Canad. Math. Bull. Vol. **53** (1), 2010 pp. 77–86
doi:10.4153/CMB-2010-017-8


# Constructing (Almost) Rigid Rings and a UFD Having Infinitely Generated Derksen and Makar-Limanov Invariants

David Finston and Stefan Maubach

*Abstract.* An example is given of a UFD which has an infinitely generated Derksen invariant. The ring is "almost rigid" meaning that the Derksen invariant is equal to the Makar-Limanov invariant. Techniques to show that a ring is (almost) rigid are discussed, among which is a generalization of Mason's ABC-theorem.

## 1 Introduction and tools

The Derksen invariant and Makar-Limanov invariant are useful tools for distinguishing nonisomorphic algebras. They have been applied extensively in the context of affine algebraic varieties. Both invariants rely on locally nilpotent derivations: for a commutative ring $R$ and a commutative $R$-algebra $A$, an $R$-linear mapping $D\colon A \to A$ is an $R$-derivation if $D$ satisfies the Leibniz rule: $D(ab) = aD(b) + bD(a)$. The derivation $D$ is locally nilpotent if for each $a \in A$ there is some $n \in \mathbb{N}$ such that $D^n(a) = 0$. When $k$ is a field of characteristic 0, a locally nilpotent $k$-derivation $D$ of the $k$-algebra $A$ gives rise to an algebraic action of the additive group of $k$, $G_a(k)$, on $A$ via

$$\exp(tD)(a) \equiv \sum_{i=0}^{\infty} \frac{t^i}{i!} d^i(a).$$

for $t \in k, a \in A$. Conversely, an algebraic action $\sigma$ of $G_a(k)$ on $A$ yields a locally nilpotent derivation via

$$\left.\frac{\sigma(t,a) - a}{t}\right|_{t=0}.$$

In this case, the kernel of $D$ denoted by $A^D$ coincides with the ring of $G_a(k)$ invariants in $A$.

The Makar-Limanov invariant of the $R$-algebra $A$, denoted $ML_R(A)$, is defined as the intersection of the kernels of all locally nilpotent $R$-derivations of $A$, while the Derksen invariant, $D_R(A)$ is defined as the smallest algebra containing the kernels of all nonzero locally nilpotent $R$-derivations of $A$. The subscript $R$ will be suppressed when it is clear from the context.

Received by the editors January 19, 2007; revised July 6, 2007.
Published electronically December 4, 2009.
The second author was funded by a Veni-grant from the Council for the Physical Sciences of the Netherlands Organisation for Scientific Research (NWO).
AMS subject classification: **14R20**, 13A50, 13N15.

In [9] the question was posed of whether the Derksen invariant of a finitely generated algebra over a field could be infinitely generated. In [14] an example was given of an infinitely generated Derksen invariant of a finitely generated $\mathbb{C}$-algebra. In fact, this example is of a form described in this paper as an "almost rigid ring", a ring for which the Derksen invariant is equal to the Makar-Limanov invariant. Despite its simplicity and the simplicity of the argument, this example has a significant drawback in that it is not a UFD. In this paper we provide a UFD example having infinitely generated invariants (it is again an almost rigid ring).

The paper is organized as follows. Section 1 consists of basic notions and examples associated with rigidity and almost rigidity. In Section 2, the focus is on rigid and almost rigid rings, with techniques to prove rigidity or almost rigidity. In Section 3, certain rings are shown to be UFDs, and these are used in Section 4 to give the UFD examples having infinitely generated Makar-Limanov and Derksen invariants.

***Notations*** If $R$ is a ring, then $R^{[n]}$ denotes the polynomial ring in $n$ variables over $R$ and $R^*$ denotes the group of units of $R$. The $R$ module of $R$-derivations of an $R$-algebra $A$ is denoted by $\mathrm{Der}_R(A)$ and the set of locally nilpotent $R$-derivations by $\mathrm{LND}_R(A)$ (the $R$ will be suppressed when it is clear from the context). We will use the letter $k$ for a field of characteristic zero, and $K$ for an algebraic closure. When $X$ is a variable in a polynomial ring or rational function field, the symbol $\partial_X$ denotes the derivative with respect to $X$. When the context is clear, $x, y, z, \ldots$ will represent residue classes of elements $X, Y, Z, \ldots$ modulo an ideal.

Let $A$ be an $R$-algebra which is an integral domain. Well-known facts that we need are included in the following.

