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Evolution of FMS and Alfven waves produced by the initial disturbance in the FMS waveguide

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Abstract. A description of the evolution of the initial disturbance in the fast magnetosonic (FMS) waveguide in transversely inhomogeneous plasma, given a weak coupling between FMS and Alfven modes, is made. It is shown that the Fourier transform of the FMS waveguide disturbance with respect to the coordinates along which plasma is homogeneous can be presented as a superposition of collective modes of the leading approximation with respect to the weak FMS–Alfven wave coupling from the initial instant of time. Frequencies of such collective modes and dependence of their structures on the coordinate along the inhomogeneity are found without taking the FMS–Alfven resonance into consideration, and the mode decrements are calculated using the perturbation technique. On the basis of such a representation of the FMS waveguide disturbance, the evolution of Alfven waves generating with waveguide mode packets produced by the initial disturbance of an arbitrary longitudinal structure is described. It is shown that the longitudinal structure of the Alfven disturbance generated by the collective mode packet is determined by the ratio between longitudinal scales of the initial disturbance and scales specified by resonance conditions (the resonance longitudinal wave number and the width of the range of the resonance longitudinal wave numbers). The structures of Alfven disturbances for the cases of such different ratios are described.

1. Introduction

In this paper we consider evolution of a disturbance that takes place at an instant of time, in a region near the surface where the Alfven velocity is minimum in the direction transverse to the magnetic field. There are conditions for the waveguide propagation of fast magnetosonic (FMS) waves in such a region. The initial disturbance therefore leads to the arising FMS disturbance whose propagation along the direction of the Alfven velocity inhomogeneity is limited. At the same time, the disturbance propagates freely along two other coordinates. As a result of the FMS–Alfven wave coupling, the waveguide disturbance is gradually mode converted into Alfven waves. After a while, it propagates as an Alfven disturbance.

Magnetohydrodynamic (MHD) disturbances whose evolution follows this scenario can take place in many inhomogeneous space plasma structures. One of these is magnetotail. The magnetotail can be considered as the FMS waveguide extended along the magnetic field (Allan and Wright 1998; Allan 2000; Mills et al. 2000; Lysak et al. 2009; Mazur et al. 2010). Disturbances associated with different processes in the distant magnetotail (e.g. processes related to reconnection) can excite FMS waves. When propagating along the waveguide, FMS waves are mode converted into Alfven waves that reach the Earth, leading to geomagnetic field disturbances and particle precipitation (Wright and Allan 2008). Disturbances in open solar structures give another example of waveguide propagation of FMS waves (Deforest and Gurman 1998; Verwichte et al. 2005).

We suggest that the initial disturbance is localized near the waveguide axis, and will consider the formation of the Alfven disturbance in the region where the resonance between FMS and Alfven waves takes place (i.e. in the opaque region for the waveguide disturbance). The initial disturbance is supposed to be absent in this region, so the Alfven disturbances are excited there only due to mode conversion of the waveguide FMS disturbance.

We start by considering evolution of the waveguide disturbance. As is well known from the study of different problems, including first the coupling of electromagnetic and plasma electrostatic oscillation (Barston 1964; Sedlachek 1971a, b), and then the coupling of FMS and Alfven waves (Uberoi 1972; Grossmann and Tataronis 1973; Tataronis and Grossmann 1973; Zhu and Kivelson 1988), the waveguide modes have discrete spectrum of real frequencies when neglecting mode conversion. In this approximation, the initial disturbance can be represented as a sum of modes whose amplitudes are constant in time. On the other hand, if the mode conversion is taken into account, the discrete spectrum is replaced by the continuous one (Barston 1964; Uberoi

1972) and the discrete frequencies separate only when $t \to \infty$ (Sedlachek 1971a, b; Grossmann and Tataronis 1973; Tataronis and Grossmann 1973; Zhu and Kivelson 1988). In this paper we suggest that the FMS–Alfven wave coupling is weak due to smallness of wave numbers corresponding to the y-coordinate (in the coordinate system x is along the inhomogeneity direction, and z is along the undisturbed magnetic field). The weakness of the FMS–Alfven wave coupling results in the slowness of the mode conversion of FMS waves into Alfven ones, i.e. smallness of the decrement of FMS waves as compared with their frequency. We show that the Fourier transform of the FMS disturbance (with respect to the coordinates along which plasma is homogeneous) can be presented as a superposition of collective modes of the leading approximation from the initial instant of time. These are the modes whose frequencies and structures with respect to the x-coordinate coincide with the frequencies and structures of waveguide modes obtained without taking account of the FMS–Alfven wave coupling. Decrements of such collective modes are calculated using the perturbation technique.

In the coordinate representation, the waveguide FMS disturbance caused by the initial disturbance of an arbitrary longitudinal structure is represented as a superposition of packets of the leading approximation collective mode with the same number. Therefore, to have description of space-time evolution of the Alfven disturbance, we need to obtain that of Alfven disturbances produced by such packets. For this purpose we use the solution to the equation that describes mode conversion of the FMS disturbance into the Alfven one in terms of the Fourier transform. Applying the inverse Fourier transform to this solution, we find the Alfven disturbance into which a certain waveguide packet is transformed.

The formulae obtained for space-time evolution of Alfven disturbances relate its parameters to the parameters of the initial disturbance and to the parameters specified by resonance conditions (resonance longitudinal wave number and width of range of resonance longitudinal wave numbers). Using these formulae, we analyze the dependence of space-time structure of the Alfven disturbance on ratios of scales of the initial disturbance to resonance scales.

2. Input equations

Let us denote the undisturbed magnetic field by $\mathbf{B}_0$. We will assume that it is homogeneous and directed along the z-axis. We will assume that the undisturbed density n_0 is inhomogeneous along x. Then the Alfven velocity $V_a = B_0/(4\pi m_i n_0)^{1/2}$ is inhomogeneous along x too. Let us denote disturbance of the magnetic field by $\mathbf{B}$ and plasma velocity in disturbance by $\mathbf{v}$. We will consider waves with the given wave number k_y.

Linear MHD disturbances are described by equations

$$\partial_t \mathbf{B} = \nabla \times [\mathbf{v} \times \mathbf{B}_0], \; \partial_t \mathbf{v} = \frac{1}{4\pi}\frac{1}{m_i n_0}[\nabla \times \mathbf{B}] \times \mathbf{B}_0.$$

From these we have

$$\partial_t^2 \mathbf{v} = \frac{1}{4\pi}\frac{1}{m_i n_0}[\nabla \times \nabla \times [\mathbf{v} \times \mathbf{B}_0]] \times \mathbf{B}_0.$$

This equation can be written in the form

$$\begin{aligned} V_a^{-2}\partial_t^2 v_x - \partial_z^2 v_x &= \partial_x \psi, \\ V_a^{-2}\partial_t^2 v_y - \partial_z^2 v_y &= ik_y \psi, \end{aligned} \tag{2.1}$$

where $\psi = \partial_x v_x + ik_y v_y$. We will pass to dimensionless variables and functions with the use of some parameters l_0 and t_0 (dimension l_0 is length and t_0 is time) $t \to tt_0$, $x \to xl_0$, $z \to zl_0$, $y \to yl_0$, and $v_x \to v_x l_0/t_0$, $v_y \to v_y l_0/t_0$, $\psi \to \psi/t_0$. By using notation k_y for $k_y l_0$ and V_a for the dimensionless Alfven velocity $V_a l_0^{-1} t_0$, we still have equations in the form of (2.1) as input ones; these are, however, for dimensionless variables and functions.

We will assume that when $t = 0$, there is a displacement and acceleration of plasma in the direction perpendicular to the magnetic field: $\mathbf{v}_\perp(t=0) = v_\perp(0)$, $\partial_t \mathbf{v}_\perp = \partial_t \mathbf{v}_\perp(0)$, where the initial disturbances $v_\perp(0)$ and $\partial_t \mathbf{v}_\perp(0)$ are functions of x and z.

Let us now perform the Fourier transform with respect to z:

$$\mathbf{v}_{\perp k} = \int_{-\infty}^{\infty} e^{-ikz} \mathbf{v}_\perp dz.$$

The inverse transform is as follows:

$$\mathbf{v}_\perp = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{ikz} \mathbf{v}_{\perp k} dk.$$

We perform the Laplace transform with respect to time:

$$\mathbf{v}_{\perp k\omega} = \int_0^{\infty} e^{i\omega t} \mathbf{v}_{\perp k} dt.$$

The inverse transform is given by

$$\mathbf{v}_{\perp k} = (2\pi)^{-1} \int_{-\infty+ic}^{\infty+ic} e^{-i\omega t} \mathbf{v}_{\perp k\omega} d\omega,$$

where the contour of integration should lay above all singular points of integrand.

