

# Joining sheath to plasma in electronegative gases at low pressures using matched asymptotic approximations

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**Abstract.** The method of matched asymptotic expansions is used to examine the structure of the plasma sheath of the positive column at low pressure in electronegative gases using the fluid model to describe the positive-ion motion. It is shown that at low negative-ion concentrations, and at high concentrations, the structure is that of a plasma joined to a thin sheath, but that for the electron/negative-ion temperature ratio  $T_e/T_n \equiv \varepsilon > 5 + \sqrt{24}$ , and for a well-defined range of  $A \equiv n_{n0}/n_{e0}$  (the central negative ion to electron density ratio) and for small Debye length, there is a more complex structure with a central negative-ion-dominated plasma surrounded by a quasiplasma in which density oscillations may occur before joining to a sheath. This is in agreement with recent computations using the same model.

## 1. Introduction

There is considerable current interest in plasmas where negative ions are numerically the dominant negative charge carriers. One problem that has received particular attention is that of joining plasma and sheath at low pressures. The first consideration of this problem in the fluid approximation was by Braithwaite and Allen (1988), who showed that the Bohm criterion is generalized to

$$U^2 = \frac{N + E}{\varepsilon N + E},$$

where  $U^2 = M_i v_i^2 / kT_e$  is the normalized ion velocity,  $v_i$  being the ion velocity,  $M_i$  the positive ion mass and  $T_e$  the electron temperature, and the *local* normalized negative ion density is  $N = n_n / n_{e0}$  and correspondingly  $E = n_e / n_{e0}$ ,  $n_{e0}$  being the electron density at the centre of the plasma. They further showed that for electron/negative ion temperature ratios  $\varepsilon \equiv T_e / T_n > 5 + \sqrt{24}$  ( $= 9.90$ ), the potential  $\phi^*$  at the plasma–sheath ‘boundary’ in the zero-Debye-length limit is a multivalued function of  $N_0$ . Braithwaite and Allen suggested that the physically appropriate value of  $\phi^*$  in all circumstances was the smallest. We shall examine that suggestion, showing its limitations.

Franklin and Snell (1992) solved the negative-ion-dominated column in plane and cylindrical geometries, and more recently obtained computed solutions to the plasma–sheath system in the fluid approximation (Franklin and Snell

1998). But their description had omissions at small Debye lengths that were not appreciated at the time.

Sato and Miyawaki (1992) sought to give a solution, albeit in the zero-Debye-length approximation, to the same problem using the Tonks–Langmuir (1929) or free-fall model. In particular, they postulated that, under certain circumstances, for  $\varepsilon > 10.8$  and a limited range of  $A \equiv n_{n0}/n_{e0}$ , there were two plasmas – an inner one dominated by the ions and an outer one (essentially an electron–positive-ion plasma) – separated by a double sheath. This was a significant step in the total understanding.

There has been a spate of recent activity extending the work referred to above and deepening the physical understanding (Kono 1999; Sheridan 1999; Sheridan et al. 1999; Franklin 1999) – all of which is basically computational.

This has prompted us to go back to the method employed by Franklin and Ockendon (1970) for the conventional positive column, namely, matched asymptotic expansions, and to include negative ions, hoping thereby to gain further physical insight. This paper, then, is complementary to those referred to above, and we believe meets our expectations. It also extends the field surveyed by Riemann (1991).

Thus the combination of different mathematical models, analytical techniques, and computational solutions can now be brought into the sort of harmony where the differences can be understood in terms of where the models fail to properly capture all of the physics involved.

The conditions under which there are separate different regions of plasma are limited in the parameter range over which they occur; however, they should be experimentally observable.

## 2. The basic equations

These are those given by Franklin and Ockendon (1970), augmented by the addition of negative ions, and in terms of normalized variables are as follows:

electron generation:

$$\frac{d}{dx}(x^\beta f h) = \lambda g x^\beta; \quad (1)$$

positive-ion motion:

$$\frac{d}{dx}(x^\beta f h^2) = x^\beta f \frac{d\phi}{dx}; \quad (2)$$

electron motion:

$$\frac{1}{g} \frac{dg}{dx} = \frac{-d\phi}{dx}; \quad (3)$$

negative-ion motion:

$$\frac{1}{n} \frac{dn}{dx} = -\varepsilon \frac{d\phi}{dx}; \quad (4)$$

Poisson's equation:

$$\alpha^2 \frac{d}{dx} \left( x^\beta \frac{d\phi}{dx} \right) = x^\beta (f - g - n). \tag{5}$$

Equations (3) and (4) give  $g = \exp(-\phi)$  and  $n = A \exp(-\varepsilon\phi)$ . Equation (4) is new, and (5) is modified.  $\beta = 0$  in plane geometry and  $\beta = 1$  in cylindrical geometry.  $g = n_e/n_{e0}$  is the normalized electron density,  $n = n_n/n_{e0}$  is the normalized negative-ion density,  $f = n_p/n_{e0}$  is the normalized positive-ion density,  $\phi = -eV/kT_e$  is the normalized potential,  $h = v_i(M_i/kT_e)^{1/2} = v_i/c_s$  is the normalized ion speed,  $\alpha^2 = \lambda_D^2/L^2$ ,  $\lambda_D$  being the Debye length and  $L$  the lateral extent of the plasma, and  $\lambda \equiv ZL/c_s$  is the eigenvalue of the problem, where  $Z$  is the electron-generation rate.

The boundary conditions at the centre ( $x = 0$ ) are

$$g = 1, \quad n = A, \quad h = 0, \quad \phi = 0, \quad \frac{d\phi}{dx} = 0, \quad \frac{df}{dx} = 0. \tag{6}$$

It follows that  $dg/dx = dn/dx = 0$  there.

The boundary condition at the wall ( $x = 1$ ) is

$$fh = g\delta, \quad \text{where} \quad \delta \equiv \left( \frac{M_i}{2\pi m_e} \right)^{1/2}, \tag{7}$$

expressing equality of the ion directed flux and the electron random flux.

Restricting ourselves to plane geometry, (2) and (5) give as a first integral

$$fh^2 = 1 - g + \frac{1}{\varepsilon}(A - n) + \frac{\alpha^2}{2} \left( \frac{d\phi}{dx} \right)^2 \tag{8}$$

### 3. The quasineutral plasma

We construct an asymptotic solution for the planar case ( $\beta = 0$ ) in the limit  $\alpha \rightarrow 0$ . The parameters  $A$  and  $\varepsilon$  will be considered as fixed (i.e. of order unity in comparison with  $\alpha$ ),  $A > 0$  and  $\varepsilon \geq 1$ .

We assume that the asymptotic expansion of the eigenvalue  $\lambda$  in the limit  $\alpha \rightarrow 0$  starts with a term of order unity:

$$\lambda = \lambda_0 + \dots \tag{9}$$

In order to describe unknown functions, various asymptotic expansions must be considered.

A straightforward asymptotic expansion, describing the quasineutral plasma, is related to the limit ( $\alpha \rightarrow 0$ ,  $x$  finite), and reads

$$f = f_1(x) + \dots, \quad h = h_1(x) + \dots, \quad \phi = \phi_1(x) + \dots \tag{10}$$

It is expected that these expansions are applicable in the region  $0 \leq x < x_s$ , where  $x_s$  is a constant ( $0 < x_s \leq 1$ ), which is not known a priori and must be determined as a part of the asymptotic solution.

