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PERIODS OF AUTOMORPHIC FORMS: THE TRILINEAR CASE

SHUNSUKE YAMANA[1,2]

[1] *Hakubi Center, Yoshida-Ushinomiya-cho, Sakyo-ku, Kyoto 606-8501, Japan*
[2] *Department of mathematics, Kyoto University, Kitashirakawa Oiwake-cho, Sakyo-ku, Kyoto, 606-8502, Japan* (yamana07@math.kyoto-u.ac.jp)

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Abstract Following Jacquet, Lapid and Rogawski, we regularize trilinear periods. We use the regularized trilinear periods to compute Fourier–Jacobi periods of residues of Eisenstein series on metaplectic groups, which has an application to the Gan–Gross–Prasad conjecture.

Keywords: Fourier–Jacobi model; trilinear form; regularization; special values of L-functions; Weil representations; Gan–Gross–Prasad conjecture

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1. Introduction

Let F be a number field with adèle ring $\mathbb{A}$. Fix a nontrivial additive character ψ of $F\backslash\mathbb{A}$. Let W_m be a symplectic space of dimension $2m$, $H = H_m = Sp(W_m)$ its symplectic group and $\mathbf{H}$ the metaplectic double cover of $H(\mathbb{A})$. As is well known, there is an $H(F)$-invariant functional Θ on the Weil representation $\omega_{W_m,\psi}$ of $\mathbf{H}$. For each $f \in \omega_{W_m,\psi}$ we define a theta function by $\Theta^\psi(h, f) = \Theta(\omega_{W_m,\psi}(h)f)$ for $h \in \mathbf{H}$.

For an automorphic form φ on $H(\mathbb{A})$ and for genuine automorphic forms φ' and φ'' on $\mathbf{H}$ the trilinear period is defined as the integral

$$I(\varphi, \varphi', \varphi'') = \int_{H(F)\backslash H(\mathbb{A})} \varphi(h)\varphi'(h)\varphi''(h)\, dh$$

whenever it converges. The integrand is defined on $H(\mathbb{A})$ as the product of two genuine automorphic forms is no longer genuine. When either φ' or φ'' is a theta function, the trilinear period is a special case of the Fourier–Jacobi period (see [6, 12] for its definition).

For finite central coverings of arbitrary connected reductive algebraic groups, we make sense of the integrals of trilinear type, via a certain regularization procedure, even when

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they may be divergent. It is worth noting that several global zeta integrals are of trilinear type (cf. [3, 7, 16]). The regularized trilinear period constructed in § 3 is an $H(\mathbb{A})$-invariant functional that agrees with the trilinear period whenever it is convergent. It provides a powerful tool for computing trilinear periods of noncuspidal automorphic forms. As an application, Proposition 6.3 computes the Fourier–Jacobi period of certain residues of Eisenstein series, which has a direct consequence for the Gan–Gross–Prasad conjecture.

For each place v of F, we denote the preimage of $H(F_v)$ in $\mathbf{H}$ by $\mathbf{H}_v$. Let π' be an irreducible genuine cuspidal automorphic representation of $\mathbf{H}$. For almost all v, there are unramified quasicharacters $\chi_{i,v}$ of $F_v^\times$ such that the local component π'_v of π' is the spherical constituent of the representation of $\mathbf{H}_v$ induced from a genuine character

$$(\mathrm{diag}[a_1, \dots, a_m, a_m^{-1}, \dots, a_1^{-1}], \zeta) \mapsto \zeta \prod_{i=1}^{m} \gamma(a_i, \psi_v)\chi_{i,v}(a_i) \prod_{1\leqslant j<k\leqslant m} (a_j, a_k)_v,$$

extended to the preimage of the standard Borel subgroup of $H(F_v)$ in $\mathbf{H}$ trivially along the subgroup of unipotent matrices, where $(\ ,\ )_v$ is the Hilbert symbol over F_v and $\gamma(\cdot, \psi_v)$ is the Weil index. Here we use the Rao cocycle to define the group law of $\mathbf{H}_v$ (cf. [13, 23]).

An irreducible automorphic representation of $\mathrm{GL}_{2m}(\mathbb{A})$ is called a weak ψ-functorial lift of π' if for almost all v its local component is the spherical constituent of the representation of $\mathrm{GL}_{2m}(F_v)$ induced from the character

$$\mathrm{diag}[a_1, \dots, a_{2m}] \mapsto \prod_{i=1}^{m} \chi_{i,v}(a_i)\chi_{i,v}(a_{2m+1-i})^{-1}.$$

If π' is globally ψ-generic and lifts ψ-weakly to an irreducible automorphic representation $\mathrm{BC}_\psi(\pi')$ of $\mathrm{GL}_{2m}(\mathbb{A})$, then $\mathrm{BC}_\psi(\pi')$ has the form $\sigma_1 \times \cdots \times \sigma_k$ or $\mathbf{1}_{\mathrm{GL}_2} \times \sigma_1 \times \cdots \times \sigma_k$, where σ_i are pairwise inequivalent irreducible cuspidal automorphic representations of general linear groups such that the exterior square L-functions $L^S(s, \sigma_i, \Lambda^2)$ have a pole at $s = 1$ and $L^S(1/2, \sigma_i) \neq 0$ (see [12, Theorem 11.2]). Any irreducible globally generic cuspidal automorphic representation π of $H(\mathbb{A})$ has a weak functorial lift to an automorphic representation $\mathrm{BC}(\pi)$ of $\mathrm{GL}_{2m+1}(\mathbb{A})$ (see [5]). We give the following definition:

$$L_\psi(s, \pi \times \pi') := L(s, \mathrm{BC}(\pi) \times \mathrm{BC}_\psi(\pi')).$$