We multiply both parts of (2.1) by $\exp(-ikz)\exp(i\omega t)$; we then integrate from $-\infty$ to ∞ over z and from 0 to ∞ over t. Denoting

$$U_\perp = -\partial_t \mathbf{v}_{\perp k}(0) + i\omega \mathbf{v}_{\perp k}(0)$$

and

$$l_a = \omega^2 V_a^{-2} - k^2,$$

we obtain the following equations from (2.1):

$$l_a v_{xk\omega} = -\partial_x \psi_{k\omega} + U_x, \tag{2.2}$$

$$l_a v_{yk\omega} = -ik_y \psi_{k\omega} + U_y, \tag{2.3}$$

$$\psi_{k\omega} = \partial_x v_{xk\omega} + ik_y v_{yk\omega}. \tag{2.4}$$

From these equations, we have for $\psi_{k\omega}$

$$\partial_x \left(l_a^{-1} \partial_x \psi_{k\omega}\right) + \left(1 - k_y^2 l_a^{-1}\right)\psi_{k\omega} = F, \tag{2.5}$$

where $F = \partial_x(l_a^{-1}U_x)+ik_y l_a^{-1}U_y$. The velocity divergence ψ describes a compressible part of the disturbance. Thus, we have (2.5) as an equation for the FMS disturbance. For the sake of definiteness, we will suppose that the Alfven velocity has one minimum when $x = 0$ and increases monotonically as function $|x|$ when $x > 0$ and $x < 0$, therefore $V_a \to \infty$ when $|x| \to \infty$. Then $l_a \to -k^2$ as $|x| \to \infty$. In this case we can choose the disturbance vanishing when $|x| \to \infty$ as boundary conditions for (2.5).

The solution to (2.5) can be represented as

$$\psi_{k\omega} = \int_{-\infty}^{\infty} F(\xi,\omega)\, G(x,\xi,\omega)\, d\xi,$$

where $G(x,\xi,\omega)$ is Green's function. It satisfies equation

$$\partial_x\left(l_a^{-1}\partial_x G(x,\xi,\omega)\right) + \left(1-k_y^2 l_a^{-1}\right) G(x,\xi,\omega) = \delta(x-\xi)$$

and boundary conditions when $|x| \to \infty$. The only variables we give as arguments for F and G are those that will take part in transforms with the use of F and G.

The inverse Laplace transform with respect to ω provides time evolution of the FMS disturbance produced by $\mathbf{v}_{\perp k}(0)$ and $\partial_t \mathbf{v}_{\perp k}(0)$. We have

$$\psi_k(x) = \frac{1}{2\pi}\int_{-\infty+ic}^{\infty+ic} e^{-i\omega t}\int_{-\infty}^{\infty} F(\xi,\omega)\, G(x,\xi,\omega)\, d\xi d\omega. \tag{2.6}$$

The Green's function can be written in the form

$$G(x,\xi,\omega) = \frac{1}{l_a^{-1}(x,\omega)\, W(x,\omega)}\, g(x,\xi,\omega),$$

$$g(x,\xi,\omega) = \psi_1(x,\omega)\,\psi_2(\xi,\omega)\,\theta(\xi-x) + \psi_2(x,\omega)\,\psi_1(\xi,\omega)\,\theta(x-\xi). \tag{2.7}$$

Functions ψ_1 and ψ_2 are the solutions to the homogeneous equation corresponding to (2.5):

$$\partial_x\left(l_a^{-1}\partial_x\psi_{k\omega}\right) + \left(1-k_y^2 l_a^{-1}\right)\psi_{k\omega} = 0. \tag{2.8}$$

The solution ψ_1 satisfies the boundary conditions when $x \to -\infty$, and the solution ψ_2 satisfies the boundary conditions when $x \to \infty$. $W(\psi_1,\psi_2)$ in (2.7) is the Wronskian of functions ψ_1 and ψ_2: $W(\psi_1,\psi_2) = \psi_1\partial_x\psi_2 - \psi_2\partial_x\psi_1$; the product $l_a^{-1}(x,\omega)W(x,\omega)$ is independent of the x-coordinate. The function θ in (2.7) is the Heaviside Unit Step Function: $\theta = 1$ when $x \geqslant 0$, $\theta = 0$ when $x < 0$.

Representation of the evolving disturbance through solution to the inhomogeneous equation for the Laplace transform of this disturbance using Green's function was applied, for instance, in Sedlachek (1971a, b), Grossmann and Tataronis (1973), and Tataronis and Grossmann (1973). An analysis of the expression of the form (2.6) in Sedlachek (1971a), Grossmann and Tataronis (1973), and Tataronis and Grossmann (1973) made it possible to determine the asymptotic behavior of the FMS disturbance as $t \to \infty$, when the FMS–Alfven wave coupling takes place. We want to obtain at first the description of temporal behavior of the FMS disturbance with given k at all moments of time $t > 0$ and then to determine temporal behavior of the Alfven disturbance with given k, using (2.3). The description of longitudinal evolution of the Alfven disturbance will be then obtained with the use of the inverse Fourier transform over k.

In order to obtain the description of temporal behavior of the FMS disturbance with the given k at all moments of time, using the inverse Laplace transform, we preliminarily transform the integrand in (2.6). Taking account of the weak FMS–Alfven wave coupling, we pass from solutions of (2.5) to solutions of the corresponding leading order equation with respect to the FMS–Alfven wave coupling.

3. Waveguide modes without FMS–Alfven wave coupling, dispersion equation, and decrement

We will describe evolution of disturbance in the case when only small k_y values are significant in the initial disturbance. In this case, the FMS–Alfven wave coupling can be considered weak, and we will make use of this fact. In this paragraph we will obtain relations for the leading approximation with respect to the FMS–Alfven wave coupling in order to use them in the next paragraph. We will denote the leading approximation by superscript (0).

We have from (2.5) the following equation for the velocity divergence at $k_y = 0$:

$$\partial_x\left(l_a^{-1}\partial_x\psi_{k\omega}^{(0)}\right) + \psi_{k\omega}^{(0)} = F^{(0)}, \tag{3.1}$$

where $F^{(0)} = \partial_x(l_a^{-1}U_x)$. This equation does not take account of the FMS–Alfven wave coupling; although it possesses a singular point, its solutions are regular. This can be easily confirmed using, for instance, the Frobenius method. Equation for $v_{xk\omega}^{(0)}$ has even simpler form; we will use its solutions to express solutions to (3.1). As the velocity divergence is determined only by the x component of velocity at $k_y = 0$,

$$\psi_{k\omega}^{(0)} = \partial_x v_{xk\omega}^{(0)}. \tag{3.2}$$

We have the following equation for $v_{xk\omega}^{(0)}$ from (3.1):

$$\partial_x^2 v_{xk\omega}^{(0)} + l_a v_{xk\omega}^{(0)} = U_x. \tag{3.3}$$

The boundary conditions for this equation are vanishing for $|v_{xk\omega}^{(0)}|$ when $|x| \to \infty$. The homogeneous equation corresponding to (3.3) –

$$\partial_x^2 v^{(0)} + l_a v^{(0)} = 0, \tag{3.4}$$

– with the boundary conditions of disturbance vanishing and given k^2 is ω^2 eigenvalue problem – the Sturm–Liouville problem over an infinite interval.

Let us denote eigenfunctions of (3.4) corresponding to Ω_n^2 eigenvalues by $\bar{v}_n^{(0)}$. It is easy to prove that problem (3.4) has the same properties as the Sturm–Liouville problem over a finite interval: eigenfunctions $\bar{v}_n^{(0)}$ corresponding to different values Ω_n^2 are orthogonal with weight V_a^{-2} and constitute a complete system on

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the real axis; eigenvalues Ω_n^2 are real. Note that there is no continuous spectrum because $V_a \to \infty$ when $|x| \to \infty$.

Eigenfunctions can be chosen as the real ones. We will assume that $\bar{v}_n^{(0)}$ are real and normalized as follows:

$$\int_{-\infty}^{\infty} V_a^{-2}(x)\,\bar{v}_n^{(0)}(x)\,\bar{v}_m^{(0)}(x)\,dx = \delta_{nm}, \qquad (3.5)$$

where δ_{nm} is the Kronecker symbol. In consequence of (3.2), eigenfunctions $\bar{\psi}_n^{(0)}$ of homogeneous equation

$$\partial_x\left(l_a^{-1}\partial_x\psi^{(0)}\right) + \psi^{(0)} = 0, \qquad (3.6)$$

corresponding to (3.3), related to $\bar{v}_n^{(0)}$ through equality

$$\bar{\psi}_n^{(0)} = \partial_x\bar{v}_n^{(0)}. \qquad (3.7)$$

Let us use notation $W(\psi_1^{(0)},\psi_2^{(0)})$ for the Wronskian of functions $\psi_1^{(0)}$ and $\psi_2^{(0)}$, which are solutions to (3.6) satisfying the boundary conditions when $x \to -\infty$ and $x \to \infty$, respectively. Let us use designation $W(v_1^{(0)},v_2^{(0)})$ for the Wronskian of functions $v_1^{(0)}$ and $v_2^{(0)}$, which are solutions to (3.4) satisfying the boundary conditions when $x \to -\infty$ and $x \to \infty$, respectively. We have

$$W\left(\psi_1^{(0)},\psi_2^{(0)}\right) = \psi_1^{(0)}\partial_x\psi_2^{(0)} - \psi_2^{(0)}\partial_x\psi_1^{(0)} \text{ and}$$

$$W\left(v_1^{(0)},v_2^{(0)}\right) = v_1^{(0)}\partial_x v_2^{(0)} - v_2^{(0)}\partial_x v_1^{(0)}.$$

As (3.2) and (3.6) yield

$$\psi_{1,2}^{(0)} = \partial_x v_{1,2}^{(0)},\ l_a^{-1}\partial_x\psi_{1,2}^{(0)} = -v_{1,2}^{(0)}, \qquad (3.8)$$

then

$$W\left(\psi_1^{(0)},\psi_2^{(0)}\right) = l_a W\left(v_1^{(0)},v_2^{(0)}\right). \qquad (3.9)$$

Functions $W(v_1^{(0)},v_2^{(0)})$ and $l_a^{-1}W(\psi_1^{(0)},\psi_2^{(0)})$ are independent of x.