Substituting (10) into (1), (2) and (5), expanding and retaining leading terms, we obtain the following system of equations:

$$f_1 \frac{dh_1}{dx} - h_1(g_1 + \varepsilon n_1) \frac{d\phi_1}{dx} = \lambda_0 g_1, \quad (11)$$

$$-f_1 h_1 \frac{dh_1}{dx} + f_1 \frac{d\phi_1}{dx} = \lambda_0 g_1 h_1, \quad (12)$$

$$f_1 = g_1 + n_1, \quad (13)$$

$$g_1 = e^{-\phi_1}, \quad n_1 = A e^{-\varepsilon \phi_1}. \quad (14)$$

The boundary conditions at  $x = 0$  assume the form

$$\phi_1 = 0, \quad \frac{d\phi_1}{dx} = 0. \quad (15)$$

The problem (11)–(14) may be solved analytically as follows. We multiply (11) by  $h_1$  and in turn add and subtract (12). Integrating the relationship thus obtained, we find

$$x\lambda_0 = \int_0^{\phi_1} \frac{f_1 - h_1^2(g_1 + \varepsilon n_1)}{2g_1 h_1} d\phi_1, \quad (16)$$

$$f_1 h_1^2 = 1 - g_1 + \frac{1}{\varepsilon}(A - n_1). \quad (17)$$

(Note that (17) can be obtained in a simpler way by expanding (8) in powers of  $\alpha$ .) Substituting the function  $h_1(\phi_1)$  defined by (17) into (16), we obtain a closed quadrature formula relating  $\phi_1$  to  $x\lambda_0$ .

The above reasoning breaks down if the numerator of the integrand in (16) (which coincides, to the accuracy of a factor, with the determinant of (11) and (12) considered as a system of two linear equations for the derivatives) vanishes and the derivative  $d\phi_1/dx$  becomes infinite. It follows that the numerator must be positive in the interval  $0 \leq x < x_s$ . Assuming continuity, we must suppose that at  $x = x_s$  the numerator is non-negative:

$$f_s \geq h_1^2(g_s + \varepsilon n_s). \quad (18)$$

Here, and later,

$$f_s = f_1(x_s), \quad g_s = g_1(x_s), \quad n_s = n_1(x_s), \quad h_s = h_1(x_s), \quad \phi_s = \phi_1(x_s).$$

With the use of (17), the inequality (18) may be written as

$$(g_s + A g_s^\varepsilon)^2 \geq (g_s + A \varepsilon g_s^\varepsilon) \left[ 1 - g_1 + \frac{1}{\varepsilon}(A - n_1) \right]. \quad (19)$$

#### 4. The sheath

An asymptotic expansion valid in a vicinity of the point  $x_s$  describes the space-charge sheath and is related to the limit  $\alpha \rightarrow 0$ ,  $y = (x - x_s)/\alpha$  finite:

$$f = f_2(y) + \dots, \quad h = h_2(y) + \dots, \quad \phi = \phi_2(y) + \dots \quad (20)$$

Substituting these expansions into (1), (2) and (5), expanding and retaining leading terms, we obtain the following system of equations:

$$\frac{d(f_2 h_2)}{dy} = 0, \tag{21}$$

$$-h_2 \frac{dh_2}{dx} + \frac{d\phi_2}{dx} = 0, \tag{22}$$

$$\frac{d^2\phi_2}{dy^2} = f_2 - g_2 - n_2, \tag{23}$$

$$g_2 = e^{-\phi_2}, \quad n_2 = A e^{-\varepsilon\phi_2}. \tag{24}$$

Asymptotic matching results in the boundary conditions

$$f_2 \rightarrow f_s, \quad h_2 \rightarrow h_s, \quad \phi_2 \rightarrow \phi_s \quad \text{as } y \rightarrow -\infty. \tag{25}$$

Equations (21) and (22) may be integrated to give

$$f_2 h_2 = C_1, \quad \phi_2 - \frac{1}{2}h_2^2 = C_2, \tag{26}$$

where  $C_1$  and  $C_2$  are constants of integration, which may be determined by means of the boundary conditions (25):

$$C_1 = \left\{ (g_s + A g_s^\varepsilon) \left[ 1 - g_s + \frac{A}{\varepsilon} (1 - g_s^\varepsilon) \right] \right\}^{1/2}, \tag{27}$$

$$C_2 = \phi_s - \frac{1 - g_s + \frac{A}{\varepsilon} (1 - g_s^\varepsilon)}{2(g_s + A g_s^\varepsilon)}. \tag{28}$$

We find

$$f_2 = (g_s + A g_s^\varepsilon) \left[ 1 + \frac{2(\phi_2 - \phi_s)(g_s + A g_s^\varepsilon)}{1 - g_s + \frac{A}{\varepsilon} (1 - g_s^\varepsilon)} \right]^{-1/2}, \tag{29}$$

$$h_2 = \left[ 2(\phi_2 - \phi_s) + \frac{1 - g_s + \frac{A}{\varepsilon} (1 - g_s^\varepsilon)}{g_s + A g_s^\varepsilon} \right]^{1/2}. \tag{30}$$

Multiplying (23) by  $d\phi_2/dy$  and integrating, we find

$$\frac{1}{2} \left( \frac{d\phi_2}{dy} \right)^2 = F, \tag{31}$$

where

$$F = \left[ 1 - g_s + \frac{A}{\varepsilon} (1 - g_s^\varepsilon) \right] \left[ 1 + \frac{2(\phi_2 - \phi_s)(g_s + A g_s^\varepsilon)}{1 - g_s + \frac{A}{\varepsilon} (1 - g_s^\varepsilon)} \right]^{1/2} - (1 - g_2) - \frac{A}{\varepsilon} (1 - g_2^\varepsilon). \tag{32}$$

Here the integration constant has been determined by means of the last boundary condition in (25), which implies that  $d\phi_2/dy \rightarrow 0$  as  $\phi \rightarrow \phi_2$ . Note that (31) may be obtained also by expanding (8) and making use of (29) and (30).

The expansion of the right-hand side of (31) in powers of  $\phi - \phi_s$  is

$$F = F_1 \frac{(\phi_2 - \phi_s)^2}{2} + F_2 \frac{(\phi_2 - \phi_s)^3}{6} + \dots, \quad (33)$$

where

$$F_1 = F_1(g_s, A, \varepsilon) = g_s + A\varepsilon g_s^\varepsilon - \frac{(g_s + Ag_s^\varepsilon)^3}{1 - g_s + \frac{A}{\varepsilon}(1 - g_s^\varepsilon)}, \quad (34)$$

$$F_2 = F_2(g_s, A, \varepsilon) = -g_s - A\varepsilon^2 g_s^\varepsilon + \frac{3(g_s + Ag_s^\varepsilon)^3}{\left[1 - g_s + \frac{A}{\varepsilon}(1 - g_s^\varepsilon)\right]^2}. \quad (35)$$

According to the inequality (19),  $F_1$  is negative or zero. The former cannot be the case, since the leading term of the expansion must be positive, as is the left-hand side of (31). It follows that the latter is the case, and the inequality (19) holds marginally:

$$(g_s + Ag_s^\varepsilon)^2 = (g_s + A\varepsilon g_s^\varepsilon) \left[1 - g_s + \frac{A}{\varepsilon}(1 - g_s^\varepsilon)\right]. \quad (36)$$

This equality represents a specific form of the Bohm criterion for the problem considered, and allows one to determine the function  $g_s(A, \varepsilon)$ , i.e. the electron density at the edge of the space-charge sheath (knowing  $g_s$ , one can determine all the other local parameters, such as  $\phi_s, f_s$ , etc.). This generalized Bohm criterion is a simple variant of that found by Braithwaite and Allen (1988), namely (18).

