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Partial regularity of strong local minimizers of quasiconvex integrals with (p, q)-growth

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We consider strictly quasiconvex integrals

$$F[u] := \int_\Omega f(Du)\,\mathrm{d}x \quad \text{for } u : \mathbb{R}^n \supset \Omega \to \mathbb{R}^N$$

in the multi-dimensional calculus of variations. For the C^2-integrand $f : \mathbb{R}^{Nn} \to \mathbb{R}$ we impose (p, q)-growth conditions

$$\gamma|\xi|^p \leqslant f(\xi) \leqslant \Gamma(1 + |\xi|^q) \quad \text{for all } \xi \in \mathbb{R}^{Nn}$$

with γ, $\Gamma > 0$ and $1 < p \leqslant q < \min\{p + 1/n, p(2n-1)/(2n-2)\}$. Under these assumptions we prove partial $C^{1,\alpha}_{\mathrm{loc}}$-regularity for strong local minimizers of F and the associated relaxed functional $\mathcal{F}$.

1. Introduction

In this paper we investigate the regularity of strong local minimizers of autonomous variational integrals

$$F[u] := \int_\Omega f(Du)\,\mathrm{d}x \tag{1.1}$$

defined on vector-valued maps $u : \Omega \to \mathbb{R}^N$, $N \geqslant 1$. Here Ω denotes a bounded open set in $\mathbb{R}^n$, $n \geqslant 2$, and $f : \mathbb{R}^{Nn} \to \mathbb{R}$ is a C^2-function satisfying suitable assumptions described below.

The existence and the partial regularity of minimizers of F are classical issues in the modern calculus of variations, and they have been studied extensively, especially over the last 20 years. Specifically, we will focus here on gradient regularity under the basic assumption that f is quasiconvex; that is

$$⨍_B f(\xi + D\varphi)\,\mathrm{d}x \geqslant f(\xi)$$

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holds for all $\xi \in \mathbb{R}^{Nn}$ and all $\varphi \in C_c^\infty(B;\mathbb{R}^N)$, where B denotes the unit ball in $\mathbb{R}^n$. Quasiconvexity, introduced by Morrey in his seminal paper [39], generalizes the classical convexity assumption in the calculus of variations and has turned out to be a key concept for both the existence and the partial regularity of minimizers. In addition, the central role of quasiconvexity in nonlinear elasticity was pointed out in the fundamental work of Ball [5].

Before presenting our theorems, let us briefly describe some previous existence and regularity results. Primarily, imposing the standard growth conditions

$$\gamma|\xi|^p \leqslant f(\xi) \leqslant \Gamma(1+|\xi|^p) \tag{1.2}$$

for some $p > 1$, Morrey [39] proved that quasiconvexity is a necessary and sufficient condition for lower semicontinuity of F with respect to weak $W^{1,p}$-convergence (see also [1,30,33,37,38]). This semicontinuity property is, in turn, via the direct method of calculus of variations, intimately linked to the existence of minimizers of F.

As for regularity, classical examples of minimizers with singularities of the gradient can be constructed [15, 41, 49, 50], even for smooth convex functionals and $n = 3$, showing that in the vectorial case everywhere regularity of minimizers in the interior of Ω does not hold. Therefore, one is led to consider partial regularity, i.e. regularity outside a negligible closed subset of Ω, called the singular set. For quasiconvex functionals and $p \geqslant 2$, partial $C^{1,\alpha}_{\mathrm{loc}}$-regularity of minimizers was first shown by Evans [21] (see [2, 4, 16, 23, 25, 31] for extensions and variants). Partial regularity in the subquadratic case $1 < p < 2$ was eventually proved in [13] (see also [3, 17, 48]).

Contrary to convex functionals, quasiconvex functionals may, in general, admit non-trivial critical points, i.e. weak solutions of the Euler equation which are not (absolutely) minimizing. Actually, Müller and Sverák [40] have even constructed examples of critical points which are nowhere C^1. This result is in sharp contrast to the partial regularity of minimizers and leads to the investigation of an intermediate notion, namely local minimizers of F in the sense of the following definition.

Definition 1.1 ($W^{1,\bar q}$ local minimizer [32]). Let $1 \leqslant p \leqslant \infty$ and $1 \leqslant \bar q \leqslant \infty$. A map $\bar u \in W^{1,p}(\Omega,\mathbb{R})$ with $F[\bar u] < \infty$ is called a $W^{1,\bar q}$ local minimizer of F if there exists some $\delta > 0$ such that

$$F[\bar u] \leqslant F[\bar u + \varphi] \quad \text{holds for all } \varphi \in W_0^{1,\bar q}(\Omega,\mathbb{R}^N)$$

with $\|D\varphi\|_{L^{\bar q}(\Omega,\mathbb{R}^{Nn})} \leqslant \delta$. In particular, we call $\bar u$ a strong local minimizer for $1 \leqslant \bar q < \infty$ and a weak local minimizer for $\bar q = \infty$.

Having introduced this notion, it is natural to ask whether non-trivial local minimizers of F exist and if they are still regular or not. Actually, the investigation of the existence (and non-existence) of local minimizers has followed previous developments [26, 27] for critical points, and has, until now, focused on the case of L^1 local minimizers with affine boundary data. For instance, if the underlying domain Ω is an annulus in $\mathbb{R}^2$, there exist non-trivial L^1 local minimizers [32, 44], while for a star-shaped Ω every L^1 local minimizer is already absolutely minimizing [51]. In fact, generalizing these ideas, Taheri [52, 53] has provided multiplicity bounds for local minimizers in terms of topological invariants of Ω.

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Clearly, the examples of L^1 local minimizers are also $W^{1,\bar{q}}$ local minimizers for every $1 \leqslant \bar{q} \leqslant \infty$. However, in the light of [32, § 2] it would be interesting to discuss whether non-trivial examples of $W^{1,\bar{q}}$ local minimizers still exist in the simple case that Ω is a ball. Moreover, in view of remark 2.5 they should ideally possess the additional feature that they are not $W^{1,p}$ local minimizers. Indeed, for $\bar{q} > p$ we are not aware of any theoretical obstruction, but no such examples seem to be present in the literature.

Assuming standard growth (1.2), the regularity theory for $W^{1,\bar{q}}$ local minimizers has been started in [32]. Let us restate this result.

THEOREM 1.2 (Kristensen and Taheri [32]). *Let $2 \leqslant p < \infty$, $1 \leqslant \bar{q} < \infty$. Assume that $f \in C^2(\mathbb{R}^{Nn})$ is uniformly strictly quasiconvex with (1.2) and that*

$$\bar{u} \in W^{1,\bar{q}}_{\mathrm{loc}}(\Omega, \mathbb{R}^N) \cap W^{1,p}(\Omega, \mathbb{R}^N)$$

is a $W^{1,\bar{q}}$ local minimizer of F. Then there exists an open set $\Omega_0 \subset \Omega$ with $|\Omega \backslash \Omega_0| = 0$ such that $\bar{u} \in C^{1,\alpha}_{\mathrm{loc}}(\Omega_0, \mathbb{R}^N)$ for every $0 < \alpha < 1$.

A similar result for weak local minimizers is given in [32] and an analogous statement in the subquadratic case $1 < p < 2$ was established in [12]. We stress that in the light of the counter-examples from [40] these theorems treat a borderline case of regularity. Finally, we mention that it would be desirable to remove the technical assumption $\bar{u} \in W^{1,\bar{q}}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ in theorem 1.2. However, at present it seems quite difficult to achieve this.

Next we turn to a generalization of (1.2). Actually, starting with a series of papers by Marcellini (see, for example, [35, 36]) an increasing interest in more flexible growth conditions than (1.2) has emerged. In this paper, we concentrate on the (p, q)-growth conditions

$$\gamma|\xi|^p \leqslant f(\xi) \leqslant \Gamma(1 + |\xi|^q) \tag{1.3}$$

with two growth exponents $1 < p \leqslant q < \infty$. In the general vectorial setting the regularity theory for (1.3) was started in [42]. Assuming that f is strictly convex and that $q < \min\{p + 1, pn/(n - 1)\}$ the authors showed partial $C^{1,\alpha}_{\mathrm{loc}}$-regularity of minimizers of F. Subsequently, considering less restrictive conditions on the growth exponents, higher integrability results for the gradient of minimizers have been given in [18, 19], and partial regularity has been established in [9]. Here, the most general condition on the exponents, appearing in [9, 18], reads $q < p(n + 2)/n$. For results concerning non-autonomous functionals we refer the reader to [8, 10, 14, 20].

The quasiconvex case is more recent. In [7, 22, 28, 29, 34, 43] the semicontinuity properties in $W^{1,p}(\Omega, \mathbb{R}^N)$ of quasiconvex functionals satisfying (1.3) have been investigated. For our approach the following notion from [7] has turned out to be crucial.

DEFINITION 1.3 ($W^{1,p}$-quasiconvexity [7]). We say that f is $W^{1,p}$-quasiconvex if and only if

$$\fint_B f(\xi + D\varphi)\,\mathrm{d}x \geqslant f(\xi)$$

holds for all $\xi \in \mathbb{R}^{Nn}$ and all $\varphi \in W^{1,p}_0(B; \mathbb{R}^N)$.

Relying on [7, 22], it has been shown in [46] that (strict) $W^{1,p}$-quasiconvexity, together with (1.3) and some restrictions on the exponents p and q, allows one to establish both existence and partial regularity of absolute minimizers of F. Precisely, the restriction on the exponents reads $q < np/(n-1)$ for the existence and

$$1 < p \leqslant q < p + \frac{\min\{2, p\}}{2n} \tag{1.4}$$

for the regularity. We refer the reader to [45] for similar results in the higher-order case.

For $n = N$, an important class of examples is given by the polyconvex integrands

$$f(\xi) = (1 + |\xi|^2)^{p/2} + h(\det \xi), \tag{1.5}$$

where h is a convex function of growth rate q/n. These integrands are of some interest in nonlinear elasticity, as pointed out in [5–7, 34]. Moreover, we recall from [7] that f from (1.5) is $W^{1,p}$-quasiconvex if and only if $p \geqslant n$ holds. Thus, in this case, the above existence and regularity results apply. Let us mention, at this stage, that polyconvex integrands with a structure related to the one in (1.5) and $p > n-1$, but with a completely different growth behaviour, have previously been treated in [24] by means of more specific methods taking into account the peculiar nature of the functional.

In the case when $p < n$ the integrands (1.5) are not $W^{1,p}$-quasiconvex and the above-mentioned results do not apply. However, this case, in which F can potentially admit discontinuous minimizers, is of particular physical interest. To extend the existence and regularity results, a relaxation method, which is closely related to the classical idea of the Lebesgue–Serrin extension, was introduced in [11, 22, 34, 47]. Precisely, we consider the relaxed functional

$$\mathcal{F}[u] := \inf\left\{ \liminf_{k\to\infty} F[u_k] : W^{1,q}_{\mathrm{loc}}(\Omega, \mathbb{R}^N) \ni u_k \rightharpoonup u \text{ weakly in } W^{1,p}(\Omega, \mathbb{R}^N) \right\} \tag{1.6}$$

for $u \in W^{1,p}(\Omega, \mathbb{R}^N)$. It is not difficult to see from this definition that the minimum of $\mathcal{F}$ is attained on every Dirichlet class [47]. Furthermore, invoking representation results from [11, 22], the approach of [46] has been carried over to minimizers of $\mathcal{F}$. Precisely, partial regularity of minimizers of $\mathcal{F}$ has been established in [47], assuming that f is strictly quasiconvex with (1.3) and (1.4).