***Lemma 1.1*** *Let $D \in \mathrm{LND}_R(A)$.*

(i) *Then $D(A^*) = 0$.*
(ii) *If $D(ab) = 0$ where $a, b$ are both nonzero, then $D(a) = D(b) = 0$.*
(iii) *If $\tilde{D} \in \mathrm{Der}_R(A)$ and $f \in A$ satisfy $f\tilde{D} \in \mathrm{LND}_R(A)$, then $\tilde{D} \in \mathrm{LND}_R(A)$ and $f \in A^{\tilde{D}}$.*

## 2 (Almost) Rigid Rings

As defined in [8, p. 196] and [2,3], a rigid ring is a ring which has no locally nilpotent derivations except the zero derivation. Examples include the rings

$$R := \mathbb{C}[X, Y, Z]/(X^a + Y^b + Z^c)$$

with $a, b, c \geq 2$ and pairwise relatively prime [6], and coordinate rings of Platonic $\mathbb{C}^*$ fiber spaces [13]. We define an almost rigid ring here as a ring whose set of locally nilpotent derivations is, in some sense, one-dimensional.

***Definition 2.1*** An $R$-algebra $A$ is called almost rigid if there is a nonzero $D \in \mathrm{LND}(A)$ such that $\mathrm{LND}(A) = A^D D$.

For a field $F$ any derivation $D$ of $F[X]$ has the form $D = f(X)\partial_X$. Thus the simplest almost rigid algebra is $F[X]$. Other examples include the algebras

$$\mathbb{C}[X, Y, Z, U, V]/(X^a + Y^b + Z^c, X^m V - Y^n U - 1)$$

with $a, b, c$ pairwise relatively prime given in [6] as counterexamples to a cancellation problem. Clearly an almost rigid algebra has its Derksen invariant equal to its Makar-Limanov invariant. The following lemma is useful in determining rigidity.

***Lemma 2.2*** *Let $D$ be a nonzero locally nilpotent derivation on a domain $A$ containing $\mathbb{Q}$. Then $A$ embeds into $K[S]$, where $K$ is some algebraically closed field of characteristic zero, in such a way that $D = \partial_S$ on $K[S]$.*

**Proof** The proof uses some well-known facts about locally nilpotent derivations. Since $D \neq 0$ is locally nilpotent, we can find an element $p$ such that $D^2(p) = 0$, $D(p) \neq 0$. Set $q := D(p)$ (and thus $q \in A^D$) and observe that $D$ extends uniquely to a locally nilpotent derivation $\tilde{D}$ of $\tilde{A} := A[q^{-1}]$. Since $\tilde{D}$ has the slice $s := p/q$ (a slice is an element $s$ such that $\tilde{D}(s) = 1$) we have $\tilde{A} = \tilde{A}^{\tilde{D}}[s]$ and $\tilde{D} = \partial_s$ (see [5, Proposition 1.3.21]). Denote by $k$ the quotient field of $\tilde{A}^{\partial/\partial s}$ (which equals the quotient field of $A^D$), noting that $D$ extends uniquely to $k[s]$. One can embed $k$ into its algebraic closure $K$, and the derivation $\partial_s$ on $K[s]$, restricted to $A \subseteq K[s]$, equals $D$. ∎

As an application, we have the following.

***Example 2.3*** Let $R := \mathbb{C}[x, y] = \mathbb{C}[X, Y]/(X^a + Y^b + 1)$ where $a, b \geq 2$. Then $R$ is rigid.

**Proof** Suppose $D \in \mathrm{LND}(R)$, $D \neq 0$. Using Lemma 2.2, we see $D$ as $\partial_S$ on $K[S] \supseteq R$. Now the following lemma (mini-Mason's) shows that $x, y$ must both be constant polynomials in $S$. But that means $D(x) = D(y) = 0$, so $D$ is the zero derivation, which is a contradiction. So the only derivation on $R$ is the zero derivation, *i.e.*, $R$ is rigid. ∎

Versions of the following lemma can be found as [8, Lemma 9.2] and [11, Lemma 2]. Here we give it the appellation "mini-Mason's" as it can be seen as a very special case of Mason's very useful original theorem. (Note that Mason's theorem is the case $n = 3$ of Theorem 2.5.)

***Lemma 2.4* (Mini-Mason)** *Let $f, g \in K[S]$ where $K$ is algebraically closed and of characteristic zero. Suppose that $f^a + g^b \in K^*$ where $a, b \geq 2$. Then $f, g \in K$.*

**Proof** Note that $\gcd(f, g) = 1$. Taking the derivative with respect to $S$ gives $af'^{a-1} = -bg'g^{b-1}$. So $f$ divides $gg'$, so $f$ divides $g'$. By the same reasoning, $g$ divides $f'$. This can only be if $f' = g' = 0$. ∎

Mason's theorem provides a very useful technique for constructing rigid rings (see [6] for an example). With appropriate care, a generalization of Mason's theorem provides more examples. In this paper, we will use [1, Theorem 3.1], which is a corollary of a generalization of Mason's theorem (see [1, Theorem 2.1]).

