Proceedings of the Royal Society of Edinburgh, page 1 of 24
DOI:10.1017/prm.2024.39

Structural stability of transonic shock flows with an external force

Shangkun Weng and Wengang Yang
School of Mathematics and Statistics, Wuhan University, Wuhan, Hubei Province 430072, People's Republic of China (skweng@whu.edu.cn; yangwg@whu.edu.cn)

(Received 2 June 2023; accepted 8 March 2024)

This paper is devoted to the structural stability of a transonic shock passing through a flat nozzle for two-dimensional steady compressible flows with an external force. We first establish the existence and uniqueness of one-dimensional transonic shock solutions to the steady Euler system with an external force by prescribing suitable pressure at the exit of the nozzle when the upstream flow is a uniform supersonic flow. It is shown that the external force helps to stabilize the transonic shock in flat nozzles and the shock position is uniquely determined. Then we are concerned with the structural stability of these transonic shock solutions when the exit pressure is suitably perturbed. One of the new ingredients in our analysis is to use the deformation-curl decomposition to the steady Euler system developed by Weng and Xin [Sci. Sinica Math., 49 (2019), pp. 307–320] to deal with the transonic shock problem.

Keywords: transonic shock; stabilization effect of external force; structural stability; deformation-curl decomposition

2020 *Mathematics Subject Classification:* 76H05; 35M12; 35L65; 76N15

1. Introduction

This paper is concerned with the transonic shock problem in a two-dimensional nozzle with an external force. In 1948, Courant and Friedrichs [9] had described the following transonic shock phenomena in a de Laval nozzle: given the appropriately large receiver pressure, if the upcoming flow is still supersonic after passing through the throat of the nozzle, to match the prescribed appropriately large exit pressure, a shock front intervenes at some place in the diverging part of the nozzle and the gas is compressed and slowed down to subsonic speed. In this paper, we will investigate such a problem for the two-dimensional compressible Euler flow exerted by proper external force in a two-dimensional flat nozzle.

The studies of transonic shock solutions for inviscid compressible flows in different kinds of nozzles have a long history. A lot of significant results have been achieved over the past two decades. People first used the quasi one-dimensional model to study the transonic shock problem [1, 9, 10, 14]. In [14], Liu has proven that the flows along the expanding portion of the nozzle are stable, which suggests that the widening nozzle played a stabilizing effect. However, our results imply that the external force also has the effect of stabilizing the shock. Next, the structural stability of multidimensional transonic shocks in flat or diverging nozzles was further investigated in [**6**, **30**, **31**] using the steady potential flows with different kinds of boundary conditions. In particular, [**30**, **31**] proved that the stability of transonic shocks for potential flows is usually ill-posed under the perturbations of the exit pressure. The structural stability of the transonic shock problem in two-dimensional divergent nozzles under the perturbations for the exit pressure was first established in [**16**] when the opening angle of the nozzle is suitably small. Later on, this restriction was removed in [**17**, **20**]. Moreover, the transonic shock in general two-dimensional straight divergent nozzles was shown in [**20**] to be structurally stable under generic perturbations for both the nozzle shape and the exit pressure. The existence and stability of transonic shock for three-dimensional axisymmetric flows without swirl in a conic straight nozzle were established in [**18**, **19**] with respect to small perturbations of the exit pressure. Compared with those results, a new decomposition method to the compressible Euler system was applied in this article.

Many researchers also considered the transonic shock problem in the flat or almost flat nozzles with the exit pressure satisfying some special constraint, see [**3**–**5**, **15**, **29**] and the references therein. Recently, there has been interesting progress on the stability and existence of transonic shock solutions to the two-dimensional and three-dimensional axisymmetric steady compressible Euler system in an almost flat finite nozzle with the receiver pressure prescribed at the exit of the nozzle (see [**11**, **12**]), where the shock position was uniquely determined. For the structural stability under the axisymmetric perturbation of the nozzle wall, a modified Lagrangian coordinate was introduced in [**26**] to deal with the corner singularities near the intersection points of the shock surface and nozzle boundary and the artificial singularity near the axis simultaneously. Most recently, the authors in [**24**, **25**] studied radially symmetric transonic flow with/without shock in an annulus. Thanks to the effect of angular velocity, it was found in [**24**] that besides the well-known supersonic-subsonic shock in a divergent nozzle as in the case without angular velocity, there exists a supersonic–supersonic shock solution, where the downstream state may change smoothly from supersonic to subsonic. Furthermore, there exists a supersonic–sonic shock solution where the shock circle and the sonic circle coincide.

The rest of this paper will be organized as follows. In the next section, we formulate the problem investigated in this article and state our main results. In § 3, we reformulate the original 2-D problem by deformation-curl decomposition developed in [**27**, **28**] so that one can rewrite system (2.1) with the velocity and the Bernoulli function. We obtain a 2×2 first-order system for the velocity field, a transport-type equation for the Bernoulli function, and the first-order ordinary differential equation for the shock after linearization. In § 4, we design an elaborate iteration scheme inspired by the works [**17**] for the non-linear system. The investigation of well-posedness and regularity for the linear system is given in the remainder part of this section. In § 5, we prove the main existence and uniqueness theorem.

2. Formulation of the problem and main results

In this section, we present the two-dimensional transonic shock problem with external force under suitable perturbation of exit pressure and the main results. The 2-D steady compressible isentropic Euler system with external force is of the form

$$\begin{cases} \partial_{x_1}(\rho u_1) + \partial_{x_2}(\rho u_2) = 0, \\ \partial_{x_1}(\rho u_1^2 + P(\rho)) + \partial_{x_2}(\rho u_1 u_2) = \rho \partial_{x_1}\Phi, \\ \partial_{x_1}(\rho u_1 u_2) + \partial_{x_2}(\rho u_2^2 + P(\rho)) = \rho \partial_{x_2}\Phi, \end{cases} \tag{2.1}$$

where $(u_1, u_2) = \mathbf{u} : \mathbb{R}^2 \to \mathbb{R}^2$ is the unknown velocity filed and $\rho : \mathbb{R}^2 \to \mathbb{R}$ is the density, and $\Phi(x_1, x_2)$ is a given potential function of external force. For the ideal polytropic gas, the equation of state is given by $P(\rho) = A\rho^\gamma$, here A and γ $(1 < \gamma < \infty)$ are positive constants. We take $A = 1$ throughout this paper for the convenience.

To this end, let's first focus on the 1-D steady compressible flow with an external force on an interval $I = [L_0, L_1]$, which is governed by

$$\begin{cases} (\bar{\rho}\bar{u})'(x_1) = 0, \\ \bar{\rho}\bar{u}\bar{u}' + \frac{\mathrm{d}}{\mathrm{d}x_1}P(\bar{\rho}) = \bar{\rho}\bar{f}(x_1), \\ \bar{\rho}(L_0) = \rho_0 > 0, \quad \bar{u}(L_0) = u_0 > 0, \end{cases} \tag{2.2}$$

where we assume that the flow state at the entrance $x_1 = L_0$ is supersonic, meaning that $u_0^2 > c^2(\rho_0) = \gamma\rho_0^{\gamma-1}$.

Denote $J = \bar{\rho}\bar{u} = \rho_0 u_0 > 0$, then it follows from (2.2) that

$$\begin{cases} \bar{\rho}(x_1) = \frac{J}{\bar{u}(x_1)}, \\ ((\bar{u})^{\gamma+1} - \gamma J^{\gamma-1})\bar{u}' = \bar{u}^\gamma \bar{f}. \end{cases} \tag{2.3}$$

Also one has

$$\bar{u}' = \frac{\bar{u}\bar{f}}{\bar{u}^2 - c^2(\bar{\rho})}, \quad \bar{\rho}' = -\frac{\bar{\rho}\bar{f}}{\bar{u}^2 - c^2(\bar{\rho})}, \tag{2.4}$$

$$\frac{\mathrm{d}}{\mathrm{d}x_1}\bar{M}^2(x_1) = \frac{(\gamma+1)\bar{M}^2}{\bar{M}^2 - 1}\frac{\bar{f}}{c^2(\bar{\rho})}, \tag{2.5}$$

where $\bar{M}(x_1) = \bar{u}(x_1)/c(\bar{\rho})$ is the Mach number.

Since $\bar{M}^2(L_0) > 1$, it follows from (2.5) that if the external force satisfies

$$\bar{f}(x_1) > 0, \quad \forall L_0 < x_1 < L_1, \tag{2.6}$$

then problem (2.2) has a global supersonic solution $(\bar{\rho}^-, \bar{u}^-)$ on $[L_0, L_1]$. If one prescribes a large enough end pressure at $x_1 = L_1$, a shock will form at some point $x_1 = L_s \in (L_0, L_1)$ and the gas is compressed and slowed down to subsonic speed, the gas pressure will increase to match the given end pressure. Mathematically, one looks for a shock $x_1 = L_s$ and smooth functions $(\bar{\rho}^\pm, \bar{u}^\pm, \bar{P}^\pm)$ defined on $I^+ = [L_s, L_1]$ and $I^- = [L_0, L_s]$ respectively, which solves (2.3) on $I^\pm$ with the jump at the shock $x_1 = L_s \in (L_0, L_1)$ satisfying the physical entropy condition $[\bar{P}(L_s)] = \bar{P}^+(L_s) - \bar{P}^-(L_s) > 0$ and the Rankine–Hugoniot conditions

$$\begin{cases} [\bar{\rho}\bar{u}](L_s) = 0, \\ [\bar{\rho}\bar{u}^2 + P(\bar{\rho})](L_s) = 0. \end{cases} \tag{2.7}$$

and also the boundary conditions

$$\rho(L_0) = \rho_0, \ u(L_0) = u_0 > 0, \tag{2.8}$$

$$\bar{P}(L_1) = P_e. \tag{2.9}$$

We will show that there is a unique transonic shock solution to the 1-D Euler system when the end pressure P_e lies in a suitable interval. Such a problem will be solved by a shooting method employing the monotonicity relation between the shock position and the end pressure.