The aim of the present paper is now to examine the regularity properties of local minimizers of quasiconvex functionals satisfying (p, q)-growth conditions. Clearly, our main interest remains the model case (1.5), for which examples of L^1 local minimizers have been provided in [53, § 4]. Precisely, our results are the following. If f is strictly $W^{1,\bar{q}}$-quasiconvex, we prove partial $C^{1,\alpha}_{\mathrm{loc}}$-regularity for $W^{1,\bar{q}}$ local minimizers $\bar{u}$ of F (see theorem 2.1). In addition, if f is only strictly quasiconvex, following the approach of [47] we prove a similar regularity theorem for $W^{1,\bar{q}}$ local minimizers $\bar{u}$ of the relaxed functional $\mathcal{F}$ (see theorem 2.2). In both cases we are mainly interested in the case $\bar{q} > p$, where, as in [32], we need to impose the technical integrability assumption $\bar{u} \in W^{1,\bar{q}}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$. Note that, as a by-product, this assumption allows us to work with $W^{1,\bar{q}}$-quasiconvexity instead of $W^{1,p}$-quasiconvexity in theorem 2.1. Our results generalize those obtained in [12, 32, 46, 47].

Finally, let us briefly comment on some technical issues. In the subquadratic case $1 < p < 2$, we improve the condition (1.4), replacing it by

$$1 < p \leqslant q < \min\left\{\frac{2n-1}{2n-2}p,\ p + \frac{1}{n}\right\}. \tag{1.7}$$

Note that in the model case (1.5) with $q = n = N = 2$ the bound (1.7) allows replacement of the condition $p > \frac{8}{5}$ from (1.4) by $p > \frac{3}{2}$. However, the reason for this improvement is mainly a technical one. Moreover, we mention that, following [12, 32], we use a blow-up argument based on the excess

$$E(x, r) = \fint_{B_r(x)} (1 + |D\bar{u} - (D\bar{u})_{x,r}|^2)^{(p-2)/2} |D\bar{u} - (D\bar{u})_{x,r}|^2 \, \mathrm{d}x \tag{1.8}$$

to prove the partial regularity. In particular, even in the case of absolute minimizers, we provide an alternative proof of the results in [46, 47], where the $\mathcal{A}$-harmonic approximation method has been used.

2. Statement of the results

In this section we state our main results concerning partial regularity of strong local minimizers. Starting with a growth and a coercivity condition we will now supply precise statements of our assumptions.

(H1) ***q*-growth.** There exists a bound $\varGamma > 0$ such that we have

$$0 \leqslant f(\xi) \leqslant \varGamma(1 + |\xi|^q) \quad \text{for every } \xi \in \mathbb{R}^{Nn}.$$

(H2) ***p*-coercivity.** There is a coercivity constant $\gamma > 0$ such that we have

$$f(\xi) \geqslant \gamma|\xi|^p \quad \text{for every } \xi \in \mathbb{R}^{Nn}.$$

Next we state two quasiconvexity conditions, which will be imposed in theorems 2.1 and 2.2, respectively.

(H3) **Strict $W^{1,\bar{q}}$-quasiconvexity.** For each $L > 0$ there is a convexity constant $\nu_L > 0$ such that we have

$$\fint_{B_r(x_0)} (f(\xi + D\varphi) - f(\xi)) \geqslant \nu_L \fint_{B_r(x_0)} (1 + |D\varphi|^2)^{(p-2)/2} |D\varphi|^2 \, \mathrm{d}x$$

for all balls $B_r(x_0) \subset \mathbb{R}^n$, for all $\xi \in \mathbb{R}^{Nn}$ with $|\xi| \leqslant L + 1$ and for all $\varphi \in W_0^{1,\bar{q}}(B_r(x_0), \mathbb{R}^N)$.

(H4) **Strict quasiconvexity.** For each $L > 0$ there is a convexity constant $\nu_L > 0$ such that we have

$$\fint_{B_r(x_0)} (f(\xi + D\varphi) - f(\xi)) \geqslant \nu_L \fint_{B_r(x_0)} (1 + |D\varphi|^2)^{(p-2)/2} |D\varphi|^2 \, \mathrm{d}x$$

for all balls $B_r(x_0) \subset \mathbb{R}^n$, for all $\xi \in \mathbb{R}^{Nn}$ with $|\xi| \leqslant L+1$ and for all $\varphi \in C^\infty_{\mathrm{c}}(B_r(x_0), \mathbb{R}^N)$.

Now we present our first main result, a regularity result for strong local minimizers of the functional F defined in (1.1).

THEOREM 2.1. *Let* $\bar{q} \in [1, \infty)$ *and*

$$1 < p \leqslant q < \min\left\{p + \frac{1}{n},\ \frac{2n-1}{2n-2}p\right\}. \tag{2.1}$$

Assume that $f \in C^2(\mathbb{R}^{Nn})$ *satisfies* (H1) *and* (H3) *and that*

$$\bar{u} \in W^{1,\bar{q}}_{\mathrm{loc}}(\Omega, \mathbb{R}^N) \cap W^{1,p}(\Omega, \mathbb{R}^N)$$

is a $W^{1,\bar{q}}$ *local minimizer of* F *in the sense of definition 1.1. Then there exists an open set* $\Omega_0 \subset \Omega$ *with* $|\Omega \setminus \Omega_0| = 0$ *such that* $\bar{u} \in C^{1,\alpha}_{\mathrm{loc}}(\Omega_0, \mathbb{R}^N)$ *for every* $0 < \alpha < 1$.

Our second main result concerns strong local minimizers of $\mathcal{F}$ from (1.6), where we define local minimizers of $\mathcal{F}$ along the lines of definition 1.1 with F replaced by $\mathcal{F}$. Before stating the result, we recall some properties of the functional $\mathcal{F}$. Assuming (H1) with $1 < p \leqslant q < np/(n-1)$, from [22, 47] we have

$$\mathcal{F}[u] \geqslant \int_\Omega Qf(Du)\,\mathrm{d}x \quad \text{for } u \in W^{1,p}(\Omega, \mathbb{R}^N), \tag{2.2}$$

$$\mathcal{F}[u] = \int_\Omega Qf(Du)\,\mathrm{d}x \quad \text{for } u \in W^{1,q}(\Omega, \mathbb{R}^N), \tag{2.3}$$

where Qf denotes the quasiconvex envelope of f. Furthermore, it has been shown in [11, 22] that $\mathcal{F}$ depends on the domain Ω like a Radon measure, whose absolutely continuous part has density $Qf(Du)$. These facts will be crucial in the proof of the following result. In particular, we mention that, by an argument of [47], they can be used to prove the validity of Euler's equation for minimizers of $\mathcal{F}$. This is an important observation for the proof of the following theorem.

THEOREM 2.2. *Let* $\bar{q} \in [1, \infty)$ *and*

$$1 < p \leqslant q < \min\left\{p + \frac{1}{n}, \frac{2n-1}{2n-2}p\right\}. \tag{2.4}$$

Assume that $f \in C^2(\mathbb{R}^{Nn})$ *satisfies* (H1), (H2) *and* (H4) *and that*

$$\bar{u} \in W^{1,\bar{q}}_{\mathrm{loc}}(\Omega, \mathbb{R}^N) \cap W^{1,p}(\Omega, \mathbb{R}^N)$$

is a $W^{1,\bar{q}}$ *local minimizer of* $\mathcal{F}$ *on* Ω. *Then there exists an open set* $\Omega_0 \subset \Omega$ *with* $|\Omega \setminus \Omega_0| = 0$ *such that* $\bar{u} \in C^{1,\alpha}_{\mathrm{loc}}(\Omega_0, \mathbb{R}^N)$ *for every* $0 < \alpha < 1$.

We highlight some features of the previous theorems.

REMARK 2.3. In some sense $W^{1,\bar{q}}$-quasiconvexity is necessary for the existence of $W^{1,\bar{q}}$ local minimizers. Precisely, adapting the proof of [53, proposition 4.1] one finds that if $\bar{u} \in W^{1,p}(\Omega, \mathbb{R}^N)$ is a $W^{1,\bar{q}}$ local minimizer of F, then for every $\varphi \in W^{1,\bar{q}}(B, \mathbb{R}^N)$ the following holds:

$$\fint_B f(D\bar{u}(y) + D\varphi(x))\,\mathrm{d}x \geqslant f(D\bar{u}(y)) \quad \text{for a.e. } y \in \Omega.$$

REMARK 2.4. If $\bar{q} \geqslant q$, then (strict) $W^{1,\bar{q}}$-quasiconvexity is equivalent to (strict) quasiconvexity. Combining this with (2.3), theorems 2.1 and 2.2 turn out to be equivalent in this case.

REMARK 2.5. In the case $\bar{q} \leqslant p$, it follows from [32, § 2] that theorem 2.1 can be reduced to the case of absolute minimizers.

REMARK 2.6. Under the additional assumption

$$\limsup_{r \to 0^+} \|Du - (Du)_{x,r}\|_{L^\infty(B_r(x), \mathbb{R}^{Nn})} < \delta,$$

theorems 2.1 and 2.2 also hold in the case of weak local minimizers, i.e. for $\bar{q} = \infty$. This generalization is straightforward, along the lines of [12, 32].

REMARK 2.7. The proofs of the theorems will show that we can choose Ω_0 such that

$$\Omega \setminus \Omega_0 \subset \Big\{ x \in \Omega : \liminf_{r \to 0+} E(x,r) > 0 \text{ or } \limsup_{r \to 0+} |(Du)_{x,r}| = \infty \Big\}$$

holds, where $E(x,r)$ is defined in (1.8).

REMARK 2.8. Under the assumptions of theorem 2.1 or 2.2, if $f \in C^\infty(\mathbb{R}^{Nn})$, then we have $u \in C^\infty(\Omega_0, \mathbb{R}^N)$. Once $C^{1,\alpha}_{\mathrm{loc}}$-regularity is proved, this higher-regularity result follows from the application of linear theory to the Euler equation.

3. Preliminaries

Throughout this paper we denote by a c a positive constant possibly varying from line to line. The dependencies of such constants will only occasionally be highlighted. We write $B_r(x)$ for the open ball with centre x and radius r in $\mathbb{R}^n$ and set $B_r := B_r(0)$ and $B := B_1$. In addition, we will use the common abbreviations

$$u_{x,r} := \fint_{B_r(x)} u\,\mathrm{d}x := \frac{1}{|B_r(x)|} \int_{B_r(x)} u\,\mathrm{d}x$$

and $u_r := u_{0,r}$ for mean values, where $|\cdot|$ denotes the n-dimensional Lebesgue measure. Moreover, for $\beta > 0$ we define the functions $V_\beta : \mathbb{R}^k \mapsto \mathbb{R}^k$ and $W_\beta : \mathbb{R}^k \mapsto \mathbb{R}^k$ by

$$V_\beta(\xi) = (1 + |\xi|^2)^{(\beta-1)/2}\xi, \qquad W_\beta(\xi) = (1 + |\xi|)^{\beta-1}\xi \tag{3.1}$$

for $\xi \in \mathbb{R}^k$, $k \in \mathbb{N}$. Since we are mostly dealing with $\beta = p/2$, where p is a fixed exponent, we use the abbreviations $V = V_{p/2}$ and $W = W_{p/2}$. Next, we will collect some useful properties of V and W. Clearly, we have $|V_\beta(\xi)| = V_\beta(|\xi|)$, $|W_\beta(\xi)| = W_\beta(|\xi|)$ and

$$c^{-1}|W_\beta(\xi)| \leqslant |V_\beta(\xi)| \leqslant c|W_\beta(\xi)|, \tag{3.2}$$

where c depends only on β. Furthermore, the functions $V_\beta(t)$ and $W_\beta(t)$ are both non-decreasing in $t \geqslant 0$ and some elementary calculations show that $|W|^2$ is convex for $1 \leqslant p < \infty$ and $|W|^{2/p}$ is convex for $1 \leqslant p \leqslant 2$ (in contrast to $|V|^2$ and $|V|^{2/p}$). Some additional properties are summarized in the following lemma.