The Wronskian $W(v_1^{(0)},v_2^{(0)})$ and the function $l_a^{-1}W(\psi_1^{(0)}, \psi_2^{(0)})$, identically equal to it, become zero at such ω^2 values that solutions $\psi_1^{(0)}$, $\psi_2^{(0)}$ coincide with $\bar{\psi}_n^{(0)}$ and $v_1^{(0)}$, $v_2^{(0)}$ coincide with $\bar{v}_n^{(0)}$, i.e. when $\omega^2 = \Omega_n^2$. Thus, equations $W(v_1^{(0)},v_2^{(0)}) = 0$ and $l_a^{-1}W(\psi_1^{(0)},\psi_2^{(0)}) = 0$ are the different forms of the dispersion equation of the leading approximation with respect to the FMS–Alfven wave coupling. Solutions to this dispersion equation determine real Ω_n^2 as functions of k^2.

Equation (3.6) has real eigenvalues $\omega^2 = \Omega_n^2$, since, unlike (2.8), there is no term with k_y^2 that could describe the FMS–Alfven wave coupling. Taking this term in (2.8) into consideration as a small correction and using the standard procedure of the perturbation theory, we can determine imaginary parts of frequencies at which there are solutions to (2.8) – $\bar{\psi}_n$– satisfying the boundary conditions of disturbance vanishing as $|x| \to \infty$. Differentiating both parts of (2.8) with respect to x and using notation $\bar{v}_n$ ($\bar{v}_n = -l_a^{-1}\partial_x\bar{\psi}_n$) in the first two terms, we obtain

$$\partial_x^2\bar{v}_n + l_a\bar{v}_n = -k_y^2\partial_x(l_a^{-1}\bar{\psi}_n).$$

The right-hand side of this equation takes account of the FMS–Alfven wave coupling. We write complex frequencies, corresponding to $\bar{\psi}_n$, as follows:

$$\omega_{n\pm} = \pm\Omega_n - i\gamma_n, \qquad (3.10)$$

where the positive value has been chosen for Ω_n at real k. The decrement value is not dependent of signs of k or of real part of ω_n. Let us write the sequence of calculations of γ_n only for $\mathrm{Re}\,\omega_n = \Omega_n$, $k > 0$. Representing $\bar{\psi}_n$ and $\bar{v}_n$ in the form

$$\bar{\psi}_n = \bar{\psi}_n^{(0)} + \bar{\psi}_n^{(1)}, \left|\bar{\psi}_n^{(1)}\right| \ll \left|\bar{\psi}_n^{(0)}\right|,$$

$$\bar{v}_n = \bar{v}_n^{(0)} + \bar{v}_n^{(1)}, \left|\bar{v}_n^{(1)}\right| \ll \left|\bar{v}_n^{(0)}\right|,$$

we have

$$\partial_x^2\bar{v}_n + l_a\left(\Omega_n^2\right)\bar{v}_n - 2i\Omega_n\gamma_n V_a^{-2}\bar{v}_n^{(0)} = -k_y^2\partial_x\left(l_a^{-1}\bar{\psi}_n^{(0)}\right).$$

Multiplication of both parts of this equation by $\bar{v}_n^{(0)}$ and integration between $-\infty$ and ∞ yields

$$2i\Omega_n\gamma_n\int_{-\infty}^{\infty} V_a^{-2}\left(\bar{v}_n^{(0)}\right)^2 dx = k_y^2\,\mathrm{Im}$$

$$\times\int_{-\infty}^{\infty}\bar{v}_n^{(0)}\partial_x\left(\left(l_a\left(\Omega_n^2\right)\right)^{-1}\bar{\psi}_n^{(0)}\right)dx.$$

Let us use normalization conditions of (3.5) and the fact that

$$\int_{-\infty}^{\infty}\bar{v}_n^{(0)}\partial_x\left(\left(l_a\left(\Omega_n^2\right)\right)^{-1}\bar{\psi}_n^{(0)}\right)dx = -\int_{-\infty}^{\infty}\left(\partial_x\bar{v}_n^{(0)}\right)^2$$

$$\times\left(l_a\left(\Omega_n^2\right)\right)^{-1}dx.$$

We also put $l_a^{-1} = V_a(2k(\omega - kV_a))^{-1}$ and $V_a = V_a(x_{n1,2}) + (\partial_x V_a)_{(x_{n1,2})}(x - x_{n1,2})$, where $x_{n1,2}$ are the coordinates of the surfaces where the resonant condition $l_a(\Omega_n^2) = 0$ is satisfied. The bypass rule of singular points when integrating over x is defined by the relation $\mathrm{Im}\,x_{n1,2} = -\mathrm{Im}\,\omega/(k(\partial_x V_a)_{(x_{n1,2})})$ and the fact that $\mathrm{Im}\,\omega > 0$ due to Laplace transform. We obtain

$$\gamma_n = \pi\frac{k_y^2}{4k^2\Omega_n}\sum_{i=1,2}\left(\left(\partial_x\bar{v}_n^{(0)}\right)^2\left|\left(V_a/(\partial_x V_a)\right)\right|\right)_{(x_{ni})}. \qquad (3.11)$$

In conclusion, we write Green's function for (3.4). Let us denote it by $G_{(x)}^{(0)}$. It can be represented as an expansion in eigenfunctions $\bar{v}_n^{(0)}$. Actually, if we suppose that the solution to equation

$$\partial_x^2 G_{(x)}^{(0)} + l_a G_{(x)}^{(0)} = \delta(x - \xi)$$

is represented in the form

$$G_{(x)}^{(0)} = \sum\nolimits_n a_n\bar{v}_n^{(0)},$$

we get

$$\sum\nolimits_n a_n\left[\frac{1}{V_a^2}\left[\omega^2 - \Omega_n^2\right]\right]\bar{v}_n^{(0)} = \delta(x - \xi).$$

Multiplication of both parts by $\bar{v}_m^{(0)}$ and integration using the normalization conditions (3.5) yield

$$G_{(x)}^{(0)} = \sum\nolimits_n\frac{1}{\left[\omega^2 - \Omega_n^2\right]}\bar{v}_n^{(0)}(x)\,\bar{v}_n^{(0)}(\xi). \qquad (3.12)$$

4. Evolution of waveguide FMS disturbance

In this paper, description of evolution of the initial disturbance for the case of weak FMS–Alfven wave coupling is based on the method that has been used to study evolution in the general case of an arbitrary coupling in Sedlachek (1971a, b), Grossmann and Tataronis (1973), and Tataronis and Grossmann (1973), and which originates from Landau's problem of electromagnetic wave damping in plasma. The key point of this method is the analytic continuation of Green's function from the region of values ω with a positive imaginary part (for which the Laplace transform has been made) to the region of values ω with a negative imaginary part, while keeping the bypass rule of the singular point obtained in the upper half-plane. The subsequent deformation of the contour of integration over ω in the inverse Laplace transform allows to obtain the asymptotic description of initial disturbance when $t \to \infty$ (Sedlachek 1971a, b; Grossmann and Tataronis 1973; Tataronis and Grossmann 1973). The difference between the method employed in this paper and the classical examination of an integral in the inverse Laplace transform is as follows: To describe evolution from the initial moment of time, we transform the integrand in (2.6) using the slowness of resonance absorption resulting from the weak FMS–Alfven wave coupling before performing its analytic continuation.