Lemma 2.1. *Suppose that the initial state (u_0, ρ_0) at $x_1 = L_0$ is supersonic and the external force f satisfying (2.6), there exist two positive constants $P_0, P_1 > 0$ such that if the end pressure $P_e \in (P_1, P_0)$, there exists a unique transonic shock solution $(\bar{u}^-, \bar{\rho}^-)$ and $(\bar{u}^+, \bar{\rho}^+)$ defined on $I^- = [L_0, L_s)$ and $I^+ = (L_s, L_1)$ respectively, with a shock located at $x_1 = L_s \in (L_0, L_1)$. In addition, the shock position $x_1 = L_s$ increases as the exit pressure P_e decreases. Furthermore, the shock position L_s approaches to L_1 if P_e goes to P_1 and L_s tends to L_0 if P_e goes to P_0.*

Proof. The existence and uniqueness of smooth supersonic flow $(\bar{u}^-, \bar{\rho}^-)$ starting from (ρ_0, u_0) on $[L_0, L_1]$ is trivial. Suppose the shock occurs at $x_1 = L_s \in (L_0, L_1)$, then it is well-known that there exists a unique subsonic state $(\bar{u}^+(L_s), \bar{\rho}^+(L_s))$ satisfying the Rankine–Hugoniot conditions (2.7) and the entropy condition. With $(\bar{u}^+(L_s), \bar{\rho}^+(L_s))$ as the initial data, equation (2.2) has a unique smooth solution $(\bar{u}^+, \bar{\rho}^+)$ on $[L_s, L_1]$. Denote $P_e = (\bar{\rho}^+(L_1))^\gamma$. In the following, we show that the monotonicity between the shock position $x_1 = L_s$ and the exit pressure $P_e = (\bar{\rho}^+(L_1))^\gamma$. $\bar{\rho}^+(L_1)$ is regarded as a function of L_s. Since $(\bar{\rho}^+\bar{u}^+)(L_s) = (\bar{\rho}^-\bar{u}^-)(L_s) = J = \rho_0 u_0 > 0$, then

$$\bar{u}^-(L_s) + \frac{J^{\gamma-1}}{(\bar{u}^-(L_s))^\gamma} = \bar{u}^+(L_s) + \frac{J^{\gamma-1}}{(\bar{u}^+(L_s))^\gamma}. \tag{2.10}$$

It follows from the second equation in (2.2) that

$$\frac{1}{2}(\bar{u}^+(L_1))^2 + \frac{\gamma}{\gamma-1}(\bar{\rho}^+(L_1))^{\gamma-1} - \bar{\Phi}(L_1) = \frac{1}{2}(\bar{u}^+(L_s))^2 + \frac{\gamma}{\gamma-1}(\bar{\rho}^+(L_s))^{\gamma-1} - \bar{\Phi}(L_s).$$

Differentiating with respect to L_s, one deduces that

$$\left(\gamma(\bar{\rho}^+(L_1))^{\gamma-2} - \frac{J^2}{(\bar{\rho}^+(L_1))^3}\right)\frac{\mathrm{d}\bar{\rho}^+(L_1)}{\mathrm{d}L_s} = \left(\gamma(\bar{\rho}^+(L_s))^{\gamma-2} - \frac{J^2}{(\bar{\rho}^+(L_s))^3}\right)\frac{\mathrm{d}\bar{\rho}^+(L_s)}{\mathrm{d}L_s} - \bar{f}(L_s) =: I. \tag{2.11}$$

Also (2.10) yields that

$$\left\{1-\frac{\gamma J^{\gamma-1}}{(\bar{u}^+(L_s))^{\gamma+1}}\right\}\frac{\mathrm{d}\bar{u}^+(L_s)}{\mathrm{d}L_s}=\left\{1-\frac{\gamma J^{\gamma-1}}{(\bar{u}^-(L_s))^{\gamma+1}}\right\}\frac{\mathrm{d}\bar{u}^-(L_s)}{\mathrm{d}L_s}=\frac{\bar{f}(L_s)}{\bar{u}^-(L_s)}.$$

Finally, we conclude that

$$\begin{aligned} I &= -\left\{\gamma(\rho^+(L_s))^{\gamma-1}-\frac{J^2}{(\rho^+(L_s))^2}\right\}\frac{1}{\bar{u}^+(L_s)}\frac{\mathrm{d}\bar{u}^+(L_s)}{\mathrm{d}L_s}-\bar{f}(L_s)\\ &= \frac{\bar{f}(L_s)(\bar{u}^+(L_s)-\bar{u}^-(L_s))}{\bar{u}^-(L_s)}<0. \end{aligned}$$

Since the coefficients

$$\gamma(\bar{\rho}^+(L_1))^{\gamma-2}-\frac{J^2}{(\bar{\rho}^+(L_1))^3}>0,$$

then (2.11) implies that the end density $\bar{\rho}^+(L_1)$ is a strictly decreasing function of the shock position $x_1 = L_s$. It follows that the end pressure $P_e = (\bar{\rho}^+(L_1))^\gamma$ is a strictly decreasing and continuous differentiable function on the shock position $x_1 = L_s$. In particular, when $L_s = L_0$ and $L_s = L_1$, there are two different end pressure P_1, P_2 with $P_0 > P_1$. Hence, by the monotonicity, one can obtain a transonic shock for the end pressure $P_e \in (P_1, P_0)$. □

Remark 2.2. Lemma 2.1 shows that the external force helps to stabilize the transonic shock in flat nozzles and the shock position is uniquely determined.

The one-dimensional transonic shock solution $(\bar{u}^\pm, \bar{\rho}^\pm)$ with a shock occurring at $x_1 = L_s$ constructed in lemma 2.1 will be called the background solution in this paper. The extension of the subsonic flow $(\bar{u}^+(x_1), \bar{\rho}^+(x_1))$ of the background solution to $L_s - \delta_0 < x_1 < L_1$ for a small positive number δ_0 will be denoted by $(\hat{u}^+(x_1), \hat{\rho}^+(x_1))$.

It is natural to focus on the structural stability of these transonic shock flows. For simplicity, we only investigate the structural stability under suitable small perturbations of the end pressure. Therefore, the supersonic incoming flow is unchanged and remains to be $(\bar{u}^-(x_1), 0, \bar{\rho}^-(x_1))$.

Assume that the possible shock curve Σ and the flow behind the shock are denoted by $x_1 = \xi(x_2)$ and $(u_1^+, u_2^+, P^+)(x)$ respectively (see figure 1). Let $\Omega^+ = \{(x_1, x_2) : \xi(x_2) < x_1 < L_1, -1 < x_2 < 1\}$ denotes the subsonic region of the flow. Then the Rankine–Hugoniot conditions on Σ gives

$$\begin{cases} [\rho u_1]-\xi'(x_2)[\rho u_2]=0,\\ [\rho u_1^2+P]-\xi'(x_2)[\rho u_1 u_2]=0,\\ [\rho u_1 u_2]-\xi'(x_2)[\rho u_2^2+P]=0. \end{cases} \tag{2.12}$$

In addition, the pressure P satisfies the physical entropy conditions

$$P^+(x) > P^-(x) \quad \text{on}\,\Sigma. \tag{2.13}$$

https://doi.org/10.1017/prm.2024.39 Published online by Cambridge University Press

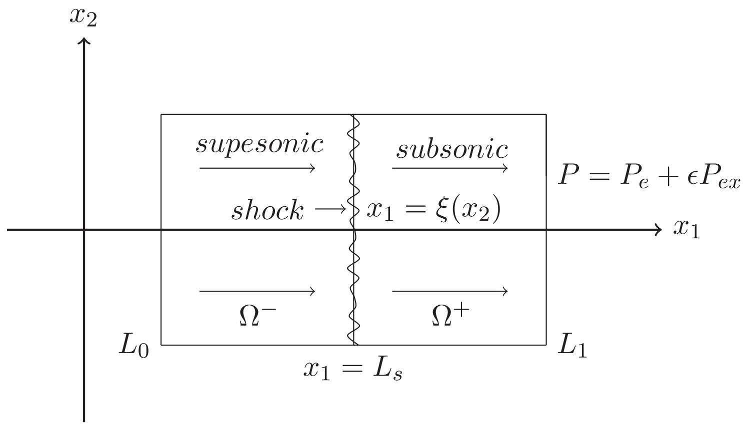

Figure 1. Nozzle.

Since the flow is tangent to the nozzle walls $x_2 = \pm 1$, then

$$u_2^+(x_1, \pm 1) = 0. \tag{2.14}$$

The end pressure is perturbed by

$$P^+(L_1, x_2) = P_e + \epsilon P_{ex}(x_2), \tag{2.15}$$

due to some technical reasons, we may readily suppose that $P_{ex}(x_2) = P_e^{1/\gamma} \hat{P}_{ex}(x_2) \in C^{2,\alpha}([-1, 1])(\alpha \in (0, 1))$ satisfies the compatibility conditions

$$\hat{P}'_{ex}(\pm 1) = 0. \tag{2.16}$$

The following theorem gives the main results of this paper.

THEOREM 2.3. *Suppose that (2.6), (2.8) and (2.9) hold. Then there exist positive constant $\epsilon_0 > 0$ such that, for all $\epsilon \in (0, \epsilon_0]$, system (2.1) with boundary conditions (2.12)–(2.15) has a unique transonic shock solution $(u_1^+(x), u_2^+(x), P^+(x); \xi(x_2))$ which admits the following properties:*
(i). The shock $x_1 = \xi(x_2) \in C^{3,\alpha}([-1, 1])$, and satisfies

$$\|\xi(x_2) - L_s\|_{C^{3,\alpha}([-1,1])} \leqslant C\epsilon, \tag{2.17}$$

where the positive constant C only depends on the background solution, the exit pressure and α.
(ii). The velocity and pressure in subsonic region $(u_1^+, u_2^+, P^+)(x) \in (C^{2,\alpha}(\bar{\Omega}^+))^3$, and there holds

$$\|(u_1^+, u_2^+, P^+)(x) - (\hat{u}, 0, \hat{P})\|_{C^{2,\alpha}(\bar{\Omega}^+)} \leqslant C\epsilon, \tag{2.18}$$

where $\Omega^+ = \{(x_1, x_2) : \xi(x_2) < x_1 < L_1, -1 < x_2 < 1\}$ is the subsonic region and $(\hat{u}, 0, \hat{P}) = (\hat{u}(x_1), 0, P(\hat{\rho}(x_1)))$ is the extended background solution.

https://doi.org/10.1017/prm.2024.39 Published online by Cambridge University Press

Our proof is influenced by the approach developed in [16, 17, 20], yet the reformulation of the problem is different from there. It is well-known that steady Euler equations are hyperbolic-elliptic coupled in the subsonic region. The entropy and Bernoulli's function are conserved along the particle path, while the pressure and the flow angle satisfy a first-order elliptic system in the subsonic region. These facts are widely used in the structural stability analysis for the transonic shock problems in flat or divergent nozzles, one may refer to [3, 7, 8, 16, 17, 20, 22, 32, 33] and the references therein. Here we resort to a different decomposition based on the deformation and curl of the velocity developed in [27, 28] for three-dimensional steady Euler and Euler–Poisson systems. The idea in that decomposition is to rewrite the density equation as a Frobenius inner product of a symmetric matrix and the deformation matrix by using Bernoulli's law. The vorticity is resolved by an algebraic equation of Bernoulli's function and the entropy. We should mention that there are several different decompositions to the three-dimensional steady Euler system [2, 4, 5, 21, 23, 32] developed by many researchers for different purposes. An interesting issue that deserves further discussion is when using the deformation-curl decomposition to deal with the transonic shock problem, the end pressure boundary condition becomes non-local since it involves the information from the shock front. However, this non-local boundary condition reduces to be local after introducing the potential function.