Lemma 3.1. *Let $\beta > 0$, $1 < p < \infty$ and $M > 0$. Then, for all $\xi, \eta \in \mathbb{R}^k$ and $t > 0$, we have*

(i) $|V(t\xi)| \leqslant \max\{t, t^{p/2}\}|V(\xi)|$,

(ii) $|V_\beta(\xi + \eta)| \leqslant c(|V_\beta(\xi)| + |V_\beta(\eta)|)$,

(iii) $(1 + |\xi|^2 + |\eta|^2)^{p/2} \leqslant c(1 + |V(\xi)|^2 + |V(\eta)|^2)$,

(iv) $|V_{p-1}(\xi)||\eta| \leqslant |V(\xi)|^2 + |V(\eta)|^2$.

Here, c depends only on β and p, respectively.

Proof. Assertions (i) and (iii) are easy to check. Part (ii) has been proved for $\frac{1}{2} < \beta < 1$ in [13, lemma 2.1] and is easily seen to hold for all $\beta > 0$. Part (iv) follows from the fact that V_{p-1} is non-decreasing. □

Next we restate an integral inequality for V (see, for instance, [46]).

Lemma 3.2. *Let $1 < p < \infty$ and $u \in W^{1,p}(\Omega, \mathbb{R}^N)$. Then we have*

$$\int_\Omega |V(Du - (Du)_\Omega)|^2 \, \mathrm{d}x \leqslant c \int_\Omega |V(Du)|^2 \, \mathrm{d}x$$

with a constant c depending only on p. □

Furthermore, we recall a Poincaré-type inequality and a Sobolev–Poincaré-type inequality for V.

Lemma 3.3. *We consider $1 < p < \infty$, a ball $B_r(x_0)$ in $\mathbb{R}^n$ and a function $u \in W^{1,p}(B_r(x), \mathbb{R}^N)$. Then, we have*

$$\int_{B_r(x_0)} \left| V\left(\frac{u - u_{x_0,r}}{r}\right)\right|^2 \mathrm{d}x \leqslant c \int_{B_r(x_0)} |V(Du)|^2 \, \mathrm{d}x. \tag{3.3}$$

In addition, setting $p^\# := 2n/(n-p) > 2$ for $1 < p < 2$, we have

$$\left(\fint_{B_r(x_0)} \left| V\left(\frac{u - u_{x_0,r}}{r}\right)\right|^{p^\#}\right)^{1/p^\#} \mathrm{d}x \leqslant c\left(\fint_{B_r(x_0)} |V(Du)|^2 \, \mathrm{d}x\right)^{1/2}. \tag{3.4}$$

The constant c depends only on n, N and p in both inequalities. □

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Here, (3.4) has been proved in [17, theorem 2] and (3.3) follows easily from the standard Poincaré inequality for $p \geqslant 2$ and from (3.4) for $1 < p < 2$. The reader should note that a weaker version of (3.4) was established in [13].

Next we restate some estimates for smoothing operators, which will be crucial for our approach. These estimates, introduced first in [22, lemma 2.2], have already been used in the regularity theory of integrals with (p, q)-growth (see [42, 46, 47]). We state them in the form of [46, lemma 6.3].

LEMMA 3.4. *Let $0 < r < s$ and $B_s \subset \Omega$. We define a linear smoothing operator*

$$T_{r,s} : W^{1,1}(\Omega; \mathbb{R}^N) \to W^{1,1}(\Omega; \mathbb{R}^N)$$

for $u \in W^{1,1}(\Omega; \mathbb{R}^N)$ and $x \in \Omega$ by

$$T_{r,s}u(x) := \fint_{B_1} u(x + \vartheta(x)y)\,\mathrm{d}y, \quad \textit{where } \vartheta(x) := \tfrac{1}{2}\max\{\min\{|x| - r,\ s - |x|\}, 0\}. \tag{3.5}$$

With this definition, for all $1 \leqslant p \leqslant q < np/(n-1)$ and all $u \in W^{1,p}(\Omega; \mathbb{R}^N)$, the following assertions are true:

$$T_{r,s}u \in W^{1,p}(\Omega; \mathbb{R}^N),$$

$$u = T_{r,s}u \quad \textit{almost everywhere on } (\Omega \setminus B_s) \cup B_r, \tag{3.6}$$

$$T_{r,s}u \in u + W_0^{1,p}(B_s \setminus \overline{B_r}; \mathbb{R}^N), \tag{3.7}$$

$$|DT_{r,s}u| \leqslant c(n) T_{r,s}|Du| \quad \textit{almost everywhere on } \Omega, \tag{3.8}$$

$$\|T_{r,s}u\|_{p;B_s\setminus B_r} \leqslant c(n,p)\|u\|_{p;B_s\setminus B_r}, \tag{3.9}$$

$$\|DT_{r,s}u\|_{p;B_s\setminus B_r} \leqslant c(n,p)\|Du\|_{p;B_s\setminus B_r}, \tag{3.10}$$

$$\|T_{r,s}u\|_{q;B_s\setminus B_r} \leqslant c(n,p,q)(s-r)^{n/q-(n-1)/p} \times \left[\sup_{t\in]r,s[} \frac{\tilde{\Xi}(t) - \tilde{\Xi}(r)}{t-r} + \sup_{t\in]r,s[} \frac{\tilde{\Xi}(s) - \tilde{\Xi}(t)}{s-t}\right]^{1/p}, \tag{3.11}$$

$$\|DT_{r,s}u\|_{q;B_s\setminus B_r} \leqslant c(n,p,q)(s-r)^{n/q-(n-1)/p} \times \left[\sup_{t\in]r,s[} \frac{\Xi(t) - \Xi(r)}{t-r} + \sup_{t\in]r,s[} \frac{\Xi(s) - \Xi(t)}{s-t}\right]^{1/p}. \tag{3.12}$$

Here we have used the abbreviations

$$\tilde{\Xi}(t) := \|u\|^p_{p;B_t} \quad \textit{and} \quad \Xi(t) := \|Du\|^p_{p;B_t}.$$

□

Further estimates of the terms on the right-hand sides of (3.11) and (3.12) can be obtained by means of the following simple lemma.

LEMMA 3.5. *Let $-\infty < r < s < \infty$ and a continuous non-decreasing function $\Xi : [r, s] \to \mathbb{R}$ be given. Then, there exist $\tilde{r} \in]r, \frac{1}{3}(2r + s)[$ and $\tilde{s} \in]\frac{1}{3}(r + 2s), s[$ for* *which the following hold:*

$$\left.\begin{aligned}\frac{\Xi(t)-\Xi(\tilde r)}{t-\tilde r} &\leqslant 4\frac{\Xi(s)-\Xi(r)}{s-r}\\ \frac{\Xi(\tilde s)-\Xi(t)}{\tilde s-t} &\leqslant 4\frac{\Xi(s)-\Xi(r)}{s-r}\end{aligned}\right\} \quad \textit{for every } t\in\,]\tilde r,\tilde s[. \tag{3.13}$$

In particular, we have

$$\tfrac{1}{3}(s-r)\leqslant \tilde s-\tilde r\leqslant s-r. \tag{3.14}$$

Proof. An elementary proof is given in [22]. □

REMARK 3.6. Assume that Ξ is absolutely continuous and non-decreasing and a set $N\subset\mathbb{R}$ of Lebesgue measure zero is given. Then, we can choose $\tilde r$ and $\tilde s$ as in lemma 3.5 even with the additional property $\tilde r,\tilde s\notin N$ (see [47, lemma 4.6]).

Finally, we state another useful lemma concerning the smoothing operator $T_{r,s}$.

LEMMA 3.7. *Let $1<p<\infty$, $0<r<s$ and $B_s(x_0)\subset\Omega$. Then, for $u\in W^{1,p}(\Omega,\mathbb{R}^N)$ we have*

$$|V(DT_{r,s}u)|^2\leqslant cT_{r,s}[|V(Du)|^2] \quad \textit{almost everywhere on } \Omega, \tag{3.15}$$

where c depends only on n and p.

Proof. Due to (3.2) it suffices to show the claim with V replaced by W. Since $|W|^2$ is a non-decreasing and convex function, using (3.5), (3.8) and Jensen's inequality, we obtain

$$|W(DT_{r,s}u)|^2\leqslant c|W(T_{r,s}|Du|)|^2\leqslant cT_{r,s}\big[|W(Du)|^2\big] \quad \text{almost everywhere on } \Omega.$$

This proves the claim. □

4. Proof of theorem 2.1

First we note that the definition (1.8) of the excess reads

$$E(x,r)=\fint_{B_r(x)}|V(D\bar u-(D\bar u)_{x,r})|^2\,\mathrm{d}x$$

in the terminology of § 3. We will establish a decay estimate for this excess in the following proposition, which we prove by an indirect blow-up argument.

PROPOSITION 4.1. *Under the assumptions of theorem 2.1 for every $L>0$ there is a constant $C>0$ with the following property: for each $0<\tau\leqslant\frac{1}{2}$ there exists a number $\varepsilon>0$ such that the conditions*

$$|(D\bar u)_{x,r}|\leqslant L,\quad r<\varepsilon \quad \textit{and} \quad E(x,r)<\varepsilon$$

for a ball $B_r(x)\subset\subset\Omega$ imply that

$$E(x,\tau r)\leqslant C\tau^2E(x,r).$$

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Proof. We argue by contradiction. Assuming the proposition to be false, there exist $L > 0$ and $0 < \tau \leqslant \frac{1}{2}$, corresponding to a constant C that will be chosen later, such that the following holds. There is a sequence of balls $B_{r_j}(x_j) \subset\subset \Omega$ with $r_j \to 0$ such that

$$|(D\bar{u})_{x_j,r_j}| \leqslant L \quad \text{and} \quad 0 < \lambda_j := \sqrt{E(x_j, r_j)} \to 0 \quad \text{as } j \to \infty, \tag{4.1}$$

but

$$E(x_j, \tau r_j) > C\tau^2 E(x_j, r_j). \tag{4.2}$$

STEP 1 (blow-up). We define $\xi_j := (D\bar{u})_{x_j,r_j}$ and

$$u_j(y) := \frac{1}{\lambda_j r_j}[\bar{u}(x_j + r_j y) - (\bar{u})_{x_j,r_j} - \xi_j r_j y] \quad \text{for } y \in B.$$

Then we have

$$\lambda_j Du_j(y) = D\bar{u}(x_j + r_j y) - \xi_j \quad \text{for } y \in B, \quad (u_j)_{0,1} = 0, \quad (Du_j)_{0,1} = 0$$

and

$$\fint_B \left|\frac{V(\lambda_j Du_j)}{\lambda_j}\right|^2 \mathrm{d}x = 1. \tag{4.3}$$