Let us write $\psi_{1,2}$ in the form

$$\psi_{1,2}(x,\omega) = \psi_{1,2}^{(0)}(x,\omega) + \psi_{1,2}^{(1)}(x,\omega).$$

As the FMS–Alfven wave coupling is weak, we can assume

$$\left|\psi_{1,2}^{(1)}(x,\omega)\right| \ll \left|\psi_{1,2}^{(0)}(x,\omega)\right|.$$

Neglecting small differences between $\psi_{1,2}(x,\omega)$ and $\psi_{1,2}^{(0)}(x,\omega)$, we obtain in (2.6)

$$G(x,\xi,\omega) = \frac{1}{l_a^{-1} W(\psi_1,\psi_2)} g^{(0)}(x,\xi,\omega), \tag{4.1}$$

where

$$g^{(0)}(x,\xi,\omega) = \psi_1^{(0)}(x,\omega)\,\psi_2^{(0)}(\xi,\omega)\,\theta(\xi - x) + \psi_2^{(0)}(x,\omega)\,\psi_1^{(0)}(\xi,\omega)\,\theta(x-\xi).$$

Function $g^{(0)}$ has no singularities due to resonance, then singular points of Green's function G in the form of (4.1) as a function of ω are dependent on the denominator of (4.1) only. So function G has poles at such ω values that the denominator in (4.1) is zero. Function $l_a^{-1} W(\psi_1,\psi_2)$ is independent of x; it equals to zero at such ω values that the Wronskian is zero. We have for zeros formulae (3.10) and (3.11). The bypass rule corresponding to the analytic continuation of ψ_1 and ψ_2 from the upper half-plane of complex ω was used to obtain (3.11). Zeros of the Wronskian are in the range of ω values with a negative imaginary part: $\omega = \omega_{n+} = \Omega_n - i\gamma_n$ and $\omega = \omega_{n-} = -\Omega_n - i\gamma_n$. Thus, integral over ω in the inverse Laplace transform with Green's function in the form of (4.1) can be reduced to the summation of contributions from the poles at $\omega = \omega_{n+}$ and $\omega = \omega_{n-}$ by closing the integration contour in the lower half-plane when $t > 0$. Prior to calculating integral over ω, we transform integral over ξ being its part. We first put $F = F^{(0)} = \partial_x(l_a^{-1} U_x)$ in it. Then we use equalities

$$\int_{-\infty}^{\infty} F^{(0)}(\xi,\omega)\,G(x,\xi,\omega)\,d\xi = -\frac{1}{l_a^{-1} W(\psi_1,\psi_2)} \times \int_{-\infty}^{\infty} l_a^{-1}(\xi,\omega)\,U_x(\xi,\omega)\,\partial_\xi g^{(0)}(x,\xi,\omega)\,d\xi \tag{4.2}$$

and

$$\partial_\xi g^{(0)}(x,\xi,\omega) = \theta(\xi - x)\,\psi_1^{(0)}(x,\omega)\,\partial_\xi \psi_2^{(0)}(\xi,\omega) + \theta(x-\xi)\,\psi_2^{(0)}(x,\omega)\,\partial_\xi \psi_1^{(0)}(\xi,\omega).$$

Taking (3.8) into consideration, we have

$$l_a^{-1}\partial_\xi g^{(0)}(x,\xi,\omega) = -\theta(\xi - x)\left(\partial_x v_1^{(0)}(x,\omega)\right) v_2^{(0)}(\xi,\omega) - \theta(x-\xi)\left(\partial_x v_2^{(0)}(x,\omega)\right) v_1^{(0)}(\xi,\omega),$$

and

$$\theta(\xi - x) v_1^{(0)}(x,\omega) v_2^{(0)}(\xi,\omega) + \theta(x-\xi) v_2^{(0)}(x,\omega) v_1^{(0)}(\xi,\omega) = W\left(v_1^{(0)}, v_2^{(0)}\right) G_{(x)}^{(0)}(x,\xi,\omega),$$

then

$$l_a^{-1}(\xi)\partial_\xi g^{(0)}(x,\xi,\omega) = -W\left(v_1^{(0)}, v_2^{(0)}\right)\partial_x G_{(x)}^{(0)}(x,\xi,\omega).$$

Notation $G_{(x)}^{(0)}$ was introduced above for the Green's function of (3.4). As $W(v_1^{(0)}(\xi), v_2^{(0)}(\xi))$ is independent of ξ, we substitute $W(v_1^{(0)}(x), v_2^{(0)}(x))$ for $W(v_1^{(0)}(\xi), v_2^{(0)}(\xi))$. Taking (3.12) into account, we get

$$l_a^{-1}(\xi)\,\partial_\xi g^{(0)}(x,\xi,\omega) = -W\left(v_1^{(0)}, v_2^{(0)}\right)\sum\nolimits_n \frac{1}{\left(\omega^2 - \Omega_n^2\right)} \times \left(\partial_x \bar{v}_n^{(0)}(x,k^2)\right)\bar{v}_n^{(0)}(\xi). \tag{4.3}$$

By substituting (4.3) in the right-hand side of (4.2) and taking account of (3.9) and (3.7), we obtain, instead of $\int_{-\infty}^{\infty} F(\xi,\omega,k)\,G(x,\xi,\omega)\,d\xi$ in (2.6), the following expression:

$$\frac{W\left(\psi_1^{(0)},\psi_2^{(0)}\right)}{W(\psi_1,\psi_2)}\sum\nolimits_n \frac{1}{\left(\omega^2 - \Omega_n^2\right)}\bar{\psi}_n^{(0)}(x) \times \int_{-\infty}^{\infty} U_x(\xi,\omega)\,\bar{v}_n^{(0)}(\xi)\,d\xi.$$

The inverse Laplace transform (2.6) is as follows:

$$\psi_k = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t}\frac{W\left(\psi_1^{(0)},\psi_2^{(0)}\right)}{W(\psi_1,\psi_2)}\sum\nolimits_n \frac{1}{\left(\omega^2 - \Omega_n^2\right)}\bar{\psi}_n^{(0)}(x) \times \int_{-\infty}^{\infty} U_x(\xi,\omega)\,\bar{v}_n^{(0)}(\xi)\,d\xi\,d\omega.$$

There are no singularities when $\omega^2 - \Omega_n^2 = 0$, since $W^{(0)}(\psi_1^{(0)}, \psi_2^{(0)}) = 0$ when $\omega^2 - \Omega_n^2 = 0$. As we can put

$$\left[\frac{dW(\psi_1, \psi_2)}{d\omega^2}\right]_{(\omega_{n\pm}^2)} = \left[\frac{dW\left(\psi_1^{(0)}, \psi_2^{(0)}\right)}{d\omega^2}\right]_{(\Omega_n^2)},$$

we have

$$\frac{W\left(\psi_1^{(0)}, \psi_2^{(0)}\right)}{W(\psi_1, \psi_2)} \frac{1}{\left(\omega^2 - \Omega_n^2\right)} = \frac{1}{\left(\omega^2 - \omega_{n\pm}^2\right)}$$

near poles of $W(\psi_1, \psi_2)$; for the denominator, we have $(\omega^2 - \omega_n^2) = \pm 2\Omega_n(\omega - \omega_{n\pm})$. We should also put $U_x(\xi, \omega_{n\pm}) = U_x(\xi, \pm\Omega_n)$. By closing contour in the lower half-plane when $t > 0$, we replace integration with summation of contributions from residues and get

$$\psi_k = \theta(t) \sum\nolimits_n \left(\psi_{kn+} e^{-i\omega_{n+}t} + \psi_{kn-} e^{-i\omega_{n-}t}\right),$$

where

$$\psi_{kn\pm} = c_{n\pm}(k^2) \bar{\psi}_n^{(0)}(x, k^2) \tag{4.4}$$

and

$$c_{n\pm}(k^2) = \mp \frac{i}{2\Omega_n} \int_{-\infty}^{\infty} U_x(\xi, \pm\Omega_n, k^2) \bar{v}_n^{(0)}(\xi, k^2) d\xi.$$

As $\psi_{kn\pm}$ will be then used in the Fourier transform of k, we write down the argument k^2, which was omitted before, in their expressions. In view of the determination of U_x, we have

$$c_{n\pm} = \frac{1}{2} \int_{-\infty}^{\infty} \left(v_{xk}(0) \pm i \frac{1}{\Omega_n} \partial_t v_{xk}(0)\right) \bar{v}_n^{(0)}(\xi, k^2) d\xi.$$

Thus, the Fourier transform of the FMS disturbance is the superposition of collective modes from the initial instant of time. Structure of these collective modes with respect to coordinates along the inhomogeneity and real parts of frequencies are determined from the leading order of (2.8), i.e. with neglecting of the FMS–Alfven wave coupling, but they have decrements due to this coupling. So these modes are collective modes of the leading approximation.

5. Evolution of Alfven disturbance

The Alfven disturbance is incompressible: its components of velocity $v_x^{(a)}$ and $v_y^{(a)}$ are related by equation $\partial_x v_x^{(a)} = -ik_y v_y^{(a)}$. It therefore suffices to determine $v_y^{(a)}$. Let us use (2.3) for this purpose. We are interested in the Alfven disturbance resulting from mode conversion of the FMS disturbance in the waveguide rather than in the Alfven disturbance caused by the initial disturbance. Thus, we suggest that there is no initial disturbance in the mode conversion region and put $U_y = 0$ there. As we calculate y-component of velocity only for the Alfven disturbance, we will not write index (a) in what follows. To abridge notation, we will consider the Alfven disturbance produced by the collective mode packet with one number n; such a disturbance will be denoted by inferior index n. Let us first determine the Alfven disturbance in k-representation. Equation (2.3) yields: $v_{yk\omega n} = -ik_y l_a^{-1} \psi_{k\omega n}$. The inverse Laplace transform yields

$$v_{ykn} = -ik_y \frac{1}{2\pi} \int_{-\infty+ic}^{\infty+ic} e^{-i\omega t} \frac{\psi_{k\omega n}}{l_a} d\omega.$$

