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ON CERTAIN COMMUTING FAMILIES OF RANK ONE OPERATORS

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1. Introduction

A study of nonselfadjoint algebras of Hilbert space operators was begun by considering special types of such algebras, namely those determined by a commuting family of rank one operators. A first step in this direction was made by Erdos in [**1**] and is continued more extensively in [**2**].

Here we examine the algebra of bounded linear operators on l^2 which have a specific set of vectors in l^2 as eigenvectors. We prove that this algebra is a maximal abelian subalgebra of $\mathscr{B}(l^2)$ determined by a commuting family of rank one operators, is topologically isomorphic to the Hilbert space l^2 and characterise those operators in it which have simple eigenvalues. Moreover, we describe the compact operators in the algebra and give a new class of compact operators which, although they have a complete system of eigenvectors, do not allow spectral synthesis.

Examples of maximal abelian reflexive algebras are given in [**1**] and [**2**]. In the sequel we give sufficient conditions for a compact operator in the algebra given in Section 6 of [**2**] to be reflexive and admit spectral synthesis. Finally we prove that none of the reflexive operators in the above mentioned algebras is subnormal or even similar to a subnormal operator and hence these examples are not covered by the results of R. F. Olin and J. E. Thomson in [**5**].

In this paper, the term *Hilbert space* will mean complex, separable, infinite dimensional Hilbert space, *subspace* will mean closed linear subspace and *operator* will mean bounded linear operator. We denote by $\mathscr{B}(H)$ the set of all operators on a Hilbert space H. The inner product is denoted by $\langle\ ,\ \rangle$. For any sets $\mathscr{A}$ of operators and $\mathscr{L}$ of subspaces we write $\operatorname{Lat}\mathscr{A}$ for the set of subspaces of H which are invariant under every member of $\mathscr{A}$, and $\operatorname{Alg}\mathscr{L}$ for the set of operators on H which leave every member of $\mathscr{L}$ invariant. We denote the commutant of $\mathscr{A}$ by $\mathscr{A}'$. If x and y are non-zero vectors, the operator $t \to \langle t, x\rangle y$ is denoted by $x \otimes y$. The strongly closed algebra generated by a commuting family $\mathscr{R}$ of rank one operators is denoted by $\mathscr{A}(\mathscr{R})$. An algebra $\mathscr{A}$ is called *reflexive* if $\mathscr{A} = \operatorname{Alg}\operatorname{Lat}\mathscr{A}$. An operator A is called *reflexive* if the weakly closed algebra generated by A and the identity I is reflexive. If V is a subset of H, the closed linear span of V will be denoted by $\operatorname{cls} V$. The range of an operator A is denoted by $\operatorname{ran} A$.

A sequence $\{x_n\}_1^\infty$ of vectors in a Hilbert space H is said to be *complete* if $\operatorname{cls}\{x_n : n \geqq 1\} = H$ and is called a *basis* of H if for every $x \in H$ there exists a unique sequence $\{a_n\}_1^\infty$ of scalars such that $x = \sum a_n x_n$. The following terminology is taken from

[4] (see also [2]). The sequence $\{x_n\}_1^\infty$ is called *minimal* if $x_n \notin \text{cls}\{x_m : n \neq m\}$ for every $n \geqq 1$. A sequence $\{x_n\}_1^\infty$ is minimal if and only if there exists a sequence $\{y_n\}_1^\infty$ bi-orthogonal to it; that is, a sequence such that $\langle x_n, y_m \rangle = 1$ for $n = m$ and $= 0$ for $n \neq m$. If $\{x_n\}_1^\infty$ is complete and minimal the bi-orthogonal sequence $\{y_n\}_1^\infty$ is unique. The sequence $\{x_n\}_1^\infty$ is said to be *strongly complete* if it is complete and minimal and for every $x \in H$, $x \in \text{cls}\{x_n : \langle x, y_n \rangle \neq 0\}$ where $\{y_n\}_1^\infty$ is the sequence bi-orthogonal to $\{x_n\}_1^\infty$. Any basis is strongly complete; the converse is false (see [2], Section 6). A vector $x \in H$ is called a *root vector* of $A \in \mathscr{B}(H)$ corresponding to the eigenvalue λ, if $(A - \lambda I)^n x = 0$ for some n. We shall say that $A \in \mathscr{B}(H)$ allows *spectral synthesis* if for any invariant subspace M of the operator A the set of root vectors of A contained in M is complete in M. A compact operator A is called *complete* if the system of all its root vectors corresponding to nonzero eigenvalues is complete in H and we shall say that A allows *strict spectral synthesis* if its restriction to any invariant subspace is a complete operator.

2. The algebra $\mathscr{R}'$

Consider the set of vectors $x_n = (1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, 0, \ldots)$, $n \geqq 1$, in l^2. Its unique bi-orthogonal sequence is $\{y_n\}_1^\infty$ where

$$y_n = (0, \ldots, 0, \underset{\substack{\uparrow \\ (n\text{th place})}}{n}, -(n+1), 0, \ldots), \quad n \geqq 1.$$

Clearly $\text{cls}\{x_n : n \geqq 1\} = l^2$, and so $\{x_n\}_1^\infty$ is complete, and obviously minimal. If $z_0 = (1, \frac{1}{2}, \frac{1}{3}, \ldots)$ then since $\langle z_0, y_n \rangle = 0$ for all n, $z_0 \notin \text{cls}\{y_n : n \geqq 1\}$ and hence $\{y_n\}_1^\infty$ is not complete. Also since $z_0 \notin \text{cls}\{x_n : \langle z_0, y_n \rangle \neq 0\}$, $\{x_n\}_1^\infty$ is not strongly complete.