Since $\lambda_j (Du_j)_{0,\tau} = (D\bar{u})_{x_j,\tau r_j} - \xi_j$, from (4.2) we obtain

$$\lambda_j^{-2} \fint_{B_\tau} |V(\lambda_j(Du_j - (Du_j)_{0,\tau}))|^2 \,\mathrm{d}x > C\tau^2. \tag{4.4}$$

Furthermore, we define

$$f_j(\xi) := \frac{f(\xi_j + \lambda_j \xi) - f(\xi_j) - Df(\xi_j)\lambda_j \xi}{\lambda_j^2} \quad \text{for } \xi \in \mathbb{R}^{Nn}.$$

Noting that $|\xi_j| \leqslant L$ we get from [2, lemma II.3] that there is a positive constant $k(L)$ with

$$\begin{aligned} |f_j(\xi)| &\leqslant k\lambda_j^{-2}|V_{q/2}(\lambda_j \xi)|^2, \\ |Df_j(\xi)| &\leqslant k\lambda_j^{-1}|V_{q-1}(\lambda_j \xi)| \end{aligned} \tag{4.5}$$

for all $\xi \in \mathbb{R}^{Nn}$. Moreover, we rewrite the quasiconvexity hypothesis $(H3)$ in the following form:

$$\nu_L \int_B \left|\frac{V(\lambda_j D\varphi)}{\lambda_j}\right|^2 \mathrm{d}x \leqslant \int_B (f_j(\xi + D\varphi) - f_j(\xi))\,\mathrm{d}x \tag{4.6}$$

for all $\xi \in \mathbb{R}^{Nn}$ with $|\lambda_j \xi| \leqslant 1$ and for all $\varphi \in W_0^{1,\bar{q}}(B, \mathbb{R}^N)$. In addition, setting

$$F_j[u] = \int_B f_j(Du)\,\mathrm{d}x,$$

the minimizing property of $\bar{u}$ can be rephrased as follows. For all $\varphi \in W_0^{1,\bar{q}}(B, \mathbb{R}^N)$ with

$$\|D\varphi\|_{L^{\bar{q}}(B,\mathbb{R}^{Nn})} \leqslant \frac{\delta}{\lambda_j r_j^{n/\bar{q}}} \tag{4.7}$$

we have

$$F_j[u_j] \leqslant F_j[u_j + \varphi]. \tag{4.8}$$

Next we claim

$$\fint_B |Du_j|^{\min\{2,p\}} \, \mathrm{d}x \leqslant c. \tag{4.9}$$

Actually, (4.9) follows immediately from (4.3) for $p \geqslant 2$. In contrast, for $p \leqslant 2$ we first deduce

$$\fint_B |V(Du_j)|^2 \, \mathrm{d}x \leqslant c$$

from (4.3) by virtue of lemma 3.1(i) and then get (4.9) by lemma 3.1(iii). Thus, passing to subsequences we may assume that for some $u \in W^{1,\min\{2,p\}}(B, \mathbb{R}^N)$ and some $\xi_\infty \in \mathbb{R}^{Nn}$ we have

$$\left.\begin{aligned} u_j &\to u && \text{weakly in } W^{1,\min\{2,p\}}(B, \mathbb{R}^N), \\ u_j &\to u && \text{strongly in } L^{\min\{2,p\}}(B, \mathbb{R}^N), \\ \lambda_j Du_j &\to 0 && \text{almost everywhere on } B, \\ \xi_j &\to \xi_\infty && \text{in } \mathbb{R}^{Nn}. \end{aligned}\right\} \tag{4.10}$$

Step 2 (linearization). In this step we will show that u is a weak solution of a linear system. Precisely, we claim

$$\int_B D^2 f(\xi_\infty)(Du, D\varphi) \, \mathrm{d}x = 0 \quad \text{for all } \varphi \in C^1_{\mathrm{c}}(B, \mathbb{R}^N). \tag{4.11}$$

Actually, the derivation of the limit equation (4.11) is well known (see, for instance, [2,9,21,32,42]) and we will only sketch it. From the minimality property of u_j in (4.8) we get the following Euler–Lagrange equation:

$$\int_B Df_j(Du_j) D\varphi \, \mathrm{d}x = 0 \quad \text{for all } \varphi \in C^1_{\mathrm{c}}(B, \mathbb{R}^N).$$

We will show that the preceding equation converges to (4.11) as $j \to \infty$. Setting $B_j^+ := \{x \in B : |\lambda_j Du_j(x)| > 1\}$ and using (4.5) and $q \leqslant p + 1$, we obtain

$$\begin{aligned} \left| \int_{B_j^+} Df_j(Du_j) D\varphi \, \mathrm{d}x \right| &\leqslant \frac{c}{\lambda_j} \int_{B_j^+} |V_{q-1}(\lambda_j Du_j)| \, \mathrm{d}x \sup_B |D\varphi| \\ &\leqslant c \sup_B |D\varphi| \lambda_j \fint_B \frac{|V(\lambda_j Du_j)|^2}{\lambda_j^2} \, \mathrm{d}x. \end{aligned}$$

By (4.3) we infer that this term vanishes as $j \to \infty$ and it remains to treat the integral over $B_j^- := \{x \in B : |\lambda_j Du_j(x)| \leqslant 1\}$. Here, noting $|B_j^+| \leqslant c\lambda_j^2 \to 0$ as in [32], we have

$$\int_{B_j^-} Df_j(Du_j)D\varphi\,\mathrm{d}x = \frac{1}{\lambda_j}\int_{B_j^-}(Df(\xi_j+\lambda_j Du_j)-Df(\xi_j))D\varphi\,\mathrm{d}x$$
$$= \int_{B_j^-}\int_0^1 D^2 f(\xi_j+t\lambda_j Du_j)\,\mathrm{d}t(Du_j, D\varphi)\,\mathrm{d}x$$
$$\to \int_B D^2 f(\xi_\infty)(Du, D\varphi)$$

and (4.11) follows. The condition (H3) implies that $D^2 f(\xi_\infty)$ is elliptic in the the sense of Legendre–Hadamard with ellipticity constant $2\nu_L$ and upper bound $K_L := \sup_{|\xi|\leqslant L} D^2 f(\xi)$. Thus, we can apply linear theory to deduce that u is C^1 on B and

$$\fint_{B_\tau} |Du-(Du)_{0,\tau}|^2\,\mathrm{d}x \leqslant c\tau^2 \tag{4.12}$$

is valid. Here, c depends only on n, N, ν_L and K_L. The remainder of the proof is now mostly devoted to showing that

$$\lambda_j^{-2}\int_{B_\tau} |V(\lambda_j(Du_j-Du))|^2\,\mathrm{d}x \to 0 \quad \text{as } j\to\infty. \tag{4.13}$$

Once we have proved (4.13) we will see that (4.12) contradicts (4.4).

Step 3 (construction of test functions and preliminary estimates). We consider

$$B_r(x_0) \subset\subset B_\sigma$$

and fix $\tau < \sigma < 1$, $0 < \alpha < 1$. We define affine functions $a_j(x) := (u_j)_{x_0,r} + (Du_j)_{x_0,r}(x-x_0)$ and set

$$v_j(x) := u_j(x) - a_j(x).$$

Moreover, we introduce the abbreviation

$$\varXi_j(t) := \lambda_j^{-2}\int_{B_t(x_0)}\left(\left|V\left(\frac{\lambda_j v_j}{(1-\alpha)r}\right)\right|^2 + |V(\lambda_j Dv_j)|^2\right)\mathrm{d}x$$

and choose for this function $\alpha r \leqslant \tilde{r}_j < \tilde{s}_j \leqslant r$ as in lemma 3.5. In particular, we have

$$\tfrac{1}{3}(1-\alpha)r \leqslant \tilde{s}_j - \tilde{r}_j \leqslant (1-\alpha)r. \tag{4.14}$$

Now we consider smooth cut-off functions $\eta_j : \mathbb{R}^n \to [0,1]$ which satisfy $\eta_j \equiv 1$ in a neighbourhood of $\overline{B_{\tilde{r}_j}(x_0)}$, $\eta_j = 0$ in a neighbourhood of $\mathbb{R}^n \setminus B_{\tilde{s}_j}(x_0)$ and $|\nabla\eta_j| \leqslant 2/(\tilde{s}_j-\tilde{r}_j)$ on $B_r(x_0)$. We define

$$\chi_j = [(1-\eta_j)v_j], \quad \psi_j := T_{\tilde{r}_j,\tilde{s}_j}\chi_j \quad \text{and} \quad \varphi_j := v_j - \psi_j,$$

where the smoothing operator T is defined in lemma 3.4. According to (3.6) and (3.7) we have $\varphi_j \in W_0^{1,p}(B_{\tilde{s}_j}(x_0), \mathbb{R}^N)$, $\varphi_j = v_j$, $\psi_j = 0$ on $B_{\tilde{r}_j}(x_0)$ and

$$Du_j - Da_j = Dv_j = D\varphi_j + D\psi_j \quad \text{on } B. \tag{4.15}$$

In addition, the product rule and (4.14) give

$$|D\chi_j| \leqslant |Dv_j| + \left|\frac{v_j}{(1-\alpha)r}\right|. \tag{4.16}$$

Next, we will derive two preparatory estimates for χ_j, namely (4.17) and (4.18). Setting

$$Y_j := \lambda_j^{-2} \int_{B_r(x_0)\setminus B_{\alpha r}(x_0)} \left(\left|V\left(\frac{\lambda_j v_j}{(1-\alpha)r}\right)\right|^2 + |V(\lambda_j Dv_j)|^2\right) \mathrm{d}x,$$

we apply in turn (3.8), lemma 3.7, (3.9) (with $p = 1$), (4.16) and lemma 3.1(ii) to get the estimate

$$\begin{aligned}
\lambda_j^{-2} \int_{B_{\tilde{s}_j}(x_0)\setminus B_{\tilde{r}_j}(x_0)} & |V(\lambda_j D\psi_j)|^2 \,\mathrm{d}x \\
&\leqslant c\lambda_j^{-2} \int_{B_{\tilde{s}_j}(x_0)\setminus B_{\tilde{r}_j}(x_0)} |T_{\tilde{r}_j,\tilde{s}_j}[|V(\lambda_j D\chi_j)|^2]| \,\mathrm{d}x \\
&\leqslant c\lambda_j^{-2} \int_{B_{\tilde{s}_j}(x_0)\setminus B_{\tilde{r}_j}(x_0)} |V(\lambda_j D\chi_j)|^2 \,\mathrm{d}x \leqslant cY_j.
\end{aligned} \tag{4.17}$$