Writing $\psi_{k\omega n}$ as

$$\psi_{k\omega n} = \int_0^{\infty} e^{i\omega\tau} \left(\psi_{kpn+} e^{-i\omega_{n+}t} + \psi_{kpn-} e^{-i\omega_{n-}t}\right) d\tau$$

and reducing integral over ω to summation of pole contributions by contour closures in the lower half-plane when $(t - \tau) > 0$ and in the upper half-plane when $(t - \tau) < 0$, we get

$$v_{ykn} = ik_y \frac{V_a}{2k} \sum\nolimits_{\pm} \psi_{kn\pm} \left[\frac{1}{kV_a - \omega_{n\pm}} \left[e^{-i\omega_{n\pm}t} - e^{-ikV_a t}\right] + \frac{1}{kV_a + \omega_{n\pm}} \left[e^{-i\omega_{n\pm}t} - e^{ikV_a t}\right]\right], \tag{5.1}$$

where $\sum_{\pm}$ denotes summation of expressions with upper and lower signs. The Alfven disturbance v_{ykn} consists of induced and eigen oscillations:

$$v_{ykn}^{(I)} = ik_y \frac{V_a}{2k} \sum\nolimits_{\pm} \psi_{kn\pm} \left[\frac{1}{kV_a - \omega_{n\pm}} + \frac{1}{kV_a + \omega_{n\pm}}\right] e^{-i\omega_{n\pm}t},$$

$$v_{ykn}^{(E)} = -ik_y \frac{V_a}{2k} \sum\nolimits_{\pm} \psi_{kn\pm} \times \left[\frac{1}{kV_a - \omega_{n\pm}} e^{-ikV_a t} + \frac{1}{kV_a + \omega_{n\pm}} e^{ikV_a t}\right].$$

When $\gamma_n t \gg 1$, induced oscillations $v_{ykn}^{(I)}$ decay and only undamped eigen Alfven oscillations $v_{ykn}^{(E)}$ remain. As we see, v_{ykn} as function of k has poles whose positions in the plane of complex k are determined by the following equations:

$$kV_a = \omega_{n\pm} \tag{5.2}$$

and

$$kV_a = -\omega_{n\pm}. \tag{5.3}$$

Let us denote the value of k^2 which is the root of equation

$$\Omega_n^2(k^2) - k^2 V_a^2(x) = 0 \tag{5.4}$$

by K_n^2. For the sake of simplicity, we assume that there is only one root for every n. We choose $K_n > 0$. It is clear that K_n is the function of only variable x.

Let us denote solutions to (5.2) by $k_{n\pm}$. Disturbances with $k_{n\pm}$ have positive phase velocity and propagate toward $z = \infty$. Solutions to (5.3) are $k = -k_{n\pm}$. Disturbances corresponding them propagate toward $z = -\infty$.

Let us write solutions to (5.2) as follows: $k_{n\pm} = \pm K_n - i\Lambda_n$. Let us write $\omega_{n\pm}(k_{n\pm})$ in the form of expansion near

$$k = \pm K_n : \omega_{n\pm}(k_{n\pm}) = \pm\Omega_{na} - \pm \left.\frac{d\Omega_n\left(K_n^2\right)}{dk}\right|_{k=\pm K_n} i\Lambda_n - i\Gamma_n.$$

We noted

$$\Omega_{na} = \Omega_n\left(K_n^2\right) \text{ and } \Gamma_n = \gamma_n\left(K_n^2\right).$$

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According to (3.11),

$$\Gamma_n = \pi \frac{k_y^2}{4K_n^2 \Omega_{na}} \sum_{i=1,2} ((\partial_x V_n)^2 \, |V_a/(\partial_x V_a)|)_{(X_{ni})},$$

where $X_{n1,2}$ are the x-coordinates of the surfaces where the condition

$$\Omega_n^2 \left(K_n^2 (x)\right) - K_n^2 (x) \, V_a^2 \ \left(X_{n1,2}\right) = 0$$

is satisfied. There are two such coordinates as we assumed above the Alfven velocity to have one minimum. One of $X_{n1,2}$ equals to the chosen x-coordinate. Thus, decrement Γ_n is the function of x only.

Let us denote $\frac{d\Omega_n}{dk}\big|_{k=K_n}$ by V_{gn}. It is clear that V_{gn} is equal to the absolute value of group velocity of resonant mode, given $k = K_n$. Since Ω_n is determined from equation $W(v_1^{(0)}, v_2^{(0)}) = 0$, where W is independent of x, we can write $W(\Omega_n^2, k^2) = 0$. Differentiating this equality with respect to k, we get

$$\frac{d\omega}{dk} = -\frac{k}{\omega}\frac{dW}{dk^2} \Big/ \frac{dW}{d\omega^2};$$

thus,

$$\left.\frac{d\left(\Omega_n\right)}{dk}\right|_{-K_n} = -V_{gn}.$$

Consequently,

$$\omega_{n\pm}(k_{n\pm}) = \pm \Omega_{na} - V_{gn} i \Lambda_n - i\Gamma_n.$$

Substitution of $\omega_{n\pm}(k_{n\pm})$ in this form into (5.2) yields $\Lambda_n = \Gamma_n/(V_a - V_{gn})$.

Thus, we have obtained that there is an ensemble of collective modes being in resonance with Alfven waves on surfaces (or, taking account of collectivity of modes, near surfaces) where the Alfven velocity is $V_a(x)$. Width of the range of longitudinal wave numbers of these modes near $k = K_n$ or $k = -K_n$ is Λ_n. It is small due to smallness of Γ_n. Therefore, the only significant contribution in the disturbance produced by nth collective mode packet is the contribution made by modes with real k such that $|k \pm K_n| \lesssim \Lambda_n$. So we can substitute $\omega_{n\pm}$ by their expansions near $k = K_n$ and $k = -K_n$ or their equivalent (by virtue of smallness of Λ_n) expansions near $k_{n\pm}$ and $-k_{n\pm}$. As we have $\omega_{n\pm} = k_{n\pm} V_a + V_{gn}(k - k_{n\pm})$ at k near to $k_{n\pm}$, and $\omega_{n\pm} = k_{n\pm} V_a - V_{gn}(k + k_{n\pm})$ at k near to $-k_{n\pm}$, we obtain

$$v_{ykn} = \frac{ik_y V_a}{2k(V_a - V_{gn})} \sum\nolimits_{\pm} \psi_{kn\pm} \times \left[\frac{\exp(-i(k_{n\pm} V_a + V_{gn}(k - k_{n\pm}))t) - \exp(-ikV_a t)}{(k - k_{n\pm})} + \frac{1}{(k + k_{n\pm})} \left(e^{-i\left(k_{n\pm} V_a - V_{gn}\left(k + k_{n\pm}\right)\right)t} - e^{ikV_a t}\right)\right]$$

instead of (5.1), after having used these expansions.

Disturbance v_{ykn} consists of waves $v_{ykn}^{(+)}$ and $v_{ykn}^{(-)}$ that propagate toward $z = \infty$ and $z = -\infty$, respectively. We have $v_{ykn}^{(+)} = v_{ykn}{}^{(I)(+)} + v_{ykn}{}^{(E)(+)}$, where

$$v_{ykn}{}^{(I)(+)} = \frac{ik_y V_a}{2k(V_a - V_{gn})} \times \sum\nolimits_{\pm} \psi_{kn\pm} \frac{\exp(-i(k_{n\pm} V_a + V_{gn}(k - k_{n\pm}))t)}{(k - k_{n\pm})},$$

$$v_{ykn}{}^{(E)(+)} = -\frac{ik_y V_a}{2k(V_a - V_{gn})} \sum\nolimits_{\pm} \psi_{kn\pm} \frac{-\exp\left(-ikV_a t\right)}{(k - k_{n\pm})};$$

and

$$v_{ykn}^{(-)} = v_{ykn}^{(I)(-)} + v_{ykn}^{(E)(-)},$$

where

$$v_{ykn}^{(I)(-)} = \frac{ik_y V_a}{2k(V_a - V_{gn})} \times \sum\nolimits_{\pm} \psi_{kn\pm} \frac{1}{(k + k_{n\pm})} e^{-i\left(k_{n\pm} V_a - V_{gn}\left(k + k_{n\pm}\right)\right)t},$$

$$v_{ykn}^{(E)(-)} = -\frac{ik_y V_a}{2k(V_a - V_{gn})} \sum\nolimits_{\pm} \psi_{kn\pm} \frac{1}{(k + k_{n\pm})} e^{ikV_a t}.$$

Induced and eigen oscillations are denoted by (I) and (E) as above.

