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INFINITE FAMILIES OF CONGRUENCES MODULO 3 AND 9 FOR BIPARTITIONS WITH 3-CORES

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Abstract

Let $A_3(n)$ denote the number of bipartitions of n with 3-cores. Recently, Lin ['Some results on bipartitions with 3-core', *J. Number Theory* **139** (2014), 44–52] established some congruences modulo 4, 5, 7 and 8 for $A_3(n)$. In this paper, we prove several infinite families of congruences modulo 3 and 9 for $A_3(n)$ by employing two identities due to Ramanujan.

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1. Introduction

A partition λ of a positive integer n is any nonincreasing sequence of positive integers whose sum is n. For a positive integer $t \geq 2$, a partition is said to be a t-core partition if its Ferrers graph does not contain a hook whose length is a multiple of t. For any nonnegative integer n, let $a_t(n)$ denote the number of t-core partitions of n. From [10], the generating function for $a_t(n)$ is given by

$$\sum_{n=0}^{\infty} a_t(n)q^n = \frac{f_t^t}{f_1}.$$

Here and throughout this paper, for any positive integer k, f_k is defined by

$$f_k := \prod_{n=1}^{\infty}(1 - q^{kn}).$$

Numerous properties of $a_t(n)$ have been extensively studied (see, for example, [2, 6, 7, 9–12, 17]).

A bipartition (λ, μ) of n is a pair of partitions (λ, μ) such that the sum of all of the parts is n. Arithmetic properties of the number of bipartitions of n have been established (see, for example, [1, 5, 8, 16]). Numerous arithmetic properties have been

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proved for bipartitions with certain restrictions on each partition (see, for example, [4, 13–15, 18]). A bipartition with t-cores is a pair of partitions (λ, μ) such that λ and μ are both t-cores. Let $A_t(n)$ denote the number of bipartitions of n with t-cores. It is easy to see that the generating function of $A_t(n)$ is

$$\sum_{n=0}^{\infty} A_t(n)q^n = \frac{f_t^{2t}}{f_1^2}. \tag{1.1}$$

Very recently, Lin [14] discovered some congruences modulo 4, 5, 7 and 8. For example, he proved that for $n \geq 0$ and $\alpha \geq 0$,

$$A_3\Big(4^{\alpha+1}n + \frac{11 \times 4^{\alpha} - 2}{3}\Big) \equiv 0 \pmod 4$$

and

$$A_3\Big(16^{\alpha+1}n + \frac{8 \times 16^{\alpha} - 2}{3}\Big) \equiv 0 \pmod 5.$$

The aim of this paper is to prove several infinite families of congruences modulo 3 and 9 for $A_3(n)$. Our main result can be stated as follows.

Theorem 1.1. *For all $\alpha \geq 0$ and $n \geq 0$,*

$$A_3\Big(64^{\alpha}n + \frac{2(64^{\alpha} - 1)}{3}\Big) \equiv A_3(n) \pmod 3, \tag{1.2}$$

$$A_3\Big(64^{(\alpha+1)}n + \frac{2^{6\alpha+5} - 2}{3}\Big) \equiv 0 \pmod 3, \tag{1.3}$$

$$A_3\Big(4^{9\alpha}n + \frac{2(4^{9\alpha} - 1)}{3}\Big) \equiv A_3(n) \pmod 9, \tag{1.4}$$

$$A_3\Big(4^{9(\alpha+1)}n + \frac{2^{18\alpha+17} - 2}{3}\Big) \equiv 0 \pmod 9. \tag{1.5}$$

Thanks to (1.4), we can deduce the following corollary.

Corollary 1.2. *For all integers $\alpha \geq 0$,*

$$A_3\Big(r_i \times 4^{9\alpha} + \frac{2(4^{9\alpha} - 1)}{3}\Big) \equiv i \pmod 9,$$

where $r_0 = 17$, $r_1 = 9$, $r_2 = 1$, $r_3 = 10$, $r_4 = 3$, $r_5 = 2$, $r_6 = 5$, $r_7 = 11$ *and* $r_8 = 4$.

2. Proofs of Theorem 1.1 and Corollary 1.2

In this section, we present a proof of Theorem 1.1. We first present the following lemma.