Thus the term in  $(\phi_2 - \phi_s)^2$  in the expansion (33) is absent and the leading one is the term in  $(\phi_2 - \phi_s)^3$ . The sign of the latter is governed by  $F_2$ . It can be verified directly that

$$F_2[g_s(A, \varepsilon), A, \varepsilon] = -g_s \frac{\partial F_1}{\partial g_s} \Big|_{g_s(A, \varepsilon), A, \varepsilon}. \quad (37)$$

Using the identity  $F_1[g_s(A, \varepsilon), A, \varepsilon] \equiv 0$ , differentiated with respect to  $A$ , we can rewrite (37) in the form

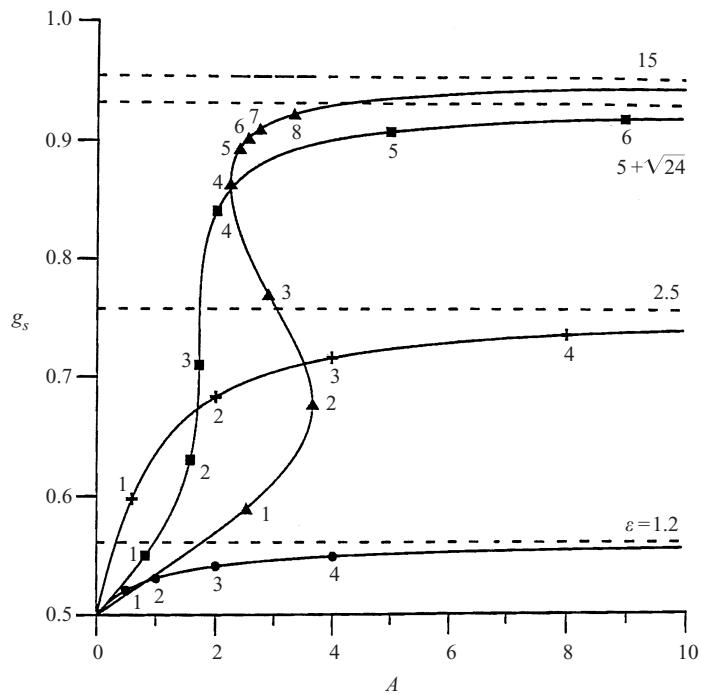
$$F_2[g_s(A, \varepsilon), A, \varepsilon] = \frac{g_s}{\partial g_s} \frac{\partial F_1}{\partial A} \Big|_{g_s(A, \varepsilon), A, \varepsilon}. \quad (38)$$

Let us proceed to find the function  $g_s(A, \varepsilon)$ . Solving (36) asymptotically in the limiting cases  $A \rightarrow 0$  and  $A \rightarrow \infty$ , we find asymptotic behaviours of the function  $g_s(A, \varepsilon)$  for small and large  $A$  respectively:

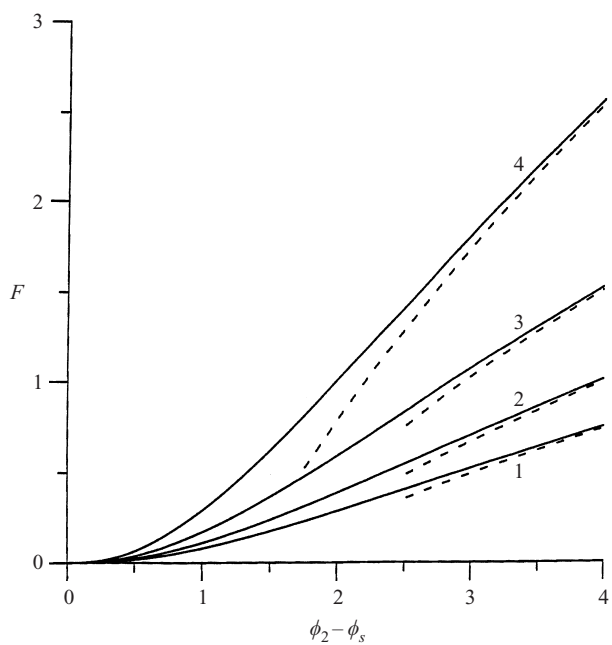
$$g_s = \frac{1}{2} + \frac{(\varepsilon - 1)^2 + 2(2^{\varepsilon-1} - 1)}{2^{\varepsilon+1}\varepsilon} A + \dots, \quad (39)$$

$$g_s = \frac{1}{2^{1/\varepsilon}} + \frac{\varepsilon^2 2^{2/\varepsilon} - 2\varepsilon + 1 - \varepsilon^2}{2^{2/\varepsilon}\varepsilon^2} \frac{1}{A} + \dots. \quad (40)$$

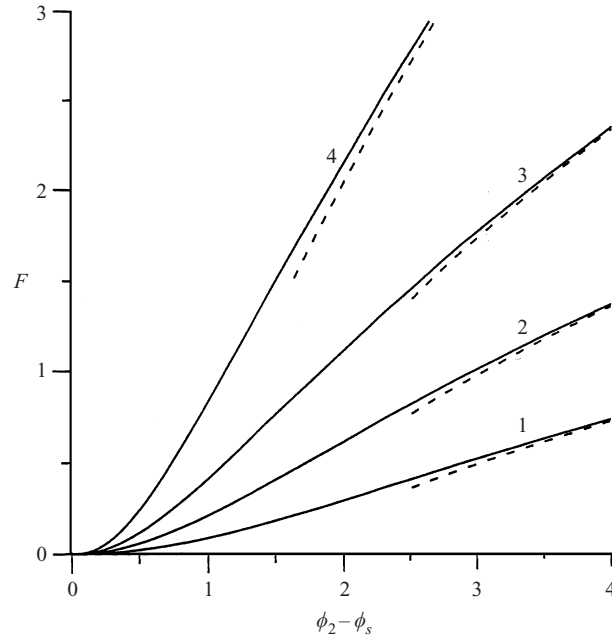
It is evident that the coefficient of  $A$  in the expansion (39) is positive for  $\varepsilon > 1$ , and it is verified by a direct numerical evaluation that the coefficient of  $1/A$



**Figure 1.** The normalized electron density at the plasma-sheath boundary  $g_s$  as a function of the normalized central negative-ion density  $A$  for different values of the electron/negative-ion temperature ratio  $\epsilon$ . The dashed asymptotes correspond to  $A \rightarrow \infty$ , while the circles, crosses, squares and triangles correspond to values identified in subsequent figures.



**Figure 2.**  $F$ , the squared electric field, as a function of potential in the sheath  $\phi_2 - \phi_s$  corresponding to the circles in Fig. 1,  $\epsilon = 1.2$ . The dashes are the asymptotes corresponding to (42) for large  $\phi_2$ .



**Figure 3.**  $F$ , the squared electric field, as a function of potential in the sheath  $\phi_2 - \phi_s$  corresponding to the crosses in Fig. 1,  $\varepsilon = 2.5$ .

in the expansion (40) is negative for  $\varepsilon > 1$ . This suggests that  $g_s$  for finite  $A$  is between the limit values  $\frac{1}{2}$  and  $(\frac{1}{2})^{1/\varepsilon}$ . Thus we seek a solution of (36) that lies in the interval  $[\frac{1}{2}, (\frac{1}{2})^{1/\varepsilon}]$ .

In order to investigate how many solutions exist, it is convenient to solve (36) for  $A$ . Since it is a quadratic equation in  $A$ , (36) has two roots. It can be shown that in the range  $\frac{1}{2} < g_s < (\frac{1}{2})^{1/\varepsilon}$ , one of the roots is positive and the other is negative. Discarding the negative root, one obtains a (single-valued) function  $A(g_s, \varepsilon)$ . The dependence of  $A$  on  $g_s$  found in such a way for several values of  $\varepsilon$  is plotted in Fig. 1. For small  $\varepsilon$ , one value of  $g_s$  corresponds to each  $A$ , while for large  $\varepsilon$ , the function  $g_s(A)$  is multivalued. It was shown by Braithwaite and Allen (1988) that a transition from one situation to another occurs at  $\varepsilon = 5 + \sqrt{24}$ .

