m\times n$ (0, 1)-matrix, $m\le n$, then* Per$(A)=0$, *if and only if A contains an $s\times t$ zero submatrix with $s+t=n+1$.*

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Proof of the Corollary. Let B be the n-square (0, 1)-matrix whose first m rows are those of A and each of the remaining rows (if any) is $(1, 1, \ldots, 1)$. Then, by Theorem 3, every diagonal of B contains a zero, if and only if B contains an $s \times t$ zero submatrix with $s+t=n+1$. But all the entries in the last $n-m$ rows of B are positive, and thus the zeros in each diagonal and the zero submatrix all belong to A. In other words, the sequence $(a_{1\sigma(1)}, \ldots, a_{m\sigma(m)})$ contains a zero for all one-one functions $\sigma:\{1, \ldots, m\} \rightarrow \{1, \ldots, n\}$, that is, $\operatorname{Per}(A)=0$, if and only if A contains an $s \times t$ submatrix with $s+t=n+1$. ∎

Proof of Theorem 2. If every row sum of A is greater than or equal to m, i.e. every row has at least m positive entries, then the number of zeros in any row cannot exceed $n-m$. Thus if A contains an $s \times t$ zero submatrix then $t \leq n-m$. But the total number of available rows is m. Therefore $s \leq m$ and

$$\begin{aligned} s+t &\leq m+(n-m) \\ &= n. \end{aligned}$$

Hence by the corollary, $\operatorname{Per}(A) \neq 0$. ∎

It follows from Theorem 2 that the condition $\operatorname{Per}(A)>0$ can be dropped from inequality (2). Similarly in other theorems on permanents of (0, 1)-matrices depending on the hypothesis $\operatorname{Per}(A)>0$ (see, e.g. [5] and [4], where, however, the results are stated in the language of sets), this condition can be dropped whenever all the row sums of A are greater than or equal to the number of rows in A.

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