Theorem 1.1. *Let π be an irreducible globally generic cuspidal automorphic representation of $H(\mathbb{A})$ and π' an irreducible genuine cuspidal automorphic representation of $\mathbf{H}$. Suppose that π' admits a weak ψ-functorial lift to an automorphic representation of $\mathrm{GL}_{2m}(\mathbb{A})$ of the form $\sigma_1 \times \cdots \times \sigma_k$, where σ_i are irreducible cuspidal automorphic representations of general linear groups such that $L^S(s, \sigma_i, \Lambda^2)$ have a pole at $s = 1$. If there are $\varphi \in \pi$, $\varphi' \in \pi'$ and $f \in \omega_{W_m,\psi}$ such that $I(\varphi, \varphi', \Theta^\psi(f)) \neq 0$, then $L_\psi(1/2, \pi \times \pi') \neq 0$.*

Ginzburg *et al.* [9] have proved this result for the case where $\mathrm{BC}(\pi)$ and $\mathrm{BC}_\psi(\pi')$ are cuspidal. The proof of Theorem 1.1 is to realize the Fourier–Jacobi periods of the cuspidal automorphic representations as an 'inner' period of Fourier–Jacobi periods of some

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residual automorphic representations. This argument, which was introduced by Ginzburg, Jiang and Rallis, has been used to study many interesting cases, for which we refer the reader to [8–11, 18]. However, their computation is roundabout. We streamline the discussion by using the regularized trilinear period, a generic uniqueness principle stated in Lemma 4.1 and a description in [12] of Jacquet modules of induced representations corresponding to Fourier–Jacobi characters.

2. General notation

We fix some general notation. It is mostly standard and can be found in [22]. Let G be a connected reductive algebraic group over a number field F with adèle ring $\mathbb{A}$. We fix a minimal F-parabolic subgroup P_0 of G, a Levi decomposition $P_0 = M_0U_0$ and a good maximal compact subgroup $K = \prod_v K_v$ of $G(\mathbb{A})$ with respect to M_0 (see [22, I.1.4]). The Levi factor M_0 is the centralizer of a maximal split torus T_0. An F-parabolic subgroup is said to be standard if it contains P_0.

Let $\mathbf{G}$ be a topological group that is a finite central covering of $G(\mathbb{A})$, i.e., there exists a surjective morphism $\mathrm{pr} : \mathbf{G} \to G(\mathbb{A})$, whose kernel $\mathbf{N}$ is a finite subgroup of the centre of $\mathbf{G}$, pr being a topological covering. For each place v of F, we denote by $\mathbf{G}_v$ the inverse image of $G(F_v)$ under pr. Clearly, $\mathbf{G}_v$ equipped with the topology induced by that of $\mathbf{G}$ is a topological group. Suppose that $G(F)$ lifts to a subgroup of $\mathbf{G}$. Fix once and for all such a lifting, which is denoted also by $G(F)$. It is known that $U_0(\mathbb{A})$ lifts canonically into $\mathbf{G}$ (see [22]). For any F-subgroup U of U_0 we still use $U(\mathbb{A})$ to denote the image of this lifting.

We reserve the letters $P = MU$ for standard parabolic subgroups of G with their standard Levi decomposition. Let $M(\mathbb{A})^1$ be the intersection of the kernels of the homomorphisms $|\chi| : M(\mathbb{A}) \to \mathbb{R}_+^\times$, where χ ranges over the lattice $X^*(M)$ of F-rational characters of M. For subgroups J of $G(\mathbb{A})$ (resp. $G(F_v)$), we use boldface letters $\mathbf{J}$ to denote their preimages in $\mathbf{G}$ (resp. $\mathbf{G}_v$). In particular, $\mathbf{K}$, $\mathbf{M}$ and $\mathbf{M}^1$ are the inverse images of K, $M(\mathbb{A})$ and $M(\mathbb{A})^1$ in $\mathbf{G}$, respectively. Let A_0 be the identity component of $T_0(\mathbb{R})$, where $\mathbb{R}$ is embedded in $\mathbb{A}$ diagonally at the archimedean places. Since A_0 is simply connected, the group A_0 lifts to a unique subgroup of $\mathbf{T}_0$, which we denote by using the same symbol.

Let $\mathfrak{a}_0^*$ be the $\mathbb{R}$-vector space spanned by the lattice $X^*(T_0)$. The dual space $\mathfrak{a}_0$ of $\mathfrak{a}_0^*$ is the $\mathbb{R}$-vector space spanned by the lattice of one-parameter subgroups in T_0. The canonical pairing on $\mathfrak{a}_0^* \times \mathfrak{a}_0$ is denoted by $\langle\ ,\ \rangle$. We write Δ_0^P and $(\Delta^\vee)_0^P$ for the sets of simple roots and simple coroots of T_0 in M with respect to $M \cap P_0$. We denote by $(\mathfrak{a}_0^P)^*$ the span of Δ_0^P and by $\mathfrak{a}_0^P$ the span of $(\Delta^\vee)_0^P$. We identify $\mathfrak{a}_P^* = X^*(M) \otimes_{\mathbb{Z}} \mathbb{R}$ with a subspace of $\mathfrak{a}_0^*$. For $Q \subset P$, there are a canonical injection $\mathfrak{a}_P^* \to \mathfrak{a}_Q^*$ and a canonical surjection $\mathfrak{a}_Q^* \to \mathfrak{a}_P^*$, and similarly for dual spaces $\mathfrak{a}_P$. We have $\mathfrak{a}_Q = \mathfrak{a}_Q^P \oplus \mathfrak{a}_P$ and $\mathfrak{a}_Q^* = (\mathfrak{a}_Q^P)^* \oplus \mathfrak{a}_P^*$, where $\mathfrak{a}_Q^P = \mathfrak{a}_Q \cap \mathfrak{a}_0^P$ and $(\mathfrak{a}_Q^P)^* = \mathfrak{a}_Q^* \cap (\mathfrak{a}_0^P)^*$. We denote by X^Q (resp. X_Q^P, X_P) the canonical projection of $X \in \mathfrak{a}_0$ onto $\mathfrak{a}_0^Q$ (resp. $\mathfrak{a}_Q^P$, $\mathfrak{a}_P$) given by the decomposition $\mathfrak{a}_0 = \mathfrak{a}_0^Q \oplus \mathfrak{a}_Q^P \oplus \mathfrak{a}_P$, and similarly for $\mathfrak{a}_0^*$. Let Δ_Q^P be the set of restrictions of elements in $\Delta_0^P \smallsetminus \Delta_0^Q$ to $\mathfrak{a}_Q^P$. For any $\alpha \in \Delta_Q^P$, we have the corresponding coroot $\alpha^\vee \in \mathfrak{a}_Q^P$. Put