In the following we examine the bounded linear operators on l^2 having the sequence $\{x_n\}_1^\infty$ as eigenvectors. The following two results are taken from [2].

Let $\mathscr{R}$ be a commuting family of rank one operators on a separable Hilbert space H. Let

$$X_0 = \text{cls}\{\text{ran}\, R : R \in \mathscr{R}\}, \quad Y_0 = \text{cls}\{\text{ran}\, R^* : R \in \mathscr{R}\}.$$

Proposition 1. *If either $X_0 = H$ or $Y_0 = H$ and $\mathscr{R}$ is closed under multiplication by non-zero scalars then $\mathscr{R}$ is maximal.*

Proposition 2. *If $\mathscr{R}$ is a maximal commuting family of rank one operators then any one of the conditions $X_0 = H$, $Y_0 = H$, $X_0 \cap Y_0 = (0)$ implies that $\mathscr{R}'$ is abelian.*

Let

$$\mathscr{R} = \{\lambda(y_n \otimes x_n), n \geqq 1, \lambda \in \mathbb{C} \backslash \{0\}\} \tag{1}$$

where $\{x_n\}_1^\infty$, $\{y_n\}_1^\infty$ are as defined above. The properties of the sequences $\{x_n\}_1^\infty$ and $\{y_n\}_1^\infty$ ensure that $\mathscr{R}$ is a commuting family and Proposition 1 shows it to be maximal. If $\mathscr{R}'$ is the commutant of $\mathscr{R}$ then, since $X_0 = \text{cls}\{x_n : n \geqq 1\} = l^2$ and $\mathscr{R}'$ is maximal abelian if and only if it is abelian, we have by Proposition 2 that $\mathscr{R}'$ is a maximal abelian subalgebra of $\mathscr{B}(l^2)$.

Let $T \in \mathscr{R}'$. Then each vector x_n is an eigenvector of T. The converse is also true. Indeed, suppose that there exists a sequence $\{\lambda_n\}_1^\infty$ of scalars such that $Tx_n = \lambda_n x_n$ where T is a bounded operator on l^2. Then

$$\begin{aligned}\langle T^* y_m - \bar{\lambda}_m y_m, x_n\rangle &= \langle y_m, \lambda_n x_n\rangle - \bar{\lambda}_m \langle y_m, x_n\rangle \\ &= (\bar{\lambda}_n - \bar{\lambda}_m)\langle y_m, x_n\rangle \\ &= 0\end{aligned}$$

for every m, n and since $\operatorname{cls}\{x_n : n \geqq 1\} = l^2$ we have $T^* y_m = \bar{\lambda}_m y_m$. Hence

$$T(y_m \otimes x_m) = y_m \otimes Tx_m = \lambda_m (y_m \otimes x_m)$$

and

$$(y_m \otimes x_m) T = T^* y_m \otimes x_m = \lambda_m (y_m \otimes x_m)$$

for all m. That is, T commutes with all members of $\mathscr{R}$ and so $T \in \mathscr{R}'$.

Let $\{\phi_n\}_1^\infty$ be the standard orthonormal basis for l^2 and for $T \in \mathscr{R}'$ consider the matrix representation of T with respect to the basis $\{\phi_n\}_1^\infty$. We can easily see, since each x_n is an eigenvector of T, that this matrix is of the form

$$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & \cdots \\ 0 & s_2 & \frac{1}{2}a_3 & \frac{1}{2}a_4 & \frac{1}{2}a_5 & \cdots \\ 0 & 0 & s_3 & \frac{1}{3}a_4 & \frac{1}{3}a_5 & \cdots \\ 0 & 0 & 0 & s_4 & \frac{1}{4}a_5 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \end{bmatrix}$$

where $\{a_n\}_1^\infty$ is a sequence of complex numbers and

$$s_n = \sum_{k=1}^{n} \frac{1}{k} a_k.$$

The following result shows for which sequences $\{a_n\}_1^\infty$ of complex numbers the corresponding operators on l^2 are bounded.

Proposition 3. *Let $\{a_n\}_1^\infty$ be a sequence of complex numbers and let $s_n = \sum_{k=1}^{n} (1/k) a_k$. If*

$$\eta_n = s_n \xi_n + \frac{1}{n} \sum_{m=n+1}^{\infty} a_m \xi_m$$

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then the map $T:l^2\to l^2$ such that

$$(\xi_1,\xi_2,\xi_3,\dots,\xi_n,\dots)\to(\eta_1,\eta_2,\eta_3,\dots,\eta_n,\dots)$$

defines a bounded linear operator on l^2 if and only if $a=\{a_n\}_1^\infty$ belongs to l^2.

Proof. Suppose that T defined as above is a bounded operator. Then, since

$$\begin{aligned}\langle T^*\phi_1,\phi_n\rangle&=\langle\phi_1,T\phi_n\rangle\\&=\bar{a}_n\end{aligned}$$

and

$$\begin{aligned}\sum_{n=1}^{\infty}|\bar{a}_n|^2&=\sum_{n=1}^{\infty}|\langle T^*\phi_1,\phi_n\rangle|^2\\&=\|T^*\phi_1\|^2<\infty\end{aligned}$$

we have that $\{\bar{a}_n\}_1^\infty\in l^2$ and hence $a\in l^2$.