Arguing in a similar way, but using (3.11) (with $p = 1$, $q = \kappa$) instead of (3.9), we find for $1 \leqslant \kappa < n/(n-1)$ that

$$\begin{aligned}
\lambda_j^{-2} & \int_{B_{\tilde{s}_j}(x_0)\setminus B_{\tilde{r}_j}(x_0)} |V(\lambda_j D\psi_j)|^{2\kappa} \\
&\leqslant c\lambda_j^{-2} \int_{B_{\tilde{s}_j}(x_0)\setminus B_{\tilde{r}_j}(x_0)} \left|T_{\tilde{r}_j,\tilde{s}_j}[|V(\lambda_j D\chi_j)|^2]\right|^{\kappa} \mathrm{d}x \\
&\leqslant c\lambda_j^{-2}(\tilde{s}_j - \tilde{r}_j)^{n-(n-1)\kappa}\Bigg(\sup_{t\in]\tilde{r}_j,\tilde{s}_j[} \frac{1}{t-\tilde{r}_j} \int_{B_t(x_0)\setminus B_{\tilde{r}_j}(x_0)} |V(\lambda_j D\chi_j)|^2 \,\mathrm{d}x \\
&\qquad\qquad + \sup_{t\in]\tilde{r}_j,\tilde{s}_j[} \frac{1}{\tilde{s}_j - t} \int_{B_{\tilde{s}_j}(x_0)\setminus B_t(x_0)} |V(\lambda_j D\chi_j)|^2 \,\mathrm{d}x\Bigg)^{\kappa} \\
&\leqslant c\lambda_j^{2\kappa-2}(\tilde{s}_j - \tilde{r}_j)^{n(1-\kappa)+\kappa}\left(\sup_{t\in]\tilde{r}_j,\tilde{s}_j[} \frac{\Xi_j(t) - \Xi_j(\tilde{r}_j)}{t-\tilde{r}_j} + \sup_{t\in]\tilde{r}_j,\tilde{s}_j[} \frac{\Xi_j(\tilde{s}_j) - \Xi_j(t)}{\tilde{s}_j - t}\right)^{\kappa}.
\end{aligned}$$

Combining the last inequality with the estimates of lemma 3.5, we obtain

$$\lambda_j^{-2} \int_{B_{\tilde{s}_j}(x_0)\setminus B_{\tilde{r}_j}(x_0)} |V(\lambda_j D\psi_j)|^{2\kappa} \leqslant c\left(\frac{\lambda_j^2 Y_j}{((1-\alpha)r)^n}\right)^{\kappa-1} Y_j. \tag{4.18}$$

Step 4 (the main estimate). In this step we will combine ideas of [32,46] to establish a key estimate. This estimate will lead to (4.13) later in the proof. Here, our first aim is to verify (4.7) for φ_j with j large, which will enable us to use (4.8). We start with the following computation and use for this purpose (4.14), (3.9) and the

Poincaré inequality:

$$\begin{aligned}\int_B |D\varphi_j|^{\bar q}\,\mathrm{d}x &\leqslant \int_{B_{\tilde r_j}(x_0)} |Dv_j|^{\bar q}\,\mathrm{d}x + \int_{B_{\tilde s_j}(x_0)\setminus B_{\tilde r_j}(x_0)} |D\psi_j|^{\bar q}\,\mathrm{d}x \\ &\leqslant \int_{B_r(x_0)} |Dv_j|^{\bar q}\,\mathrm{d}x + c\int_{B_{\tilde s_j}(x_0)\setminus B_{\tilde r_j}(x_0)} |D\chi_j|^{\bar q}\,\mathrm{d}x \\ &\leqslant \int_{B_r(x_0)} |Dv_j|^{\bar q}\,\mathrm{d}x \\ &\quad + c\biggl(\int_{B_{\tilde s_j}(x_0)\setminus B_{\tilde r_j}(x_0)} |Dv_j|^{\bar q}\,\mathrm{d}x + \int_{B_{\tilde s_j}(x_0)\setminus B_{\tilde r_j}(x_0)} \biggl(\frac{|v_j|}{(1-\alpha)r}\biggr)^{\bar q}\mathrm{d}x\biggr) \\ &\leqslant c\biggl(1+\frac{1}{1-\alpha}\biggr)^{\bar q}\int_{B_r(x_0)} |Dv_j|^{\bar q}\,\mathrm{d}x.\end{aligned}$$

Changing coordinates in view of $B_{rr_j}(x_j + r_j x_0) \subset B_{r_j}(x_j)$, we obtain

$$\|D\varphi_j\|_{L^{\bar q}(B,\mathbb{R}^{Nn})} \leqslant \frac{c}{\lambda_j r_j^{n/\bar q}}\biggl(1+\frac{1}{1-\alpha}\biggr)\biggl(\int_{B_{r_j}(x_j)} |D\bar u|^{\bar q}\,\mathrm{d}x\biggr)^{1/\bar q}.$$

Therefore, the condition (4.7) is fulfilled if

$$c\biggl(1+\frac{1}{1-\alpha}\biggr)\biggl(\int_{B_{r_j}(x_j)} |D\bar u|^{\bar q}\,\mathrm{d}x\biggr)^{1/\bar q} \leqslant \delta$$

holds and this is satisfied for sufficiently large j, say for $j \geqslant j_1(\alpha)$. Furthermore, from the definition of a_j and (4.9) we see that $|Da_j| \leqslant cr^{-n}$. Hence, there exists a $j_2(r)$ such that $|\lambda_j Da_j| \leqslant 1$ holds for all $j \geqslant j_2(r)$. We define $j_0(\alpha, r) := \max\{j_1(\alpha), j_2(r)\}$. Then, for $j \geqslant j_0(\alpha, r)$, we use (4.6), (4.8) and (4.15) to get

$$\begin{aligned}\nu_L\int_{B_{\tilde r_j}(x_0)}\biggl|\frac{V(\lambda_j Dv_j)}{\lambda_j}\biggr|^2\mathrm{d}x \leqslant \nu_L\int_{B_{\tilde s_j}(x_0)}\biggl|\frac{V(\lambda_j D\varphi_j)}{\lambda_j}\biggr|^2\mathrm{d}x \\ &\leqslant \int_{B_{\tilde s_j}(x_0)} (f_j(Da_j + D\varphi_j) - f_j(Da_j))\,\mathrm{d}x \\ &= \int_{B_{\tilde s_j}(x_0)} (f_j(Du_j - D\psi_j) - f_j(Du_j))\,\mathrm{d}x \\ &\quad + \int_B (f_j(Du_j) - f_j(Du_j - D\varphi_j))\,\mathrm{d}x \\ &\quad + \int_{B_{\tilde s_j}(x_0)} (f_j(Da_j + D\psi_j) - f_j(Da_j))\,\mathrm{d}x \\ &\leqslant \int_{B_{\tilde s_j}(x_0)} (f_j(Du_j - D\psi_j) - f_j(Du_j))\,\mathrm{d}x \\ &\quad + \int_{B_{\tilde s_j}(x_0)} (f_j(Da_j + D\psi_j) - f_j(Da_j))\,\mathrm{d}x. \qquad (4.19)\end{aligned}$$

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Recalling $\psi_j = 0$ on $B_{\tilde r_j}(x_0)$, we estimate the right-hand side by using inequality (4.5) and lemma 3.1(ii):

$$\begin{aligned}
\nu_L \int_{B_{\tilde r_j}(x_0)} \left| \frac{V(\lambda_j D v_j)}{\lambda_j} \right|^2 \mathrm{d}x
&\leqslant \int_{B_{\tilde s_j}(x_0)} \int_0^1 (Df_j(Da_j + tD\psi_j) - Df_j(Du_j - tD\psi_j)) D\psi_j \,\mathrm{d}t\,\mathrm{d}x \\
&\leqslant c \int_{B_{\tilde s_j}(x_0)\setminus B_{\tilde r_j}(x_0)} \biggl(\left| \frac{V_{q-1}(\lambda_j Da_j)}{\lambda_j} \right| |D\psi_j| \\
&\qquad + \left| \frac{V_{q-1}(\lambda_j Dv_j)}{\lambda_j} \right| |D\psi_j| + \left| \frac{V_{q/2}(\lambda_j D\psi_j)}{\lambda_j} \right|^2 \biggr) \mathrm{d}x \\
&=: c(\mathrm{I} + \mathrm{II} + \mathrm{III})
\end{aligned}$$

with the obvious labelling.

Estimation of III. We estimate the last integral by lemma 3.1(iii), (4.17) and (4.18) with $\kappa = q/p < n/(n-1) = 1 + 1/(n-1)$:

$$\begin{aligned}
\mathrm{III} &= \int_{B_{\tilde s_j(x_0)}\setminus B_{\tilde r_j}(x_0)} \left| \frac{V_{q/2}(\lambda_j D\psi_j)}{\lambda_j} \right|^2 \mathrm{d}x \\
&\leqslant \lambda_j^{-2} \int_{B_{\tilde s_j(x_0)}\setminus B_{\tilde r_j}(x_0)} (1 + |\lambda_j D\psi_j|^2)^{(p/2)(q-p)/p} |V(\lambda_j D\psi_j)|^2 \,\mathrm{d}x \\
&\leqslant c\lambda_j^{-2} \int_{B_{\tilde s_j}(x_0)\setminus B_{\tilde r_j}(x_0)} (1 + |V(\lambda_j D\psi_j)|^2)^{(q-p)/p} |V(\lambda_j D\psi_j)|^2 \,\mathrm{d}x \\
&\leqslant c\lambda_j^{-2} \int_{B_{\tilde s_j}(x_0)\setminus B_{\tilde r_j}(x_0)} (|V(\lambda_j D\psi_j)|^2 + |V(\lambda_j D\psi_j)|^{2q/p}) \,\mathrm{d}x \\
&\leqslant c\biggl(Y_j + \biggl(\frac{\lambda_j^2 Y_j}{((1-\alpha)r)^n} \biggr)^{(q/p)-1} Y_j \biggr) \\
&\leqslant c\biggl(Y_j + \biggl(\frac{\lambda_j^2 Y_j}{((1-\alpha)r)^n} \biggr)^{1/(n-1)} Y_j \biggr).
\end{aligned}$$

Estimation of II. We estimate the integral II, distinguishing the cases $p > 2(n-1)/n$ and $p \leqslant 2(n-1)/n$.

Case 1 ($p > 2(n-1)/n$). In this case, (2.1) reads $q < p + 1/n$ and we have $\frac{1}{2}p + 1 < p + 1/n$. Hence, enlarging q if necessary, we may assume that $q \geqslant \frac{1}{2}p + 1$ (without destroying $q < p + 1/n$). Next we give a pointwise estimation of the integrand in II. For $|\lambda_j Dv_j| \leqslant 1$ with Young's inequality we obtain

$$\begin{aligned}
&(1 + |\lambda_j Dv_j|^2)^{(q-2)/2} |Dv_j||D\psi_j| \\
&\qquad = \lambda_j^{-2} (1 + |\lambda_j Dv_j|^2)^{(p-2)/4 + (2q-p-2)/4} |\lambda_j Dv_j||\lambda_j D\psi_j|
\end{aligned}$$

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$$\begin{aligned}&\leqslant c\lambda_j^{-2}(1+|\lambda_j Dv_j|^2)^{(p-2)/4}|\lambda_j Dv_j||\lambda_j D\psi_j|\\&\leqslant c\biggl(\biggl|\frac{V(\lambda_j Dv_j)}{\lambda_j}\biggr|^2+\lambda_j^{-2}|\lambda_j D\psi_j|^2\biggr),\end{aligned}$$

while for $|\lambda_j Dv_j| > 1$ a similar computation yields

$$\begin{aligned}&(1+|\lambda_j Dv_j|^2)^{(q-2)/2}|Dv_j||D\psi_j|\\&\quad=\lambda_j^{-2}(1+|\lambda_j Dv_j|^2)^{((p-2)/2)(q-1)/p+(2q-p-2)/2p}|\lambda_j Dv_j||\lambda_j D\psi_j|\\&\quad\leqslant c\lambda_j^{-2}(1+|\lambda_j Dv_j|^2)^{((p-2)/2)(q-1)/p}|\lambda_j Dv_j|^{2(q-1)/p}|\lambda_j D\psi_j|\\&\quad\leqslant c\biggl(\biggl|\frac{V(\lambda_j Dv_j)}{\lambda_j}\biggr|^2+\lambda_j^{-2}|\lambda_j D\psi_j|^{p/(p+1-q)}\biggr).\end{aligned}$$