3. Reformulation of the problem

Different from previous works on transonic shock problems [3, 7, 8, 16, 17, 20], we will use the deformation-curl decomposition developed in [27, 28] for steady Euler system to decompose the original system (2.1) into an equivalent system (3.3), where the hyperbolic quantity B and elliptic quantities u_1, u_2 are effectively decoupled in subsonic regions. To this end, define the Bernoulli's function

$$B = \frac{1}{2}|\mathbf{u}|^2 + h(\rho) - \Phi, \tag{3.1}$$

where $h(s) = \frac{\gamma}{\gamma-1}s^{\gamma-1}$ is the enthalpy function. Hence, the density can be expressed by the Bernoulli function and velocity field as

$$\rho = H(B, \Phi, |\mathbf{u}|^2) = \left[\frac{\gamma-1}{\gamma}\left(B + \Phi - \frac{1}{2}|\mathbf{u}|^2\right)\right]^{1/\gamma-1}. \tag{3.2}$$

Consequently, the 2-D Euler system (2.1) with unknown function (u_1, u_2, P) is equivalent to the following system

$$\begin{cases} \displaystyle\sum_{i,j=1}^{2}(c^2(H)\delta_{ij} - u_iu_j)\partial_i u_j + u_1\partial_1\Phi + u_2\partial_2\Phi = 0, \\ \partial_1 u_2 - \partial_2 u_1 = -\frac{\partial_2 B}{u_1}, \\ u_1\partial_1 B + u_2\partial_2 B = 0, \end{cases} \tag{3.3}$$

with unknown function (u_1, u_2, B).

https://doi.org/10.1017/prm.2024.39 Published online by Cambridge University Press

The shock curve is determined by

$$\xi'(x_2) = \frac{[\rho u_1 u_2]}{[\rho u_2^2 + P]}(\xi(x_2), x_2), \quad x_2 \in (-1, 1). \tag{3.4}$$

Furthermore, it follows from the R-H conditions (2.12) that

$$\begin{cases} [\rho u_1] = \frac{[\rho u_2][\rho u_1 u_2]}{[\rho u_2^2 + P]}, \\ [\rho u_1^2 + P(\rho)] = \frac{([\rho u_1 u_2])^2}{[\rho u_2^2 + P]}. \end{cases} \tag{3.5}$$

A direct computation by using (3.5) shows that on $x_1 = \xi(x_2)$

$$\begin{aligned} &(\rho^+(\xi(x_2), x_2) - \bar{\rho}^+(L_s), u_1^+(\xi(x_2), x_2) - \bar{u}^+(L_s)) \\ &\quad = (h_1, h_2)(\rho^-(\xi(x_2)) - \bar{\rho}^-(L_s), u^-(\xi(x_2)) - \bar{u}^-(L_s), (u_2^+(\xi(x_2), x_2))^2) \end{aligned} \tag{3.6}$$

here $h_i(0,\,0,\,0) = 0$ for $i = 1,\,2$. In addition, we have

$$\begin{cases} \dfrac{\partial h_1}{\partial(\rho^- - \bar{\rho}^-)}|_{(0,0,0)} = 2\bar{u}^-(L_s)\dfrac{\bar{u}^+ - \bar{u}^-}{(\bar{u}^+(L_s))^2 - c^2(\bar{\rho}^+(L_s))} + 1, \\ \dfrac{\partial h_1}{\partial(u^- - \bar{u}^-)}|_{(0,0,0)} = 2\bar{\rho}^-(L_s)\dfrac{\bar{u}^+ - \bar{u}^-}{(\bar{u}^+(L_s))^2 - c^2(\bar{\rho}^+(L_s))}, \\ \dfrac{\partial h_1}{\partial(u_2^+)^2}|_{(0,0,0)} = \dfrac{(\bar{\rho}^+(L_s)\bar{u}^+(L_s))^2}{\bar{P}^+(L_s) - \bar{P}^-(L_s)}\dfrac{1}{(\bar{u}^+(L_s))^2 - c^2(\bar{\rho}^+(L_s))}, \end{cases} \tag{3.7}$$

and

$$\begin{cases} \dfrac{\partial h_2}{\partial(\rho^- - \bar{\rho}^-)}|_{(0,0,0)} = -(\gamma - 1)\dfrac{\bar{P}^+(L_s) - \bar{P}^-(L_s)}{(\bar{\rho}^+(L_s))^2\bar{u}^+(L_s)}\dfrac{\bar{u}^+\bar{u}^-}{(\bar{u}^+(L_s))^2 - c^2(\bar{\rho}^+(L_s))}, \\ \dfrac{\partial h_2}{\partial(u^- - \bar{u}^-)}|_{(0,0,0)} = \dfrac{2\bar{\rho}^-(L_s)\bar{u}^+(L_s)}{\bar{\rho}^+(L_s)}\dfrac{\bar{u}^-\bar{u}^+}{(\bar{u}^+(L_s))^2 - c^2(\bar{\rho}^+(L_s))} + \dfrac{\bar{\rho}^-(L_s)}{\bar{\rho}^+(L_s)}, \\ \dfrac{\partial h_2}{\partial(u_2^+)^2}|_{(0,0,0)} = \dfrac{\bar{\rho}^+(L_s)\bar{u}^+(L_s)}{\bar{P}^+(L_s) - \bar{P}^-(L_s)}\dfrac{c^2(\bar{\rho}^+(L_s))}{(\bar{u}^+(L_s))^2 - c^2(\bar{\rho}^+(L_s))}. \end{cases} \tag{3.8}$$

By substituting (3.6) into (3.1), we conclude that there is a function h_3 such that

$$\begin{aligned} B^+(\xi(x_2), x_2) - \bar{B}^+(L_s) &= h_3(\rho^-(\xi(x_2)) \\ &\quad - \bar{\rho}^-(L_s), u^-(\xi(x_2)) - \bar{u}^-(L_s), (u_2^+(\xi(x_2), x_2))^2). \end{aligned} \tag{3.9}$$

Thus, theorem 2.3 is established as long as we solve problem (3.3)–(3.4) with boundary conditions (3.5), (2.14)–(2.15). In order to deal with the free boundary value problem (3.3)–(3.4), we introduce the following transformation to reduce it into a

https://doi.org/10.1017/prm.2024.39 Published online by Cambridge University Press

fixed boundary value problem. Setting

$$y_1 = \frac{x_1 - \xi(x_2)}{L_1 - \xi(x_2)}(L_1 - L_s) + L_s, \quad y_2 = x_2, \tag{3.10}$$

then, the domain $\Omega^+ = \{(x_1,\, x_2) : \xi(x_2) < x_1 < L_1,\, -1 < x_2 < 1\}$ is changed into

$$Q = \{(y_1, y_2) : L_s < y_1 < L_1, -1 < y_2 < 1\}. \tag{3.11}$$

The inverse change variable gives

$$x_1 = \xi(y_2) + \frac{L_1 - \xi(y_2)}{L_1 - L_s}(y_1 - L_s) = y_1 + \frac{L_1 - y_1}{L_1 - L_s}(\xi(y_2) - L_s), \quad x_2 = y_2.$$

We now set for $y \in Q$

$$(\tilde{u}_j, \tilde{\rho}, \tilde{B}, \tilde{\Phi})(y_1, y_2) = (u_j, \rho, B, \Phi)\left(\frac{L_1 - \xi(y_2)}{L_1 - L_s}(y_1 - L_s) + \xi(y_2), y_2\right), \quad j = 1, 2.$$

Shock equation (3.4) becomes

$$\begin{aligned} \xi'(y_2) &= \frac{(\rho u_1 u_2)(\xi(y_2), y_2)}{P^+(\rho)(\xi(y_2), y_2) - P^-(\xi(y_2)) + \rho(u_2)^2(\xi(y_2), y_2)}, \\ &= \frac{(\tilde{\rho}\tilde{u}_1\tilde{u}_2)(L_s, y_2)}{P^+(\tilde{\rho})(L_s, y_2) - P^-(\xi(y_2)) + \tilde{\rho}(\tilde{u}_2)^2(L_s, y_2)}, \; y_2 \in (-1, 1), \end{aligned} \tag{3.12}$$

and system (3.3) is changed into

$$(c^2(\tilde{\rho}) - \tilde{u}_1^2)\frac{L_1 - L_s}{L_1 - \xi(y_2)}\partial_{y_1}\tilde{u}_1 + c^2(\tilde{\rho})\partial_{y_2}\tilde{u}_2 + \tilde{u}_1\frac{L_1 - L_s}{L_1 - \xi(y_2)}\partial_{y_1}\tilde{\Phi} = F_1(\tilde{\mathbf{u}}, \tilde{B}), \tag{3.13}$$

$$\begin{aligned} &\frac{L_1 - L_s}{L_1 - \xi(y_2)}\partial_{y_1}\tilde{u}_2 - \partial_{y_2}\tilde{u}_1 - \frac{y_1 - L_1}{L_1 - \xi(y_2)}\xi'(y_2)\partial_{y_1}\tilde{u}_1 + \frac{\partial_{y_2}\tilde{B}}{\tilde{u}_1} = F_2(\tilde{\mathbf{u}}, \tilde{B}), \\ &\tilde{u}_1\frac{L_1 - L_s}{L_1 - \xi(y_2)}\partial_{y_1}\tilde{B} + \tilde{u}_2\partial_{y_2}\tilde{B} + \frac{y_1 - L_1}{L_1 - \xi(y_2)}\xi'(y_2)\tilde{u}_2\partial_{y_1}\tilde{B} = 0, \end{aligned} \tag{3.14}$$

where

$$\begin{aligned} F_1(\tilde{\mathbf{u}}, \tilde{B}) &= \tilde{u}_2^2\partial_{y_2}\tilde{u}_2 - (c^2(\tilde{\rho}) - \tilde{u}_2^2)\frac{y_1 - L_1}{L_1 - \xi(y_2)}\xi'(y_2)\partial_{y_1}\tilde{u}_2 + \tilde{u}_1\tilde{u}_2\frac{L_1 - L_s}{L_1 - \xi(y_2)}\partial_{y_1}\tilde{u}_2 \\ &+ \tilde{u}_2\tilde{u}_1(\partial_{y_2}\tilde{u}_1 + \frac{y_1 - L_1}{L_1 - \xi(y_2)}\xi'(y_2)\partial_{y_1}\tilde{u}_1) - \tilde{u}_2(\partial_{y_2}\tilde{\Phi} + \frac{y_1 - L_1}{L_1 - \xi(y_2)}\xi'(y_2)\partial_{y_1}\tilde{\Phi}), \\ F_2(\tilde{\mathbf{u}}, \tilde{B}) &= -\frac{y_1 - L_1}{L_1 - \xi(y_2)}\frac{\xi'(y_2)}{\tilde{u}_1}\partial_{y_1}\tilde{B}. \end{aligned}$$

Consider the perturbed functions $v_i(y_1,\, y_2)$, $i = 1,\, 2,\, 3,\, 4$, as

$$\begin{aligned} &v_1(y_1, y_2) = \tilde{u}_1(y_1, y_2) - \bar{u}^+(y_1), \quad v_2(y_1, y_2) = \tilde{u}_2(y_1, y_2), \\ &v_3(y_1, y_2) = \tilde{B}(y_1, y_2) - \bar{B}^+, \quad v_4(y_2) = \xi(y_2) - L_s, \end{aligned}$$

https://doi.org/10.1017/prm.2024.39 Published online by Cambridge University Press

and define the vector functions

$$V(y_1, y_2) = (v_1(y_1, y_2), v_2(y_1, y_2), v_3(y_1, y_2), v_4(y_2)). \tag{3.15}$$