Conversely, let $a=\{a_n\}_1^\infty\in l^2$ and let D be the diagonal operator defined by $D\phi_n=s_n\phi_n$, $n\geqq 1$. Then

$$\begin{aligned}|s_n|&=\left|\sum_{k=1}^{n}\frac{1}{k}a_k\right|\\&\leqq\left(\sum_{k=1}^{n}\frac{1}{k^2}\right)^{1/2}\left(\sum_{k=1}^{n}|a_k|^2\right)^{1/2}\\&\leqq\frac{\pi\sqrt{6}}{6}\|a\|.\end{aligned}\tag{2}$$

Hence $\{s_n\}_1^\infty$ is a bounded sequence and consequently D is a bounded operator. So it is enough to show that $A=T-D$ is bounded. But A maps $x=(\xi_1,\xi_2,\xi_3,\dots,\xi_n,\dots)$ into $\{\eta_1,\eta_2,\eta_3,\dots,\eta_n,\dots)$ where

$$\eta_n=\frac{1}{n}\sum_{m=n+1}^{\infty}a_m\xi_m.$$

Therefore

$$\begin{aligned}\|Ax\|^2&=\sum_{n=1}^{\infty}\frac{1}{n^2}\left|\sum_{m=n+1}^{\infty}a_m\xi_m\right|^2\\&\leqq\sum_{n=1}^{\infty}\frac{1}{n^2}\left(\sum_{m=n+1}^{\infty}|a_m|^2\right)\left(\sum_{m=n+1}^{\infty}|\xi_m|^2\right)\end{aligned}$$

$$\leqq \|a\|^2 \|x\|^2 \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$= \frac{\pi^2}{6} \|a\|^2 \|x\|^2$$

and so $\|A\| \leqq (\pi\sqrt{6}/6)\|a\|$. Hence T is bounded and using (2) we get

$$\begin{aligned} \|T\| &\leqq \|D\| + \|S\| \\ &\leqq \sup_n |s_n| + \frac{\pi\sqrt{6}}{6}\|a\| \\ &\leqq \frac{\pi\sqrt{6}}{6}\|a\| + \frac{\pi\sqrt{6}}{6}\|a\| \\ &= \frac{\pi\sqrt{6}}{3}\|a\|. \end{aligned} \tag{3}$$

Corollary 4. *The algebra $\mathscr{R}'$ and the Hilbert space l^2 are topologically isomorphic (where $\mathscr{R}'$ is considered with the norm topology).*

Proof. Proposition 3 shows that there exists a linear one-to-one map ψ from l^2 onto $\mathscr{R}'$. So we have to show that both ψ and ψ^{-1} are bounded. If T corresponds to $a \in l^2$ and $\bar{a} = \{\bar{a}_n\}_1^\infty$, then

$$\begin{aligned} \|T\bar{a}\|^2 &= \sum_{n=1}^{\infty} \left| s_n \bar{a}_n + \frac{1}{n} \sum_{m=n+1}^{\infty} a_m \bar{a}_m \right|^2 \\ &\geqq \left| s_1 \bar{a}_1 + \sum_{m=2}^{\infty} |a_m|^2 \right|^2 \\ &= \|a\|^4 \end{aligned}$$

and hence

$$\begin{aligned} \|T\| &= \sup\{\|Tx\|, x \in l^2, \|x\| = 1\} \\ &\geqq \frac{\|T\bar{a}\|}{\|a\|} \\ &\geqq \|a\|. \end{aligned} \tag{4}$$

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Comparing (3) and (4) we have

$$\|a\| \leqq \|T\| \leqq \frac{\pi\sqrt{6}}{3}\|a\|$$

which implies the continuity of ψ and ψ^{-1}.

Remark. Let $\mathcal{R}$ be a commuting family of rank one operators and let $\mathcal{A}(\mathcal{R})$ be the strongly closed algebra generated by $\mathcal{R}$. It is proved in [2] that:

(i) $I \in \mathcal{A}(\mathcal{R})$ implies $\operatorname{cls}\{\operatorname{ran} R : R \in \mathcal{R}\} = \operatorname{cls}\{\operatorname{ran} R^* : R \in \mathcal{R}\} = H$ where I is the identity operator.

(ii) $\mathcal{A}(\mathcal{R})$ is maximal abelian if and only if $I \in \mathcal{A}(\mathcal{R})$.

Now if $\mathcal{R}$ is as in (1), then the corresponding strongly closed algebra $\mathcal{A}(\mathcal{R})$ is not maximal, since otherwise $I \in \mathcal{A}(\mathcal{R})$ and we must have $\operatorname{cls}\{y_n : n \geqq 1\} = l^2$ which is not true. Hence $\mathcal{A}(\mathcal{R})$ is a proper subset of $\mathcal{R}'$.

Next we describe the compact operators of $\mathcal{R}'$.

Proposition 5. *Let T be the operator on l^2 determined by the sequence $\{a_n\}_1^\infty$ as in Proposition 3. Then T is compact if and only if $s_n \to 0$ as $n \to \infty$.*

Proof. Suppose T is compact. Then, since each s_n is an eigenvalue of T, we have $s_n \to 0$ as $n \to \infty$.