***Theorem 2.5*** *Let $f_1, f_2, \dots, f_n \in K[S]$, where $K$ is an algebraically closed field containing $\mathbb{Q}$. Assume $f_1^{d_1} + f_2^{d_2} + \cdots + f_n^{d_n} = 0$. Additionally, assume that for every $1 \le i_1 < i_2 < \cdots < i_s \le n$,*

$$f_{i_1}^{d_{i_1}} + f_{i_2}^{d_{i_2}} + \cdots + f_{i_s}^{d_{i_s}} = 0 \Longrightarrow \gcd\{f_{i_1}, f_{i_2}, \dots, f_{i_s}\} = 1.$$

*Then*

$$\sum_{i=1}^{n} \frac{1}{d_i} \le \frac{1}{n-2}$$

*implies that all $f_i$ are constant.*

***Example 2.6*** Let $R := \mathbb{C}[X_1, X_2, \dots, X_n]/(X_1^{d_1} + X_2^{d_2} + \cdots + X_n^{d_n})$ where $d_1^{-1} + d_2^{-1} + \cdots + d_n^{-1} \le \frac{1}{n-2}$. Then $R$ is a rigid ring.

The proof will follow from the more general lemma.

***Lemma 2.7*** *Let $A$ be a finitely generated $\mathbb{Q}$ domain. Consider a subset*

$$\mathcal{F} = \{F_1, F_2, \dots, F_m\}$$

*of $A$ and positive integers $d_1, \dots d_n$ satisfying:*

(i) *$P := F_1^{d_1} + F_2^{d_2} + \dots + F_m^{d_m}$ is a prime element of $A$.*

(ii) *No nontrivial subsum of $F_1^{d_1}, F_2^{d_2}, \dots, F_m^{d_m}$ lies in $(P)$.*

*Additionally, assume that $d_1^{-1} + d_2^{-1} + \cdots + d_n^{-1} \le \frac{1}{n-2}$. Set $R := A/(P)$ and let $D \in \mathrm{LND}(R)$. With $f_i \in R$ equal to the residue class of $F_i$, we have $D(f_i) = 0$ for all $1 \le i \le n$.*

**Proof** Suppose $D \in \mathrm{LND}(R)$, where $D \neq 0$. Using Lemma 2.2 with $K$ an algebraic closure of the quotient field of $R^D$, we realize $D$ as $\partial_S$ on $K[S] \supseteq R$. In particular, $f_1(S)^{d_1} + f_2(S)^{d_2} + \cdots + f_m(S)^{d_m} = 0$. By hypothesis there cannot be a subsum $f_{i_1}^{d_{i_1}} + f_{i_2}^{d_{i_2}} + \cdots + f_{i_s}^{d_{i_s}} = 0$. Applying Theorem 2.5, we find that all $f_i$ are constant. ■

This lemma also helps in constructing almost rigid rings not of the form $R^{[1]}$ with $R$ rigid.

***Example 2.8*** [14] Define

$$R := \mathbb{C}[a, b] = \mathbb{C}[A, B]/(A^3 - B^2) \quad \text{and} \quad S := R[X, Y, Z]/(Z^2 - a^2(aX + bY)^2 - 1).$$

Then $\mathrm{LND}(S) = S^D D$, where $D := b\partial_X - a\partial_Y$.

The following is an example of a rigid unique factorization domain. The proof of the UFD property is deferred to the next section.

***Example 2.9*** Let $n \ge 3$, and in $\mathbb{C}[X_1, X_2, \dots, X_n, Y_1, Y_2, \dots, Y_n]$ set

$$P := X_1^{d_1} + X_2^{d_2} + \cdots + X_n^{d_n} + L_2^{e_2} + L_3^{e_3} + \cdots + L_n^{e_n}$$

and

$$R := \mathbb{C}[X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_n]/(P),$$

where $L_i := X_iY_1 - X_1Y_i$ and $d_1, d_2, \ldots, d_n, e_2, e_3, \ldots, e_n$ are chosen so that

$$d_1^{-1} + d_2^{-1} + \cdots + d_n^{-1} + e_2^{-1} + e_3^{-1} + \cdots + e_n^{-1} \leq 1/(2n-3),$$

and the images in $R$ of the $X_i$ are prime elements, *e.g.*, the $d_i$ are relatively prime. Denote by $x_i, y_{i,} l_i$ the images of $X_i, Y_{i,} L_i$ in $R$. Then $R$ is an almost rigid UFD, and $\operatorname{LND}(R) = R^D D$ where $D(x_i) = 0, D(y_i) = x_i$.

**Proof** An elementary argument shows that $R$ is a domain. View

$$P \in \mathbb{C}[X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_{n-1}][Y_n].$$

The residue of $P$ modulo $(Y_1, Y_2, \ldots, Y_{n-1})$ has the same degree in $Y_n$ as $P$ and is clearly irreducible.