It follows from (3.12) that the shock satisfies

$$v_4'(y_2) = \frac{\tilde{\rho}(\bar{u} + v_1)v_2(L_s, y_2)}{P^+(\tilde{\rho})(L_s, y_2) - P^-(\xi(y_2)) + \tilde{\rho}(\tilde{u}_2)^2(L_s, y_2)}. \tag{3.16}$$

Through a direct computation, one can derive from (3.9) and (3.14) that the Bernoulli function satisfies a transport-type equation

$$\begin{cases} [(\bar{u}^+ + v_1)(L_1 - L_s) + v_2(y_1 - L_1)v_4'(y_2)]\partial_{y_1} v_3 + v_2(L_1 - v_4 - L_s)\partial_{y_2} v_3 = 0, \\ v_3(L_s, y_2) = b_3 v_4(y_2) + R_3(y_2), \end{cases} \tag{3.17}$$

where

$$b_3 = \frac{\bar{\rho}^-(L_s) - \bar{\rho}^+(L_s)}{\bar{\rho}^+(L_s)} \bar{f}(L_s), \tag{3.18}$$

and $R_3(y_2) = R_3(V(L_s,\, y_2)) = O(|V(L_s,\, y_2)|)^2$ is an error term of second order. We may readily drop superscribe $+$ on the background solutions if there is no risk of confusion. And the first-order system for v_1, v_2 is given by,

$$\begin{cases} (c^2(\bar{\rho}^+) - (\bar{u}^+)^2)\partial_{y_1} v_1 + c^2(\bar{\rho}^+)\partial_{y_2} v_2 + B_1(y_1)v_1 \\ \quad + B_3(y_1)v_3 + B_4(y_1)v_4(y_2) = F_3(V, \nabla V), \\ \partial_{y_1} v_2 - \partial_{y_2} v_1 + \frac{L_1 - y_1}{L_1 - L_s}\bar{u}' v_4' + \frac{\partial_{y_2} v_3}{\bar{u}} = F_4(V, \nabla V), \end{cases} \tag{3.19}$$

where F_3, F_4 represent the remainder term of the second order with respect to V and ∇V, and

$$B_1(y_1) = \bar{f}(y_1) - (\gamma + 1)\bar{u}\bar{u}' = \frac{\gamma \bar{u}^2 + c^2(\bar{\rho}^+)}{c^2(\bar{\rho}^+) - \bar{u}^2} \bar{f} > 0,$$

$$B_3(y_1) = (\gamma - 1)\bar{u}',$$

$$B_4(y_1) = \frac{1}{L_1 - L_s}[(\gamma - 1)\bar{f}(y_1)(L_1 - y_1)\bar{u}' - \bar{u}\bar{f} + \bar{u}\bar{f}'(y_1)(L_1 - y_1)].$$

It's obvious that the second formula in (3.6) gives the boundary condition of v_1 on the entrance $x_1 = L_s$. Meanwhile, formula (3.2) after changing the variable becomes

$$\begin{aligned} c^2(\tilde{\rho})(y) = \gamma \tilde{\rho}^{\gamma - 1} &= (\gamma - 1)\left(\tilde{B} - \frac{1}{2}|\tilde{\mathbf{u}}|^2 + \bar{\Phi}\right) \\ &= (\gamma - 1)\left(\bar{B}^+ + v_3 - \frac{1}{2}(\bar{u}^+ + v_1)^2 - \frac{1}{2}v_2^2 + \bar{\Phi}\right) \\ &= c^2(\bar{\rho}^+) + (\gamma - 1)\left(v_3 - \bar{u}^+ v_1 - \frac{1}{2}v_1^2 - \frac{1}{2}v_2^2\right), \end{aligned} \tag{3.20}$$

https://doi.org/10.1017/prm.2024.39 Published online by Cambridge University Press

which together with (2.15) gives the boundary condition of v_1 on the exit $x_1 = L_1$. Hence, the boundary conditions to system (3.19) read as follow

$$\begin{cases} v_1(L_s, y_2) = b_2 v_4(y_2) + R_2(y_2), \\ v_1(L_1, y_2) = \frac{1}{\bar{u}(L_1)}(v_3(L_1, y_2) - \epsilon \hat{P}_{ex}(y_2)) + R_4(y_2), \\ v_2(y_1, \pm 1) = 0, \end{cases} \tag{3.21}$$

here

$$b_2 = \frac{\bar{\rho}^-(L_s)\bar{u}^+(L_s)}{\bar{\rho}^+(L_s)[(\bar{u}^+(L_s))^2 - c^2(\bar{\rho}^+(L_s))]} \bar{f}(L_s) < 0, \tag{3.22}$$

and $R_2(y_2) = R_2(V(L_s, y_2)) = O(|V(L_s, y_2)|)^2, R_4(y_2) = R_4(V(L_1, y_2)) = O(|V (L_1, y_2)|)^2$ are error terms of second order. Based on our reformulation, theorem 2.3 follows from the following results.

Theorem 3.1. *Under the same assumptions as in theorem 2.3, there exists a positive constant $\epsilon_0 > 0$ such that for each $\epsilon \in (0, \epsilon_0]$, system (3.16)–(3.21) has a unique solution $V \in (C^{2,\alpha}(\bar{Q}))^3 \times C^{3,\alpha}([-1, 1])$ satisfying the following estimate*

$$\sum_{i=1}^{3} \|v_i\|_{C^{2,\alpha}(\bar{Q})} + \|v_4\|_{C^{3,\alpha}([-1,1])} \leqslant C\epsilon, \tag{3.23}$$

where the constant C depends only on the background solution, the exit pressure and $\alpha \in (0, 1)$.

4. Iteration scheme and the linear system

In the first part of this section, we construct an iteration scheme for the non-linear system, and the problem is reduced to the solvability of corresponding linear systems. Indeed, it turns out that the linear system is a non-local elliptic equation of second order with a free parameter denoting the relative location of the shock position on the wall $x_2 = -1$. Then we study the existence, uniqueness and regularity for this linear system in the remainder part of this section.

4.1. Iteration scheme

Inspired by [**17**], we will develop an iteration scheme to prove theroem 3.1. Consider the Banach space

$$\mathcal{V}_\delta := \Bigg\{ V : \sum_{i=1}^{3} \|v_i\|_{C^{2,\alpha}(\bar{Q})} + |v_4|_{C^{3,\alpha}[-1,1]} \leqslant \delta; \partial_{y_2} v_j(y_1, \pm 1) = 0,$$
$$j = 1, 3; v_2(y_1, \pm 1) = \partial^2_{y_2} v_2(y_1, \pm 1) = 0; v_4'(\pm 1) = v_4^{(3)}(\pm 1) = 0 \Bigg\}, \tag{4.1}$$

here $\delta > 0$ is a small constant to be determined later. For a fix $\hat{V} \in \mathcal{V}_\delta$, equivalently, we have the following quantity

$$(\hat{v}_1, \hat{v}_2, \hat{B}, \hat{\rho}, \hat{P}, \hat{\xi})(y).$$

https://doi.org/10.1017/prm.2024.39 Published online by Cambridge University Press

We now define the linearized scheme to problem (3.16)–(3.21) as follows.

Firstly, thanks to (3.16), v_4 is determined by

$$v_4'(y_2) = b_0 v_2(L_s, y_2) + F_5(\hat{V})(L_s, y_2), \tag{4.2}$$

where

$$b_0 = \frac{(\bar{\rho}\bar{u})(L_s)}{P^+(\bar{\rho})(L_s) - P^-(L_s)} > 0,$$

$$F_5(y_2) = \left\{ \frac{\hat{\rho}\hat{u}_1}{P^+(\hat{\rho})(L_s, y_2) - P^-(\hat{\xi}(y_2)) + \hat{\rho}(\hat{u}_2)^2(L_s, y_2)} - b_0 \right\} v_2(L_s, y_2),$$

hence, one can express the shock as

$$v_4(y_2) = v_4(-1) + \int_{-1}^{y_2} b_0 v_2(L_s, \tau)\,\mathrm{d}\tau + R_5(y_2), \tag{4.3}$$

where $R_5(y_2) = \int_{-1}^{y_2} F_5(\tau)\,\mathrm{d}\tau$ is a error term of second order. Due to $\hat{V} \in \mathcal{V}_\delta$, we have

$$F_5(\pm 1) = F_5''(\pm 1) = 0,\ \|F_5\|_{C^{k,\alpha}[-1,1]} \leqslant C\delta \|\hat{v}_2\|_{C^{k,\alpha}(\bar{Q})}, \quad k = 0, 1, 2. \tag{4.4}$$

Secondly, using (3.17), we get the linearized transport equation for v_3:

$$[(\bar{u} + \hat{v}_1)(L_1 - L_s) + \hat{v}_2(y_1 - L_1)\hat{v}_4'(y_2)]\partial_{y_1} v_3 + \hat{v}_2(L_1 - \hat{v}_4 - L_s)\partial_{y_2} v_3 = 0 \text{ in } Q,$$

with initial data

$$v_3(L_s, y_2) = b_3 v_4(y_2) + R_3(y_2).$$

Thus, it can be solved by characteristic methods. Let $y_2(s;\beta)$ be the characteristics going through $(y_1,\, y_2)$ with $y_2(L_s) = \beta$, i.e.

$$\begin{cases} \dfrac{\mathrm{d}y_2}{\mathrm{d}s}(s;\beta) = \dfrac{\hat{v}_2(L_1 - \hat{v}_4 - L_s)}{(\bar{u} + \hat{v}_1)(L_1 - L_s) + \hat{v}_2(y_1 - L_1)\hat{v}_4'(y_2)}, \quad L_s < s < L_1, \\ y_2(L_s) = \beta. \end{cases} \tag{4.5}$$

It is noted that β can be also regarded as a function of $y = (y_1,\, y_2)$, this is denoted by $\beta = \beta(y)$, which leads to

$$v_3(y_1, y_2) = v_3(L_s, \beta(y)) = b_3 v_4(y_2) + F_6(y), \tag{4.6}$$

where

$$F_6(y) = b_3 \int_{y_2}^{\beta(y)} v_4'(\tau)\,\mathrm{d}\tau + R_3(\hat{V}(L_s, \beta(y)))$$

is an error term of second order. Furthermore, we have

$$\partial_{y_2} F_6(y_1, \pm 1) = 0,$$

$$\|F_6\|_{C^{k,\alpha}(\bar{Q})} \leqslant C\delta \left(\sum_{i=1}^{3} \|\hat{v}_i\|_{C^{k,\alpha}(\bar{Q})} + \|\hat{v}_4\|_{C^{k+1,\alpha}(\bar{Q})} \right), \quad k = 0, 1, 2. \tag{4.7}$$