Conversely, if D is the diagonal operator defined by $D\phi_n = s_n\phi_n$, where $\{\phi_n\}_1^\infty$ is the usual basis for l^2, then $s_n \to 0$ as $n \to \infty$ implies that D is compact. Let $A = T - D$. It is sufficient to show that A is compact. Define A_N, $N \geqq 2$ by $A_N x = y$ where, if $x = \{\xi_n\}_1^\infty$ and $y = \{\eta_n\}_1^\infty$,

$$\eta_n = \begin{cases} \dfrac{1}{n} \displaystyle\sum_{m=n+1}^{N} a_m \xi_m & n \leqq N-1 \\ 0 & n \geqq N \end{cases}$$

For every N, A_N is finite rank operator and $(A - A_N)x = y$ where $\eta_n = (1/n)\sum_{m=n+1}^\infty b_m \xi_m$ and

$$b_m = \begin{cases} 0 & m \leqq N \\ a_m & m \geqq N+1. \end{cases}$$

If $b = \{b_n\}_1^\infty$ then by (3) in the proof of Proposition 3

$$\|A - A_N\| \leqq \frac{\pi\sqrt{6}}{3}\|b\|$$

$$=\frac{\pi\sqrt{6}}{3}\left(\sum_{m=N+1}^{\infty}|a_m|^2\right)^{1/2}.$$

Now $a\in l^2$ implies $\sum_{m=N+1}^{\infty}|a_m|^2\to 0$ as $N\to\infty$ and therefore $A_N\to A$ in norm as $N\to\infty$. Hence A is a norm limit of finite rank operators and therefore it is compact.

We can easily find a compact operator in $\mathscr{R}'$. Let $a_1=1$ and $a_n=-1/(n-1)$, $n\geqq 2$. Then $s_n=1/n$ and hence $s_n\to 0$ as $n\to\infty$. So by Proposition 5 the operator T corresponding to the sequence $\{a_n\}_1^\infty$ is compact.

Corollary 6. *Let T be the operator on l^2 determined by the sequence $\{a_n\}_1^\infty$ as in Proposition 3. Then T is compact if and only if the vector $a=\{a_n\}_1^\infty$ is orthogonal to the vector $z_0=(1,\frac{1}{2},\frac{1}{3},\ldots)$.*

Proof. Obviously $s_n\to 0$ as $n\to\infty$ if and only if $\sum_{k=1}^{\infty}(1/k)a_k=0$ which is equivalent to the fact that a is orthogonal to z_0.

Remark. A simple calculation shows that for any $T\in\mathscr{R}'$ the vector z_0 is an eigenvector of T with corresponding eigenvalue $\sum_{k=1}^{\infty}(1/k)a_k$, where $\{a_n\}_1^\infty$ is the sequence determining the operator T.

Let

$$z_n=z_0-x_n$$
$$=\Big(0,0,\ldots,\underset{\substack{\uparrow\\(n+1)\text{th place}}}{\frac{1}{n+1}},\frac{1}{n+2},\ldots\Big)$$

We have the following:

Proposition 7. *Let T be an operator on l^2 determined by the sequence $a=\{a_n\}_1^\infty$. Then $z_n=z_0-x_n$ is an eigenvector of T if and only if a is orthogonal to z_n. When z_n is an eigenvector of T the corresponding eigenvalue is $s_n=\sum_{k=1}^{n}(1/k)a_k$.*

Proof. Since $x_n=z_0-z_n$, $Tz_0=(\sum_{k=1}^{\infty}(1/k)a_k)z_0$ *and* $Tx_n=s_nx_n$ we have

$$Tz_n=Tz_0-Tx_n$$
$$=\left(\sum_{k=1}^{\infty}\frac{1}{k}a_k\right)z_0-\left(\sum_{k=1}^{n}\frac{1}{k}a_k\right)x_n$$
$$=\left(\sum_{k=1}^{n}\frac{1}{k}a_k\right)z_n+\left(\sum_{k=n+1}^{\infty}\frac{1}{k}a_k\right)z_0$$
$$=s_nz_n+\left(\sum_{k=n+1}^{\infty}\frac{1}{k}a_k\right)z_0. \tag{5}$$

The last equality shows that $Tz_n = s_n z_n$ if and only if $(\sum_{k=n+1}^{\infty} (1/k) a_k) z_0 = \{0\}$; equivalently $\sum_{k=n+1}^{\infty} (1/k) a_k = 0$. This is also equivalent to a being orthogonal to z_n, and the proof is complete.

Proposition 8. *Let T be an operator on l^2 determined by the sequence $a = \{a_n\}_1^{\infty}$. If $s_m \neq s_n$ for $m \neq n$ and the vector $a = \{a_n\}_1^{\infty}$ is not orthogonal to any of the vectors z_n, $n \geqq 1$ then T has simple eigenvalues.*