To obtain formula describing longitudinal structure of the disturbance, we should perform the inverse Fourier transform of function v_{ykn} over k. Let us first do it for $v_{ykn}{}^{(I)(+)}$. We have

$$v_{yn}^{(I)(+)} = \frac{ik_y V_a}{4\pi(V_a - V_{gn})} \sum\nolimits_{\pm} \int_{-\infty}^{\infty} e^{ikz} c_{n\pm} \overline{\varphi}_n^{(0)}(x, k^2) \times \left[\frac{1}{k(k - k_{n\pm})} e^{\left[-ik_{n\pm} V_a - iV_{gn}\left(k - k_{n\pm}\right)\right]t}\right] dk. \quad (5.5)$$

In (5.5), we wrote $\psi_{kn\pm}$ like in (4.4)

Let us write $\partial_t v_{xk}(0)$ and $v_{xk}(0)$ in the form of the Fourier transform: $v_{xk}(t = 0) = \int_{-\infty}^{\infty} e^{-ikl} v_x(0)\, dl$ and $\partial_t v_{xk}(0) = \int_{-\infty}^{\infty} e^{-ikl} \partial_t v_x(0)\, dl$. Substitution of these integrals in $c_{n\pm}$ yields

$$c_{n\pm} = \int_{-\infty}^{\infty} e^{-ikl} \widetilde{c}_{n\pm} dl,$$

where

$$\widetilde{c}_{n\pm} = \frac{1}{2}\left(\int_{-\infty}^{\infty} v_x(0) \pm i \frac{1}{\Omega_n(k^2)} \partial_t v_x(0)\right) \overline{v}_n^{(0)}(\xi, k^2) d\xi.$$

As there are initial disturbances $\partial_t v_x(0)$ and $v_x(0)$ on the right-hand side, $\widetilde{c}_{n\pm}$ functions are dependent on the longitudinal coordinate (l), and presence of Ω_n and $\overline{v}_n^{(0)}$ implies dependence of $\widetilde{c}_{n\pm}$ on k^2, so $\widetilde{c}_{n\pm}$ are the functions of l and k^2. Using $\widetilde{c}_{n\pm}$, instead of (5.5) we get

$$v_{yn}^{(I)(+)} = \frac{ik_y V_a}{4\pi(V_a - V_{gn})} \times \sum\nolimits_{\pm} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{ik(z-l)} \widetilde{c}_{n\pm} dl \, \overline{\varphi}_n^{(0)}(x, k^2) \times \left[\frac{1}{k(k - k_{n\pm})} e^{\left[-ik_{n\pm} V_a - iV_{gn}(k - k_{n\pm})\right]t}\right] dk.$$

To perform k integration, we use the fact that the integrand has poles at $k = k_{n\pm}$. As $k_{n\pm} = \pm K_n - i\Lambda_n$, $\Lambda_n > 0$, the poles are in the lower part of the complex plane. Let us close the contour of k integration below, including the poles, when $(z - l - V_{gn}t) < 0$; and close the contour of integration in the upper half-plane when $(z - l - V_{gn}t) > 0$. Therefore, the integral over k is reduced to the summation of residue contribution. We can put $\Omega_n(k_{n\pm}^2) = \Omega_{na}$, and $\bar{\psi}_n^{(0)}(x, k_{n\pm}^2) = \bar{\psi}_n^{(0)}(x, K_n^2)$, $\bar{v}_n^{(0)}(x, k_{n\pm}^2) = \bar{v}_n^{(0)}(x, K_n^2)$ at the poles at $k = k_{n\pm}$ due to smallness of the imaginary part of $k_{n\pm}$. As K_n^2 is a function of only variable x, $\bar{\psi}_n^{(0)}(x, K_n^2)$ and $\bar{v}_n^{(0)}(x, K_n^2)$ are functions of only variable x too. Let us denote these by Ψ_n and V_n, respectively. These describe structure of the disturbances of velocity divergence and the x-component of velocity in resonant collective modes with respect to the x-coordinate. Let us denote $\tilde{c}_{n\pm}$ by $C_{n\pm}$ when $k^2 = K_n^2$. We have

$$C_{n\pm} = \frac{1}{2}\int_{-\infty}^{\infty} v_x(0)\, V_n(\xi)\, d\xi \pm i\frac{1}{2\Omega_{na}}\int_{-\infty}^{\infty} \partial_t v_x(0)\, V_n(\xi)\, d\xi. \tag{5.6}$$

Thus, we get the following equality for induced oscillations:

$$v_{yn}^{(I)(+)} = \frac{k_y V_a}{2K_n\left(V_a - V_{gn}\right)}\Psi_n \times \int_{(z-V_{gn}t)}^{\infty} \left(e^{ik_{n+}(z-l-V_a t)}C_{n+} - e^{ik_{n-}(z-l-V_a t)}C_{n-}\right) dl.$$

By analogy, we get

$$v_{yn}^{(E)(+)} = -\frac{k_y V_a}{2K_n\left(V_a - V_{gn}\right)}\Psi_n \times \int_{(z-V_a t)}^{\infty} \left(e^{ik_{n+}(z-l-V_a t)}C_{n+} - e^{ik_{n-}(z-l-V_a t)}C_{n-}\right) dl \tag{5.7}$$

for eigen oscillations. As $V_{gn} < V_a$, the entire disturbance $v_{yn}^{(+)} = v_{yn}^{(I)(+)} + v_{yn}^{(E)(+)}$ propagating toward $z = \infty$ is described by

$$v_{yn}^{(+)} = -\frac{k_y V_a}{2K_n\left(V_a - V_{gn}\right)}\Psi_n \times \int_{(z-V_a t)}^{(z-V_{gn}t)} \left(e^{ik_{n+}(z-l-V_a t)}C_{n+} - e^{ik_{n-}(z-l-V_a t)}C_{n-}\right) dl.$$

The substitution C_{n+} and C_{n-} from (5.7) yields

$$v_{yn}^{(+)} = -i\frac{k_y}{2K_n}\frac{V_a}{\left(V_a - V_{gn}\right)}\Psi_n \times \int_{(z-V_a t)}^{(z-V_{gn}t)} e^{\Lambda_n(z-l-V_a t)}\left(C_{n1}\left(\sin\left(K_n\left(z - l - V_a t\right)\right)\right) + \frac{1}{\Omega_{na}}C_{n2}\left(\cos\left(K_n\left(z - l - V_a t\right)\right)\right)\right) dl. \tag{5.8}$$

where

$$C_{n2} = \int_{-\infty}^{\infty} \partial_t v_x(0)\, V_n(\xi)\, d\xi \text{ and } C_{n1} = \int_{-\infty}^{\infty} v_x(0)\, V_n(\xi)\, d\xi.$$

Coefficients C_{n1} and C_{n2} are the coefficients of expansion of the initial disturbance in eigenfunction V_n; these are the functions of z. Thus, packet (5.8) consists of two components: the first is determined by the initial plasma shift at velocity $v_x(0)$ at the initial moment of time; and the second is determined by the acceleration $\partial_t v_x(0)$ that plasma obtained when shifting.

Formula (5.8) describes the formation, growth, and propagation of the Alfven disturbance. When $t = 0$, there is no Alfven disturbance; it starts growing when the integration interval increases with increasing t. Factor $\exp(\Lambda_n(z - l - V_a t))$, decreasing with increasing integration variable from its lower limit, constrains the domain of integration, which contributes significantly to the integral. Thus, given $\Lambda_n(V_a - V_{gn})t \gtrsim 1$ (i.e. $\Gamma_n t \gtrsim 1$), the upper limit of integration in (5.8) can be substituted by ∞; this yields (5.7) (i.e. eigen oscillations only). Therefore, formula (5.7) describes the Alfven disturbance after vanishing of its generating waveguide disturbance.

Let us use (5.8)) to study the dependence of space-time structure of the Alfven wave packet on relation between scales of the initial disturbance and longitudinal scales specified by resonance conditions. There are two latter scales. The first is specified by the resonant wave number K_n. This is wavelength λ_n of resonant collective modes, $\lambda_n \sim K_n^{-1}$. The second is specified by the width of the range of the resonance longitudinal wave numbers. This is the scale of the order of Λ_n^{-1}.

Let us consider the initial disturbances with different ratios of their scales to the resonance scales.

(1) The Alfven disturbance from the initial disturbance of large longitudinal scale:

Let us consider the case when the initial disturbance, being a function of the longitudinal coordinate, is a harmonics with wave number k_0 and with envelope of a large longitudinal scale so that C_{n1} and C_{n2} can be represented in the form

$$C_{n1} = \cos(k_0 z)\, A_{n1}\left(y, \frac{(z - z_0)}{S_n}\right),$$

$$C_{n2} = \cos(k_0 z)\, A_{n2}\left(y, \frac{(z - z_0)}{S_n}\right),$$

where $S_n \gg \Lambda_n^{-1}$. If the initial disturbance is the product of two functions each of which depends on only one coordinate x or z, the scale S_n is the same at all values of n. Factor $\exp(\Lambda_n(z - l - V_a t))$ in the integrand decreases with increasing integration variable from its lower limit; dimensions of the domain of integration (contributing significantly to the integral) are thus limited to the dimensions of about Λ_n^{-1}. As $S_n \gg \Lambda_n^{-1}$, we can neglect variation of A_{n1} and A_{n2} on the scale of the domain of integration and put in the integrand $A_{n1,2}(y, \frac{(l-z_0)}{S_n}) = A_{n1,2}(y, \frac{(z-V_a t-z_0)}{S_n})$.