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Lemma 2.1. *Let a and b be two integers. If*

$$\sum_{n=0}^{\infty} c(n)q^n \equiv af_1^{16} + bq\frac{f_2^{24}}{f_1^8} \pmod 9, \tag{2.1}$$

then

$$\sum_{n=0}^{\infty} c(4n+2)q^n \equiv (5a+8b)f_1^{16} + 4aq\frac{f_2^{24}}{f_1^8} \pmod 9. \tag{2.2}$$

Proof. The following relations are consequences of dissection formulas of Ramanujan collected in Entry 25 in Berndt's book [3, page 40]:

$$f_1^4 = \frac{f_4^{10}}{f_2^2 f_8^4} - 4q\frac{f_2^2 f_8^4}{f_4^2} \tag{2.3}$$

and

$$\frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}. \tag{2.4}$$

Substituting (2.3) and (2.4) into (2.1),

$$\begin{aligned}\sum_{n=0}^{\infty} c(n)q^n &\equiv a\Big(\frac{f_4^{10}}{f_2^2 f_8^4} - 4q\frac{f_2^2 f_8^4}{f_4^2}\Big)^4 + bqf_2^{24}\Big(\frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}\Big)^2 \\ &\equiv a\frac{f_4^{40}}{f_2^8 f_8^{16}} + (b-7a)q\frac{f_4^{28}}{f_2^4 f_8^8} + (6a+8b)q^2 f_4^{16} \\ &\quad + (7b-4a)q^3 f_2^4 f_4^4 f_8^8 + 4aq^4\frac{f_2^8 f_8^{16}}{f_4^8} \pmod 9. \end{aligned} \tag{2.5}$$

Extracting the terms with even powers of q on both sides of (2.5), then replacing q^2 by q,

$$\sum_{n=0}^{\infty} c(2n)q^n \equiv a\frac{f_2^{40}}{f_1^8 f_4^{16}} + (6a+8b)qf_2^{16} + 4aq^2\frac{f_1^8 f_4^{16}}{f_2^8} \pmod 9. \tag{2.6}$$

Substituting (2.3) and (2.4) into (2.6),

$$\begin{aligned}\sum_{n=0}^{\infty} c(2n)q^n &\equiv a\frac{f_2^{40}}{f_4^{16}}\Big(\frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}\Big)^2 + (6a+8b)qf_2^{16} \\ &\quad + 4aq^2\frac{f_4^{16}}{f_2^8}\Big(\frac{f_4^{10}}{f_2^2 f_8^4} - 4q\frac{f_2^2 f_8^4}{f_4^2}\Big)^2 \\ &\equiv a\frac{f_2^{12} f_4^{12}}{f_8^8} + (5a+8b)qf_2^{16} + 7aq^2\frac{f_2^{20} f_8^8}{f_4^{12}} \\ &\quad + 4aq^2\frac{f_4^{36}}{f_2^{12} f_8^8} + 4aq^3\frac{f_4^{24}}{f_2^8} + aq^4\frac{f_4^{12} f_8^8}{f_2^4} \pmod 9. \end{aligned} \tag{2.7}$$

Congruence (2.2) follows from (2.7). This completes the proof. □

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We are now ready to prove Theorem 1.1 by using Lemma 2.1.

PROOF OF THEOREM 1.1. Setting $t = 3$ in (1.1),

$$\sum_{n=0}^{\infty} A_3(n)q^n = \frac{f_3^6}{f_1^2}. \tag{2.8}$$

By the binomial theorem, it is easy to check that

$$f_3^6 \equiv f_1^{18} \pmod 9. \tag{2.9}$$

Combining (2.8) and (2.9),

$$\sum_{n=0}^{\infty} A_3(n)q^n \equiv f_1^{16} \pmod 9. \tag{2.10}$$

Setting $a = 1$, $b = 0$ in (2.1) and using Lemma 2.1 and (2.10), we see that

$$\sum_{n=0}^{\infty} A_3(4n+2)q^n \equiv 5f_1^{16} + 4q\frac{f_2^{24}}{f_1^8} \pmod 9. \tag{2.11}$$