Proof. It is enough to prove that the only eigenvectors of T are the non-zero scalar multiples of the vectors x_n, $n \geqq 1$ and z_0. Suppose $Tx = \lambda x$ with $x = (\xi_1, \xi_2, \xi_3, \ldots)$, $x \notin \operatorname{cls}\{z_0\}$ and let r be the smallest positive integer such that $r\xi_r \neq (r+1)\xi_{r+1}$. Then

$$\lambda \xi_r = s_r \xi_r + \frac{1}{r} \sum_{m=r+1}^{\infty} a_m \xi_m.$$

Equivalently

$$\lambda r \xi_r = r s_r \xi_r + \sum_{m=r+1}^{\infty} a_m \xi_m. \tag{6}$$

Also

$$\lambda \xi_{r+1} = s_{r+1} \xi_{r+1} + \frac{1}{r+1} \sum_{m=r+2}^{\infty} a_m \xi_m$$

$$= s_r \xi_{r+1} + \frac{1}{r+1} \sum_{m=r+1}^{\infty} a_m \xi_m.$$

Equivalently

$$\lambda (r+1) \xi_{r+1} = (r+1) s_r \xi_{r+1} + \sum_{m=r+1}^{\infty} a_m \xi_m. \tag{7}$$

Subtracting (7) from (6) we get

$$\lambda (r\xi_r - (r+1)\xi_{r+1}) = s_r (r\xi_r - (r+1)\xi_{r+1})$$

which implies $\lambda = s_r$. Also

$$\lambda \xi_{r+2} = s_{r+2} \xi_{r+2} + \frac{1}{r+2} \sum_{m=r+3}^{\infty} a_m \xi_m$$

$$= s_{r+1} \xi_{r+2} + \frac{1}{r+2} \sum_{m=r+2}^{\infty} a_m \xi_m.$$

Equivalently

$$\lambda(r+2)\xi_{r+2}=(r+2)s_{r+1}\xi_{r+2}+\sum_{m=r+2}^{\infty}a_m\xi_m. \tag{8}$$

Subtracting (8) from (7) we have

$$\lambda[(r+1)\xi_{r+1}-(r+2)\xi_{r+2}]=s_r(r+1)\xi_{r+1}+a_{r+1}\xi_{r+1}-(r+2)s_{r+1}\xi_{r+2}$$

that is

$$\lambda[(r+1)\xi_{r+1}-(r+2)\xi_{r+2}]=s_{r+1}[(r+1)\xi_{r+1}-(r+2)\xi_{r+2}].$$

But $\lambda=s_r$ and by hypothesis $s_r\neq s_{r+1}$. Therefore $(r+1)\xi_{r+1}=(r+2)\xi_{r+2}$. Using the fact that $s_r\neq s_{r+k}$, $k\geqq 1$ by induction we get

$$\xi_{r+k}=\frac{r+1}{r+k}\xi_{r+1}, \quad k\geqq 1. \tag{9}$$

Now from (7) and $\lambda=s_r$ we have $\sum_{m=r+1}^{\infty}a_m\xi_m=0$ and from this, using (9)

$$\sum_{m=r+1}^{\infty}a_m\frac{r+1}{m}\xi_{r+1}=0$$

and so $\xi_{r+1}(\sum_{m=r+1}^{\infty}(1/m)a_m)=0$. Since by hypothesis $\sum_{m=r+1}^{\infty}(1/m)a_m\neq 0$ we have $\xi_{r+1}=0$ and consequently $\xi_{r+k}=0$ for all $k\geqq 1$. Therefore x is a scalar multiple of $x_r=(1,\frac{1}{2},\frac{1}{3},\ldots,(1/r),0,\ldots)$.

Remark. The condition $s_n\neq s_m$ for $m\neq n$ implies that the vector $a=\{a_n\}_1^{\infty}$ could be orthogonal to at most one of the vectors z_n, $n\geqq 1$, for if a is orthogonal to z_n and z_m with $n>m$, say, then

$$\begin{aligned}0&=\sum_{k=m+1}^{\infty}\frac{1}{k}a_k\\&=\sum_{k=m+1}^{n}\frac{1}{k}a_k+\sum_{k=n+1}^{\infty}\frac{1}{k}a_k\\&=\sum_{k=m+1}^{n}\frac{1}{k}a_k\\&=s_n-s_m\end{aligned}$$

which implies $s_n=s_m$.

We have the following:

Corollary 9. *Let T be an operator on l^2 determined by the sequence $\{a_n\}_1^\infty$. If $s_n \neq s_m$ for $m \neq n$ then the only eigenvectors of T are the non-zero scalar multiples of the vectors z_0, x_n, $n \geqq 1$ and possibly one of the vectors z_n, $n \geqq 1$.*

Proof. Use Propositions 7 and 8 and previous remark.

Remark. If T is a compact operator in $\mathscr{R}'$ then $Tz_0 = 0$ and hence $\ker(T)$ is not trivial. If T satisfies also the conditions of Proposition 8 then $\ker(T)$ is the subspace generated by the vector z_0. Indeed, let $Tx = 0$ for some $x \in l^2$, $0 \neq x = (\xi_1, \xi_2, \xi_3, \ldots)$. Then

$$\eta_n = s_n \xi_n + \frac{1}{n} \sum_{m=n+1}^{\infty} a_m \xi_m = 0 \quad \text{for all} \quad n \geqq 1.$$

So $\eta_1 = \eta_2$ implies $a_1 (\xi_2 - \frac{1}{2}\xi_1) = 0$. Since $\{a_n\}_1^\infty$ is orthogonal to z_0 we must have $a_1 \neq 0$ otherwise $\{a_n\}_1^\infty$ will be orthogonal to z_1 contradicting our hypothesis. Therefore $\xi_2 = \frac{1}{2}\xi_1$. Also since $\{a_n\}_1^\infty$ is not orthogonal to any of z_n, $n \geqq 1$, we have $s_n \neq 0$ for every $n \geqq 1$. Hence an induction argument shows that $\xi_n = (1/n)\xi_1$ for all $n \geqq 1$. That is, x is a multiple of z_0.