It remains to determine the velocity v_1, v_2 and the shock position difference $v_4(-1)$ on the wall $x_2 = -1$. Substituting (4.3) and (4.6) into (3.19) and (3.21), we get the

https://doi.org/10.1017/prm.2024.39 Published online by Cambridge University Press

following linearized system for v_1, v_2 with an unknown parameter $v_4(-1)$:

$$\begin{cases} \partial_{y_1} v_1 + \dfrac{1}{1-\bar{M}^2}\partial_{y_2} v_2 + \dfrac{B_1(y_1)}{c^2(\bar{\rho}^+)-(\bar{u}^+)^2} v_1 \\ \quad + \dfrac{B_3(y_1)b_3 + B_4(y_1)}{c^2(\bar{\rho}^+)-(\bar{u}^+)^2}(v_4(-1) + \int_{-1}^{y_2} b_0 v_2(L_s,\tau)\,\mathrm{d}\tau) = G_1(y), \\ \partial_{y_1} v_2 - \partial_{y_2}[v_1 - \lambda(y_1)(v_4(-1) + \int_{-1}^{y_2} b_0 v_2(L_s,\tau)\,\mathrm{d}\tau)] = G_2(y), \end{cases} \tag{4.8}$$

and the boundary conditions

$$\begin{cases} v_1(L_s, y_2) = b_2(v_4(-1) + \int_{-1}^{y_2} b_0 v_2(L_s,\tau)\,\mathrm{d}\tau) + R_6(y_2), \\ v_1(L_1, y_2) = \dfrac{b_3}{\bar{u}(L_1)}(v_4(-1) + \int_{-1}^{y_2} b_0 v_2(L_s,\tau)\,\mathrm{d}\tau) - \dfrac{\epsilon \hat{P}_{ex}(y_2)}{\bar{u}(L_1)} + R_7(y_2), \\ v_2(y_1, \pm 1) = 0, \end{cases} \tag{4.9}$$

where

$$\begin{aligned} \lambda(y_1) &= \frac{L_1 - y_1}{L_1 - L_s}\bar{u}' + \frac{b_3}{\bar{u}}, \\ R_6(y_2) &= b_2 R_5(y_2) + R_2(y_2), \\ R_7(y_2) &= \frac{b_3 R_5(y_2) + F_6(L_1, y_2)}{\bar{u}(L_1)} + R_4(y_2), \\ G_1(y) &= \frac{F_3(\hat{V}, \nabla\hat{V}) - B_3(y_1)R_5(y_2) - B_3(y_1)F_6(y) - B_4(y_1)R_5(y_2)}{(c^2(\bar{\rho}^+) - (\bar{u}^+)^2)}, \\ G_2(y) &= F_4(\hat{V}, \nabla\hat{V}) - \frac{\partial_{y_2} F_6(y)}{\bar{u}} + \lambda(y_1)\partial_{y_2} R_5(y_2). \end{aligned}$$

It follows from (4.4), (4.7) and a simple calculation that

$$\begin{aligned} &\partial_{y_2} G_1(y_1, \pm 1) = 0,\ G_2(y_1, \pm 1) = 0, \\ &\|G_i\|_{C^{k-1,\alpha}(\bar{Q})} \leqslant C\delta \|\hat{V}\|_{C^{k,\alpha}(\bar{Q})}, \quad i, k = 1, 2, \end{aligned}$$

and

$$\begin{aligned} &R_6'(\pm 1) = 0,\ R_7'(\pm 1) = 0, \\ &\|(R_6, R_7)\|_{C^{k,\alpha}[-1,1]} \leqslant C\delta \|\hat{V}\|_{C^{k,\alpha}(\bar{Q})},\ k = 0, 1, 2. \end{aligned} \tag{4.10}$$

The second equation in (4.8) implies that there is a potential function $\phi(y)$ that satisfies

$$\begin{cases} \partial_{y_2}\phi = v_2,\ \ \phi(L_s, -1) = 0, \\ \partial_{y_1}\phi = v_1 - \lambda(y_1)(v_4(-1) + \int_{-1}^{y_2} b_0 v_2(L_s,\tau)\,\mathrm{d}\tau) + \int_{-1}^{y_2} G_2(y_1,\tau)\,\mathrm{d}\tau, \end{cases} \tag{4.11}$$

https://doi.org/10.1017/prm.2024.39 Published online by Cambridge University Press

it follows that v_1, v_2 can be represented by

$$\begin{cases} v_2 = \partial_{y_2}\phi, \\ v_1 = \partial_{y_1}\phi + \lambda(y_1)[v_4(-1) + b_0\phi(L_s, y_2)] - \int_{-1}^{y_2} G_2(y_1,\tau)\,\mathrm{d}\tau. \end{cases} \tag{4.12}$$

Substituting (4.12) into the first equation in (4.8), we conclude that ϕ satisfies the following non-local elliptic equation of the second with an unknown constant $v_4(-1)$

$$\begin{aligned} &\partial_{y_1}^2\phi + \frac{1}{1-\bar{M}^2}\partial_{y_2}^2\phi + \lambda_1(y_1)\partial_{y_1}\phi + \lambda_0(y_1)b_0\left(\frac{v_4(-1)}{b_0} + \phi(L_s, y_2)\right) \\ &\quad = G_1(y) + \lambda_1(y_1)\int_{-1}^{y_2} G_2(y_1,\tau)\,\mathrm{d}\tau + \partial_{y_1}\int_{-1}^{y_2} G_2(y_1,\tau)\,\mathrm{d}\tau, \end{aligned} \tag{4.13}$$

where

$$\begin{aligned} \lambda_1(y_1) &= \frac{B_1(y_1)}{c^2(\bar{\rho}) - \bar{u}^2} > 0, \\ \lambda_0(y_1) &= \frac{1}{c^2(\bar{\rho}) - \bar{u}^2}[(c^2(\bar{\rho}) - \bar{u}^2)\lambda' + B_1\lambda + B_3b_3 + B_4]. \end{aligned}$$

Similarly, substituting (4.12) into boundary conditions (4.9), combined with the boundary condition of ϕ in (4.11), we have

$$\begin{cases} \partial_{y_1}\phi(L_s, y_2) = b_0(b_2 - \lambda(L_s))\left(\frac{v_4(-1)}{b_0} + \phi(L_s, y_2)\right) + R_6(y_2) + \int_{-1}^{y_2} G_2(L_s,\tau)\,\mathrm{d}\tau, \\ \partial_{y_1}\phi(L_1, y_2) = -\frac{\epsilon\hat{P}_{ex}(y_2)}{\bar{u}(L_1)} + R_7(y_2) + \int_{-1}^{y_2} G_2(L_1,\tau)\,\mathrm{d}\tau, \\ \partial_{y_2}\phi(y_1, \pm 1) = \phi(L_s, -1) = 0. \end{cases} \tag{4.14}$$

So far, we have reduced problem (4.8)–(4.9) into a non-local elliptic equation of ϕ with an unknown constant $v_4(-1)$. Hence, it is sufficient to establish the solvability and regularity of problem (4.13)–(4.14) to study the original problem. We are going to do it in the next subsection.

4.2. A non-local elliptic equation with a free constant

In this section, we prove the existence, uniqueness and regularity of problem (4.13). To this end, we consider the following more concise form of second-order elliptic system with an unknown constant κ

$$\begin{cases} \partial_{y_1}^2\phi + a_2(y_1)\partial_{y_2}^2\phi + a_1(y_1)\partial_{y_1}\phi - a_0(y_1)(\kappa + \phi(L_s, y_2)) = \partial_{y_1}f, \text{ in } Q, \\ \partial_{y_1}\phi(L_s, y_2) - a_3(\kappa + \phi(L_s, y_2)) = g_1(y_2), \\ \partial_{y_1}\phi(L_1, y_2) = g_2(y_2), \\ \partial_{y_2}\phi(y_1, \pm 1) = \phi(L_s, -1) = 0, \end{cases} \tag{4.15}$$

https://doi.org/10.1017/prm.2024.39 Published online by Cambridge University Press

where the smooth coefficients $a_i(y_1)$, i=0, 1, 2 and the constant a_3 satisfy

$$a_1(y_1) < C_1,\ 0 < C_0 < a_i(y) < C_1, \quad i = 0, 2, 3, \tag{4.16}$$

and the parameter κ is a constant to be determined with the solution itself.

The first lemma implies that the inhomogeneous problem corresponding to system (4.15) without the unknown constant has a unique weak solution.

LEMMA 4.1. *Suppose that $f \in L^2(Q)$ and $g_i \in L^2(-1,\,1)$, $i = 1,\,2$, then there exists a suitable large positive constant K, such that the following inhomogeneous second-order elliptic equation*

$$\begin{cases} -\partial_{y_1}^2\phi - a_2(y_1)\partial_{y_2}^2\phi - a_1(y_1)\partial_{y_1}\phi + K\phi + a_0(y_1)\phi(L_s, y_2) = \partial_{y_1} f, \ \ in\, Q, \\ \partial_{y_1}\phi(L_s, y_2) - a_3\phi(L_s, y_2) = g_1(y_2), \\ \partial_{y_1}\phi(L_1, y_2) = g_2(y_2), \\ \partial_{y_2}\phi(y_1, \pm 1) = 0, \end{cases} \tag{4.17}$$

admits a unique weak solution $\phi \in H^1(Q)$ satisfying

$$\|\phi\|_{H^1(Q)} \leqslant C(\|(g_1, g_2)\|_{L^2(-1,1)} + \|f\|_{L^2(Q)}), \tag{4.18}$$

for some positive constant $C > 0$.

Proof. For ϕ, $\psi \in H^1(Q)$, define the bilinear form

$$\begin{aligned} \mathcal{B}[\phi, \psi] = &\int_Q \partial_{y_1}\phi\partial_{y_2}\psi \mathrm{d}y + \int_Q a_2(y_1)\partial_{y_2}\phi\partial_{y_2}\psi \mathrm{d}y - \int_Q a_1(y_1)\psi\partial_{y_1}\phi \mathrm{d}y \\ &+ K\int_Q \phi\psi \mathrm{d}y + \int_Q a_0(y_1)\phi(L_s, y_2)\psi \mathrm{d}y + a_3\int_{-1}^1 \phi(L_s, y_2)\psi(L_s, y_2)\mathrm{d}y_2, \end{aligned}$$

and the linear functional on $H^1(Q)$

$$l(\psi) = \int_{-1}^1 g_2(y_2)\psi(L_1, y_2)\mathrm{d}y_2 - \int_{-1}^1 g_1(y_2)\psi(L_s, y_2)\mathrm{d}y_2 - \int_Q \partial_{y_1} f\psi \mathrm{d}y.$$

It's obviously that the linear functional $l(\psi)$ on $H^1(Q)$ is continuous, i.e.

$$|l(\psi)| \leqslant C(\|(g_1, g_2)\|_{L^2(-1,1)} + \|f\|_{L^2(Q)})\|\psi\|_{H^1(Q)}, \tag{4.19}$$

where we have used the trace theorem. So, what we need to do is just verify that the conditions of the Lax–Milgram Theorem are satisfied for the bilinear form $\mathcal{B}$. The boundedness of $\mathcal{B}_K$ is trivial, we will show that $\mathcal{B}_K$ is also coercive. Denote

https://doi.org/10.1017/prm.2024.39 Published online by Cambridge University Press

$\Lambda = \min\{1, C_0\} > 0$, then

$$\Lambda \int_Q |\nabla\phi|^2 \mathrm{d}y + K \int_Q |\phi|^2 \mathrm{d}y \leqslant \mathcal{B}[\phi,\phi] + \int_Q a_1(y_1)\phi\partial_{y_1}\phi \mathrm{d}y \\ - \int_Q a_0(y_1)\phi(L_s, y_2)\phi(y_1, y_2)\mathrm{d}y - a_3 \int_{-1}^{1} |\phi(L_s, y_2)|^2 \mathrm{d}y_2,$$

Cauchy's inequality gives

$$\int_Q a_1(y_1)\phi\partial_{y_1}\phi \mathrm{d}y \leqslant C_1\epsilon \int_Q |\nabla\phi|^2 \mathrm{d}y + \frac{C_1}{4\epsilon}\int_Q |\phi|^2 \mathrm{d}y,$$

and

$$\int_Q a_0(y_1)\phi(L_s, y_2)\phi(y_1, y_2)\mathrm{d}y \leqslant C_{tr}(L_1 - L_s)C_1\epsilon \int_Q |\nabla\phi|^2 \mathrm{d}y + \frac{C_1}{4\epsilon}\int_Q |\phi|^2 \mathrm{d}y.$$

Then, fix ϵ_0 such that $C_1\epsilon_0(1 + (L_1 - L_s)C_{tr}) < \Lambda/2$, and choosing $K = \max\{\Lambda, C_1/\epsilon_0\}$, thanks to the positivity of a_3, we obtain

$$\mathcal{B}[\phi,\phi] \geqslant \frac{\Lambda}{2}\|\phi\|_{H^1(Q)},$$

the unique existence follows from the Lax–Milgram Theorem and (4.19) gives the estimates (4.18). Thus, the proof is complete. □

The unique existence of regular solution to non-local system (4.15) is established in the following proposition.