It is also elementary that $P$ does not divide any sum of a nonempty subset of $\{x_i^{d_i}, l_j^{ej} : 1 \leq i \leq n, 2 \leq j \leq n\}$. Suppose that $\sum_{i=1}^n \epsilon_i X_i^{d_i} + \sum_{j=2}^n \nu_j L_i^{e_i}$ is divisible by $P$ with $\epsilon_i, \nu_j \in \{0,1\}$ and not all $\epsilon_i, \nu_j = 0$. Lemma 2.7 yields that for any $E \in \operatorname{LND}(R)$ we have $E(x_i) = 0$, and $E(l_i) = 0$. So $x_1E(y_i) = x_iE(y_1)$. Since $R$ is a UFD, we can write $E(y_i) = \alpha x_i$ for some $\alpha \in R$. So $E = \alpha D$ where $D$ is as in the statement. ∎

## 3 Factoriality of Brieskorn–Catalan–Fermat Rings for $n \geq 5$

Because of their resemblance to rings arising in Fermat's last theorem, the Catalan conjecture, and to the coordinate rings of Brieskorn hypersurfaces, we will call the rings $\mathbb{C}[X_1, X_2, \ldots]/(X_1^{d_1} + X_2^{d_2} + \cdots + X_n^{d_n})$ Brieskorn–Catalan–Fermat (BCF) rings. Our examples depend on the factoriality of certain BCF rings. While the next observation is undoubtedly well known, a proof is included, since we could not find an explicit one in the literature.

***Theorem 3.1*** *If $n \geq 5$ and $d_i \geq 2$ for all $1 \leq i \leq n$, then*

$$\mathbb{C}[X_1, X_2, \ldots]/(X_1^{d_1} + X_2^{d_2} + \cdots + X_n^{d_n})$$

*is a UFD.*

The result follows from the next two theorems.

***Theorem 3.2*** ([7, Corollary 10.3]) *Let $A = A_0 + A_1 + \cdots$ be a graded noetherian Krull domain such that $A_0$ is a field. Let $\mathfrak{m} = A_1 + A_2 + \cdots$. Then $\operatorname{Cl}(A) \cong \operatorname{Cl}(A_{\mathfrak{m}})$, where $\operatorname{Cl}$ is the class group.*

***Theorem 3.3*** ([10]) *A local noetherian ring $(A, \mathfrak{m})$ with characteristic $A/\mathfrak{m} = 0$ and an isolated singularity is a UFD if its depth is $\geq 3$ and the embedding codimension is $\leq \dim(A) - 3$.*

https://doi.org/10.4153/CMB-2010-017-8 Published online by Cambridge University Press

**Proof of Theorem 3.1** Write

$$A = \mathbb{C}[x_1, x_2, \dots, x_n] = \mathbb{C}[X_1, X_2, \dots]/(X_1^{d_1} + X_2^{d_2} + \cdots + X_n^{d_n}).$$

Since the Krull dimension of $A$ is at least 4 and **Spec** $A$ has only the origin as a singularity, $A_p$ is regular for every height one prime ideal $p$. In addition **Spec** $A$ is an affine hypersurface, so is Cohen Macaulay. In particular, $A$ satisfies Serre's $R_1$ and $S_2$ conditions at each maximal ideal, so that $A$ is integrally closed, *i.e.*, a Krull ring.

Note that by giving appropriate positive weights to the $X_i$, the ring $A$ is graded, and $\mathfrak{m} := A_1 + A_2 + \cdots = (x_1, x_2, \dots, x_n), A_0 = \mathbb{C}$. Now $A$ satisfies the requirements of 3.2, so it is sufficient to show that $A_\mathfrak{m}$ is a UFD (note that "$A$ ia a UFD" is equivalent to "$\mathrm{Cl}(A) = \{0\}$"). As noted above, the ring $A$ is defined by one homogeneous equation, so that $A_\mathfrak{m}$ is a complete intersection, hence Cohen–Macauley, and the depth of $A_\mathfrak{m}$ is equal to its Krull dimension. Thus the depth of $A_\mathfrak{m}$ is $n-1$, which is $\geq 3$ since $n \geq 5$. Now, one can see $A_\mathfrak{m}$ as a quotient of the polynomial ring localized at the maximal ideal $(X_1, X_2, \dots, X_n)$. Clearly $A_\mathfrak{m}$ has embedding codimension 1. Now $\dim(A) - 3 = n - 4$, so, if $n \geq 5$, we have that the embedding codimension of $A$ equals $1 \leq \dim(A) - 3$. Finally, since $n \geq 5$, the criteria of Theorem 3.3 are met, and $A_\mathfrak{m}$ is a UFD. ■

The following lemma of Nagata is a very useful tool for proving factoriality.