Now we give a new class of compact operators which have simple eigenvalues and a complete sequence of eigenvectors and do not allow strict spectral synthesis. We shall use the following result from [**4**].

Theorem 10. *Let A be a compact operator all of whose non-zero eigenvalues are simple, and let $\{x_n\}_1^\infty$ be the corresponding sequence of eigenvectors. The operator A allows strict spectral synthesis if and only if $\{x_n\}_1^\infty$ is strongly complete. If* $\ker(A) = 0$ *the word "strict" can be omitted.*

Corollary 11. *If T is a compact operator in $\mathscr{R}'$ satisfying the conditions of Proposition* 8, *then T does not allow strict spectral synthesis.*

Proof. Immediate by Theorem 10 and Proposition 8.

3. A reflexivity result

Let $\{\phi_n\}_1^\infty$ be, as usual, the standard orthonormal basis for l^2. Put

$$f_n = \sum_{m=1}^{n} \phi_m \quad \text{and} \quad e_n = \phi_n - \phi_{n+1} \quad \text{for each} \quad n \geqq 1.$$

Then the sequences $\{f_n\}_1^\infty$ and $\{e_n\}_1^\infty$ are bi-orthogonal and each is complete and minimal. Moreover it is shown in [**2**] that $\{f_n\}_1^\infty$ is strongly complete and hence so is $\{e_n\}_1^\infty$. Also if

$$\mathscr{R} = \{\lambda(e_n \otimes f_n) : \lambda \in \mathbb{C} \backslash \{0\}, n \geqq 1\}$$

and $\mathscr{A}(\mathscr{R})$ is the strongly closed algebra generated by $\mathscr{R}$, then $\mathscr{A}(\mathscr{R}) = \mathscr{R}'$ is maximal abelian. We shall use the following result from [**2**].

Proposition 12. *Let $\{a_n\}_1^\infty$ be a sequence of complex numbers and let $s_n = \sum_{m=1}^n a_m$. If*

$$\eta_n = s_n \xi_n + \sum_{m=n+1}^{\infty} a_m \xi_m,$$

then the mapping $(\xi_1, \xi_2, \xi_3, \ldots, \xi_n, \ldots) \to (\eta_1, \eta_2, \eta_3, \ldots, \eta_n, \ldots)$ defines a bounded linear operator A on l^2 if and only if

(i) *the sequence $\{s_n\}_1^\infty$ is bounded,*

and

(ii) $\sup_n n \sum_{m=n+1}^{\infty} |a_m|^2 < \infty$.

The operator A is compact if and only if

(iii) *$s_n \to 0$ as $n \to \infty$;*

and

(iv) *$n \sum_{m=n+1}^{\infty} |a_m|^2 \to 0$ as $n \to \infty$.*

The matrix picture of this new operator is

$$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & \cdots \\ 0 & s_2 & a_3 & a_4 & a_5 & \cdots \\ 0 & 0 & s_3 & a_4 & a_5 & \cdots \\ 0 & 0 & 0 & s_4 & a_5 & \cdots \\ 0 & 0 & 0 & 0 & s_5 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

It is shown in [2] that the norm of A is at most

$$\sup_n |s_n| + \left(6 \sup_n n \sum_{m=n+1}^{\infty} |a_m|^2 \right)^{1/2}.$$

Proposition 13. *Let A be a compact operator on l^2 corresponding to a sequence $\{a_n\}_1^\infty$ as in Proposition 12. If the sequence $\{s_n\}_1^\infty$ of partial sums is real, strictly monotonic and $s_n \leqq M/\sqrt{n}$, $n \geqq 1$ where M is a positive constant then the operator A is reflexive and admits spectral synthesis.*

Proof. We may assume $s_n > 0$ for all n since we can consider $-A$ instead of A. Define $K_n = \sum_{m=1}^n s_m R_m$ for every $n \in \mathbb{N}$, where $R_m = e_m \otimes f_m$, $m \in \mathbb{N}$. Then $K_n \in \mathscr{A}(\mathscr{R})$ for every $n \in \mathbb{N}$ and for each integer $n \geqq 1$, K_n corresponds, via the definition in Proposition 12, to the sequence $\{a'_m\}_1^\infty$ with

$$a'_m = \begin{cases} a_m & m \leqq n \\ -s_n & m = n+1 \\ 0 & m > n+1 \end{cases}$$

If $\{s'_m\}_1^\infty$ is the corresponding sequence of partial sums, then

$$s'_m = \begin{cases} s_m & m \leqq n \\ 0 & m > n \end{cases}.$$

It follows from Proposition 12 that for each n

$$\|K_n\| \leqq \sup_k |s'_k| + \sqrt{6}\left\{\sup_k k \,.\sum_{m=k+1}^{\infty} |a'_m|^2\right\}^{1/2}$$

$$= \sup_{k \leqq n} |s_k| + \sqrt{6}\left\{\sup_{k \leqq n}\left[k \,.\sum_{m=k+1}^{n} |a_m|^2 + k|s_n|^2\right]\right\}^{1/2}. \tag{10}$$