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$(\Delta^\vee)_Q^P = \{\alpha^\vee \mid \alpha \in \Delta_Q^P\}$. We then define $(\hat{\Delta}^\vee)_Q^P$ and $\hat{\Delta}_Q^P$ to be the bases dual to Δ_Q^P and $(\Delta^\vee)_Q^P$, respectively. When $P = G$, we omit the superscript ^{G}. We write τ_P for the characteristic function of the subset

$$\{X \in \mathfrak{a}_0 \mid \langle \alpha, X \rangle > 0 \text{ for all } \alpha \in \Delta_P\}$$

and $\hat{\tau}_P$ for the characteristic function of the subset

$$\{X \in \mathfrak{a}_0 \mid \langle \varpi, X \rangle > 0 \text{ for all } \varpi \in \hat{\Delta}_P\}.$$

The height function $H : G(\mathbb{A}) \to \mathfrak{a}_0$ is characterized by the condition $e^{\langle \chi, H(umk) \rangle} = |\chi(m)|$ for all $\chi \in X^*(M_0)$, $m \in M_0(\mathbb{A})$, $u \in U_0(\mathbb{A})$ and $k \in K$. The map H restricts to an isomorphism $A_0 \to \mathfrak{a}_0$. We denote its inverse map by $X \mapsto e^X$. We write $\rho_0 \in \mathfrak{a}_0^*$ for half the sum of positive roots of T_0 in G and denote its projections to $\mathfrak{a}_P^*$ by ρ_P. Recall that $e^{2\langle \rho_P, H(p) \rangle} = \delta_P(p)$ for $p \in P(\mathbb{A})$, where δ_P denotes the modulus function of $P(\mathbb{A})$. We sometimes view functions on $G(\mathbb{A})$ as functions on $\mathbf{G}$ by composing them with $\mathbf{pr}$.

3. Regularization of trilinear forms

Let $\mathscr{A}_P(\mathbf{G})$ be the space of automorphic forms on $U(\mathbb{A})P(F)\backslash\mathbf{G}$, i.e., smooth, $\mathbf{K}$-finite and $\mathfrak{z}$-finite functions on $U(\mathbb{A})P(F)\backslash\mathbf{G}$ of moderate growth, where $\mathfrak{z}$ is the centre of the universal enveloping algebra of the complexified Lie algebra of the product of the archimedean localizations of G (cf. [22, I.2.17]). When $P = G$, we omit the subscript $_G$. The constant term map from $\mathscr{A}(\mathbf{G})$ to $\mathscr{A}_P(\mathbf{G})$ is denoted by $\varphi \mapsto \varphi_P$. An automorphic form ϕ on $U(\mathbb{A})P(F)\backslash\mathbf{G}$ can be written as

$$\phi(ue^X mk) = \sum_i Q_i(X)\phi_i(mk)e^{\langle \lambda_i + \rho_P, X \rangle}$$

for $u \in U(\mathbb{A})$, $X \in \mathfrak{a}_P$, $m \in \mathbf{M}^1$ and $k \in \mathbf{K}$, where $\lambda_i \in \mathfrak{a}_P^* \otimes_{\mathbb{R}} \mathbb{C}$, Q_i are nonzero polynomials on $\mathfrak{a}_P$ and ϕ_i are nonzero automorphic forms satisfying $\phi_i(e^X g) = \phi_i(g)$ for all $X \in \mathfrak{a}_P$ and $g \in \mathbf{G}$. We denote the set of distinct exponents λ_i by $\mathscr{E}_P(\phi)$.

For a quadratic extension E of F, Jacquet *et al.* [15, 19] have defined a mixed truncation operator Λ_m^T, which carries automorphic forms on $G(E)\backslash G(\mathbb{A}_E)$ to functions on $G(F)\backslash G(\mathbb{A})$ of rapid decay, for sufficiently regular $T \in \mathfrak{a}_0$. The author [27] regularized a twisted analogue of the trilinear period, using this mixed truncation. By taking $E = F \oplus F$, we obtain a mixed truncation that carries automorphic forms on $G(F) \times G(F)\backslash G(\mathbb{A}) \times G(\mathbb{A})$ to rapidly decreasing functions on $G(F)\backslash G(\mathbb{A})^1$ as a special case. We can easily adapt this construction to the covering groups $\mathbf{G} \times \mathbf{G}$.