***Lemma 3.4*** **(Nagata)** *Let $A$ be a domain, and $x \in A$ is prime. If $A[x^{-1}]$ is a UFD, then $A$ is a UFD.*

***Lemma 3.5*** *$R$ as in Example* 2.9 *is a UFD.*

**Proof** Note that with our hypotheses on the exponents,

$$X_2^{d_2} + X_3^{d_3} + \cdots + X_n^{d_n} + (X_2Y_1)^{e_2} + (X_3Y_1)^{e_3} + \cdots + (X_nY_1)^{e_n}$$

is irreducible, so $R/(x_1)$ is a domain. Using Lemma 3.4, it is enough to show that $R[x_1^{-1}]$ is a UFD. Define $m_i := y_i - \frac{x_i}{x_1} y_1$ for $2 \leq i \leq n$, and

$$S := \mathbb{C}[x_1, x_2, \dots, x_n, m_2, m_3, \dots, m_n].$$

Then $R[x^{-1}] = S[x_1^{-1}][Y_1]$, where $Y_1$ is algebraically independent over $S[x_1^{-1}]$. It is now enough to prove that $S$ is a UFD. But this follows from Theorem 3.1, since $n \geq 3$. ■

# 4 A UFD Having Infinitely Generated Invariants

## 4.1 Definitions

***Definition 4.1*** In $\mathbb{C}^{[7]} = \mathbb{C}[X, Y, Z, S, T, U, V]$, set

$$L_1 := Y^3S - X^3T, \quad L_2 := Z^3S - X^3U,$$

$$L_3 := Y^2Z^2S - XV, \quad P := X^{d_1} + Y^{d_2} + Z^{d_3} + L_1^{d_4} + L_2^{d_5} + L_3^{d_6},$$

https://doi.org/10.4153/CMB-2010-017-8 Published online by Cambridge University Press

where the $d_i \geq 2$ are integers. Define

$$A := \mathbb{C}[x, y, z, s, t, u, v] = \mathbb{C}[X, Y, Z, S, T, U, V]/(P),$$

and let $R$ be the subring $\mathbb{C}[x, y, z]$.

The elements $s, t, u, v$ in $A$ form a regular sequence; in particular they are algebraically independent.

***Definition 4.2*** $E := X^3\partial_S + Y^3\partial_T + Z^3\partial_U + X^2Y^2Z^2\partial_V$. Note that $E$ is locally nilpotent and $P \in \ker(E)$. Thus $E$ induces a well-defined element of $\mathrm{LND}(A)$ denoted by $D$.

## 4.2 The Factoriality of $A$

For a 6-tuple of positive integers $\mathbf{d} = (d_1, d_2, \ldots, d_6)$ as above in the definition of $A$, define

$$Q(\mathbf{d}) := Y^{d_2} + Z^{d_3} + (Y^3S)^{d_4} + (Z^3S)^{d_5} + (Y^2Z^2S)^{d_6}.$$

***Proposition 4.3*** *If $Q(\mathbf{d})$ is irreducible in $\mathbb{C}[Y, Z, S]$, then $A$ is a UFD.*

**Proof** Assume that $Q(\mathbf{d})$ is irreducible. Note that

$$A/(x) \cong \mathbb{C}[Y, Z, S, T, U, V]/(Q(\mathbf{d}))$$

so that $x$ is prime. By Lemma 3.4, it is enough to show that $A[x^{-1}]$ is a UFD. Now define

$$M_1 := T - \frac{Y^3}{X^3}S, \quad M_2 := U - \frac{Z^3}{X^3}S, \quad M_3 := V - \frac{Y^2Z^2}{X}S.$$

Write $m_i$ for the image of $M_i$ in $A[x^{-1}]$, and let $B = \mathbb{C}[x, y, z, m_1, m_2, m_3][x^{-1}]$. Since $D(s) = x^3$, $s/x^3$ is a slice for the extension of $D$ to $A[x^{-1}] = B[s]$, with $s$ transcendental over $B$. Consider

$$C := \mathbb{C}[X, Y, Z, M_1, M_2, M_3]/(X^{d_1} + Y^{d_2} + Z^{d_3} + M_1^{d_4} + M_2^{d_5} + M_3^{d_6}).$$

This ring is a UFD by Theorem 3.1, so $C[x^{-1}] = B$ is also a UFD, from which we deduce that $B[s] = A[x^{-1}]$ is a UFD. ■

The polynomial $Q(\mathbf{d})$ is irreducible for infinitely many positive integer choices of the $d_i$; take, for example, $\gcd(d_2, d_3) = 1$ and $d_2 \geq \max(3d_4, 2d_6)$.

## 4.3 $A$ Is Not Finitely Generated

In this section, we assume that $d_1, \ldots, d_6$ are such that $Q(\mathbf{d})$ is irreducible, *i.e.*, $A$ is a UFD, and such that $d_1^{-1} + d_2^{-1} + \cdots + d_6^{-1} \leq \frac{1}{4}$ (note that by necessity $d_1, d_2, d_3 \geq 4$). The following lemma shows that $A$ is an almost rigid ring.