After  $g_s$  has been found, one can determine a solution of (31) by means of the formula

$$y = \int \frac{d\phi_2}{\sqrt{2F}}. \quad (41)$$

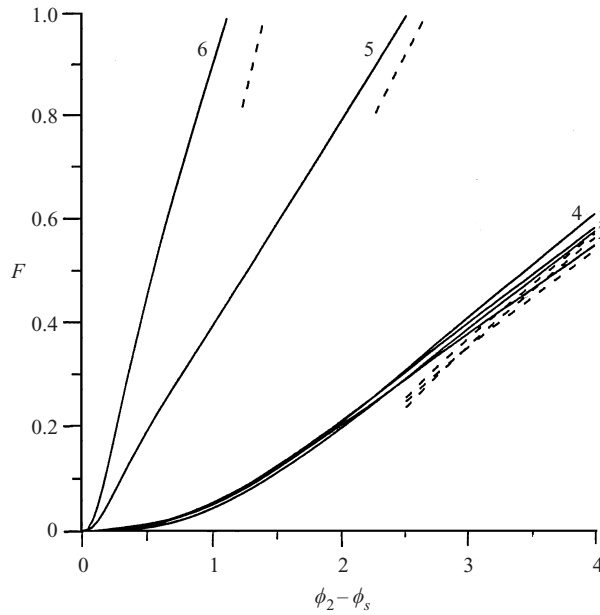
The above reasoning holds if the function  $F$  is positive at least in some range of  $\phi_2$  above  $\phi_s$ . It can be verified by direct numerical evaluation that

$$\left. \frac{\partial F_1}{\partial A} \right|_{g_s(A, \varepsilon), A, \varepsilon} > 0 \quad \text{for all } A \text{ and } \varepsilon > 1.$$

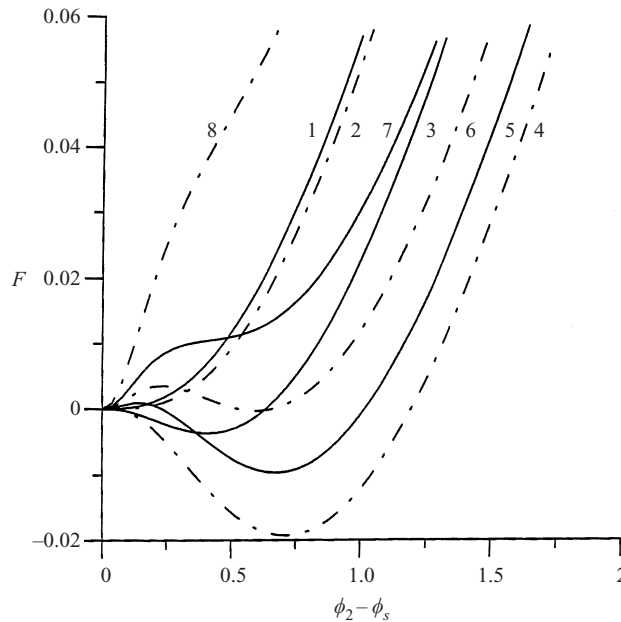
It follows that the leading term of the expansion (33) is positive for  $\phi_2 > \phi_s$  if the function  $g_s(A)$  is increasing at the point in question (for  $\varepsilon$  fixed), and is negative otherwise.

Thus, if  $g_s(A)$  is decreasing there, the function  $F$  is negative, at least for  $\phi_2$





**Figure 4.**  $F$ , the squared electric field, as a function of potential in the sheath  $\phi_2 - \phi_s$  corresponding to the squares in Fig. 1,  $\varepsilon = 5 + \sqrt{24}$ .



**Figure 5.**  $F$ , the squared electric field, as a function of potential in the sheath  $\phi_2 - \phi_s$  corresponding to the triangles in Fig. 1,  $\varepsilon = 15$ .

slightly exceeding  $\phi_s$ , and the above reasoning does not hold. If  $g_s(A)$  is increasing there, the function  $F$  is positive for  $\phi_2$  slightly exceeding  $\phi_s$ , and the question arises whether it remains positive up to the wall, i.e. for large enough  $\phi_2$ . As examples, graphs of the function  $F$  for several values marked in Fig. 1

by circles, crosses, rectangles and triangles are given in Figs 2–5. The dashed lines represent asymptotic behaviour of this function for  $\phi_2 \gg 1$ , governed by the formula

$$F \approx \left[ 1 - g_s + \frac{A}{\varepsilon} (1 - g_s^\varepsilon) \right] \left[ 1 + \frac{2(\phi_2 - \phi_s)(g_s + Ag_s^\varepsilon)}{1 - g_s + \frac{A}{\varepsilon} (1 - g_s^\varepsilon)} \right]^{1/2} - 1 - \frac{A}{\varepsilon}. \quad (42)$$

One can conclude from Figs 2–4 that if  $\varepsilon \leq 5 + \sqrt{24}$ , the function  $F$  is positive for all  $\phi_2 > \phi_s$ . If  $\varepsilon > 5 + \sqrt{24}$  (Fig. 5), the function  $F$  on the lower increasing section of the function  $g_s(A)$  is also positive for all  $\phi_2 > \phi_s$  (lines 1 and 2). On the falling section of  $g_s(A)$ , the function  $F$  is negative in a certain range of  $\phi_2$  adjacent to the point  $\phi_2 = \phi_s$ , and becomes positive for higher  $\phi_2$  (lines 3 and 4). At the beginning of the upper increasing section of  $g_s(A)$ ,  $F$  is first positive, then negative, and then positive again (line 5). As  $A$  increases, the width of the negative interval decreases and eventually becomes zero (line 6). For larger  $A$ ,  $F$  is positive for all  $\phi_2$  (lines 7 and 8). Note that in all the cases, the behaviour of  $F$  shown by the graphs for  $\phi_2$  slightly exceeding  $\phi_s$  agrees with the considerations derived analytically above.

## 5. Solution as a whole

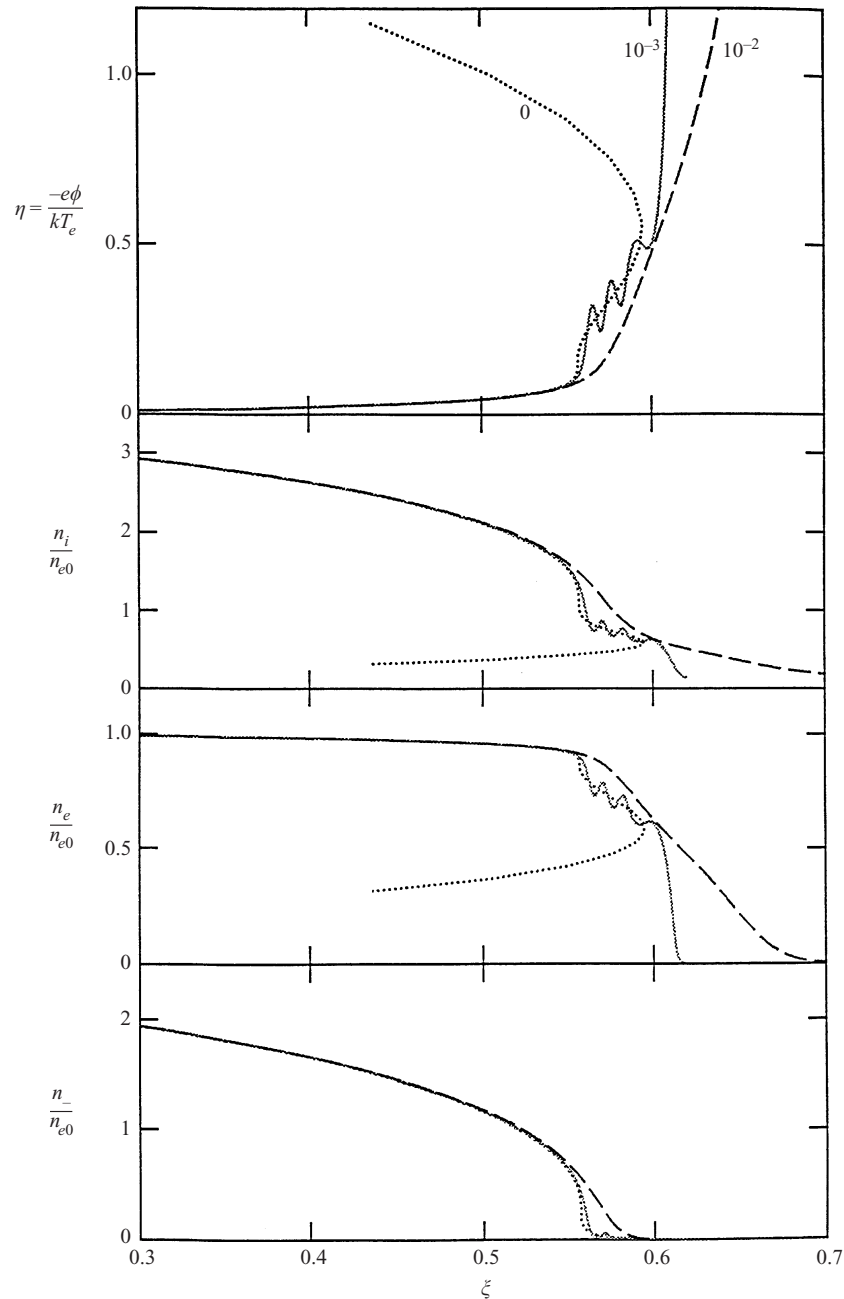
### 5.1. The case of an asymptotically thin sheath with a monotonic potential distribution