Thus, using $p/(p+1-q)\geqslant 2$, $q<p+1/n$, (4.17) and (4.18) (with $\kappa = 1/(p+1-q) < n/(n-1)$) we argue essentially as for III:

$$\begin{aligned}\mathrm{II}&\leqslant c\int_{B_{\tilde{s}_j}(x_0)\setminus B_{\tilde{r}_j}(x_0)}\biggl|\frac{V(\lambda_j Dv_j)}{\lambda_j}\biggr|^2\,\mathrm{d}x\\&\qquad+c\lambda_j^{-2}\int_{B_{\tilde{s}_j}(x_0)\setminus B_{\tilde{r}_j}(x_0)}(|\lambda_j D\psi_j|^2+|\lambda_j D\psi_j|^{p/(p+1-q)})\,\mathrm{d}x\\&\leqslant c\int_{B_{\tilde{s}_j}(x_0)\setminus B_{\tilde{r}_j}(x_0)}\biggl(\biggl|\frac{V(\lambda_j Dv_j)}{\lambda_j}\biggr|^2+\biggl|\frac{V(\lambda_j D\psi_j)}{\lambda_j}\biggr|^2\\&\qquad\qquad+\lambda_j^{-2}|V(\lambda_j D\psi_j)|^{2/(p+1-q)}\biggr)\,\mathrm{d}x\\&\leqslant c\biggl(Y_j+\biggl(\frac{\lambda_j^2 Y_j}{((1-\alpha)r)^n}\biggr)^{(1/(p+1-q))-1}Y_j\biggr)\\&\leqslant c\biggl(Y_j+\biggl(\frac{\lambda_j^2 Y_j}{((1-\alpha)r)^n}\biggr)^{1/(n-1)}Y_j\biggr).\end{aligned}$$

Case 2 ($p\leqslant 2(n-1)/n$). In this case (2.1) reads $q<(2n-1)p/(2n-2)$ and we have, in particular, $q\leqslant \frac{1}{2}p+1$. Again we will give an estimate for the integrand in II. In the case when $|Dv_j|\leqslant|D\psi_j|$, since V_{q-1} is non-decreasing we find

$$\biggl|\frac{V_{q-1}(\lambda_j Dv_j)}{\lambda_j}\biggr||D\psi_j|\leqslant\biggl|\frac{V_{q/2}(\lambda_j D\psi_j)}{\lambda_j}\biggr|^2$$

and in the case when $|Dv_j|>|D\psi_j|$ with Young's inequality and $q\leqslant\frac{1}{2}p+1$ we get

$$\begin{aligned}\biggl|\frac{V_{q-1}(\lambda_j Dv_j)}{\lambda_j}\biggr||D\psi_j|&=\lambda_j^{-2}(1+|\lambda_j Dv_j|^2)^{(p-2)/4+(2q-p-2)/4}|\lambda_j Dv_j||\lambda_j D\psi_j|\\&\leqslant\lambda_j^{-2}|V_{(2q-p)/2}(\lambda_j D\psi_j)||V(\lambda_j Dv_j)|\\&\leqslant\lambda_j^{-2}(|V_{(2q-p)/2}(\lambda_j D\psi_j)|^2+|V(\lambda_j Dv_j)|^2).\end{aligned}$$

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Arguing essentially as supplied above and using $q/p \leqslant (2q-p)/p < n/(n-1)$, we derive the following estimate for II in this case:

$$\begin{aligned}\mathrm{II} &\leqslant c\biggl(Y_j + \biggl(\frac{\lambda_j^2 Y_j}{((1-\alpha)r)^n}\biggr)^{((2q-p)/p)-1} Y_j + \biggl(\frac{\lambda_j^2 Y_j}{((1-\alpha)r)^n}\biggr)^{(q/p)-1} Y_j\biggr)\\ &\leqslant c\biggl(Y_j + \biggl(\frac{\lambda_j^2 Y_j}{((1-\alpha)r)^n}\biggr)^{1/(n-1)} Y_j\biggr).\end{aligned}$$

Estimation of I. It remains to control I. Here, employing $|\lambda_j Da_j| \leqslant 1$ for $j \geqslant j_0(\alpha, r)$, lemma 3.1(iv) and (4.17), we get

$$\begin{aligned}\mathrm{I} &\leqslant c\int_{B_{\tilde{s}_j}(x_0)\setminus B_{\tilde{r}_j}(x_0)} \biggl|\frac{V_{p-1}(\lambda_j Da_j)}{\lambda_j}\biggr| |D\psi_j|\,\mathrm{d}x\\ &\leqslant c\int_{B_{\tilde{s}_j}(x_0)\setminus B_{\tilde{r}_j}(x_0)} \biggl(\biggl|\frac{V(\lambda_j Da_j)}{\lambda_j}\biggr|^2 + \biggl|\frac{V(\lambda_j D\psi_j)}{\lambda_j}\biggr|^2\biggr)\,\mathrm{d}x\\ &\leqslant c\biggl(\int_{B_r(x_0)\setminus B_{\alpha r}(x_0)} \biggl|\frac{V(\lambda_j Da_j)}{\lambda_j}\biggr|^2\,\mathrm{d}x + Y_j\biggr).\end{aligned}$$

Collecting the estimates for I–III, we have proved that

$$\begin{aligned}&\int_{B_{\alpha r}(x_0)} \biggl|\frac{V(\lambda_j Dv_j)}{\lambda_j}\biggr|^2\,\mathrm{d}x\\ &\qquad\leqslant c\biggl(Y_j + \biggl(\frac{\lambda_j^2 Y_j}{((1-\alpha)r)^n}\biggr)^{1/(n-1)} Y_j\biggr) + c\int_{B_r(x_0)\setminus B_{\alpha r}(x_0)} \biggl|\frac{V(\lambda_j Da_j)}{\lambda_j}\biggr|^2\,\mathrm{d}x.\end{aligned}$$

By the Poincaré-type inequality (3.3), lemmas 3.1(i) and 3.2 and (4.3), we have

$$\begin{aligned}Y_j &\leqslant \lambda_j^{-2}\int_{B_r(x_0)} \biggl(|V(\lambda_j Dv_j)|^2 + \biggl|V\biggl(\frac{\lambda_j v_j}{(1-\alpha)r}\biggr)\biggr|^2\biggr)\,\mathrm{d}x\\ &\leqslant \biggl(1 + \frac{c}{(1-\alpha)^{\max\{2,p\}}}\biggr)\int_{B_r(x_0)} \biggl|\frac{V(\lambda_j Dv_j)}{\lambda_j}\biggr|^2\,\mathrm{d}x\\ &\leqslant \frac{c}{(1-\alpha)^{\max\{2,p\}}}.\end{aligned}$$

Combining the last two inequalities we find

$$\begin{aligned}&\int_{B_{\alpha r}(x_0)} \biggl|\frac{V(\lambda_j Dv_j)}{\lambda_j}\biggr|^2\,\mathrm{d}x\\ &\qquad\leqslant c\biggl(Y_j + \int_{B_r(x_0)\setminus B_{\alpha r}(x_0)} \biggl|\frac{V(\lambda_j Da_j)}{\lambda_j}\biggr|^2\,\mathrm{d}x\biggr) + c_{\alpha,r}\lambda_j^{2/(n-1)},\end{aligned}$$

where $c_{\alpha,r} > 0$ is a fixed constant depending, in particular, on α and r. The reader should note that, contrarily, the constants c in the preceding estimates do not depend on α or r. Applying lemma 3.1(ii), we deduce

$$\begin{aligned}
&\int_{B_{\alpha r}(x_0)} \left|\frac{V(\lambda_j(Du_j - Du))}{\lambda_j}\right|^2 \mathrm{d}x \\
&\quad\leqslant c\biggl[\int_{B_{\alpha r}(x_0)} \left|\frac{V(\lambda_j Dv_j)}{\lambda_j}\right|^2 \mathrm{d}x + \int_{B_r(x_0)} \left|\frac{V(\lambda_j(Du - Da_j))}{\lambda_j}\right|^2 \mathrm{d}x\biggr] \\
&\quad\leqslant c\biggl[\int_{B_r(x_0)\setminus B_{\alpha r}(x_0)} \left|\frac{V(\lambda_j(Du_j - Du))}{\lambda_j}\right|^2 \mathrm{d}x + \int_{B_r(x_0)} \left|\frac{V(\lambda_j(Du - Da_j))}{\lambda_j}\right|^2 \mathrm{d}x \\
&\qquad\qquad + \int_{B_r(x_0)\setminus B_{\alpha r}(x_0)} \left|\frac{V(\lambda_j Da_j)}{\lambda_j}\right|^2 \mathrm{d}x \\
&\qquad\qquad + \lambda_j^{-2} \int_{B_r(x_0)} \left|V\left(\frac{\lambda_j v_j}{(1-\alpha)r}\right)\right|^2 \mathrm{d}x\biggr] + c_{\alpha,r}\lambda_j^{2/(n-1)}.
\end{aligned}$$

By Widman's hole-filling trick, that is, adding

$$c\int_{B_{\alpha r}(x_0)} \left|\frac{V(\lambda_j(Du_j - Du))}{\lambda_j}\right|^2 \mathrm{d}x$$

on both sides, we finally arrive at the main estimate,

$$\begin{aligned}
&\int_{B_{\alpha r}(x_0)} \left|\frac{V(\lambda_j(Du_j - Du))}{\lambda_j}\right|^2 \mathrm{d}x \\
&\qquad\leqslant \theta\int_{B_r(x_0)} \left|\frac{V(\lambda_j(Du_j - Du))}{\lambda_j}\right|^2 \mathrm{d}x + \int_{B_r(x_0)} \left|\frac{V(\lambda_j(Du - Da_j))}{\lambda_j}\right|^2 \mathrm{d}x \\
&\qquad\quad + \int_{B_r(x_0)\setminus B_{\alpha r}(x_0)} \left|\frac{V(\lambda_j Da_j)}{\lambda_j}\right|^2 \mathrm{d}x \\
&\qquad\quad + \lambda_j^{-2} \int_{B_r(x_0)} \left|V\left(\frac{\lambda_j v_j}{(1-\alpha)r}\right)\right|^2 \mathrm{d}x + c_{\alpha,r}\lambda_j^{2/(n-1)} \qquad (4.20)
\end{aligned}$$

for $j \geqslant j_0(\alpha, r)$ with $\theta = c/(1+c) < 1$. We stress that θ does not depend on α or r.