PROPOSITION 4.2. *For any $f \in C^{1,\alpha}(\bar{Q})$, $g_i \in C^{\alpha}(\bar{Q})$, there is a unique weak solution (ϕ, κ), such that $\phi \in H^1(Q)$ and the following estimate holds*

$$\|\phi\|_{H^1(Q)} + |\kappa| \leqslant C(\|f\|_{C^{\alpha}(\bar{Q})} + |(g_1, g_2)|_{C^{1,\alpha}[-1,1]}). \tag{4.20}$$

Moreover, if the compatibility conditions

$$\partial_{y_2} f(y_1, -1) = \partial_{y_2} f(y_1, 1) = 0,\ g_i'(-1) = g_i'(1) = 0,\ i = 1, 2, \tag{4.21}$$

are fulfilled, then $\phi \in C^{2,\alpha}(\bar{Q})$

$$\|\phi\|_{C^{1,\alpha}(\bar{Q})} \leqslant C(\|f\|_{C^{\alpha}(\bar{Q})} + |(g_1, g_2)|_{C^{\alpha}[-1,1]} + \|\phi\|_{H^1(Q)} + |\kappa|), \tag{4.22}$$

and

$$\|\phi\|_{C^{2,\alpha}(\bar{Q})} \leqslant C(\|f\|_{C^{1,\alpha}(\bar{Q})} + |(g_1, g_2)|_{C^{1,\alpha}[-1,1]} + \|\phi\|_{H^1(Q)} + |\kappa|). \tag{4.23}$$

for some positive constant $C > 0$.

Proof. The proof is divided into two steps.

Step 1: Regularity of weak solutions. We will use the symmetric extension methods to exclude the possible singularities that may appear at the corner, which

https://doi.org/10.1017/prm.2024.39 Published online by Cambridge University Press

implies that the weak solution $\phi \in H^1(Q)$ to system (4.15) is essentially more regular. To this end, introduce the notation

$$Q^* := \{(y_1, y_2) : L_s < y_1 < L_1, -2 < y_2 < 2\}.$$

and define the extended function $\phi^*(y)$ on Q^* as

$$\phi^*(y) = \begin{cases} \phi(y_1, 2 - y_2),\ 1 < y_2 < 2, \\ \phi(y_1, y_2), -1 < y_2 < 1, \\ \phi(y_1, -2 - y_2), -2 < y_2 < -1. \end{cases} \tag{4.24}$$

Then the extended function ϕ^* satisfies

$$\begin{cases} \partial^2_{y_1}\phi^* + a_2(y_1)\partial^2_{y_2}\phi^* + a_1(y_1)\partial_{y_1}\phi^* - a_0(y_1)(\kappa + \phi^*(L_s, y_2)) = \partial_{y_1} f^*, \text{in } Q^*, \\ \partial_{y_1}\phi^*(L_s, y_2) - a_3(\kappa + \phi^*(L_s, y_2)) = g_1^*(y_2), \\ \partial_{y_1}\phi^*(L_1, y_2) = g_2^*(y_2), \\ \partial_{y_2}\phi^*(y_1, \pm 1) = \phi^*(L_s, -1) = 0, \end{cases} \tag{4.25}$$

Using the standard interior and the boundary estimates for the second-order linear elliptic equations in [**13**], we obtain that $\phi^*(y) \in C^{1,\alpha}(Q^*)$ and

$$\|\phi^*\|_{C^{1,\alpha}(Q^*)} \leqslant C(\|f^*\|_{C^{1,\alpha}(Q^*)} + \|(g_1, g_2)\|_{C^{1,\alpha}[-2,2]} + \|\phi^*(L_s, y_2)\|_{L^2[-2,2]} + |\kappa|), \tag{4.26}$$

which implies that $\phi^*(L_s,\, y_2) \in C^{1,\alpha}[-2,\, 2]$. Use estimate (4.26) again to conclude that $\phi^*(y) \in C^{2,\alpha}(Q^*)$ and

$$\|\phi^*\|_{C^{1,\alpha}(Q^*)} \leqslant C(\|f^*\|_{C^{1,\alpha}(Q^*)} + \|(g_1, g_2)\|_{C^{1,\alpha}[-2,2]} + \|\phi^*(0, y_2)\|_{L^2[-2,2]} + |\kappa|). \tag{4.27}$$

Then (4.22) and (4.23) follow immediately.

Step 2: Existence and uniqueness of weak solutions. Due to the linearity, any solution ϕ to problem (4.15) can be decomposed as $\phi = \phi_1 + \phi_2$, where ϕ_i, i=1,2, satisfy the following equation respectively

$$\begin{cases} \partial^2_{y_1}\phi_1 + a_2(y_1)\partial^2_{y_2}\phi_1 + a_1(y_1)\partial_{y_1}\phi_1 - a_0(y_1)\phi_1(L_s, y_2) = \partial_{y_1} f, \text{ in } Q, \\ \partial_{y_1}\phi_1(L_s, y_2) - a_3\phi_1(L_s, y_2) = g_1(y_2), \\ \partial_{y_1}\phi_1(L_1, y_2) = g_2(y_2), \\ \partial_{y_2}\phi_1(y_1, \pm 1) = 0, \end{cases} \tag{4.28}$$

and

$$\begin{cases} \partial^2_{y_1}\phi_2 + a_2(y_1)\partial^2_{y_2}\phi_2 + a_1(y_1)\partial_{y_1}\phi_2 - a_0(y_1)(\kappa + \phi_2(L_s, y_2)) = 0, \text{ in } Q, \\ \partial_{y_1}\phi_2(L_s, y_2) - a_3(\kappa + \phi_2(L_s, y_2)) = 0, \\ \partial_{y_1}\phi_2(L_1, y_2) = \partial_{y_2}\phi_2(y_1, \pm 1) = 0, \\ \phi_2(L_s, -1) = -\phi_1(L_s, -1). \end{cases} \tag{4.29}$$

https://doi.org/10.1017/prm.2024.39 Published online by Cambridge University Press

Combing lemma 4.1 with the Fredholm alternative theorem, one can easily derive that (4.28) has a unique $H^1(Q)$ solution ϕ_1 which satisfies (4.20). On the other hand, one can prove that the only weak solution to (4.29) must be $(\phi_2, \kappa) = (-\phi_1(L_s, -1), \phi_1(L_s, -1))$ by applying the maximum principle. For the detailed proof, one can refer to lemma 4.1 in [17]. Thus, the proof is complete. □

At this point, we can easily illustrate the well-posedness to reformulated problem (4.13)–(4.14).

Lemma 4.3. *Problem (4.13)–(4.14) has a unique solution $(\phi, \kappa) \in C^{2,\alpha}(\bar{Q}) \times \mathbb{R}$ satisfying*

$$\begin{aligned} \|\phi\|_{C^{k,\alpha}(\bar{Q})} + |\kappa| \leqslant C(&\|(G_1, G_2)\|_{C^{k-1,\alpha}(\bar{Q})} \\ &+ \|(R_6(y_2), R_7(y_2))\|_{C^{1,\alpha}[-1,1]} + \|\epsilon \hat{P}_{ex}(y_2)\|_{C^{1,\alpha}[-1,1]}), \quad k = 1, 2, \end{aligned} \tag{4.30}$$

for some positive constant C.

Proof. It suffices to verify solvability condition (4.16) for problem (4.13)–(4.14). A direct but tedious computation shows that

$$\begin{aligned} a_0(y_1) &= -b_0\lambda_0(y_1) = -\frac{2c^2(\bar{\rho}^+)\bar{f} b_0 b_3}{\bar{u}(c^2(\bar{\rho}^+) - \bar{u}^2)^2} > 0, \\ a_1(y_1) &= \lambda_1(y_1) = \frac{\gamma\bar{u}^2 + c^2(\bar{\rho}^+)}{(c^2(\bar{\rho}^+) - \bar{u}^2)^2}\bar{f} > 0, \\ a_2(y_1) &= \frac{1}{1 - \bar{M}^2} > 0, \\ a_3 &= b_0(b_2 - \lambda(L_s)) = b_0\frac{c^2(\bar{\rho}^+)(\bar{\rho}^+ - \bar{\rho}^-)\bar{f}(L_s)}{\bar{\rho}^+\bar{u}(c^2(\bar{\rho}^+) - \bar{u}^2)} > 0, \end{aligned}$$

since the background solution is subsonic and smooth, the upper bound is trivial. Hence, proposition 4.2 implies that there exists a unique weak solution (ϕ, κ), and estimate (4.30) follows from (4.22) and (4.23). □

In view of the analysis of problem (4.13)–(4.14), the well-posedness of equation (4.8)–(4.9) follows.