***Lemma 4.4*** *Any locally nilpotent derivation on $A$ is a multiple of $D$.*

https://doi.org/10.4153/CMB-2010-017-8 Published online by Cambridge University Press

**Proof** Let $\triangle$ be a nonzero LND on $A$. By Lemma 2.7, since we assumed $\sum_{i=1}^{6} d_i \leq \frac{1}{4}$, we see that $x, y, z, l_1, l_2, l_3$ must be in $A^{\triangle}$. So $\triangle(l_1) = 0$, therefore $x^3\triangle(t) = y^3\triangle(s)$, and thus $\triangle(S) = x^3\alpha$ for some $\alpha \in A$ (since $A$ is a UFD). Using $\triangle(l_1) = \triangle(l_2) = \triangle(l_3) = 0$, this yields $\triangle(T) = y^3\alpha$, $\triangle(U) = z^3\alpha$, $\triangle(V) = x^2y^2z^2\alpha$, *i.e.*, $\triangle = \alpha D$. ∎

***Lemma 4.5*** $A^D \subseteq (x, y, z)A + R$.

The proof of this lemma is the technical heart of this paper, as the reader will no doubt notice.

**Proof** Let

$$\mathcal{J} := (X^3, Y^3, Z^3, X^2Y^2Z^2)(X, Y, Z)\mathbb{C}^{[7]},$$

$$H := (x, y, z)A \supseteq J := (x^3, y^3, z^3, x^2y^2z^2)H.$$

Both $J$ and $H$ are $D$-stable ideals of $A$. Denote by $\overline{D}$ the locally nilpotent derivation induced by $D$ on $\overline{A} := A/J$, $\overline{H} := H/J$, and $\overline{R}$ the image of $R$ in $\overline{A}$. Note that $\overline{A}^{\overline{D}} \supseteq (\overline{R} + \overline{H})$. We will prove $\overline{A}^{\overline{D}} \subseteq (\overline{R} + \overline{H})$ (*i.e.*, $\overline{A}^{\overline{D}} = \overline{R} + \overline{H}$). This will imply that $A^D + J \subseteq H + J + R$, and the required result then follows since $J \subseteq H$.

To that end assume there exists $h \in \overline{A}^{\overline{D}}$ with $h \notin \overline{H} + \overline{R}$. Once we arrive at a contradiction, we are done. Note that since $P \in \mathcal{J}$, we have

$$\overline{A} \cong (\mathbb{C}^{[7]}/(P))/(\mathcal{J}/(P)) \cong \mathbb{C}^{[7]}/\mathcal{J}$$
$$\cong \overline{R}[S, T, U, V],$$

the latter a polynomial ring over $\overline{R}$. With $\bar{x}, \bar{y}, \bar{z}, \bar{s}, \bar{t}, \bar{u}, \bar{v}$ denoting as usual the images of $X, Y, Z, S, T, U, V$ in $\overline{A}$, write $\overline{A} = \overline{R}[\bar{s}, \bar{t}, \bar{u}, \bar{v}]$, a polynomial ring over $\overline{R}$. If $a, b, c, d \in \mathbb{N}$, write $\mathcal{T}_{(a,b,c,d)} := s^a t^b u^c v^d$. Then $\overline{A}$ is a free $\overline{R}$-module with the basis $\{\mathcal{T}_{(a,b,c,d)} \mid a, b, c, d \in \mathbb{N}\}$. We can write

$$h = \sum_{(a,b,c,d)\in\mathbb{N}^4} r_{(a,b,c,d)}\mathcal{T}_{(a,b,c,d)},$$

where the $r_{(a,b,c,d)} \in \overline{R}$. We refer to $\mathcal{T}_{(a,b,c,d)}$ as a term and $r\mathcal{T}_{(a,b,c,d)}$ as a monomial where $r \in \bar{R}$. For $h$ written as above, we say that a term $\mathcal{T}_{(a,b,c,d)}$ appears or is contained in $h$ if $r_{(a,b,c,d)} \neq 0$. In case $r_{(a,b,c,d)} \in \overline{H}$, then $D(r_{(a,b,c,d)}\mathcal{T}_{(a,b,c,d)}) = 0$ and we can replace $h$ by $h - r_{(a,b,c,d)}\mathcal{T}_{(a,b,c,d)}$. We can also replace $h$ by $h - r_{(0,0,0,0)}$. Using this argument, we can assume that $h$ satisfies $r_{(a,b,c,d)} = 0$ or $r_{(a,b,c,d)} \notin \overline{H}$ for each $(a, b, c, d) \in \mathbb{N}^4$, and $r_{(0,0,0,0)} = 0$. Because of the assumption that $h \notin \overline{R} + \overline{H}$, we still have $h \neq 0$.