Step 5 (strong convergence). Recalling that u is C^1 on B, it follows from (4.3) that

$$\lambda_j^{-2}\int_{B_\sigma} |V(\lambda_j(Du_j - Du))|^2 \,\mathrm{d}x$$

remains bounded as $j \to \infty$. Thus, there exists a non-negative Radon measure μ on $\overline{B_\sigma}$ such that, passing to subsequences again, we have

$$\lambda_j^{-2}|V(\lambda_j(Du_j - Du))|^2\mathcal{L}^n \overset{*}{\rightharpoonup} \mu \quad \text{weakly in the sense of measures on } \overline{B_\sigma}.$$

Introducing the affine function $a(x) := (u)_{x_0,r} + (Du)_{x_0,r}(x - x_0)$ we obviously have $a_j \to a$ and $\lambda_j^{-1}V(\lambda_j Da_j) \to Da$. Furthermore, setting $v := u - a$, we claim that

$$\int_{B_r(x_0)} \lambda_j^{-2}\left|V\left(\frac{\lambda_j(v_j - v)}{(1-\alpha)r}\right)\right|^2 \mathrm{d}x \to 0. \qquad (4.21)$$

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For $1 < p < 2$ we will now prove (4.21) following an argument of [13]. We choose $t \in (0,1)$ such that $\frac{1}{2} = t + (1-t)/p^{\#}$ holds (recall that $p^{\#} = 2n/(n-p) > 2$) and apply the interpolation inequality and the Sobolev type inequality (3.4) to get

$$\begin{aligned}\int_{B_r(x_0)} \left|\frac{V(\lambda_j(v_j - v))}{\lambda_j}\right|^2 \mathrm{d}x \\ &\leqslant \lambda_j^{2(t-1)} \left(\int_{B_r(x_0)} |v_j - v| \,\mathrm{d}x\right)^{2t} \left(\int_{B_r(x_0)} |V(\lambda_j(v_j - v))|^{p^{\#}} \,\mathrm{d}x\right)^{2(1-t)/p^{\#}} \\ &\leqslant \left(\int_{B_r(x_0)} |v_j - v| \,\mathrm{d}x\right)^{2t} \left(\int_{B_r(x_0)} \left|\frac{V(\lambda_j(Dv_j - Dv))}{\lambda_j}\right|^2 \mathrm{d}x\right)^{(1-t)}.\end{aligned}$$

By (4.3) and (4.10) the right-hand side converges to 0 for $j \to \infty$ and (4.21) is verified for $p < 2$. For $p \geqslant 2$, (4.21) follows from (4.10) and (4.3) by a simpler argument and we omit further details.

Returning to the general case, for every measurable subset A of $\overline{B_\sigma}$ we have

$$\begin{aligned}\mu(\operatorname{int} A) &\leqslant \liminf_{j\to\infty} \int_A \lambda_j^{-2} |V(\lambda_j(Du_j - Du))|^2 \,\mathrm{d}x \\ &\leqslant \limsup_{j\to\infty} \int_A \lambda_j^{-2} |V(\lambda_j(Du_j - Du))|^2 \,\mathrm{d}x \leqslant \mu(\bar{A}).\end{aligned}$$

Keeping this in mind and passing to the limit in (4.20), we obtain

$$\begin{aligned}\mu(B_{\alpha r}(x_0)) \leqslant \theta\mu(\overline{B_r(x_0)}) + \int_{B_r(x_0)} |Dv|^2 \,\mathrm{d}x + (1-\alpha^n) r^n |Da|^2 \\ + c \int_{B_r(x_0)} \left|\frac{v}{(1-\alpha)r}\right|^2 \mathrm{d}x.\end{aligned}$$

Here, for the treatment of the fourth term on the right-hand side we have used lemma 3.1(ii) and (4.21). Since $0 < \alpha < 1$ is arbitrary, we can, by virtue of a continuity argument, replace $\mu(B_{\alpha r}(x_0))$ by $\mu(\overline{B_{\alpha r}(x_0)})$ on the left-hand side of the previous inequality. Hence, dividing by r^n, we have established that

$$\alpha^n \frac{\mu(\overline{B_{\alpha r}(x_0)})}{\alpha^n r^n} \leqslant \theta \frac{\mu(\overline{B_r(x_0)})}{r^n} + \varepsilon_1(r) + |Da|^2(1-\alpha^n) + \frac{\varepsilon_2(r)}{(1-\alpha)^2}, \tag{4.22}$$

where we have set

$$\varepsilon_1(r) = \frac{1}{r^n} \int_{B_r(x_0)} |Dv|^2 \quad \text{and} \quad \varepsilon_2(r) = \frac{c}{r^{n+2}} \int_{B_r(x_0)} |v|^2 \,\mathrm{d}x.$$

Since u is C^1, we have $\varepsilon_1(r) + \varepsilon_2(r) \to 0$ and $Da \to Du(x_0)$ as $r \to 0^+$. Next we claim that

$$\liminf_{r\to 0^+} \frac{\mu(\overline{B_r(x_0)})}{r^n} = 0. \tag{4.23}$$

To prove (4.23), following [32], we first suppose that

$$\limsup_{r\to 0^+} \frac{\mu(\overline{B_r(x_0)})}{r^n} > 0.$$

Then, by an argument of [32, pp. 78–79] we can pass $r \to 0^+$ in (4.22), arriving at

$$\alpha^n \leqslant \theta + |Du(x_0)|^2(1-\alpha^n) \limsup_{r\to 0^+} \frac{r^n}{\mu(\overline{B_r(x_0)})}$$

for all $0 < \alpha < 1$. Thus, passing $\alpha \to 1^-$ (recall that $\theta < 1$ is independent of α) we get (4.23) in any case and for all $x_0 \in B_\sigma$. Hence, following [32] again, by Vitali's covering theorem we deduce that

$$\mu(\overline{B_\tau}) = 0,$$

which, in turn, implies the strong convergence stated in (4.13).

STEP 6 (conclusion). Noting that $(Du_j)_{0,\tau} \to (Du)_{0,\tau}$, from lemma 3.1(ii), (4.12) and (4.13) we deduce that

$$\begin{aligned}
\lim_{j\to\infty} \frac{1}{\lambda_j^2} \fint_{B_\tau} |V(\lambda_j(Du_j - (Du_j)_{0,\tau}))|^2 \,\mathrm{d}x \\
&\leqslant \lim_{j\to\infty} \frac{c}{\lambda_j^2} \fint_{B_\tau} [|V(\lambda_j(Du_j - Du))|^2 + |V(\lambda_j(Du - (Du)_{0,\tau}))|^2 \\
&\qquad + |V(\lambda_j((Du)_{0,\tau} - (Du_j)_{0,\tau}))|^2] \,\mathrm{d}x \\
&= c \fint_{B_\tau} |Du - (Du)_{0,\tau}|^2 \,\mathrm{d}x \\
&\leqslant C^* \tau^2
\end{aligned}$$

for some constant $C^* > 0$. Finally, the last inequality contradicts (4.4) if we choose $C = C^* + 1$ and the proof is finished. The reader should note that C and C^* depend only on n, N, p, ν_L and K_L. □

Once proposition 4.1 is established, theorem 2.1 follows by a well-known iteration argument and Campanato's integral characterization of the Hölder continuity. For further details see, for instance, [13, 21].

5. Proof of theorem 2.2

In this section we present the proof of theorem 2.2, modifying the proof of theorem 2.1 along the lines of [47].

First we recall some simple estimates for the non-degenerate p-energy

$$e_p(\xi) := (1 + |\xi|^2)^{p/2} \tag{5.1}$$

(see, for example, [47] for a proof).

LEMMA 5.1. *For $1 < p < \infty$, $L > 0$, $\xi \in \mathbb{R}^{Nn}$ with $|\xi| \leqslant L+1$, a ball $B_r(x_0)$ in $\mathbb{R}^n$ and $\varphi \in W_0^{1,p}(B_\rho(x_0), \mathbb{R}^N)$ we have*

$$\begin{aligned}
C_1^{-1} \int_{B_r(x_0)} |V(D\varphi)|^2 \,\mathrm{d}x &\leqslant \int_{B_r(x_0)} [e_p(\xi + D\varphi) - e_p(\xi)] \,\mathrm{d}x \\
&\leqslant C_1 \int_{B_r(x_0)} |V(D\varphi)|^2 \,\mathrm{d}x
\end{aligned}$$

for some constant $C_1 > 0$ depending only on p and L. □

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In the following lemmas we collect several properties of the relaxed functional $\mathcal{F}$. These lemmas have been proposed in [47] and rely heavily on (2.2), (2.3) and the measure and integral representation results obtained in [11,22]. Later in this section we will also apply (2.3) and the measure representation result [22, theorem 3.1] explicitly. Next we give a reformulation of [47, lemma 7.1]. Note that the growth condition imposed on Df in [47] follows from (H1) and the quasiconvexity of f.

Lemma 5.2. *We suppose that $f \in C^1$ is quasiconvex with (H1) and $1 < p \leqslant q < \min\{p+1, np/(n-1)\}$. Then, for $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ with $\mathcal{F}[u] < \infty$ and $\psi \in W^{1,p/(p+1-q)}(\Omega, \mathbb{R}^N)$ we have*

$$\mathcal{F}[u+\psi] - \mathcal{F}[u] = F[u+\psi] - F[u].$$

As in [47, lemma 7.3] we see that lemma 5.2 implies the validity of Euler's equation for $W^{1,\bar{q}}$ local minimizers of $\mathcal{F}$.

Lemma 5.3 (Euler's equation). *Let $1 \leqslant \bar{q} \leqslant \infty$. We suppose that $f \in C^1$ is quasiconvex with (H1) and $1 < p \leqslant q < \min\{p+1, np/(n-1)\}$. Then, every $W^{1,\bar{q}}$ local minimizer $\bar{u} \in W^{1,p}(\Omega, \mathbb{R}^N)$ of $\mathcal{F}$ is a weak solution of the Euler equation of F, i.e.*

$$\int_\Omega Df(D\bar{u}) D\varphi \,\mathrm{d}x = 0 \quad \textit{for all } \varphi \in C^\infty_{\mathrm{c}}(\Omega, \mathbb{R}^N).$$

Now we introduce the additional notation

$$\mathcal{F}[u; O] := \inf\left\{ \liminf_{k\to\infty} \int_O f(Du_k)\,\mathrm{d}x : W^{1,q}_{\mathrm{loc}}(O, \mathbb{R}^N) \ni u_k \rightharpoonup u \text{ weakly in } W^{1,p}(O, \mathbb{R}^N) \right\}$$

for open subsets O of Ω. We will need the next two lemmas, which can also be found in [47].

Lemma 5.4 ($W^{1,p}$-quasiconvexity). *Assume (H1) and (H2) with $1 < p \leqslant q < np/(n-1)$. Then, the following $W^{1,p}$-quasiconvexity condition holds for $\mathcal{F}$: for every ball $B_r(x_0)$ in $\mathbb{R}^n$, every $\xi \in \mathbb{R}^{Nn}$ and every $\varphi \in W^{1,p}(B_r(x_0), \mathbb{R}^N)$ with compact support in $B_r(x_0)$ we have*

$$\mathcal{F}[l_\xi + \varphi; B_r(x_0)] \geqslant \mathcal{F}[l_\xi; B_r(x_0)], \tag{5.2}$$

where we have set $l_\xi(x) := \xi x$.

Lemma 5.5 (additivity property). *Assume (H1) and (H2) with $1 < p \leqslant q < np/(n-1)$. We consider a ball $B_s(x_0) \subset\subset \Omega$ and $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ such that the boundary regularity condition*

$$\limsup_{\varepsilon\searrow 0} \frac{1}{\varepsilon} \int_{B_{s+\varepsilon}(x_0)\setminus B_{s-\varepsilon}(x_0)} |Du|^p \,\mathrm{d}x < \infty \tag{5.3}$$

holds. Then we have

$$\mathcal{F}[u; \Omega] = \mathcal{F}[u; B_s(x_0)] + \mathcal{F}[u; \Omega \setminus \overline{B_s(x_0)}].$$

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After these preparations we turn to the proof of theorem 2.2. As for theorem 2.1, it suffices to establish the following proposition, whose statement is completely analogous to proposition 4.1.