For $\varphi, \varphi' \in \mathscr{A}(\mathbf{G})$, we define a mixed truncation by

$$\Lambda_m^T(\varphi \otimes \varphi')(g) = \sum_P (-1)^{\dim \mathfrak{a}_P^G} \sum_{\gamma \in P(F)\backslash G(F)} \varphi_P(\gamma g)\varphi'_P(\gamma g)\hat{\tau}_P(H(\gamma g) - T)$$

for $g \in \mathbf{G}$. More generally, we define a partial mixed truncation by

$$\Lambda_m^{T,P}(\phi \otimes \phi')(g) = \sum_{Q \subset P} (-1)^{\dim \mathfrak{a}_Q^P} \sum_{\delta \in Q(F)\backslash P(F)} \phi_Q(\delta g)\phi'_Q(\delta g)\hat{\tau}_Q^P(H(\delta g) - T)$$

for functions ϕ, ϕ' on $U(\mathbb{A})P(F)\backslash\mathbf{G}$.

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Lemma 3.1. *Let $\varphi, \varphi' \in \mathscr{A}(\mathbf{G})$. Then $\Lambda_m^T(\varphi \otimes \varphi')$ is rapidly decreasing on $G(F)\backslash \mathbf{G}^1$.*

Proof. The proof follows almost verbatim that of [19, Lemma 8.2.1(1)]. □

Fix a character $\chi = (\chi_1, \chi_2, \chi_3)$ of $\mathbf{N}^3$ that satisfies $\prod_{i=1}^3 \chi_i(\zeta) = 1$ for $\zeta \in \mathbf{N}$. Let $\mathscr{A}(\mathbf{G}^3)_\chi$ denote the space of triplets $(\varphi_1, \varphi_2, \varphi_3)$ of automorphic forms on $\mathbf{G}$ satisfying

$$\varphi_i(\zeta g) = \chi_i(\zeta)\varphi_i(g) \quad (\zeta \in \mathbf{N}, g \in \mathbf{G}, i = 1, 2, 3).$$

The regularized period of a product of $(\varphi_1, \varphi_2, \varphi_3) \in \mathscr{A}(\mathbf{G}^3)_\chi$ can be defined in terms of the convergent integral

$$\int_{G(F)\backslash G(\mathbb{A})^1} \Lambda_m^T(\varphi_1 \otimes \varphi_2)(g)\varphi_3(g)\, dg, \tag{3.1}$$

where the integrand is defined on $G(\mathbb{A})$ by the assumption on χ. Lemma 3.1 ensures the convergence of this integral. The following result is a generalization of [19, Proposition 8.4.1] and can be proved in the same way (cf. [14, Proposition 3.7] and [27, Proposition 3.2]).

Proposition 3.2. *Integral (3.1) is a function of the form $\sum_\lambda p_\lambda(T)e^{\langle \lambda, T\rangle}$, where p_λ is a polynomial in T and λ can be taken from the set*

$$\bigcup_P \{\lambda_1 + \lambda_2 + \lambda_3 + \rho_P \mid \lambda_i \in \mathscr{E}_P(\varphi_{i,P}) \ (i = 1, 2, 3)\}.$$

Definition 3.3. Let $\mathscr{A}_0(\mathbf{G}^3)_\chi$ be the subspace of triplets $(\varphi_1, \varphi_2, \varphi_3) \in \mathscr{A}(\mathbf{G}^3)_\chi$ such that the polynomial corresponding to the zero exponent of (3.1) is constant. For $(\varphi_1, \varphi_2, \varphi_3) \in \mathscr{A}_0(\mathbf{G}^3)_\chi$, we define the regularized period $I(\varphi_1, \varphi_2, \varphi_3)$ as its value $p_0(T)$. We also write

$$I(\varphi_1, \varphi_2, \varphi_3) = \int^*_{G(F)\backslash G(\mathbb{A})^1} \varphi_1(g)\varphi_2(g)\varphi_3(g)\, dg.$$

A regularization of integrals of exponential polynomial functions over cones is discussed and denoted by the symbol $\int^\#$ in [15, § 2]. Let $\mathscr{A}(\mathbf{G}^3)^*_\chi$ be the space of all triplets $(\varphi, \varphi', \varphi'') \in \mathscr{A}(\mathbf{G}^3)_\chi$ that satisfy

$$\langle \lambda + \lambda' + \lambda'' + \rho_P, \varpi^\vee \rangle \neq 0 \quad (\varpi^\vee \in \hat{\Delta}_P^\vee, \lambda \in \mathscr{E}_P(\varphi_P), \lambda' \in \mathscr{E}_P(\varphi'_P), \lambda'' \in \mathscr{E}_P(\varphi''_P))$$

for all proper parabolic subgroups P of G. If $(\varphi, \varphi', \varphi'') \in \mathscr{A}(\mathbf{G}^3)^*_\chi$, then the #-integral

$$I_P^T(\varphi, \varphi', \varphi'') = \int^\#_{P(F)\backslash G(\mathbb{A})^1} \Lambda_m^{T,P}(\varphi \otimes \varphi')(g)\varphi''(g)\tau_P(H(g) - T)\, dg$$

is defined as the triple integral

$$\int_K \int_{M(F)\backslash M(\mathbb{A})^1} \int^\#_{\mathfrak{a}_P^G} \Lambda_m^{T,P}(\varphi \otimes \varphi')(e^X mk)\varphi''_P(e^X mk)e^{-2\langle \rho_P, X\rangle}\tau_P(X - T)\, dX\, dm\, dk.$$

Proposition 3.4 (cf. [27, Proposition 3.4]).

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(1) $\mathscr{A}(\mathbf{G}^3)^*_\chi \subset \mathscr{A}_0(\mathbf{G}^3)_\chi$.

(2) *If* $(\varphi, \varphi', \varphi'') \in \mathscr{A}(\mathbf{G}^3)^*_\chi$, *then*

$$I(\varphi, \varphi', \varphi'') = \sum_P I^T_P(\varphi, \varphi', \varphi'').$$

In particular, the right-hand side is independent of T.

(3) *The regularized period defines a* $G(\mathbb{A})^1$*-invariant linear functional on* $\mathscr{A}(\mathbf{G}^3)^*_\chi$.