Now we will define a degree on $\overline{A}$: assign degree 0 to elements of $\overline{R}$, and order the terms $\mathcal{T}_{(a,b,c,d)}$ with the graded lexicographic ordering:

$$\mathcal{T}_{(a,b,c,d)} < \mathcal{T}_{(\alpha,\beta,\gamma,\delta)} \iff \begin{cases} a+b+c+d < \alpha+\beta+\gamma+\delta; \\ a+b+c+d = \alpha+\beta+\gamma+\delta,\ d < \delta; \\ a+b+c+d = \alpha+\beta+\gamma+\delta,\ d = \delta,\ c < \gamma; \\ a+b+c+d = \alpha+\beta+\gamma+\delta,\ d = \delta,\ c = \gamma,\ b < \beta. \end{cases}$$

By assumption there exists a term $\mathcal{T}_{(a,b,c,d)}$ of lowest order appearing in $h$ with $r_{(a,b,c,d)} \notin \overline{H} + \overline{R}$, and we have $r_{(a,b,c,d)} \notin \overline{H} + \overline{R}$. Note that $(a, b, c, d) \neq (0, 0, 0, 0)$. Say $M := r\mathcal{T}_{(a,b,c,d)}$, where $r = r_{(a,b,c,d)} \in \overline{R}\backslash\overline{H}$.

Let us assume $d > 0$. Consider $\overline{D}(M)$. It has the nonzero monomial summand $d\overline{x}^2\overline{y}^2\overline{z}^2 r\mathcal{T}_{(a,b,c,d-1)}$. Our claim is that then this monomial must appear in $\overline{D}(h)$. Let us explain this claim: $\overline{D}(\mathcal{T}_{(\alpha,\beta,\gamma,\delta)})$ contains at most the terms $\mathcal{T}_{(\alpha-1,\beta,\gamma,\delta)}$, $\mathcal{T}_{(\alpha,\beta-1,\gamma,\delta)}$, $\mathcal{T}_{(\alpha,\beta,\gamma-1,\delta)}$, and $\mathcal{T}_{(\alpha,\beta,\gamma,\delta-1)}$. If $\mathcal{T}_{(\alpha,\beta,\gamma,\delta)}$ is a term of higher degree than $\mathcal{T}_{(a,b,c,d)}$, then $\overline{D}(\mathcal{T}_{(\alpha,\beta,\gamma,\delta)})$ contains only monomials which have higher degree than $\mathcal{T}_{(a,b,c,d-1)}$. So, the nonzero monomial $d\overline{x}^2\overline{y}^2\overline{z}^2 r\mathcal{T}_{(a,b,c,d-1)}$ is not cancelled out, and thus $\overline{D}(h) \neq 0$, which is a contradiction.

Let us skip the cases $d = 0, c > 0$ and $d = 0, c = 0, b > 0$, as they will be slightly simpler than the next case and not essentially different.

Let us assume $b = c = d = 0, a > 0$. Now $\overline{D}(M)$ contains the monomial $a\overline{x}^3 r\mathcal{T}_{(a-1,0,0,0)}$. This monomial can appear in $\overline{D}(r_{(\alpha,\beta,\gamma,\delta)}\mathcal{T}_{(\alpha,\beta,\gamma,\delta)})$ if $\alpha + \beta + \gamma + \delta = a + b + c + d = a$. In fact, since $\overline{D}(h) = 0$ and looking at the immediate successor terms of $\mathcal{T}_{(a-1,0,0,0)}$ with respect to our ordering, we do a calculation and see that

$$a\overline{x}^3 r + \overline{y}^3 r_{(a-1,1,0,0)} + \overline{z}^3 r_{(a-1,0,1,0)} + \overline{x}^2\overline{y}^2\overline{z}^2 r_{(a-1,0,0,1)} = 0.$$

But since $r \notin \overline{H}$, this leads to a contradiction (as all terms are in the ideal $(\overline{y}, \overline{z})$ except $a\overline{x}^3 r$ which is not).

The case that $a = b = c = d = 0$ is already excluded, so in all cases we have a contradiction, and we are done. (The assumption that $h \notin \overline{H} + \overline{R}$, was wrong, and thus $h \in \overline{H} + \overline{R}$ as claimed.) ■

***Lemma 4.6*** *For each $n \in \mathbb{N}$, there exists $F_n \in A^D$ which satisfies $F_n = xV^n + f_n$ where $f_n \in \sum_{i=0}^{n-1} R[s, t, u]v^i \subset A$.*

**Proof** It is shown in several places (see [4, 12], or [5, p. 231]) that already on $\mathbb{C}^{[7]}$ there exist such $\tilde{F}_n$ that are in the kernel of the derivation $E$ (they are key to the proof that the kernel of $E$ is not finitely generated as a $\mathbb{C}$-algebra, and therefore yields a counterexample to Hilbert's 14th problem). By taking for $F_n$ the image of $\tilde{F}_n$ in $A$, we obtain the desired kernel elements. ■