The above solutions for the quasineutral plasma and for the sheath can be simply combined if  $\varepsilon \leq 5 + \sqrt{24}$ , in which case the sheath solution given by (41) can be extended up to the wall and  $x_s = 1$ . In other words, the near-wall space-charge sheath is asymptotically thin in this case and the potential distribution in it is monotonic.

If  $\varepsilon > 5 + \sqrt{24}$ , the same situation takes place in the cases  $A < A_4$  and  $A > A_6$ , where  $A_4 = A_4(\varepsilon)$  and  $A_6 = A_6(\varepsilon)$  are values of  $A$  referring to the respective points marked in Fig. 1. Note that in the case  $A > A_6$ , several values of  $g_s$  exist, and the question arises as to which value should be taken. Since the function  $g_1(x)$  in the range  $0 \leq x \leq x_s$  decreases from unity downwards, the first value to be encountered is the largest one. Thus  $g_1(x_s)$  equals the largest of the values of  $g_s(A)$ , i.e. the value on the upper increasing section of  $g_s(A)$ .

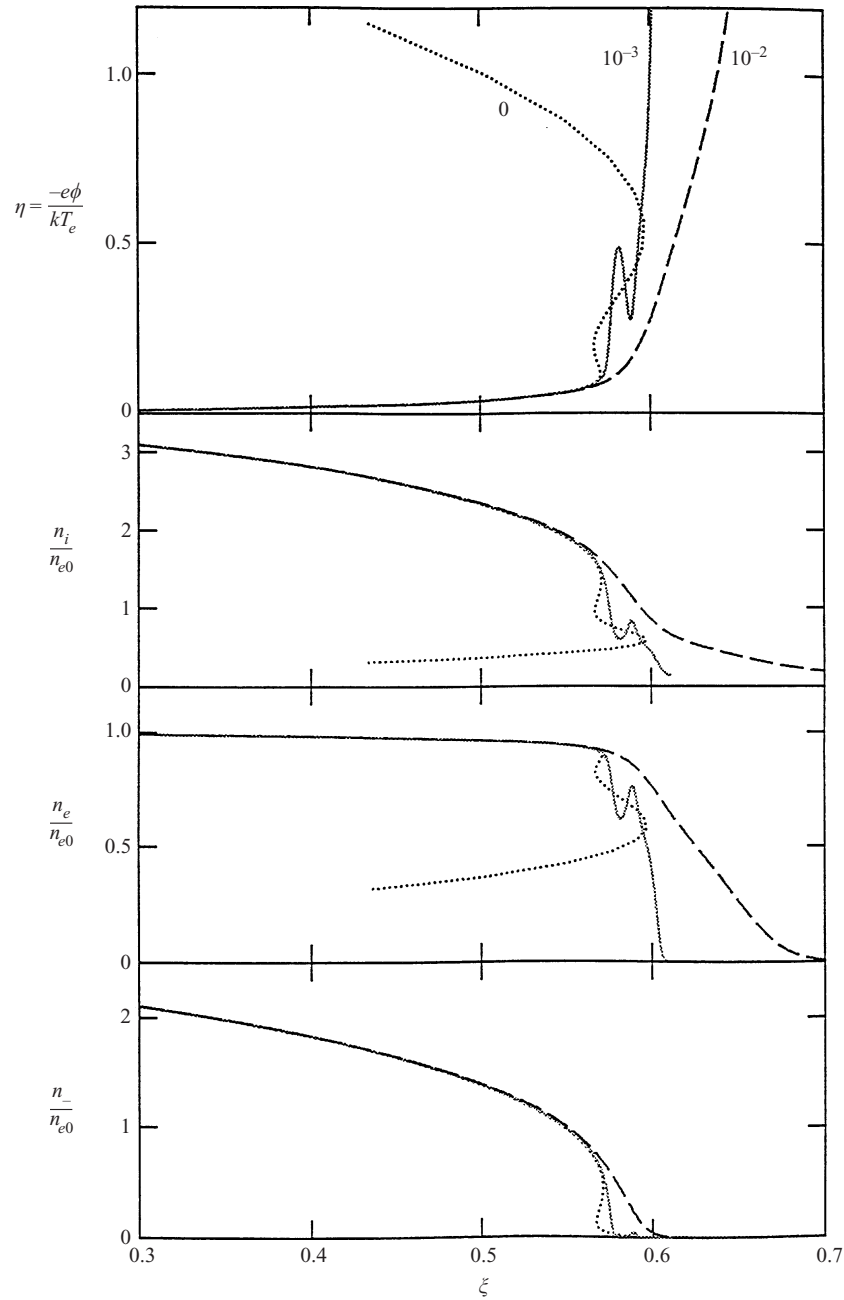
### 5.2. The case of a ‘sheath’ with oscillations

If  $A_4 < A < A_6$ , the solution described by (41) breaks down at  $\phi_2 = \phi_z$ , where  $\phi_z$  is the first positive zero of the function  $F(\phi_2)$ , and therefore cannot be extended up to the wall. The behaviour of  $F(\phi_2)$  in the vicinity of the point  $\phi_2 = \phi_z$  is linear; hence the integral on the right-hand side of (41) converges at  $\phi_2 = \phi_z$ . Thus the solution (41) breaks down at finite  $y$ , which is an indication of  $\phi_2(y)$  being non-monotonic: on passing the point at which  $\phi = \phi_z$ , the solution switches from the branch described by (41) to the branch described by an equation similar to (41) but with a minus sign before the integral. Thus we have a solution in which  $\phi_2$  increases from  $\phi_s$  (at  $y \rightarrow -\infty$ ) to  $\phi_z$ , and then decreases back to  $\phi_s$  (at  $y \rightarrow \infty$ ). The increasing and falling sections of the function  $\phi_2(y)$  are symmetrical. The problem, however, is that no quasineutral plasma with



**Figure 6.** Spatial distributions of potential and charged-particle densities for  $A = 2.24280$  and  $\varepsilon = 15$ , corresponding to point 4 of Fig. 1, for  $\alpha' = 0$  (dotted),  $\alpha' = 10^{-3}$  (solid) and  $\alpha' = 10^{-2}$  (dashed).