(4) *Let* $(\varphi, \varphi', \varphi'') \in \mathscr{A}(\mathbf{G}^3)_\chi$. *If the function* $g \mapsto \varphi(g)\varphi'(g)\varphi''(g)$ *is integrable over* $G(F)\backslash G(\mathbb{A})^1$, *then* $(\varphi, \varphi', \varphi'') \in \mathscr{A}_0(\mathbf{G}^3)_\chi$ *and*

$$I(\varphi, \varphi', \varphi'') = \int_{G(F)\backslash G(\mathbb{A})^1} \varphi(g)\varphi'(g)\varphi''(g)\, dg.$$

Proof. The proof of the first two assertions follows almost verbatim those of [15, Theorem 9(ii)] and [19, Proposition 8.4.1]. The third assertion can be proved in the same way as in the proof of [15, Theorem 9(i)]. The last assertion can be proved by the same technique as in the proof of [14, Proposition 3.8, Corollary 3.10]. We omit the details. □

Let $\mathscr{A}(\mathbf{G}^3)^{**}_\chi$ be the space of all triplets $(\varphi, \varphi', \varphi'') \in \mathscr{A}(\mathbf{G}^3)_\chi$ that satisfy

$$\langle \lambda + \lambda' + \lambda'' + \rho_P, \varpi^\vee \rangle \neq 0 \quad (\varpi^\vee \in (\hat{\Delta}^\vee)^P_Q, \lambda \in \mathscr{E}_Q(\varphi_Q), \lambda' \in \mathscr{E}_Q(\varphi'_Q), \lambda'' \in \mathscr{E}_Q(\varphi''_Q))$$

for all pairs of parabolic subgroups $Q \subset P$ of G. Clearly, $\mathscr{A}(\mathbf{G}^3)^{**}_\chi$ is a subspace of $\mathscr{A}(\mathbf{G}^3)^*_\chi$. If $(\varphi, \varphi', \varphi'') \in \mathscr{A}(\mathbf{G}^3)^{**}_\chi$, then the regularized integral

$$\begin{aligned} &\int^*_{P(F)\backslash G(\mathbb{A})^1} \varphi_P(g)\varphi'_P(g)\varphi''_P(g)\hat{\tau}_P(H(g) - T)\, dg \\ &= \int_K \int^*_{M(F)\backslash M(\mathbb{A})^1} \int^\#_{\mathfrak{a}^G_P} \varphi_P(e^X mk)\varphi'_P(e^X mk)\varphi''_P(e^X mk)\hat{\tau}_P(X - T)e^{-2\langle \rho_P, X\rangle}\, dX\, dm\, dk \end{aligned}$$

is well defined for every P. We use the following general formula for periods of truncated automorphic forms in terms of regularized periods of their constant terms.

Proposition 3.5. *If* $(\varphi, \varphi', \varphi'') \in \mathscr{A}(\mathbf{G}^3)^{**}_\chi$, *then*

$$\begin{aligned} &\int_{G(F)\backslash G(\mathbb{A})^1} \Lambda^T_m(\varphi \otimes \varphi')(g)\varphi''(g)\, dg \\ &= \sum_P (-1)^{\dim \mathfrak{a}^G_P} \int^*_{P(F)\backslash G(\mathbb{A})^1} \varphi_P(g)\varphi'_P(g)\varphi''_P(g)\hat{\tau}_P(H(g) - T)\, dg. \end{aligned}$$

Proof. The proof is the same as that of [27, Proposition 3.5], which is a generalization of [15, Theorem 10]. □

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4. Fourier–Jacobi models

In this section, we assume F to be a nonarchimedean local field of characteristic zero with residue field of order q. We fix a nontrivial additive character ψ on F. Denote the group of kth roots of unity in $\mathbb{C}^\times$ by μ_k. Let $(\ ,\ ) : F^\times \times F^\times \to \mu_2$ denote the Hilbert symbol for F. We write ν for the normalized absolute value on F, viewed as a character of any general linear group over F via composition with det.

Let $(W, \langle\ ,\ \rangle)$ be a symplectic space over F of dimension $2n$, $G = Sp(W)$ its symplectic group and $\mathbf{G}$ the metaplectic double cover of G. We fix maximal isotropic subspaces X_n and Y_n of W, in duality, with respect to $\langle\ ,\ \rangle$. Fix a complete flag in X_n:

$$\{0\} = X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_n,$$

and choose a basis $\{e_1, \ldots, e_n\}$ of X_n such that X_a is spanned by $e_1, \ldots, e_a$ for $1 \leqslant a \leqslant n$. Let $\{f_1, \ldots, f_n\}$ be the basis of Y_n, which is dual to the fixed basis of X_n, i.e., $(e_i, f_j) = \delta_{i,j}$ for $1 \leqslant i, j \leqslant n$, where $\delta_{i,j}$ denotes the Kronecker delta. We write Y_a for the subspace of Y_n spanned by $f_1, \ldots, f_a$. Then one has the polar decomposition

$$W = X_a \oplus W_{n-a} \oplus Y_a,$$

where W_{n-a} is the orthogonal complement of $X_a + Y_a$ in W. We denote by P_a the parabolic subgroup of G stabilizing X_a, by U_a its unipotent radical and by M_a the Levi subgroup of P_a stabilizing the decomposition above. Recall that H_{n-a} is the symplectic group of $(W_{n-a}, \langle\ ,\ \rangle)$. We identify $\mathrm{GL}(X_a) \simeq \mathrm{GL}_a$ with the subgroup of M_a consisting of the elements that act as the identity on W_{n-a}. Then $M_a \simeq \mathrm{GL}(X_a) \times H_{n-a}$.