We thus get

$$v_{yn}^{(+)} = -i\frac{k_y}{2K_n}V_a\Psi_n \times \left(A_{n1}\left(y, \frac{(z - V_at - z_0)}{S_n}\right)I_1 + \frac{1}{\Omega_{na}}A_{n2}\left(y, \frac{(z - V_at - z_0)}{S_n}\right)I_2\right), \quad (5.9)$$

where

$$I_1 = \frac{1}{2\left(V_a - V_{gn}\right)}\exp\left(-\Gamma_n t\right) \times \left(\frac{\left(-|k_0| + K_n\right)\cos\left(\varphi_0 + \phi_n\right) - \Lambda_n\sin\left(\varphi_0 + \phi_n\right)}{\left(\left(|k_0| - K_n\right)^2 + \Lambda_n^2\right)} + \frac{\left(|k_0| + K_n\right)\cos\left(\varphi_0 - \phi_n\right) + \Lambda_n\sin\left(\varphi_0 - \phi_n\right)}{\left(\left(|k_0| + K_n\right)^2 + \Lambda_n^2\right)}\right) - \frac{1}{2\left(V_a - V_{gn}\right)}\left[\frac{-|k_0| + K_n}{\left(|k_0| - K_n\right)^2 + \Lambda_n^2} + \frac{|k_0| + K_n}{\left(|k_0| + K_n\right)^2 + \Lambda_n^2}\right]\cos\varphi_0 - \frac{1}{2}\left[\frac{-\Lambda_n}{\left(|k_0| - K_n\right)^2 + \Lambda_n^2} + \frac{\Lambda_n}{\left(|k_0| + K_n\right)^2 + \Lambda_n^2}\right]\sin\varphi_0,$$

$$I_2 = \frac{1}{2\left(V_a - V_{gn}\right)}\exp\left(-\Gamma_n t\right) \times \left(\frac{-\Lambda_n\cos\left(\varphi_0 + \phi_n\right) + \left(|k_0| - K_n\right)\sin\left(\varphi_0 + \phi_n\right)}{\left(|k_0| - K_n\right)^2 + \Lambda_n^2} + \frac{-\Lambda_n\cos\left(\varphi_0 - \phi_n\right) + \left(|k_0| + K_n\right)\sin\left(\varphi_0 - \phi_n\right)}{\Lambda_n^2 + \left(|k_0| + K_n\right)^2}\right) + \frac{1}{2\left(V_a - V_{gn}\right)}\left(\frac{\Lambda_n\cos\varphi_0 - \left(|k_0| - K_n\right)\sin\varphi_0}{\left(|k_0| - K_n\right)^2 + \Lambda_n^2} + \frac{\Lambda_n\cos\varphi_0 - \left(k_0 + K_n\right)\sin\varphi_0}{\left(|k_0| + K_n\right)^2 + \Lambda_n^2}\right).$$

We denoted $|k_0|(z - V_at) = \varphi_0, K_n(V_{gn} - V_a)t = \phi_n$.

Formulae for I_1 and I_2 describe the formation and growth of Alfven waves in the packet as follows: when $t = 0$, functions I_1 and I_2 are zero, since the induced oscillations (terms with factor $\exp(-\Gamma_n t)$) and eigen oscillations (terms without this factor) compensate each other; damping of induced oscillations leads to increase in total disturbance; when $\Gamma_n t \gtrsim 1$, induced oscillations become small, and only eigen oscillations remain.

Region of the Alfven wave localization with respect to the longitudinal coordinate is specified by A_{n1} and A_{n2} in (5.9). As these are the functions of the longitudinal coordinate and time only through $(z - V_at)$, A_{n1} and A_{n2} describe propagation of the Alfven wave packet at the local Alfven velocity. Besides, according to (5.9), envelope structure of the Alfven wave packet with respect to the longitudinal coordinate is determined by envelope structure of the initial disturbance; thus, the envelope of packet has longitudinal scale S_n. If the initial disturbance is a harmonic disturbance with $k = k_0$ without envelope (i.e. $A_{n1,2}$ are independent of z), Alfven waves propagating in opposite directions make up a standing wave with respect to the longitudinal coordinate.

According to expressions for I_1 and I_2, the disturbance is localized along the x-coordinate near the surface where $|k_0| = K_n$. Using the inequality

$$\frac{1}{\left(|k_0| - K_n\right)^2 + \Lambda_n^2} \ll \frac{1}{\left(|k_0| + K_n\right)^2 + \Lambda_n^2},$$

we can rewrite I_1 and I_2 in the form

$$I_1 = I\left(\varphi_0, \phi_n\right) \text{ and } I_2 = I\left(\varphi_0 - \frac{\pi}{2}, \phi_n - \frac{\pi}{2}\right),$$

$$I = \exp\left(-\Gamma_n t\right) \times \left[\frac{\left(-|k_0| + K_n\right)\cos\left(\varphi_0 + \phi_n\right) - \Lambda_n\sin\left(\varphi_0 + \phi_n\right)}{2\left(V_a - V_{gn}\right)\left(\left(|k_0| - K_n\right)^2 + \Lambda_n^2\right)}\right] + \frac{\left(|k_0| - K_n\right)\cos\varphi_0 + \Lambda_n\sin\varphi_0}{2\left(V_a - V_{gn}\right)\left(\left(|k_0| - K_n\right)^2 + \Lambda_n^2\right)}.$$

On the surface $x = x_n$, where equality $|k_0| = K_n$ is satisfied, I takes on the following values:

$$I = \frac{1}{2\Gamma_n}\left(-\exp\left(-\Gamma_n t\right)\sin\left(\varphi_0 + \phi_n\right) + \sin\varphi_0\right).$$

We have $K_n(x) = |k_0| + K_n'(x_n)(x - x_n)$ near x_n; thus, the width of the layer where the Alfven disturbance is localized is $\Delta_n \sim \Lambda_n(x_n)/|K_n'(x_n)|$.

(2) The Alfven disturbance from the initial disturbance of small longitudinal scale:

Let us first consider the case when the initial disturbance takes place only on surface $z = z_0$. Suppose $C_{n1} = \alpha_{n1}\delta\left(z - z_0\right)$ and $C_{n2} = \alpha_{n2}\delta\left(z - z_0\right)$, we obtain from (5.8)

$$v_{yn}^{(+)} = -i\frac{k_yV_a}{2K_n(V_a - V_{gn})}\Psi_n e^{\Lambda_n(z - z_0 - V_at)} \times \theta\left(-\left(z - z_0 - V_at\right)\right)\theta\left(\left(z - z_0 - V_{gn}t\right)\right) \times \left(C_{n1}\left(\sin\left(K_n\left(z - z_0 - V_at\right)\right)\right) + \frac{1}{\Omega_{na}}C_{n2}\left(\cos\left(K_n\left(z - z_0 - V_at\right)\right)\right)\right).$$

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Formation of the Alfven disturbance and expansion of its region along the longitudinal coordinate is described by product of θ functions. Product of θ functions also describes propagation of the leading edge of the Alfven wave packet (let us denote its coordinate by z_1) with the Alfven velocity along the longitudinal coordinate, $z_1 = z_0 + V_a t$. Since the beginning (when $t = 0$), the disturbance has a finite amplitude on surface $z = z_1$, and the subsequent energy loss by the waveguide disturbance does not lead to its increase. There is, however, an increase in dimensions of the Alfven wave packet along the longitudinal coordinate: It increases with time from zero to $L = (V_a - V_{gn})t$. When $\Lambda_n(V_a - V_{gn})t \gtrsim 1$ (i.e. when $\Gamma_n t \gtrsim 1$, after the damping of resonant collective modes), the packet stops increasing in longitudinal size. Consequently, the maximum longitudinal scale of the packet is $L_{\max} \sim \Lambda_n$, and the form of the envelope as a function of the longitudinal coordinate is determined by the factor $\theta(-(z - z_0 - V_a t))e^{\Lambda_n(z-z_0-V_a t)}$ when $\Gamma_n t \gtrsim 1$.

Let us now consider the case of the initial disturbance localized on a small but finite scale s along the longitudinal coordinates: $C_{n1} = \alpha_{n1}(x, y, \frac{z-z_0}{s_n})$ and $C_{n2} = \alpha_{n2}(x, y, \frac{z-z_0}{s_n})$. If the initial disturbance is the product of functions of x and z, then $s_n = s$. We assume that the functions $a_{n1}\ (x, y, \frac{z-z_0}{s_n})$ and $a_{n2}(x, y, \frac{z-z_0}{s_n})$ have scale s_n along the longitudinal coordinate in the sense that these may be considered as zero when $|\frac{z-z_0}{s_n}| \gtrsim 1$. Let us suppose that scale s_n is much smaller than the resonance wavelength: $K_n s_n \ll 1$. In this case, we can put that all integrand functions, except for $\alpha_{n1,2}$, are equal to their values when $l = z_0$.
Consequently,

$$v_{yn}^{(+)} = -ik_y V_a \frac{\Psi_n}{2K_n\left(V_a - V_{gn}\right)} e^{\Lambda_n(z-z_0-V_a t)} \times\left(\cos\left(K_n\left(z - z_0 - V_a t\right)\right)\frac{1}{\Omega_{na}}\widetilde{a}_{n2} + \sin K_n\left(z - z_0 - V_a t\right)\widetilde{a}_{n1}\right), \tag{5.10}$$

where

$$\widetilde{a}_{n1,2} = \int_{(z-V_a t)}^{(z-V_{gn}t)} a_{n1,2}dl.$$

As $a_{n1,2}\ (x, y, \frac{z-z_0}{s_n})$ can be put to zero when $|\frac{z-z_0}{s}| \gtrsim 1$, then $\widetilde{a}_{n1,2} = 0$ when $z - z_0 \gtrsim V_a t + s_n$ and $z - z_0 \lesssim V_{gn}t - s_n$. When the induced oscillations are damped, i.e. when $\Gamma_n t \gtrsim 1$, we get