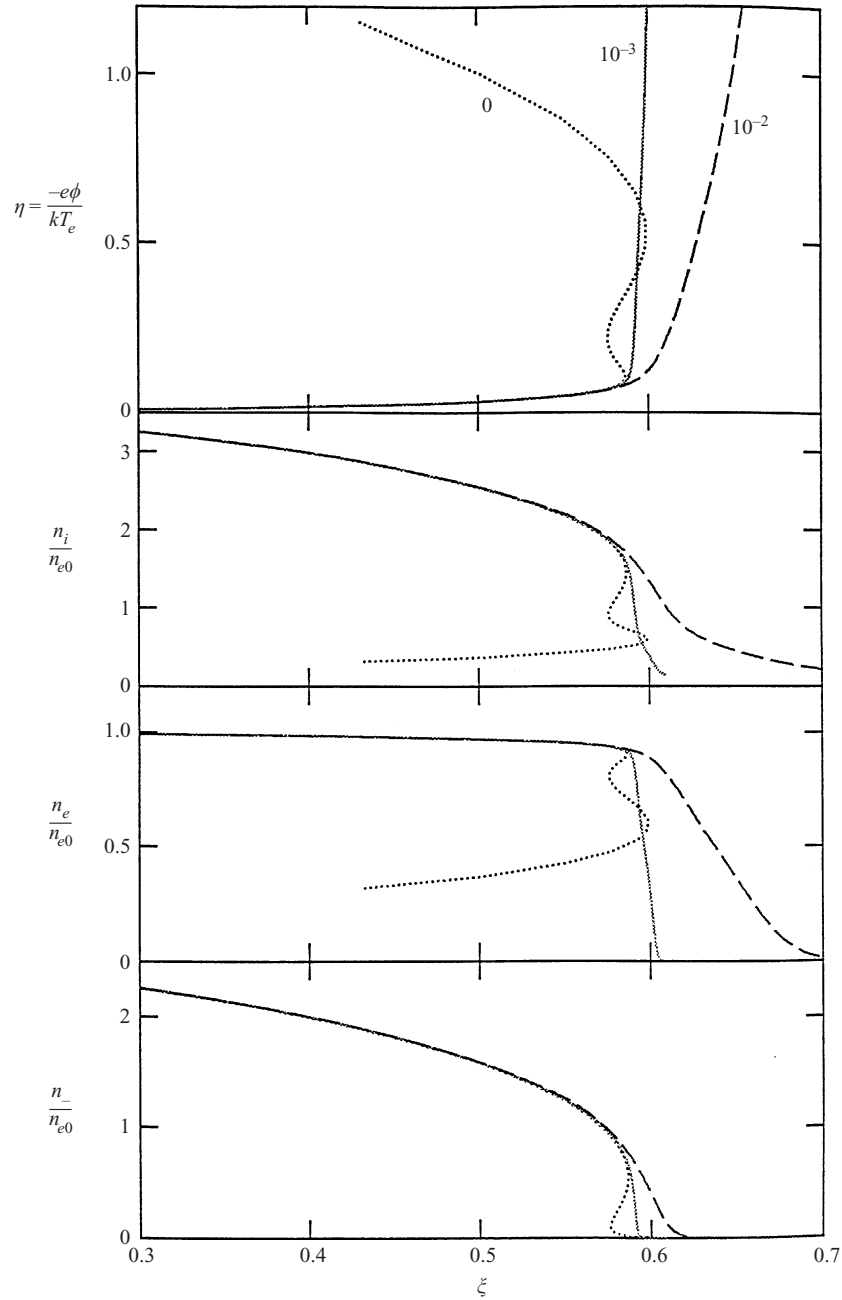
$h > 0$  can follow such a sheath: since  $d\phi_1/dx$  has the same sign as  $f_1 - h_1^2(g_1 + \varepsilon n_1)$ , a quasineutral solution that starts in an asymptotic vicinity of the upper increasing branch of the function  $g_s(A)$  cannot leave this vicinity. Thus the only possibility is that the sheath of the considered type is followed by the same one,



**Figure 7.** Spatial distributions of potential and charged-particle densities for  $A = 2.39883$  and  $\varepsilon = 15$ , corresponding to point 5 of Fig. 1, for  $\alpha' = 0$  (dotted),  $\alpha' = 10^{-3}$  (solid) and  $\alpha' = 10^{-2}$  (dashed).

which is in turn followed by the same one, etc. In other words, one must conclude that the potential distribution is oscillating in this case.

Note, however, that owing to the (second-order) effect of ionization, the falling section of the first sheath is not exactly symmetrical with the rising



**Figure 8.** Spatial distributions of potential and charged-particle densities for  $A = 2.54557$  and  $\varepsilon = 15$ , corresponding to point 6 of Fig. 1, for  $\alpha' = 0$  (dotted),  $\alpha' = 10^{-3}$  (solid) and  $\alpha' = 10^{-2}$  (dashed).

section, the second sheath will be slightly different from the first one, etc. In other words, the oscillations will not be exactly periodic. One should expect the overall effect of the ionization in the ‘sheath’ to be a drift of the solution in the direction of increasing potential. Eventually, the solution will enter the region

in which the function  $F$  is positive, and will change from oscillatory to monotonically growing, after which it can be extended up to the wall.

Thus one should expect the space-charge ‘sheath’ in the case  $A_4 < A < A_6$  to be essentially different from the asymptotically thin near-wall sheath with the monotonic potential distribution considered in the preceding subsection: space charge is essential throughout the region  $x_s \leq x \leq 1$ , and the potential distribution in this region is first oscillatory, then monotonically growing.

### 5.3. Exceptional cases

We now discuss the cases of  $A$  asymptotically close to  $A_4$  or  $A_6$ . In the first case, the quasineutral solution described by (16) may be extended down to the lower increasing branch of the function  $g_s(A)$ . This solution is separated from the wall by the asymptotically thin near-wall sheath with the monotonic potential distribution described by (41). Besides, there is an asymptotically thin internal space-charge sheath in the vicinity of the point at which  $g = g_4$ . However, the potential is, to a first approximation, constant across this sheath.

In the case of  $A$  asymptotically close to  $A_6$ , the behaviour of  $F(\phi_2)$  in the vicinity of its first positive zero is quadratic, and the integral on the right-hand side of (41) diverges. Thus we have a solution in which  $\phi_2$  varies monotonically from  $\phi_s$  at  $y \rightarrow -\infty$  to  $\phi_z$  at  $y \rightarrow \infty$ . Obviously, such a solution describes a double layer.

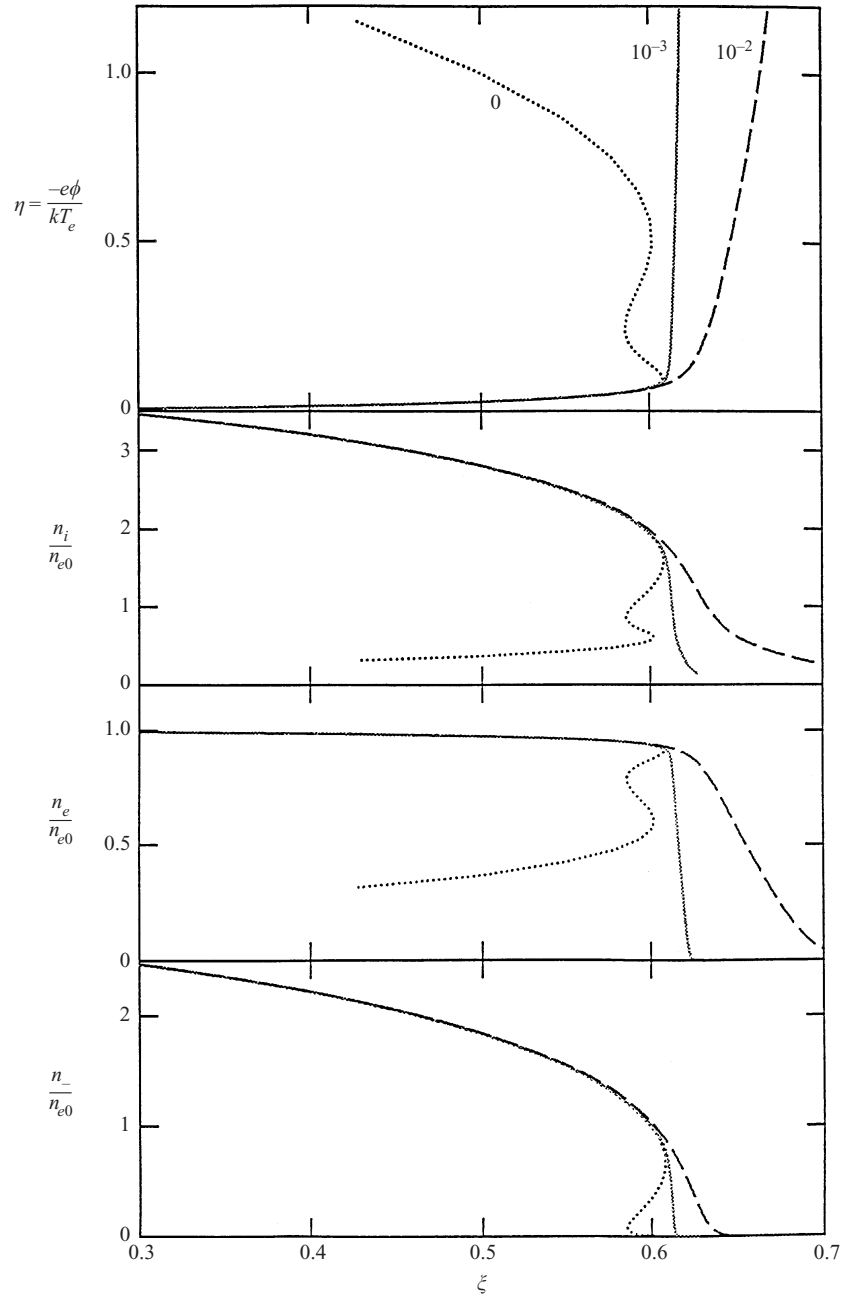
In order to understand what is a solution describing the quasineutral plasma after the double layer, the question arises as to the position of the point  $e^{-\phi_z}$  with respect to the curve  $g_s(A)$ . If it is above the falling branch then the quasineutral solution will be attracted by the upper increasing branch, one more double sheath will appear, etc.; the solution is periodic and cannot be extended up to the wall. If the point  $e^{-\phi_z}$  is below the falling branch then the quasineutral solution will be attracted by the lower increasing branch. When it reaches the lower increasing branch, the asymptotically thin sheath with a monotonic potential distribution described by (41) appears, and the solution can be extended up to the wall.