If we apply Lemma 2.1 repeatedly, starting from (2.11),

$$\sum_{n=0}^{\infty} A_3(16n+10)q^n \equiv 3f_1^{16} + 2q\frac{f_2^{24}}{f_1^8} \pmod 9, \tag{2.12}$$

$$\sum_{n=0}^{\infty} A_3(64n+42)q^n \equiv 4f_1^{16} + 3q\frac{f_2^{24}}{f_1^8} \pmod 9, \tag{2.13}$$

$$\sum_{n=0}^{\infty} A_3(256n+170)q^n \equiv 8f_1^{16} + 7q\frac{f_2^{24}}{f_1^8} \pmod 9, \tag{2.14}$$

$$\sum_{n=0}^{\infty} A_3(1024n+682)q^n \equiv 6f_1^{16} + 5q\frac{f_2^{24}}{f_1^8} \pmod 9, \tag{2.15}$$

$$\sum_{n=0}^{\infty} A_3(4096n+2730)q^n \equiv 7f_1^{16} + 6q\frac{f_2^{24}}{f_1^8} \pmod 9, \tag{2.16}$$

$$\sum_{n=0}^{\infty} A_3(16384n+10922)q^n \equiv 2f_1^{16} + q\frac{f_2^{24}}{f_1^8} \pmod 9, \tag{2.17}$$

$$\sum_{n=0}^{\infty} A_3(65536n+43690)q^n \equiv 8q\frac{f_2^{24}}{f_1^8} \pmod 9 \tag{2.18}$$

and

$$\sum_{n=0}^{\infty} A_3(262144n+174762)q^n \equiv f_1^{16} \pmod 9. \tag{2.19}$$

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In view of (2.10) and (2.13), we see that for $n \geq 0$,

$$A_3(64n + 42) \equiv A_3(n) \pmod{3}. \tag{2.20}$$

Congruence (1.2) follows from (2.20) and mathematical induction.

By (2.12), we see that

$$\sum_{n=0}^{\infty} A_3(16n + 10)q^n \equiv 2q \frac{f_2^{24}}{f_1^8} \pmod{3}. \tag{2.21}$$

Substituting (2.4) into (2.21),

$$\begin{aligned}\sum_{n=0}^{\infty} A_3(16n + 10)q^n &\equiv 2q f_2^{24} \Big(\frac{f_4^{14}}{f_2^{14} f_8^4} + 4q \frac{f_4^2 f_8^4}{f_2^{10}} \Big)^2 \\ &\equiv 2q \frac{f_4^{28}}{f_2^4 f_8^8} + q^2 f_4^{16} + 2q^3 f_2^4 f_4^4 f_8^8 \pmod{3},\end{aligned}$$

which implies that for $n \geq 0$,

$$A_3(64n + 10) \equiv 0 \pmod{3}. \tag{2.22}$$

Replacing n by $64n + 10$ in (1.2) and employing (2.22), we arrive at (1.3).

It follows from (2.10) and (2.19) that for $n \geq 0$,

$$A_3(262144n + 174762) \equiv A_3(n) \pmod{9}. \tag{2.23}$$

Congruence (1.4) follows from (2.23) and mathematical induction.

Substituting (2.4) into (2.18),

$$\begin{aligned}\sum_{n=0}^{\infty} A_3(65536n + 43690)q^n &\equiv 8q f_2^{24} \Big(\frac{f_4^{14}}{f_2^{14} f_8^4} + 4q \frac{f_4^2 f_8^4}{f_2^{10}} \Big)^2 \\ &\equiv 8q \frac{f_4^{28}}{f_2^4 f_8^8} + q^2 f_4^{16} + 2q^3 f_2^4 f_4^4 f_8^8 \pmod{9},\end{aligned}$$

which implies that for $n \geq 0$,

$$A_3(262144n + 43690) \equiv 0 \pmod{9}.$$

Replacing n by $262144n + 43690$ in (1.4), we get (1.5).

To conclude this paper, we give a proof of Corollary 1.2.

Proof of Corollary 1.2. Setting $n = r_i$ in (1.4) and then employing the facts $A_3(1) = 2$, $A_3(2) = 5$, $A_3(3) = 4$, $A_3(4) = 8$, $A_3(5) = 6$, $A_3(9) = 10$, $A_3(10) = 21$, $A_3(11) = 16$, $A_3(17) = 18$, we can deduce Corollary 1.2. This completes the proof. □

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