We consider the set $\widetilde{\mathrm{GL}}_a \simeq \mathrm{GL}_a \times \mu_2$ with multiplication

$$(g_1, \zeta_1) \cdot (g_2, \zeta_2) = (g_1 g_2, \zeta_1 \zeta_2 (\det g_1, \det g_2)).$$

We can define a genuine character of $\widetilde{\mathrm{GL}}_1$ by

$$\gamma_\psi((\lambda, \zeta)) = \zeta \gamma(\lambda, \psi)^{-1}$$

with

$$\gamma(\lambda, \psi) = \gamma(\psi_\lambda)/\gamma(\psi)$$

for $\lambda \in F^\times$ and $\zeta \in \mu_2$, where $\psi_\lambda(x) = \psi(\lambda x)$ and $\gamma(\psi) \in \mu_8$ is a Weil constant associated with ψ (see [23]). Using this character, we obtain a bijection, which depends on the choice of the additive character ψ, between the set of equivalence classes of admissible representations of GL_a and that of genuine admissible representations of $\widetilde{\mathrm{GL}}_a$ via $\sigma \mapsto \sigma_\psi$, where

$$\sigma_\psi((g, \zeta)) = \gamma_\psi((\det g, \zeta))\sigma(g).$$

The preimage $\mathbf{P}_a$ of P_a in $\mathbf{G}$ is of the form $\mathbf{P}_a = \mathbf{M}_a \cdot U_a$, where

$$\mathbf{M}_a \simeq \widetilde{\mathrm{GL}}_a \times_{\mu_2} \mathbf{H}_{n-a}.$$

For a smooth representation σ of $\mathrm{GL}(X_a)$ and a genuine smooth representation π' of $\mathbf{H}_{n-a}$, we can form a normalized induced representation $\mathrm{Ind}_{\mathbf{P}_a}^{\mathbf{G}}(\sigma_\psi \boxtimes \pi')$ of $\mathbf{G}$. We use the symbol ind for unnormalized induction and c-ind for compactly supported induction. For

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$1 \leqslant i \leqslant a$, we write $\sigma^{(i)}$ for the ith derivative of σ. Although $\mathbf{G}$ is not a linear group, many basic results concerning the induction and Jacquet functors remain valid. For a justification of this, the reader can consult [13].

For $0 \leqslant a \leqslant n$, we write $\mathcal{W}_a$ (resp. $\mathcal{N}_a$) for the unipotent radical of the parabolic subgroup of G (resp. $\mathrm{GL}(X_a)$) stabilizing the flag $\{0\} \subset X_1 \subset X_2 \subset \cdots \subset X_a$. When $0 < a < n$, let $\mathcal{H}(W_{n-a})$ be the Heisenberg group associated with the symplectic space $(W_{n-a}, 2\langle\ ,\ \rangle)$ and $\Omega_{W_{n-a},\psi}$ the Weil representation of $\mathcal{H}(W_{n-a}) \rtimes \mathbf{H}_{n-a}$ associated with ψ. Having $\mathcal{W}_{a-1}\backslash\mathcal{W}_a \simeq \mathcal{H}(W_{n-a})$, we use the same symbol to denote the pull-back of $\Omega_{W_{n-a},\psi}$ to $\mathcal{W}_a \rtimes \mathbf{H}_{n-a}$. We shall denote the restriction of $\Omega_{W_{n-a},\psi}$ to $\mathbf{H}_{n-a}$ by $\omega_{W_{n-a},\psi}$.

For $0 \leqslant \ell < n-1$, we define a character ψ_ℓ of $\mathcal{W}_{\ell+1}$, which factors through the quotient $\mathcal{W}_{\ell+1} \to U_{\ell+1}\backslash\mathcal{W}_{\ell+1} \simeq \mathcal{N}_{\ell+1}$, by setting

$$\psi_\ell(u) = \psi(\langle ue_2, f_1\rangle + \langle ue_3, f_2\rangle + \cdots + \langle ue_{\ell+1}, f_\ell\rangle), \quad u \in \mathcal{W}_{\ell+1}.$$

For a smooth representation π' of $\mathbf{G}$, we write $J_{\psi_\ell}(\pi' \otimes \Omega_{W_{n-\ell-1},\psi})$ for the Jacquet module of $\pi' \otimes \Omega_{W_{n-\ell-1},\psi}$ with respect to the group $\mathcal{W}_{\ell+1}$ and its character ψ_ℓ, regarded as a representation of the symplectic group $H_{n-\ell-1}$.

Lemma 4.1. *Put $m = n - a$. Let σ, π and $\mathcal{E}$ be smooth representations of finite lengths of $\mathrm{GL}(X_a)$, H_m and $\mathbf{G}$, respectively. Then*

$$\begin{aligned}&\dim_{\mathbb{C}} \mathrm{Hom}_G(\mathrm{ind}_{P_a}^G(\sigma\nu^s \boxtimes \pi) \otimes \omega_{W,\psi} \otimes \mathcal{E}, \mathbb{C}) \\ &\quad\leqslant \dim_{\mathbb{C}} \sigma^{(a)} \cdot \dim_{\mathbb{C}} \mathrm{Hom}_{H_m}(J_{\psi_{a-1}}(\mathcal{E} \otimes \Omega_{W_m,\psi}), \pi^\vee)\end{aligned} \tag{4.1}$$

except for finitely many q^{-s}.