PROPOSITION 5.6. *Under the assumptions of theorem 2.2, for every $L > 0$ there is a constant $C > 0$ with the following property: for each $0 < \tau \leqslant \frac{1}{2}$ there exists a number $\varepsilon > 0$ such that the conditions*

$$|(D\bar{u})_{x,r}| \leqslant L, \quad r < \varepsilon \quad \textit{and} \quad E(x,r) < \varepsilon$$

for a ball $B_r(x) \subset\subset \Omega$ imply that

$$E(x,\tau r) < C\tau^2 E(x,r).$$

Sketch of the proof. We argue by contradiction. Assuming the proposition to be wrong, we proceed by blow-up as for proposition 4.1 and we will highlight only the necessary modifications in the proof. First we note that, by lemma 5.3, the Euler equation used in step 2 is available. Thus, the remaining modifications, which will be outlined now, concern only the handling of the quasiconvexity hypothesis and the minimizing property. We use the nomenclature of the proof of proposition 4.1 but with the following difference: we choose $\tilde{r}_j$, $\tilde{s}_j$ as in remark 3.6, avoiding the set

$$N_j := \left\{ t \in]\alpha r, r[: t \mapsto \int_{B_t(x_0)} |Du_j|^p \,\mathrm{d}x \text{ is not differentiable at } t \right\}.$$

Thus, u_j satisfies the condition (5.3) near $\partial B_{\tilde{s}_j}(x_0)$. As explained in the proof of [47, lemma 7.13] it is easy to see that the same condition holds for $a_j + \varphi_j$ and $u_j - \varphi_j$. We will use this fact later when applying lemma 5.5. Next we will rewrite the quasiconvexity hypothesis (H4) in an adequate form for our purposes. To this aim we introduce the auxiliary integrand

$$g(\xi) := f(\xi) - \frac{\nu_L}{C_1} e_p(\xi) \quad \text{for } \xi \in \mathbb{R}^{Nn},$$

where e_p is defined in (5.1) and C_1, γ and ν_L denote the constants from lemma 5.1, (H2) and (H4). Moreover, for $W^{1,p}$-functions w we set

$$\begin{aligned}
G[w] &:= \int_\Omega g(Dw)\,\mathrm{d}x, \\
\mathcal{G}[w] &:= \inf \left\{ \liminf_{k\to\infty} G[w_k] : W^{1,q}_{\mathrm{loc}}(\Omega,\mathbb{R}^N) \ni w_k \rightharpoonup w \text{ weakly in } W^{1,p}(\Omega,\mathbb{R}^N) \right\}, \\
\mathcal{F}_j[w] &:= \inf \left\{ \liminf_{k\to\infty} F_j[w_k] : W^{1,q}_{\mathrm{loc}}(B,\mathbb{R}^N) \ni w_k \rightharpoonup w \text{ weakly in } W^{1,p}(B,\mathbb{R}^N) \right\},
\end{aligned}$$

and we will also use the obvious modifications of this notation for open subsets O of Ω and B, respectively. Following an argument from the proof of [47, lemma 7.13] it is not difficult to see from the definitions of $\mathcal{F}$ and $\mathcal{G}$ that we have

$$\mathcal{G}[w;O] \leqslant \mathcal{F}[w;O] - \frac{\nu_L}{C_1} \int_O e_p(Dw)\,\mathrm{d}x$$

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for all open subsets O of Ω and all $w \in W^{1,p}(O, \mathbb{R}^N)$. In addition, from (H1) and (H2) we see that g satisfies the growth conditions

$$\left(\gamma - \frac{2^{p/2}\nu_L}{C_1}\right)|\xi|^p - \frac{2^{p/2}\nu_L}{C_1} \leqslant g(\xi) \leqslant \Gamma(1 + |\xi|^q)$$

for all $\xi \in \mathbb{R}^{Nn}$. Imposing the condition $2^{p/2}\nu_L < C_1\gamma$, which is clearly not restrictive, we infer that g satisfies (H1) and (H2) up to an additive constant. Obviously, this is sufficient to allow the application of lemma 5.4 to $\mathcal{G}$. Furthermore, we deduce from (H4) and lemma 5.1 that g is quasiconvex at all ξ with $|\xi| \leqslant L + 1$. In particular, recalling (2.3) this gives $\mathcal{G}[l_\xi; B_{r_j}(x_j)] = |B_{r_j}(x_j)|Qg(\xi) = |B_{r_j}(x_j)|g(\xi)$ for these ξ, where we have used the notation l_ξ from lemma 5.4. Consequently, applying lemma 5.4 to $\mathcal{G}$ we get

$$\begin{aligned} 0 &\leqslant \mathcal{G}[l_\xi + \varphi; B_{r_j}(x_j)] - \mathcal{G}[l_\xi; B_{r_j}(x_j)] \\ &\leqslant \mathcal{F}[l_\xi + \varphi; B_{r_j}(x_j)] - |B_{r_j}(x_j)|f(\xi) - \frac{\nu_L}{C_1}\int_{B_{r_j}(x_j)} [e_p(\xi + D\varphi) - e_p(\xi)]\,\mathrm{d}x \end{aligned}$$

for $|\xi| \leqslant L + 1$ and all $\varphi \in W^{1,p}(B_{r_j}(x_j), \mathbb{R}^N)$ with compact support in $B_{r_j}(x_j)$. By lemma 5.1 we conclude that

$$\int_{B_{r_j}(x_j)} |V(D\varphi)|^2\,\mathrm{d}x \leqslant c(\mathcal{F}[l_\xi + \varphi; B_{r_j}(x_j)] - |B_{r_j}(x_j)|f(\xi))$$

for all $\xi \in \mathbb{R}^{Nn}$ with $|\xi| \leqslant L + 1$ and all $\varphi \in W^{1,p}(B_{r_j}(x_j), \mathbb{R}^N)$ with compact support in $B_{r_j}(x_j)$. Rescaling gives us[1]

$$\int_B \left|\frac{V(\lambda_j D\varphi)}{\lambda_j}\right|^2 \mathrm{d}x \leqslant c(\mathcal{F}_j[l_\xi + \varphi; B] - |B|f_j(\xi)),$$

for all $\xi \in \mathbb{R}^{Nn}$ with $|\lambda_j\xi| \leqslant 1$ and for all $\varphi \in W^{1,p}(B, \mathbb{R}^N)$ with compact support in B. Next, we recall $\varphi_j \in W_0^{1,p}(B_{\tilde{s}_j}(x_0), \mathbb{R}^N)$ and $\mathcal{F}_j[a_j; O] = |O|f_j(Da_j)$ for all open subsets O of Ω; see step 3 in the proof of proposition 4.1 and (2.3). Using these facts, lemma 5.5 and the choice of $\tilde{s}_j$, we deduce from the previous inequality that

$$\int_{B_{\tilde{s}_j}(x_0)} \left|\frac{V(\lambda_j D\varphi_j)}{\lambda_j}\right|^2 \mathrm{d}x \leqslant c(\mathcal{F}_j[a_j + \varphi_j; B_{\tilde{s}_j}(x_0)] - |B_{\tilde{s}_j}(x_0)|f_j(Da_j)) \qquad (5.4)$$

for $j \geqslant j_2(r)$. In the following we will use the quasiconvexity hypothesis in the form (5.4). Next we turn to a reformulation of the minimizing property: we have assumed that $\bar{u}$ is a $W^{1,\bar{q}}$ local minimizer of $\mathcal{F}$ on Ω. By the measure representation theorem [22, theorem 3.1] this is easily seen to imply

$$\mathcal{F}[\bar{u}; B_{r_j}(x_j)] \leqslant \mathcal{F}[\bar{u} - \varphi; B_{r_j}(x_j)]$$

[1] Actually, this can be verified by a straightforward computation using the definitions of $\mathcal{F}_j$, F_j, f_j $\mathcal{F}$, F and the fact that the integral of the linear term in the definition of f_j is weakly continuous.

for all functions $\varphi \in W^{1,\bar q}(B_{r_j}(x_j), \mathbb{R}^N)$ with compact support in $B_{r_j}(x_j)$ and $\|D\varphi\|_{L^{\bar q}(B_{r_j}(x_j),\mathbb{R}^{Nn})} \leqslant \delta$. Rescaling as before, we get

$$\mathcal{F}_j[u_j; B] \leqslant \mathcal{F}_j[u_j - \varphi; B]$$

for all $\varphi \in W^{1,\bar q}(B, \mathbb{R}^N)$ with compact support in B and

$$\|D\varphi\|_{L^{\bar q}(B,\mathbb{R}^{Nn})} \leqslant \frac{\delta}{\lambda_j r_j^{n/\bar q}}.$$

Recalling that

$$\|D\varphi_j\|_{L^{\bar q}(B,\mathbb{R}^{Nn})} \leqslant \frac{\delta}{\lambda_j r_j^{n/\bar q}} \quad \text{for } j \geqslant j_1(\alpha)$$

(see step 3 in the proof of proposition 4.1), from lemma 5.5 and the choice of $\tilde s_j$ we get

$$\mathcal{F}_j[u_j; B_{\tilde s_j}(x_0)] \leqslant \mathcal{F}_j[u_j - \varphi_j; B_{\tilde s_j}(x_0)]. \tag{5.5}$$

Finally, recalling (4.15) and combining (5.4) and (5.5) we find

$$\begin{aligned}\int_{B_{\alpha r}(x_0)} \left|\frac{V(\lambda_j D\varphi_j)}{\lambda_j}\right|^2 \mathrm{d}x \leqslant c(&\mathcal{F}_j[u_j - \psi_j; B_{\tilde s_j}(x_0)] - \mathcal{F}_j[u_j; B_{\tilde s_j}(x_0)] \\ &+ \mathcal{F}_j[a_j + \psi_j; B_{\tilde s_j}(x_0)] - |B_{\tilde s_j}(x_0)| f_j(Da_j)).\end{aligned}$$

Since $q \leqslant p/(p+1-q) < np/(n-1)$ we see that

$$\psi_j \in W^{1,p/(p+1-q)}(B_{\tilde s_j}(x_0), \mathbb{R}^N) \subset W^{1,q}(B_{\tilde s_j}(x_0), \mathbb{R}^N)$$

from the estimates of lemma 3.4. Thus, we can apply lemma 5.2 and (2.3) to simplify the right-hand side of the preceding formula, deriving

$$\begin{aligned}\int_{B_{\alpha r}(x_0)} \left|\frac{V(\lambda_j D\varphi_j)}{\lambda_j}\right|^2 \mathrm{d}x \leqslant c\bigg(&\int_{B_{\tilde s_j}(x_0)} (f_j(Du_j - D\psi_j) - f_j(Du_j))\,\mathrm{d}x \\ &+ \int_{B_{\tilde s_j}(x_0)} (f_j(Da_j + D\psi_j) - f_j(Da_j))\,\mathrm{d}x\bigg)\end{aligned}$$

for $j \geqslant j_0(\alpha, r)$. Since the last inequality coincides with the estimate in (4.19) we can now argue exactly as in the proof of proposition 4.1. □

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