Since A is a compact bounded operator in $\mathscr{A}(\mathscr{R})$, there exists a positive constant M_1 such that $k\sum_{m=k+1}^{\infty}|a_m|^2 < M_1$ for all $k \geqq 1$. Also by hypothesis, if $k \leqq n$,

$$k|s_n|^2 \leqq n s_n^2$$

$$\leqq n(M/\sqrt{n})^2 = M^2.$$

Hence (10) implies

$$\|K_n\| \leqq M + \sqrt{6}[M_1 + M^2]^{1/2}.$$

That is, the sequence $\{K_n\}_1^\infty$ of operators is norm bounded. Also for $n > m$

$$K_n f_m = s_m f_m$$

$$= A f_m$$

and so far each fixed m, the sequence $\{K_n f_m\}_1^\infty$ converges to Af_m. But the sequence $\{f_m\}_1^\infty$ is complete in l^2 and $\{K_n\}_1^\infty$ is norm bounded. This implies that $\{K_n\}_1^\infty$ converges strongly to A. Indeed, let $x \in l^2$. Then for a given $\varepsilon > 0$ there exists an integer r such that

$$\left\|x - \sum_{i=1}^{r} \lambda_i f_i\right\| < \varepsilon \quad \text{where} \quad \lambda_i \in \mathbb{C}, \quad i = 1, 2, \ldots, r.$$

Let $n > r$. Then

$$\|K_n x - Ax\| = \left\|K_n x - \sum_{i=1}^{r} \lambda_i K_n f_i + \sum_{i=1}^{r} \lambda_i K_n f_i - Ax\right\|$$

$$\leqq \|K_n\| \left\|x - \sum_{i=1}^{r} \lambda_i f_i\right\| + \left\|\sum_{i=1}^{r} \lambda_i A f_i - Ax\right\|$$

$$\leqq \varepsilon(\|K_n\| + \|A\|)$$

which implies $\|K_n x - Ax\| \to 0$ as $n \to \infty$, since $\{K_n\}_1^\infty$ is norm bounded. Hence

$$A = \sum_{n=1}^{\infty} s_n R_n \quad \text{in the strong operator topology.}$$

We show now that the strongly closed algebra $\mathscr{A}$ generated by A and the identity is equal to $\mathscr{A}(\mathscr{R})$. It is enough to show that $R_n \in \mathscr{A}$ for every $n \in \mathbb{N}$. Fix $x \in l^2$. Then, since $A = \sum_{n=1}^\infty s_n R_n$ (strongly) and $R_m R_n = \delta_{mn} R_n$, we have

$$\begin{aligned} \left\| \left(\frac{A}{s_1}\right)^k x - R_1 x \right\| &= \left\| \frac{1}{s_1} \sum_{n=2}^{\infty} \left(\frac{s_n}{s_1}\right)^{k-1} s_n R_n x \right\| \\ &\leqq \frac{1}{s_1} \left(\frac{s_n}{s_1}\right)^{k-1} \left\| \sum_{n=2}^{\infty} s_n R_n x \right\|. \end{aligned} \tag{11}$$

Since $\sum_{n=1}^\infty s_n R_n x = Ax$ the sequence $\{\sum_{n=k}^\infty s_n R_n x\}_{k=1}^\infty$ converges to zero and so it is bounded. Hence the right hand side of (11) tends to zero as $k \to \infty$. This implies that $R_1 \in \mathscr{A}$. Now if we put $A_1 = A - s_1 R_1$ then

$$\left\| \left(\frac{A_1}{s_2}\right)^k x - R_2 x \right\| \leqq \frac{1}{s_2} \left(\frac{s_3}{s_2}\right)^{k-1} \left\| \sum_{n=3}^{\infty} s_n R_n x \right\|$$

which implies $\|(A_1/s_2)^k x - R_2 x\| \to 0$ as $k \to \infty$ and therefore $R_2 \in \mathscr{A}$.

Using induction we get $R_n \in \mathscr{A}$ for all $n \in \mathbb{N}$ and so $\mathscr{A} = \mathscr{A}(\mathscr{R})$. Since $\mathscr{R}$ is a commuting family, we have $\operatorname{lat} \mathscr{R} = \operatorname{lat} \mathscr{A}(\mathscr{R})$ and since $\mathscr{A}(\mathscr{R})$ is maximal abelian Theorem 5.3 in [2] implies

$$\mathscr{A}(\mathscr{R}) = \operatorname{Alg\,Lat} \mathscr{R} = \operatorname{Alg\,Lat} \mathscr{A}(\mathscr{R}).$$

Hence $\mathscr{A}(\mathscr{R})$ is reflexive and so is $\mathscr{A}$. Finally from Corollary 6.5 of [2] it is obvious that A admits spectral synthesis.

Corollary 14. *Let A be a compact operator on l^2 determined by the sequence $\{a_n\}_1^\infty$ as in Proposition 12. If $\{s_n\}_1^\infty$ is real, strictly monotonic and $\sum_{n=1}^\infty s_n < \infty$ then A is reflexive operator and admits spectral synthesis.*

Proof. Since we can suppose $s_n > 0$, $n \geqq 1$ and since then $ns_n \leqq \sum_{k=1}^n s_k$ and $\sum_{k=1}^\infty s_k < \infty$ there exists a constant $M > 0$ such that $ns_n \leqq M$ for all n. Now use Proposition 13.