Proof. The proof is analogous to that of [6, Theorem 16.1] (cf. [7, § 11]). Throughout the proof we shall use the phrase 'almost always' to mean for all but finitely many q^{-s}. The Frobenius reciprocity [1, Theorem 2.29] shows that space (4.2) is isomorphic to

$$\mathrm{Hom}_{P_a}((\delta_{P_a}^{-1}\sigma\nu^s \boxtimes \pi) \otimes \omega_{W,\psi} \otimes \mathcal{E}, \mathbb{C}).$$

For $1 \leqslant i \leqslant a$, we put

$$Q_{a,i} = (\mathscr{P}_{a,i} \times H_m) \ltimes U_a,$$

where the group $\mathscr{P}_{a,i}$ consists of the elements in $\mathrm{GL}(X_a)$ that stabilize the flag $X_{a-i} \subset X_{a-i+1} \subset \cdots \subset X_{a-1}$ and fix e_j modulo X_{j-1} for $a-i+1 \leqslant j \leqslant a$. There is an exact sequence

$$0 \to \text{c-ind}_{\mathbf{Q}_{a,1}}^{\mathbf{P}_a} \gamma_\psi \nu^{1/2} \boxtimes \Omega_{W_m,\psi} \to \omega_{W,\psi}|_{\mathbf{P}_a} \to \gamma_\psi \nu^{1/2} \boxtimes \omega_{W_m,\psi} \to 0 \tag{4.2}$$

(see the proof of [6, Theorem 16.1]), where the action of $\mathbf{Q}_{a,1}$ on $\Omega_{W_m,\psi}$ is via the Weil representation of $U_a \rtimes \mathbf{H}_m$, and U_a acts on $\gamma_\psi\nu^{1/2} \boxtimes \omega_{W_m,\psi}$ trivially. We see that the space

$$\begin{aligned}&\mathrm{Hom}_{P_a}((\delta_{P_a}^{-1}\sigma\nu^s \boxtimes \pi) \otimes (\gamma_\psi\nu^{1/2} \boxtimes \omega_{W_m,\psi}) \otimes \mathcal{E}, \mathbb{C}) \\ &\quad\simeq \mathrm{Hom}_{M_a}((\delta_{P_a}^{-1}\sigma_\psi\nu^{s+1/2} \boxtimes (\pi \otimes \omega_{W_m,\psi})) \otimes J_{P_a}(\mathcal{E}), \mathbb{C})\end{aligned}$$

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is almost always zero by comparing central characters, where J_{P_a} denotes the Jacquet functor with respect to U_a and its trivial character. By tensoring (4.2) with $(\delta_{P_a}^{-1}\sigma\nu^s\boxtimes\pi)\otimes\mathcal{E}$ and then applying the functor $\mathrm{Hom}_{P_a}(-,\mathbb{C})$, we conclude that space (4.2) is almost always a subspace of

$$\begin{aligned}&\mathrm{Hom}_{P_a}((\delta_{P_a}^{-1}\sigma\nu^s\boxtimes\pi)\otimes\text{c-ind}_{\mathbf{Q}_{a,1}}^{\mathbf{P}_a}\gamma_\psi\nu^{1/2}\boxtimes\Omega_{W_m,\psi}\otimes\mathcal{E},\mathbb{C})\\&\simeq\mathrm{Hom}_{Q_{a,1}}((\delta_{P_a}^{-1}\sigma\nu^s\boxtimes\pi)\otimes(\delta_{P_a}\delta_{Q_{a,1}}^{-1}\gamma_\psi\nu^{1/2}\boxtimes\Omega_{W_m,\psi})\otimes\mathcal{E},\mathbb{C})\\&\simeq\mathrm{Hom}_{Q_{a,1}}(\delta_{Q_{a,1}}^{-1}\sigma_\psi\nu^{s+1/2}\boxtimes(\pi\otimes\Omega_{W_m,\psi})\otimes\mathcal{E},\mathbb{C}),\end{aligned}$$

where we have used the Frobenius reciprocity again.

Next, we appeal to a result of Bernstein and Zelevinsky (see [2, § 3.5]) according to which the restriction of σ to $\mathscr{P}_{a,1}$ has a filtration

$$\sigma=\sigma_1\supset\sigma_2\supset\cdots\supset\sigma_a\supset\sigma_{a+1}=\{0\}$$

such that $\sigma_j/\sigma_{j+1}\simeq(\Phi^+)^{j-1}\Psi^+\sigma^{(j)}$ for $j=1,2,\ldots,a$. For the definition of the functors Φ^+ and Ψ^+, the reader should consult [2]. Let $X_{a,i}$ be the subspace of X_a spanned by vectors $e_{a-i+1},e_{a-i+2},\ldots,e_a$, and define the subgroup of $\mathrm{GL}(X_{a,i})$ by $\mathcal{N}_{a,i}=\mathcal{N}_a\cap\mathrm{GL}(X_{a,i})$. Then

$$(\Phi^+)^{j-1}\Psi^+\sigma^{(j)}\simeq\text{c-ind}_{\mathscr{P}_{a,j}}^{\mathscr{P}_{a,1}}|\mathrm{det}|^{j/2}\sigma^{(j)}\boxtimes\psi_{a-1}|_{\mathcal{N}_{a,j}}$$

for $j=1,2,\ldots,a$. We apply the Frobenius reciprocity again to get

$$\begin{aligned}&\mathrm{Hom}_{Q_{a,1}}(\delta_{Q_{a,1}}^{-1}\nu^{s+1/2}(\Phi^+)^{j-1}\Psi^+\sigma_\psi^{(j)}\boxtimes(\pi\otimes\Omega_{W_m,\psi})\otimes\mathcal{E},\mathbb{C})\\&\simeq\mathrm{Hom}_{Q_{a,j}}(\delta_{Q_{a,j}}^{-1}\nu^{s+(j+1)/2}\sigma_\psi^{(j)}\boxtimes\psi_{a-1}|_{\mathcal{N}_{a,j}}\boxtimes(\pi\otimes\Omega_{W_m,\psi})\otimes J_{P_{a-j}}(\mathcal{E}),\mathbb{C}).\end{aligned}$$