Thus the point  $e^{-\phi_z}$  in a solution extendable up to the wall should be below the falling branch of the curve  $g_s(A)$ . Furthermore, it cannot be between the falling and lower increasing branches, since in such a case the coefficient  $F_1$  in the expansion similar to (33) but around the point  $\phi_2 = \phi_z$  would be negative, which does not make sense; cf. the behaviour of curve 6 in Fig. 5 in the vicinity of the positive zero. It follows that the point  $e^{-\phi_z}$  should be below the lower increasing branch. According to the results of the numerical evaluation, the latter is indeed the case, for  $\varepsilon = 15$ :  $e^{-\phi_z} \approx 0.50$ .

Thus the asymptotic structure of a solution in the case of  $A$  asymptotically close to  $A_6$  includes the inner plasma, the double layer, the outer plasma, and the near-wall sheath.

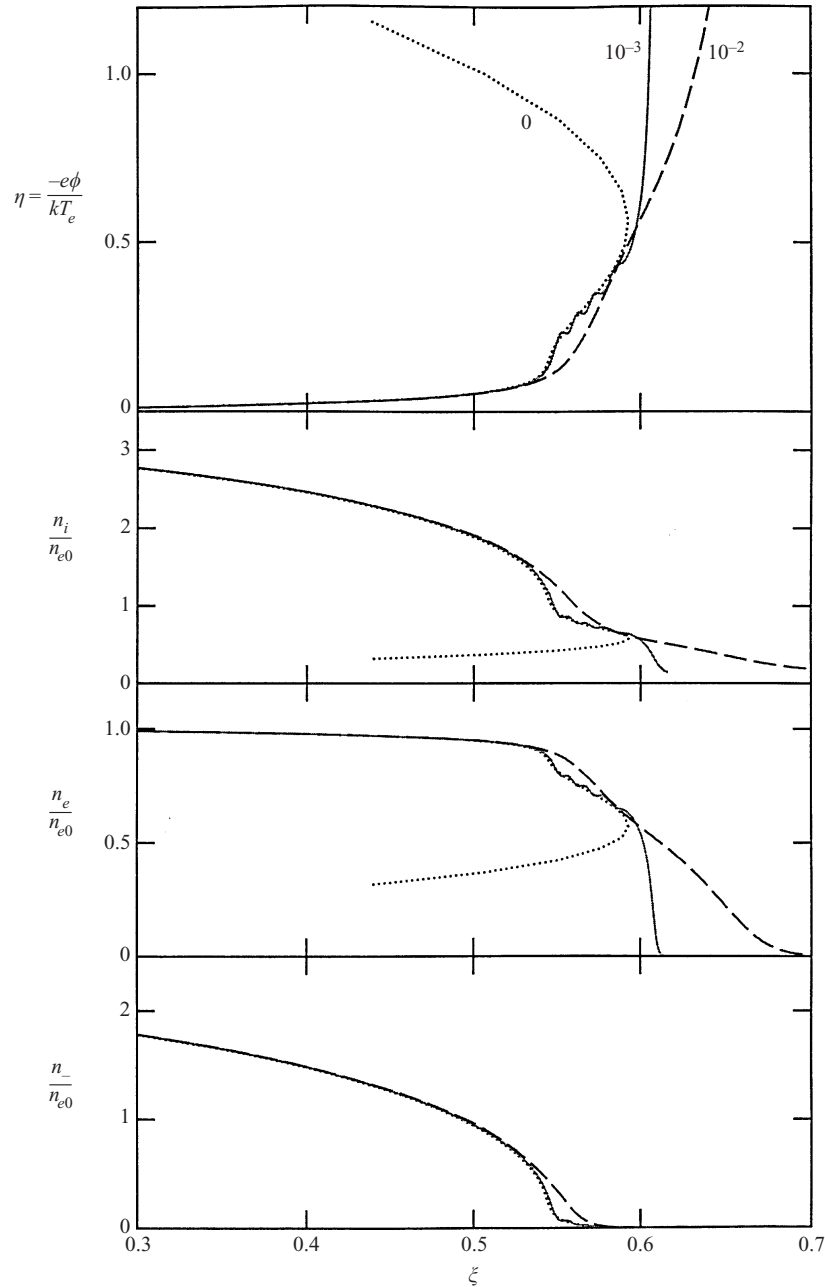
### 5.4. $\alpha$ not asymptotically small

In the real physical situation, the Debye length is finite, and for negative-ion-dominated plasmas may be relatively large, since the electron density is relatively low. This would then have the effect of smearing the structure just described, and accordingly we display in Figs 6–9 computed solutions for points



**Figure 9.** Spatial distributions of potential and charged-particle densities for  $A = 2.74106$  and  $\varepsilon = 15$ , corresponding to point 7 of Fig. 1, for  $\alpha' = 0$  (dotted),  $\alpha' = 10^{-3}$  (solid) and  $\alpha' = 10^{-2}$  (dashed).