***Corollary 4.7*** *$A^D$ is not finitely generated as a $\mathbb{C}$-algebra.*

**Proof** Suppose $A^D = R[g_1, \dots, g_s]$ for some $g_i \in A$. Since $A^D \subseteq R + (x, y, z)$ by Lemma 4.5, we can assume that all $g_i \in (x, y, z)$. Let $\mathcal{F}_n(A) := \sum_{i=0}^{n-1} R[S, T, U]V^i$, which is a subset of $A$. Choose $n$ such that $g_i \in \mathcal{F}_n(A)$ for all $1 \leq i \leq s$. Now $F_n \in \mathcal{F}_n(A) \cap A^D$. Then $F_n = P(g_1, \dots, g_s)$ for some $P \in R^{[s]}$. Compute modulo $(x, y, z)^2$. Since each $g_i \in (x, y, z)$, we have

$$P(g_1, \dots, g_n) \equiv r_1 g_1 + \cdots + r_n g_n \bmod (x, y, z)^2$$

for some $r_i \in R$. So $F_n \in Rg_1 + \cdots + Rg_n + (x, y, z)^2$. In particular,

$$F_n \in \mathcal{F}_n(A) + (x, y, z)^2.$$

https://doi.org/10.4153/CMB-2010-017-8 Published online by Cambridge University Press

Notice that $F_n - xV^n \in \mathcal{F}_n(A) \subseteq \mathcal{F}_n(A) + (x,y,z)^2$, so that $xV^n \in \mathcal{F}_n(A) + (x,y,z)^2$. But this is obviously not the case, contradicting the assumption that $A^D = R[g_1,\dots,g_s]$ for some $g_i \notin R$. Thus $A^D$ is not finitely generated as an $R$-algebra, *a fortiori* as a $\mathbb{C}$-algebra. ■

Using Lemma 4.4 we know that there is only one kernel of a nontrivial LND on $A$, so the following result is obvious.

***Corollary 4.8*** *$ML(A) = \mathrm{Der}(A) = A^D$ is not finitely generated.*

## References


[1] M. de Bondt, *Another generalisation of Mason's ABC-theorem.* arXiv:math.GM0707.0434v2.

[2] A. Crachiola, *On the AK-Invariant of Certain Domains.* Ph.D. thesis, Wayne State University, May 2004.

[3] A. Crachiola and L. Makar-Limanov, *On the rigidity of small domains.* J. Algebra **284**(2005), no. 1, 1–12. doi:10.1016/j.jalgebra.2004.09.015

[4] J. K. Deveney and D. R. Finston, *$G_a$-actions on $\mathbb{C}^3$ and $\mathbb{C}^7$.* Comm. Algebra **22**(1994), 6295–6302. doi:10.1080/00927879408825190

[5] A. van den Essen, *Polynomial Automorphisms and the Jacobian Conjecture.* Progress in Mathematics 190. Birkhäuser-Verlag, Basel, 2000.

[6] D. Finston and S. Maubach, *The automorphism group of certain factorial threefolds and a cancellation problem.* Israel J. Math. **163**(2008), 369–381. doi:10.1007/s11856-008-0016-3

[7] R. Fossum, *The Divisor Class Group of a Krull Domain.* Ergebniss der Mathematik und ihrer Grenzgebiete 74. Springer-Verlag, New York, 1973.

[8] G. Freudenburg, *Algebraic Theory of Locally Nilpotent Derivations*, Encyclopaedia of Mathematical Sciences 136. Springer-Verlag, Berlin, 2006.

[9] R V. Gurjar, K. Masuda, M. Miyanishi, and P. Russell, *Affine lines on affine surfaces and the Makar-Limanov invariant.* Canad. J. Math. **60**(2008), no. 1, 109–139. doi:10.4153/CJM-2008-005-8

[10] R. Hartshorne and A. Ogus, *On the factoriality of local rings of small embedding codimension.* Comm. Algebra **1**(1974), 415–437. doi:10.1080/00927877408548627

[11] L. Makar-Limanov, *Again $x + x^2y + z^2 + t^3 = 0$.* In: Affine Algebraic Geometry. Contemp. Math. 369. American Mathematical Society, Providence, RI, 2005, pp. 177–182.

[12] S. Maubach, *Hilbert's 14th Problem and Related Topics.* Master's thesis, University of Nijmegen, 1998.

[13] K. Masuda and M. Miyanishi, *Étale endomorphisms of algebraic surfaces with $\mathbf{G}_m$ actions.* Math. Ann. **319**(2001) no. 3, 493–516. doi:10.1007/PL00004445

[14] S. Maubach, *Infinitely generated Derksen and Makar-Limanov invariant.* Osaka J. Math. **44**(2007), no. 4, 883–886.



*Department of Mathematics, New Mexico State University, Las Cruces, NM, USA*
*e-mail*: dfinston@nmsu.edu

*Department of Mathematics, Radboud University Nijmegen, Nijmegen, The Netherlands*
*e-mail*: s.maubach@math.ru.nl