Lemma 4.4. *Problem (4.8)–(4.9) admits a unique solution $(v_1, v_2, v_4(-1)) \in (C^{2,\alpha}(\bar{Q}))^2 \times \mathbb{R}$ satisfying*

$$\begin{aligned} \|(v_1, v_2)\|_{C^{k,\alpha}(\bar{Q})} + |v_4(-1)| \leqslant C(&\|(G_1, G_2)\|_{C^{k-1,\alpha}(\bar{Q})} \\ &+ \|(R_6(y_2), R_7(y_2))\|_{C^{k,\alpha}[-1,1]} + \|\epsilon \hat{P}_{ex}(y_2)\|_{C^{k,\alpha}[-1,1]}), \quad k = 1, 2, \end{aligned} \tag{4.31}$$

and the compatibility conditions

$$\partial_{y_2} v_1(y_1, \pm 1) = 0, \quad \partial_{y_2} v_2(y_1, \pm 1) = \partial_{y_2}^2 v_2(y_1, \pm 1) = 0. \tag{4.32}$$

https://doi.org/10.1017/prm.2024.39 Published online by Cambridge University Press

Proof. Combine (4.12) and (4.30), we conclude that there is a unique solution $(v_1, v_2, v_4(-1)) \in (C^{1,\alpha}(\bar{Q}))^2 \times \mathbb{R}$ such that

$$\begin{aligned}\|(v_1, v_2)\|_{C^{\alpha}(\bar{Q})} + |\mathbf{v}_4(-1)| \leqslant C(&\|(G_1, G_2)\|_{C^{\alpha}(\bar{Q})} \\ &+ \|(R_6(y_2), R_7(y_2))\|_{C^{1,\alpha}[-1,1]} + \|\epsilon \hat{P}_{ex}(y_2)\|_{C^{1,\alpha}[-1,1]}).\end{aligned} \tag{4.33}$$

The similar estimates also hold true for $\|(v_1, v_2)\|_{C^{1,\alpha}(\bar{Q})}$, but we can derive an even better estimate by rewriting (4.8) into an elliptic equation of v_1. To this end, we first rewrite system (4.8) as

$$\begin{cases} \partial_{y_1} v_1 + \frac{1}{1-\bar{M}^2} \partial_{y_2} v_2 + \lambda_1(y_1) v_1 = \mathcal{G}_1(y), \\ \partial_{y_1} v_2 - \partial_{y_2} v_1 = \mathcal{G}_2(y), \\ v_1(L_s, y_2) = \mathcal{R}_6(y_2), \\ v_1(L_1, y_2) = \mathcal{R}_7(y_2), \\ v_2(y_1, \pm 1) = 0, \end{cases} \tag{4.34}$$

where

$$\begin{aligned}
\mathcal{G}_1(y) &= G_1(y) - \frac{B_3(y_1) b_3 + B_4(y_1)}{(c^2(\bar{\rho}^+) - (\bar{u}^+)^2)} (v_4(-1) + \int_{-1}^{y_2} b_0 v_2(L_s, \tau)\, \mathrm{d}\tau), \\
\mathcal{G}_2(y) &= G_2(y) + b_0 v_2(L_s, y_2), \\
\mathcal{R}_6(y_2) &= b_2 (v_4(-1) + \int_{-1}^{y_2} b_0 v_2(L_s, \tau)\, \mathrm{d}\tau) + R_6(y_2), \\
\mathcal{R}_7(y_2) &= \frac{b_3}{\bar{u}(L_1)} (v_4(-1) + \int_{-1}^{y_2} b_0 v_2(L_s, \tau)\, \mathrm{d}\tau) - \frac{\epsilon \hat{P}_{ex}(y_2)}{\bar{u}(L_1)} + R_7(y_2),
\end{aligned}$$

and

$$\partial_{y_2} \mathcal{G}_1(y_1, \pm 1) = 0,\ \mathcal{G}_2(y_1, \pm 1) = 0,\ \mathcal{R}_6'(\pm 1) = 0,\ \mathcal{R}_7'(\pm 1) = 0. \tag{4.35}$$

Note that (4.12) and the boundary conditions of ϕ imply that

$$v_2(y_1, \pm 1) = 0,\ \partial_{y_2} v_1(y_1, \pm 1) = 0. \tag{4.36}$$

A simple cancellation yields to

$$\begin{cases} \partial_{y_1}((1-\bar{M}^2)\partial_{y_1} v_1) + \partial_{y_2}^2 v_1 + \partial_{y_1}(\lambda_1(1-\bar{M}^2) v_1) = \partial_{y_1}((1-\bar{M}^2)\mathcal{G}_1) - \partial_{y_2} \mathcal{G}_2, \\ v_1(L_s, y_2) = \mathcal{R}_6(y_2),\ v_1(L_1, y_2) = \mathcal{R}_7(y_2), \\ \partial_{y_2} v_1(y_1, \pm 1) = 0. \end{cases} \tag{4.37}$$

Since the boundary conditions in (4.37) are compatible at the corners, we obtain the following estimates for v_1 by using the symmetric extension defined by (4.24)

$$\begin{aligned}\|v_1\|_{C^{k,\alpha}(\bar{Q})} \leqslant C(&\|v_2\|_{C^{k-1,\alpha}(\bar{Q})} + |v_4(-1)| + \|(G_1, G_2)\|_{C^{k-1,\alpha}(\bar{Q})} \\ &+ \|\hat{P}_{ex}\|_{C^{k,\alpha}[-1,1]} + \|(R_6, R_7)\|_{C^{k,\alpha}[-1,1]}), \quad k = 1, 2.\end{aligned} \tag{4.38}$$

https://doi.org/10.1017/prm.2024.39 Published online by Cambridge University Press

These estimates together with (4.34) also imply that

$$\begin{aligned}\|v_2\|_{C^{k,\alpha}(\bar{Q})} \leqslant C(&\|v_2\|_{C^{k-1,\alpha}(\bar{Q})} + |v_4(-1)| + \|(G_1, G_2)\|_{C^{k-1,\alpha}(\bar{Q})}\\ &+ \|\hat{P}_{ex}\|_{C^{k,\alpha}[-1,1]} + \|(R_6, R_7)\|_{C^{k,\alpha}[-1,1]}), \quad k = 1, 2.\end{aligned} \tag{4.39}$$

Hence, (4.31) follows from (4.33) and (4.38)–(4.39). Finally, differentiating the first equation in (4.34) with respect to x_2 and combining with $\partial_{y_2}\mathcal{G}_1(y_1, \pm 1) = 0$ and $\partial_{y_2} v_1(y_1, \pm 1) = 0$, we obtain

$$\partial_{y_2}^2 v_2(y_1, \pm 1) = 0,$$

which gives compatibility condition (4.32). Thus, the proof is complete. □

5. A priori estimates and proofs of main results

In this section, we will use the Banach contraction mapping theorem to prove theorem 3.1. Given any $\hat{V} \in \mathcal{V}_\delta$, we could establish some *a priori* estimates to the linearized problems defined in subsection 4.1, and construct a contractible mapping from $\mathcal{V}_\delta$ into itself so that there exists a unique fixed point, which is the solutions obtained in theorem 3.1 and the proof of theorem 3.1 will be finished.

Lemma 4.4 implies that there is a unique solution $(v_1, v_2, v_4(-1)) \in (C^{2,\alpha}(\bar{Q}))^2 \times \mathbb{R}$ to system (4.8)–(4.9) satisfying

$$\begin{aligned}&\|(v_1, v_2)\|_{C^{2,\alpha}(\bar{Q})} + |v_4(-1)| \leqslant C(\|(G_1, G_2)\|_{C^{1,\alpha}(\bar{Q})}\\ &\quad + \|(R_6(y_2), R_7(y_2))\|_{C^{2,\alpha}[-1,1]} + \|\epsilon \hat{P}_{ex}(y_2)\|_{C^{2,\alpha}[-1,1]})\\ &\leqslant C(\epsilon + \delta^2),\end{aligned} \tag{5.1}$$

and compatibility condition (4.32) holds true.

The shock curve v_4 is given by (4.3), which satisfies

$$\|v_4\|_{C^{3,\alpha}[-1,1]} \leqslant C(|v_4(-1)| + \|v_2\|_{C^{2,\alpha}(\bar{Q})} + \|F_5\|_{C^{2,\alpha}(\bar{Q})}) \leqslant C(\epsilon + \delta^2). \tag{5.2}$$

Moreover, it follows from (4.32) and (4.4) that

$$v_4'(\pm 1) = 0 = v_4^{(3)}(\pm 1). \tag{5.3}$$

It remains to solve v_3. Due to (4.6), combining with (4.7) and (5.3), we obtain the estimate

$$\|v_3\|_{C^{2,\alpha}(\bar{Q})} \leqslant C(\|v_4\|_{C^{2,\alpha}[-1,1]} + \|F_6\|_{C^{2,\alpha}(\bar{Q})}) \leqslant C(\epsilon + \delta^2), \tag{5.4}$$

and the compatibility condition

$$\partial_{y_2} v_3(y_1, \pm 1) = 0. \tag{5.5}$$

Taking $\delta = O(1)\epsilon$, then for any given $\hat{V} \in \mathcal{V}_\delta$ we can define a continuous mapping $T : \mathcal{V}_\delta \to \mathcal{V}_\delta$ as

$$T\hat{V} = V, \tag{5.6}$$

due to the iteration scheme introduced in the previous section and estimates (5.1)–(5.5). Finally, we show that the mapping is also contractible in the space $(C^{1,\alpha}(\bar{Q}))^3 \times C^{2,\alpha}[-1, 1]$.

https://doi.org/10.1017/prm.2024.39 Published online by Cambridge University Press

For arbitrarily given two states $\hat{V}_i = (\hat{v}_1^i,\, \hat{v}_2^i,\, \hat{v}_3^i,\, \hat{v}_4^i) \in \mathcal{V}_\delta$, $i = 1,\, 2$ with the corresponding fluid variable $(\hat{u}_1^i,\, \hat{u}_2^i,\, \hat{B}^i,\, \hat{\xi}^i)$, set

$$V_i = T\hat{V}_i, \quad i = 1, 2,$$

where $V_i = (v_1^i,\, v_2^i,\, v_3^i,\, v_4^i)$. For the convenience, we denote $\hat{W} = \hat{V}_1 - \hat{V}_2$ and $W = V_1 - V_2$, or equivalently,

$$\hat{w}_k = \hat{v}_k^1 - \hat{v}_k^2, \quad w_k = v_k^1 - v_k^2, \quad 1 \leqslant k \leqslant 4.$$

Equation (4.2) implies that

$$w_4' = b_0 w_2 + O(\epsilon) \sum_{i=1}^{4} \hat{w}_i, \tag{5.7}$$

which yields that

$$\|w_4'\|_{C^{1,\alpha}[-1,1]} \leqslant C\|w_2\|_{C^{1,\alpha}(\bar{Q})} + C\epsilon \left(\sum_{i=1}^{3} \|\hat{w}_i\|_{C^{1,\alpha}(\bar{Q})} + \|\hat{w}_4\|_{C^{1,\alpha}[-1,1]} \right). \tag{5.8}$$

It follows from (4.6) that

$$w_3 = b_3 w_4 - b_3 \int_{\beta_1(y)}^{\beta_2(y)} (\hat{v}_4^1(\tau))' \,\mathrm{d}\tau + b_3 \int_{y_2}^{\beta_2(y)} \hat{w}_4'(\tau) \,\mathrm{d}\tau + O(\epsilon) \sum_{i=1}^{3} \hat{w}_i, \tag{5.9}$$

where β_i, $i = 1,\, 2$ is the initial position such that the corresponding characteristic $y_2^i(s,\, \beta_i)$ going through $(y_1,\, y_2)$ with $y_2^i(L_s) = \beta_i$. It is easy to verify that

$$\|\beta_1(y) - \beta_2(y)\|_{C^{1,\alpha}(\bar{Q})} \leqslant C(\|\hat{w}_1\|_{C^{1,\alpha}(\bar{Q})} + \|\hat{w}_2\|_{C^{1,\alpha}(\bar{Q})} + \|\hat{w}_4\|_{C^{2,\alpha}[-1,1]}), \tag{5.10}$$

thus,

$$\|w_3\|_{C^{1,\alpha}(\bar{Q})} \leqslant C\|w_4\|_{C^{1,\alpha}[-1,1]} + C\epsilon \left(\sum_{i=1}^{3} \|\hat{w}_i\|_{C^{1,\alpha}(\bar{Q})} + \|\hat{w}_4\|_{C^{2,\alpha}[-1,1]} \right). \tag{5.11}$$