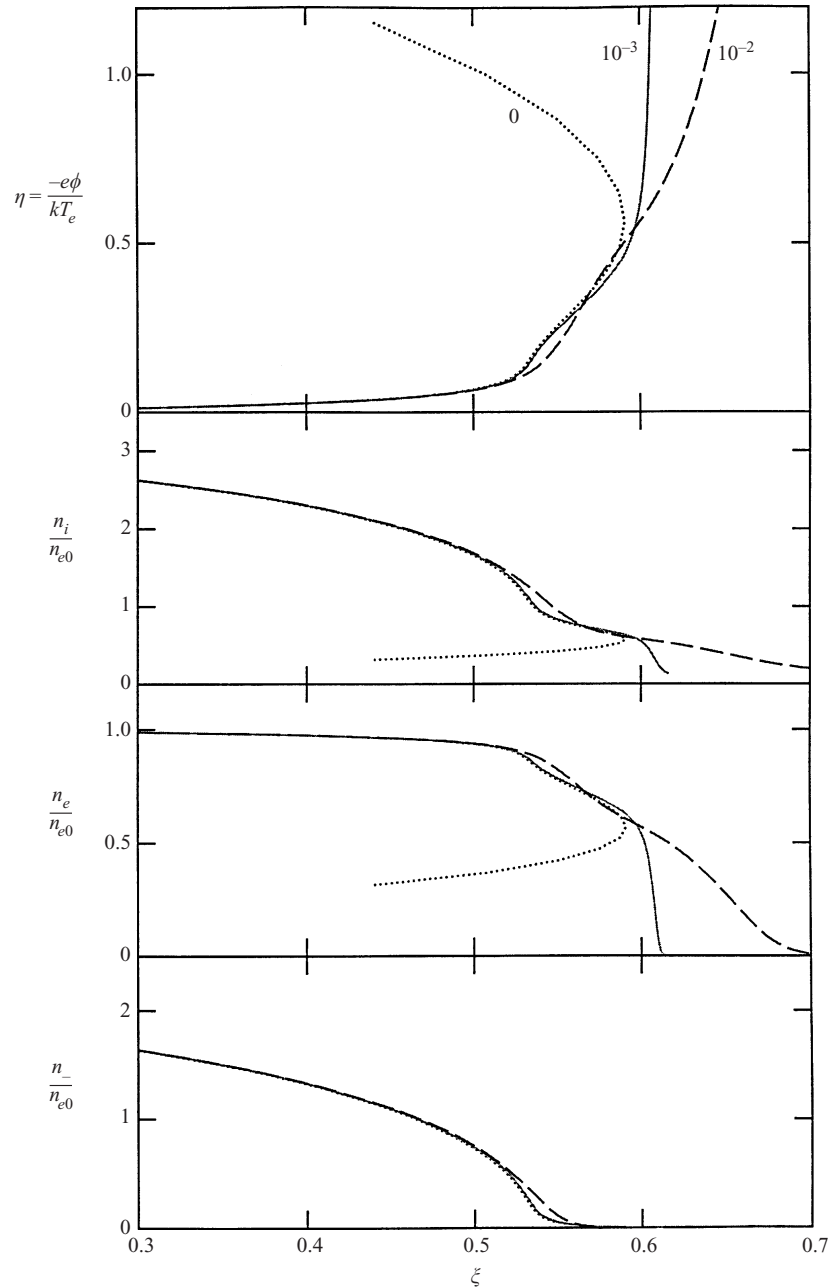
4, ..., 7 of Fig. 1, giving potential and charged particle distributions for  $\alpha' = 0$ ,  $10^{-3}$  and  $10^{-2}$  and  $\alpha' \equiv \alpha\lambda_0$ . Figures 10 and 11 are for values of  $A < A_4$ , showing the negative ion core contracting and the oscillations disappearing as  $A$  decreases.



**Figure 10.** Spatial distributions of potential and charged-particle densities for  $A = 2.09692$  and  $\varepsilon = 15$ , for  $\alpha' = 0$  (dotted),  $\alpha' = 10^{-3}$  (solid) and  $\alpha' = 10^{-2}$  (dashed).

These figures illustrate the structures described in the preceding sections in the case where  $\alpha' = 10^{-3}$ , while for  $\alpha' = 10^{-2}$  all trace of structure has disappeared. The results given in Franklin and Snell (1998) were for values of  $\alpha' = 0, 0.01, 0.03$  and  $0.1$ , so that it is not surprising that they failed to describe the detail found by others and elucidated above.





**Figure 11.** Spatial distributions of potential and charged-particle densities for  $A = 1.96053$  and  $\varepsilon = 15$ , for  $\alpha' = 0$  (dotted),  $\alpha' = 10^{-3}$  (solid) and  $\alpha' = 10^{-2}$  (dashed).

### 6. Concluding remarks

Since  $d^2\phi_2/dy^2 = dF/d\phi_2$ , regions of increase or decrease of the function  $F(\phi_2)$  in Figs 2–5 correspond to the positive and negative local space charge respectively. One can conclude that the space charge is positive throughout the (near-wall) sheath in all the cases if  $\varepsilon \leq 5 + \sqrt{24}$ .

If  $\varepsilon > 5 + \sqrt{24}$ , the space charge in the (near-wall) sheath is positive in the cases  $A < A_4$  and  $A \gtrsim A_7$ . In the case  $A_6 < A \lesssim A_7$ , the space charge is positive in the inner part of the sheath (i.e. the part adjacent to the inner plasma) and in the near-wall part, and is negative in the central part of the sheath. The double layer that appears in the case of  $A$  asymptotically close to  $A_6$  includes the (inner) part with the positive space charge and the (outer) part where the charge is negative. In the case  $A_4 < A < A_6$ , the space charge is oscillating in the inner part of the sheath and is positive in the near-wall part.

We have shown that *a double layer separating two plasmas is an exceptional case rather than the rule*. The reason is obvious: if the negative space charge is too small, the function  $F(\phi_2)$  does not vanish, i.e. the electric field remains positive at a point that is a minimum of the function  $F(\phi_2)$ , at which the space charge vanishes. In other words, the negative space charge is insufficient to shield the electric field created by the positive charges. If the negative space charge is too large,  $F(\phi_2)$  vanishes with a non-zero derivative, i.e. the space charge is still negative at the point where the field vanishes; the electric field turns negative on passing this point, and oscillations occur. *It is only in that case where the field and the negative space charge vanish simultaneously that a double layer separating two plasmas occurs.*

Thus the treatments of Sato and Miyawaki (1992) and Franklin (1999) have strictly limited validity, but nevertheless have contributed to a deeper understanding of the physics involved. On the other hand, the picture that emerges from this present treatment confirms the description of Kono (1999) of the probe-plasma sheath.

It has been shown that the method of matched asymptotic expansions allows one to obtain an asymptotic solution in most cases. An exception is the (relatively narrow) range  $A_4 < A < A_6$  in the range  $\varepsilon > 5 + \sqrt{24}$ , when the method of multiple scales (Nayfeh 1973) should be employed in order to obtain an asymptotic solution. On the other hand, the method of matched asymptotic expansions is sufficient to study the solution qualitatively and to identify the physics involved in the latter case as well.

There is a companion problem that produces identical mathematics, namely a plasma with two populations of electrons at different temperatures  $T_h$  and  $T_c$ . If we take  $T_h \equiv T_e$  and  $T_c = T_n$  or  $T_h/T_c = \varepsilon$  and  $n_c/n_h = A$  then everything carries over. However, in the plasmas that have so far been explored experimentally, it would appear that  $T_h/T_c > 9.90$  is difficult to achieve with both true temperatures rather than average energies.

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#### **References**

- Braithwaite, N. St J. and Allen, J. E. 1988 *J. Phys. D: Appl. Phys.* **21**, 1733–1737.  
 Franklin, R. N. 1999 *J. Phys. D: Appl. Phys.* **32** L71–74.

- Franklin, R. N. and Oekendon, J. 1970 *J. Plasma Phys.* **4**, 371–385.
- Franklin, R. N. and Snell, J. 1992 *J. Phys. D: Appl. Phys.* **25**, 453–457.
- Franklin, R. N. and Snell, J. 1998 *J. Phys. D: Appl. Phys.* **31**, 2532–2542.
- Kono, A. 1999 *J. Phys. D: Appl. Phys.* **32**, 1357–1363.
- Nayfeh, A. H. 1973 *Perturbation Methods*. Wiley, New York.
- Riemann, K.-U. 1991 *J. Phys. D: Appl. Phys.* **24**, 493–518.
- Sato, K. and Miyawaki, F. 1992 *Phys. Fluids* **B4**, 1247–1254.
- Sheridan, T. E. 1999 *J. Phys. D: Appl. Phys.* **82**, 1761–1767.
- Sheridan, T. E., Braithwaite, N. St. J. and Boswell, R. W. 1999 *Plasma Sources Sci. Technol.* **8**, 457–462.
- Tonks, L. and Langmuir, I. 1929 *Phys. Rev.* **34**, 876–922.