Remark. It is shown in [2] that the sequence $\{G_k\}_1^\infty$, where

$$\begin{aligned} G_k &= \frac{1}{k} \sum_{m=1}^{k} \sum_{n=1}^{m} e_n \otimes f_n \\ &= \frac{1}{k} \sum_{m=1}^{k} \sum_{n=1}^{m} R_n \end{aligned}$$

tends strongly to the identity I. Since $G_k \in \mathscr{A}(\mathscr{R})$, $k \geqq 1$ for any $A \in \mathscr{A}(\mathscr{R})$ the sequence $\{AG_k\}_1^\infty$ converges strongly to A. In particular if A is a compact operator in $\mathscr{A}(\mathscr{R})$ the sequence $\{AG_k\}_1^\infty$ converges to A in the norm topology. (see [**6**, Corollary 4.4, p. 25]). Hence every compact operator in $\mathscr{A}(\mathscr{R})$ is a uniform limit of finite rank operators in the algebra.

4. Subnormality and the algebra $\mathscr{A}(\mathscr{F})$

Let $\mathscr{F}$ be a set of vectors in a separable Hilbert space H and let $\mathscr{A}(\mathscr{F})$ be the algebra of bounded linear operators on H having the set $\mathscr{F}$ of vectors as eigenvectors. That is,

$$\mathscr{A}(\mathscr{F}) = \{A \in \mathscr{B}(H)\text{:for all } f \in \mathscr{F}, \text{ there exists } \lambda_f \in \mathbb{C} \text{ with } Af = \lambda_f f\}.$$

It is clear that $\mathscr{A}(\mathscr{F})$ is a weakly (and hence a strongly) closed subalgebra of $\mathscr{B}(H)$ containing the identity operator I.

A necessary condition for an operator $A \in \mathscr{A}(\mathscr{F})$ with simple eigenvalues to be subnormal is that $\mathscr{F}$ is orthogonal. To see this, suppose A is a subnormal operator in $\mathscr{A}(\mathscr{F})$ with simple eigenvalues. Then A has a normal extension. In other words there exists a normal operator B on a Hilbert space K such that the Hilbert space H is a subspace of K, invariant under B and the restriction of B to H is the operator A. Each eigenvalue for A is also an eigenvalue for B with the same corresponding eigenvector. Since the eigenvectors of a normal operator corresponding to different eigenvalues are orthogonal, the set $\mathscr{F}$ must be an orthogonal set. Also since H is separable $\mathscr{F}$ is at most countable.

Now consider the algebras $\mathscr{A}(\phi)$, where ϕ is the set of all characteristic functions $\phi_\alpha = \chi_{[\alpha, 1]}$, $0 \leqq \alpha < 1$ in $L^p[0,1]$, $(1 < p < \infty)$ (see [**1**], p. 80), and $\mathscr{A}(\mathscr{F})$ with $\mathscr{F} = \{f_n : n \geqq 1\}$ where $f_n = \sum_{m=1}^n \phi_m$ and $\{\phi_m\}_1^\infty$ the standard basis for l^2, as in Section 3. Then $\mathscr{A}(\mathscr{F}) = \mathscr{R}' = \mathscr{A}(\mathscr{R})$. Since ϕ is uncountable and the vectors $\{f_n : n \geqq 1\}$ are not mutually orthogonal it follows from the previous discussion that none of the known reflexive operators in the algebras $\mathscr{A}(\phi)$ and $\mathscr{A}(\mathscr{F}) = \mathscr{A}(\mathscr{R})$ is subnormal.

It is obvious that an operator A is reflexive if and only if $S^{-1}AS$ is reflexive for some bounded invertible operator S. In the sequel we shall show that none of our reflexive operators in the algebras $\mathscr{A}(\phi)$ and $\mathscr{A}(\mathscr{R})$ is similar to a subnormal operator.

Generally, if A is a reflexive operator similar to a subnormal one then there exists an invertible operator S such that SAS^{-1} is subnormal. Suppose that λ is an eigenvalue of A with corresponding eigenvector x_λ. Then,

$$(SAS^{-1})(Sx_\lambda) = SAx_\lambda = \lambda Sx_\lambda.$$

That is, Sx_λ is an eigenvector of SAS^{-1} with corresponding eigenvalue λ. Therefore if A has simple eigenvalues, the vectors

$$\{Sx : x \text{ is an eigenvector for } A\}$$

are mutually orthogonal.

Now let us consider the algebras $\mathscr{A}(\phi)$ and $\mathscr{A}(\mathscr{F})$. If S is an invertible operator then

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the set $\{S\phi_\alpha : \phi_\alpha \in \phi, \alpha \in [0,1)\}$ is uncountable and so it is not orthogonal. Also the set of vectors $\{Sf_n : f_n \in \mathscr{F}, n \geqq 1\}$ is not orthogonal. For otherwise $\{(Sf_n/\|Sf_n\|) : n \geqq 1\}$ will be a complete orthonormal set. But then

$$\left\{S^{-1}\left(\frac{Sf_n}{\|Sf_n\|}\right) : n \geqq 1\right\} = \left\{\frac{f_n}{\|Sf_n\|} : n \geqq 1\right\}$$

must be an unconditional (permutable) basis for l^2 (see [3], Theorem 2.2, p. 315). This is impossible, by Theorem 3.1, p. 20, of [7]. Therefore there is no reflexive operator in any of the algebras $\mathscr{A}(\phi)$ and $\mathscr{A}(\mathscr{F})$ with simple eigenvalues similar to a subnormal operator.

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