It is straightforward to show that w_1, w_2 satisfies

$$\begin{cases} \partial_{y_1} w_1 + \frac{1}{1-\bar{M}^2} \partial_{y_2} w_2 + \lambda_1(y_1) w_1 + \lambda_2(y_1)(w_4(-1) + \int_{-1}^{y_2} b_0 w_2(L_s, \tau) \,\mathrm{d}\tau) \\ \quad = \sum_{i=1}^4 (O(\epsilon)\hat{w}_i + O(\epsilon)\nabla\hat{w}_i) + O(\epsilon)(\beta_1 - \beta_2) + O(1) \int_{y_2}^{\beta_2(y)} \hat{w}_4'(\tau) \,\mathrm{d}\tau, \\ \partial_{y_1} w_2 - \partial_{y_2}[w_1 - \lambda(y_1)(w_4(-1) + \int_{-1}^{y_2} b_0 w_2(L_s, \tau) \,\mathrm{d}\tau)] \\ = \sum_{i=1}^4 (O(\epsilon)\hat{w}_i + O(\epsilon)\nabla\hat{w}_i) + O(\epsilon)\partial_{y_2}(\beta_1 - \beta_2) + O(1)\partial_{y_2} \int_{y_2}^{\beta_2(y)} \hat{w}_4'(\tau) \,\mathrm{d}\tau, \end{cases} \tag{5.12}$$

and the boundary conditions

$$\begin{cases} w_1(L_s, y_2) = b_2(\hat{w}_4(-1) + \int_{-1}^{y_2} b_0 \hat{w}_2(L_s, \tau) \,\mathrm{d}\tau) + \sum_{i=1}^4 O(\epsilon)\hat{w}_i, \\ w_1(L_1, y_2) = \frac{b_3}{\bar{u}(L_1)}(\hat{w}_4(-1) + \int_{-1}^{y_2} b_0 \hat{w}_2(L_s, \tau) \,\mathrm{d}\tau) + \sum_{i=1}^4 O(\epsilon)\hat{w}_i \\ \quad + O(\epsilon)(\beta_1 - \beta_2)(L_1, y_2) + O(1) \int_{y_2}^{\beta_2(L_1, y_2)} \hat{w}_4'(\tau) \,\mathrm{d}\tau, \\ w_2(y_1, \pm 1) = 0, \end{cases} \tag{5.13}$$

https://doi.org/10.1017/prm.2024.39 Published online by Cambridge University Press

where

$$\lambda_2(y_1) = \frac{B_3(y_1)b_3 + B_4(y_1)}{c^2(\bar{\rho}^+) - (\bar{u}^+)^2}.$$

Then, applying estimate (4.31) to system (5.12)–(5.13) with $k = 1$ and together with (5.10), we obtain

$$\|(w_1, w_2)\|_{C^{1,\alpha}(\bar{Q})} + |w_4(-1)| \leqslant C\epsilon \left(\sum_{i=1}^{3} \|\hat{w}_i\|_{C^{1,\alpha}(\bar{Q})} + \|\hat{w}_4\|_{C^{2,\alpha}[-1,1]} \right). \quad (5.14)$$

Finally, collecting all these estimates above leads to

$$\sum_{i=1}^{3} \|w_i\|_{C^{1,\alpha}(\bar{Q})} + \|w_4\|_{C^{2,\alpha}[-1,1]} \leqslant C\epsilon \left(\sum_{i=1}^{3} \|\hat{w}_i\|_{C^{1,\alpha}(\bar{Q})} + \|\hat{w}_4\|_{C^{2,\alpha}[-1,1]} \right). \quad (5.15)$$

By (5.15), there is a small constant ϵ_0 such that for all $\epsilon \in (0, \epsilon_0]$, the mapping T defined by (5.6) is contractible in the Banach space $(C^{1,\alpha}(\bar{Q}))^3 \times C^{2,\alpha}[-1, 1]$. Therefore, there exists a unique solution V in $(C^{1,\alpha}(\bar{Q}))^3 \times C^{2,\alpha}[-1, 1]$. Due to lemma 4.4 and *a priori* estimates (5.2), (5.4), we know that V also belongs to $\mathcal{V}_\delta$. It follows that V satisfies estimates (3.23). Thus, the proof of theorem 3.1 is complete. Theorem 2.3 is a direct inference of theorem 3.1. We omit the details.

Data availability statement

Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

Acknowledgements

Weng is partially supported by the National Natural Science Foundation of China 12071359, 12221001. The authors would like to thank the referees for their useful comments and suggestions.

Competing interest

None.

References

1 L. Bers. *Mathematical Aspects of Subsonic and Transonic Gas Dynamics* (John Wiley & Sons Inc, Chapman & Hall Ltd, New York/London, 1958).

2 C. Chen and C. Xie. Three dimensional steady subsonic Euler flows in bounded nozzles. *J. Differ. Equ.* **256** (2014), 3684–3708.

3 S. Chen. Stability on transonic shock fronts in two-dimensional Euler systems. *Trans. Am. Math. Soc.* **357** (2005), 287–308.

4 S. Chen. Transonic shocks in 3-D compressible flow passing a duct with a general section for Euler systems. *Trans. Am. Math. Soc.* **360** (2008), 5265–5289.

5 S. Chen, H.Yuan. Transonic shocks in compressible flow passing a duct for three-dimensional Euler systems. *Arch. Ration. Mech. Anal.* **187** (2008), 523–556.

6 G. Chen and M. Feldman. Multidimensional transonic shocks and free boundary problems for nonlinear equations of mixed type. *J. Amer. Math. Soc.* **16** (2003), 461–494.

7 G. Chen, J. Chen and K. Song. Transonic nozzle flows and free boundary problems for the full Euler equations. *J. Differ. Equ.* **229** (2006), 92–120.

8 G. Chen, J. Chen and M. Feldman. Transonic shocks and free boundary problems for the full Euler equations in infinite nozzles. *J. Math. Pures Appl. (9)* **88** (2007), 191–218.

9 R. Courant and K. O. Friedrichs. *Supersonic Flow and Shock Waves* (Interscience Publishers, Inc., New York, 1948).

10 P. Embid, J. Goodman and A. Majda. Multiple steady states for 1-D transonic flow. *SIAM J. Sci. Stat. Comput.* **5** (1984), 21–41.

11 B. Fang and Z. Xin. On admissible locations of transonic shock fronts for steady euler flows in an almost flat finite Nozzle with prescribed receiver pressure. *Comm. Pure Appl. Math.* **74** (2021), 1493–1544.

12 B. Fang and X. Gao. On admissible positions of transonic shocks for steady euler flows in a 3-d axisymmetric cylindrical Nozzle. *J. Differ. Equ.* **288** (2021), 62–117.

13 D. Gilbarg and N. Tudinger, Elliptic partial differential equations of second order, Vol. 224 of Grundlehren der Mathematischen Wissenschaften, 2nd ed. (Springer, Berlin-New York, 1983).

14 T. Liu. Nonlinear stability and instability of transonic flows through a nozzle. *Commun. Math. Phys.* **83** (1982), 243–260.

15 L. Liu and H. Yuan. Stability of cylindrical transonic shocks for the two-dimensional steady compressible Euler system. *J. Hyperbolic Differ. Equ.* **5** (2008), 347–379.

16 J. Li, Z. Xin and H. Yin. On transonic shocks in a nozzle with variable end pressures. *Commun. Math. Phys.* **291** (2009), 111–150.

17 J. Li, Z. Xin and H. Yin. A free boundary value problem for the full steady compressible Euler system and two dimensional transonic shocks in a large variable nozzle. *Math. Res. Lett.* **16** (2009), 777–786.

18 J. Li, Z. Xin and H. Yin. The existence and monotonicity of a three-dimensional transonic shock in a finite nozzle with axisymmetric exit pressure. *Pacific J. Math.* **247** (2010), 109–161.

19 J. Li, Z. Xin and H. Yin. On transonic shocks in a conic divergent nozzle with axi-symmetric exit pressures. *J. Differ. Equ.* **48** (2010), 423–469.

20 J. Li, Z. Xin and H. Yin. Transonic shocks for the full compressible Euler system in a general two-dimensional De Laval nozzle. *Arch. Rational Mech. Anal.* **207** (2013), 533-581.

21 L. Liu, G. Xu and H. Yuan. Stability of spherically symmetric subsonic flows and transonic shocks under multidimensional perturbations. *Adv. Math.* **291** (2016), 696–757.

22 D. Serre. Perfect fluid flow in two independent space variables. Reflection of a planar shock by a compressive wedge. *Arch. Rational Mech. Anal.* **132** (1995), 15–36.

23 S. Weng. A new formulation for the 3-D Euler equations with an application to subsonic flows in a cylinder. *Indiana Univ. Math. J.* **64** (2015), 1609–1642.

24 S. Weng, Z. Xin and H. Yuan. Steady compressible radially symmetric flows with nonzero angular velocity in an annulus. *J. Differ. Equ.* **286** (2021), 433–454.

25 S. Weng, Z. Xin and H. Yuan. On some smooth symmetric transonic flows with nonzero angular velocity and vorticity. *Math. Models Methods Appl. Sci.* **31** (2021), 2773–2817.

26 S. Weng, C. Xie and Z. Xin. Structural stability of the transonic shock problem in a divergent three-dimensional axisymmetric perturbed nozzle. *SIAM J. Math. Anal.* **53** (2021), 279–308.

27 S. Weng and Z. Xin. A deformation-curl decomposition for three dimensional steady Euler equations (in Chinese). *Sci. Sin. Math.* **49** (2019), 307–320.doi: 10.1360/N012018-00125

28 S. Weng. A deformation-curl-Poisson decomposition to the three dimensional steady Euler-Poisson system with applications. *J. Differ. Equ.* **267** (2019), 6574–6603.

29 Z. Xin, W. Yan and H. Yin. Transonic shock problem for the Euler system in a nozzle. *Arch. Rational Mech. Anal.* **194** (2009), 1–47.

30 Z. Xin and H. Yin. Transonic shock in a nozzle I, 2-D case. *Commun. Pure Appl. Math.* **LVIII** (2005), 999–1050.

31 Z. Xin and H. Yin. 3-Dimensional transonic shock in a nozzle. *Pac. J. Math.* **236** (2008), 139–193.

32 Z. Xin and H. Yin. Transonic shock in a curved nozzle, 2-D and 3-D complete Euler systems. *J. Differ. Equ.* **245** (2008), 1014–1085.

33 H. Yuan. On transonic shocks in two-dimensional variable-area ducts for steady Euler system. *SIAM J. Math. Anal.* **38** (2006), 1343–1370.