$$\widetilde{a}_{n1,2} = \int_{(z-V_a t)}^{\infty} a_{n1,2}dl.$$

The integrals on interval $z : |z - V_a t| \lesssim s_n$ vary from 0 to some finite values that they have when $z < V_a t - s_n$. Consequently, width of the packet's leading edge is of the order of s_n and its structure is given by $\widetilde{a}_{n1}$ and $\widetilde{a}_{n2}$. If scale s_n is much smaller than Λ_n^{-1}, but greater or of about their longitudinal wavelength ($K_n s_n \gtrsim 1$), then only exponents can be put to be equal to their values when $l = z_0$. We thus get

$$v_{yn}^{(+)} = -ik_y \frac{1}{2K_n}\Psi_n \frac{V_a}{\left(V_a - V_{gn}\right)} e^{\Lambda_n(z-z_0-V_a t)} \times\left(\left(\widetilde{a}_{n1}^{(c)} + \frac{1}{\Omega_{na}}\widetilde{a}_{n2}^{(s)}\right)\sin\left(K_n\left(z - z_0 - V_a t\right)\right) - \left(\widetilde{a}_{n1}^{(s)} - \frac{1}{\Omega_{na}}\widetilde{a}_{n2}^{(c)}\right)\cos\left(K_n\left(z - z_0 - V_a t\right)\right)\right),$$

where

$$\widetilde{a}_{n1,2}^{(c)} = \int_{(z-z_0-V_a t)}^{(z-z_0-V_{gn}t)} C_{n1,2}\cos\left(K_n\eta\right)d\eta,$$

$$\widetilde{a}_{n1,2}^{(s)} = \int_{(z-z_0-V_a t)}^{(z-z_0-V_{gn}t)} C_{n1,2}\sin\left(K_n\eta\right)d\eta$$

and $\eta = l - z_0$. When the induced oscillations are damped ($\Gamma_n t \gtrsim 1$), we get

$$\widetilde{a}_{n1,2}^{(c)} = \int_{(z-z_0-V_a t)}^{\infty} \cos\left(K_n\eta\right)a_{n1,2}\left(x, y, \eta\right)d\eta,$$

$$\widetilde{a}_{n1,2}^{(s)} = \int_{(z-z_0-V_a t)}^{\infty} \sin\left(K_n\eta\right)a_{n1,2}\left(x, y, \eta\right)d\eta.$$

Functions $\widetilde{a}_{n1,2}^{(c)}$ and $\widetilde{a}_{n1,2}^{(s)}$ determine structure of the leading edge of the Alfven wave packet. The leading edge has scale s_n.

In conclusion, we note that we assumed that resonance equation (5.4) had only one root. It is clear from above that the solutions obtained (both (5.8) and its corollaries) correspond to the one root, which is not necessarily the only one; if there are several roots, the solutions corresponding to different roots should be summed up.

6. Conclusion

We have obtained description of evolution of the initial disturbance in the FMS waveguide: initiation of the waveguide FMS disturbance, its dumpimg due to mode conversion into Alfven waves, and arising and growth of the Alfven disturbance.

We have shown that the Fourier transform of the FMS disturbance with respect to the coordinates along which plasma is inhomogeneous can be presented, starting from the initial instant of time, as a superposition of collective modes of the leading approximation with respect to the weak FMS–Alfven wave coupling. Frequency of such a mode and its structural dependence on the coordinate along the inhomogeneity are found without considering the FMS–Alfven coupling, and the mode decrement is calculated using the perturbation technique.

Using such a representation, we described evolution of the Alfven disturbance mode converting from the FMS waveguide disturbance produced by the initial disturbance of an arbitrary longitudinal structure. Such an initial disturbance is divided into the superposition of waveguide packets, each of which consists of collective modes with one number n and different wave numbers. Mode conversion of these packets of collective modes into Alfven waves leads to formation of Alfven wave packets.

We have obtained an analytic description of the evolving Alfven wave packet as a function of time, the coordinate along direction of inhomogeneity (x), and the coordinate along the undisturbed magnetic field (z). The formulae obtained are used to study the dependence of space-time structure of the Alfven wave packet on relation between scales of the initial disturbance and longitudinal scales specified by resonance conditions. There are two latter scales. The first is specified by resonant wave number K_n – it is a wavelength of collective modes resonantly absorbed on surfaces where the Alfven velocity is equal to $V_a(x)$. The second is specified by the width of the range of resonance longitudinal wave numbers and is of the order of Λ_n^{-1}. The resonance width with respect to k is related to time decrement of resonant modes Γ_n by equality $\Lambda_n = \Gamma_n/(V_a(x)-V_{gn}(x))$, where $V_{gn}(x)$ is the absolute value of group velocity of resonant modes. Decrements Γ_n and Λ_n, as well as K_n, are the functions of the x-coordinate only.

When the initial disturbance as function of the longitudinal coordinate is a harmonic (e.g. the entire dependence of the longitudinal coordinate can be specified as $\cos(k_0 z)$), each waveguide packet has a resonance surface $x = x_n$ whereon the wave number of resonant modes coincides with k_0. In the course of time inversely proportional to time decrement $\Gamma_n(x_n)$, the waveguide packet with number n mode converts into standing Alfven wave (with respect to the longitudinal coordinate) in the layer near surface $x = x_n$. The layer width is determined by longitudinal decrement $\Lambda_n(x_n)$. Amplitude of the Alfven wave is inversely proportional to time decrement $\Gamma_n(x_n)$.

If the initial disturbance as a function of the longitudinal coordinate is a harmonic with wave number k_0 and with envelope whose longitudinal scale is much larger than Λ_n^{-1}, then structure of the Alfven disturbance with respect to coordinate k_0 coincides with that in the case of harmonic disturbance (i.e. it divides into layers localized near surfaces $x = x_n$). Envelopes of Alfven packets propagate with the Alfven velocity in opposite directions along the longitudinal coordinate. Longitudinal scales of envelopes are determined by the coefficients of expansion of the initial disturbance in eigenfunctions $V_n(x)$ that correspond to resonant collective modes.

If the longitudinal scale of the initial disturbance is much smaller, then Λ_n^{-1}, the waveguide packet converted into the Alfven wave packet whose amplitude damps exponentially behind its leading edge with exponential is equal to Λ_n. Thus, the packet's longitudinal size is equal to Λ_n^{-1}. Width of the leading edge, wherein the envelope increases from zero to maximum, is finite; its scale is determined by the scale of the initial disturbance.

References

Allan, W. 2000 Hydromagnetic wave propagation and coupling in a magnetotail waveguide. *J. Geophys. Res.* **105**, 317–328.

Allan, W. and Wright, A. N. 1998 Hydromagnetic wave propagation and coupling in a magnetotail waveguide. *J. Geophys. Res.* **103**, 2359–2368.

Barston, E. M. 1964 Electrostatic oscillations in inhomogeneous cold plasmas. *Ann. Phys.* **29**, 282.

Deforest, C. E. and Gurman, J. B. 1998 Observation of quasi-periodic compressive waves in solar polar plumes. *ApJL* **501**, L217.

Grossmann, W. and Tataronis, J. A. 1973 Decay of MHD waves by phase mixing II: the Theta-Pinch in cylindrical geometry. *Z. Physik* **261**, 217–236.

Lysak, R. L., Song, Y. and Jones, T. W. 2009 Propagation of Alfven waves in the magnetotail during substorms. *Ann. Geophys.* **27**, 2237–2246.

Mazur, N. G., Fedorov, E. N. and Pilipenko, V. A. 2010 MHD waveguides in space plasma. *Plasma Phys. Rep.* **36**(7), 609–626.

Mills, K. J. and Wright, A. N. 2000 Trapping and excitation of modes in the magnetotail. *Phys. Plasmas* **7**, 1572.

Sedlachek, Z. 1971a Electrostatic oscillations in cold inhomogeneous plasma. 1. Differential equation approach. *J. Plasma Phys.* **5**, 239.

Sedlachek, Z. 1971b Electrostatic oscillations in cold inhomogeneous plasma. 2. Integral equation approach. *J. Plasma Phys.* **6**, 187.

Tataronis, J. A. and Grossmann, W. 1973 Decay of MHD waves by phase mixing. I. The sheet-pinch in plane geometry. *Z. Physik* **261**, 203–216.

Uberoi, C. 1972 Alfven waves in inhomogeneous magnetic fields. *Phys. Fluids* **15**, 1673.

Verwichte, E., Nakariakov, V. M. and Cooper, F. C. 2005 Transverse waves in a post-flare supra arcade. *Astron. Astrophys.* **430**(7), L65–L68.

Wright, A. N. and Allan, W. 2008 Simulations of Alfven waves in the geomagnetic tail and their auroral signatures. *J. Geophys. Res.* **113**, A02206.

Zhu, X. M. and Kivelson, M. G. 1988 Analytic formulation and quantitative solutions of the coupling ULF problemy. *J. Geophys. Res.* **93**, 8602.