We see by comparing central characters that this space is almost always zero unless $j=a$, in which case it is isomorphic to

$$\mathrm{Hom}_{\mathcal{W}_a\rtimes H_m}(\pi\otimes\psi_{a-1}\otimes\Omega_{W_m,\psi}\otimes\mathcal{E},\mathbb{C})^{\oplus t}\simeq\mathrm{Hom}_{H_m}(J_{\psi_{a-1}}(\mathcal{E}\otimes\Omega_{W_m,\psi}),\pi^\vee)^{\oplus t},$$

where we put $t=\dim_{\mathbb{C}}\sigma^{(a)}$. □

5. Certain residual automorphic representations

Back to the global setup, we take a weak Witt decomposition $W=X_a\oplus W_{n-a}\oplus Y_a$ and the stabilizer P_a of the totally isotropic subspace X_a for $1\leqslant a\leqslant n$ as in § 4. Recall that $\mathbf{P}_a$ and $\mathbf{H}_{n-a}$ denote the preimages of $P_a(\mathbb{A})$ and $H_{n-a}(\mathbb{A})$ in $\mathbf{G}$, respectively.

For a cuspidal automorphic representation ρ of $\mathbf{M}$, we write $\mathscr{A}_P^\rho(\mathbf{G})$ for the subspace of functions $\phi\in\mathscr{A}_P(\mathbf{G})$ such that for all $k\in\mathbf{K}$ the function $m\mapsto e^{-\langle\rho_P,H(m)\rangle}\phi(mk)$ belongs to the space of ρ. Put $J_a=(\delta_{i,a+1-j})\in\mathrm{GL}_a(F)$. We define $|a(g)|$ by writing $g=muk$ with $m=(\mathrm{diag}[b,h,J_a\,{}^t b^{-1}J_a],\zeta)\in\mathbf{M}_a$, $b\in\mathrm{GL}_a(\mathbb{A})$, $h\in H_{n-a}(\mathbb{A})$, $\zeta\in\{\pm1\}$, $u\in U_a(\mathbb{A})$ and $k\in\mathbf{K}$, and taking $|a(g)|=|\det b|$. When $P=P_a$ and $\phi\in\mathscr{A}_{P_a}^\rho(\mathbf{G})$, we form an Eisenstein series by

$$E(g,\phi,z)=\sum_{\gamma\in P_a(F)\backslash G(F)}\phi(\gamma g)|a(\gamma g)|^z.$$

The series converges absolutely for $\Re z>\frac{2n-a+1}{2}$ and admits a meromorphic continuation to the whole plane.

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Proposition 5.1 [9, Proposition 3.2]. *Let π be an irreducible globally generic cuspidal automorphic representation of $H_{n-a}(\mathbb{A})$ and σ an irreducible cuspidal automorphic representation of* $\mathrm{GL}_a(\mathbb{A})$. *For $\phi \in \mathscr{A}_{P_a}^{\sigma\boxtimes\pi}(G)$, the Eisenstein series $E(\phi, z)$ has at most a simple pole at $z = \frac{1}{2}$. Moreover, it has a pole at $z = \frac{1}{2}$ as ϕ varies if and only if $L(1/2, \sigma \times \mathrm{BC}(\pi)) \neq 0$ and $L(s, \sigma, \Lambda^2)$ has a pole at $s = 1$.*

For $\phi \in \mathscr{A}_{P_a}^{\sigma\boxtimes\pi}(G)$, we define the residue of the Eisenstein series to be the limit $\mathcal{E}(\phi) = \lim_{z\to 1/2}\left(z - \frac{1}{2}\right)E(\phi, z)$. Let $\mathcal{E}(\sigma, \pi)$ be the residual automorphic representation of $G(\mathbb{A})$ generated by these residues.

Put $m = n - a$. Let π' be an irreducible genuine cuspidal automorphic representation of $\mathbf{H}_m$. Suppose that π' admits a weak ψ-functorial lift $\mathrm{BC}_\psi(\pi')$ to $\mathrm{GL}_{2m}(\mathbb{A})$ of the form $\sigma_1 \times \cdots \times \sigma_k$, i.e., an automorphic representation $\mathrm{Ind}_{P(\mathbb{A})}^{\mathrm{GL}_{2m}(\mathbb{A})}(\sigma_1 \boxtimes \cdots \boxtimes \sigma_k)$, where P is a parabolic subgroup of GL_{2m} of type $(a_1, \ldots, a_k)$ and σ_i is an irreducible cuspidal automorphic representation of $\mathrm{GL}_{a_i}(\mathbb{A})$ such that $L^S(s, \sigma_i, \Lambda^2)$ has a pole at $s = 1$.

Remark 5.2. The condition that $L^S(s, \sigma_i, \Lambda^2)$ has a pole at $s = 1$ implies that a_i is even and that σ_i is self-dual and has trivial central character (see [17]). Moreover, σ_i is a Langlands functorial lift from an irreducible globally generic cuspidal automorphic representation of $\mathrm{SO}_{a_i+1}(\mathbb{A})$ (see [5, 12]).

Let σ be an irreducible cuspidal automorphic representation of $\mathrm{GL}_a(\mathbb{A})$. We set $\sigma_\psi((g, \zeta)) = \zeta\sigma(g)\prod_v \gamma(\det g_v, \psi_v)^{-1}$ for $g \in \mathrm{GL}_a(\mathbb{A})$ and $\zeta \in \mu_2$. For every idèle $a \in \mathbb{A}^\times$ the product $\prod_v \gamma(a_v, \psi_v)$ is well defined in view of [23, Proposition A.11]. Then $\rho = \sigma_{\psi^{-1}} \boxtimes \pi'$ can be viewed as a cuspidal automorphic representation of $\mathbf{M}_a$ and one can form the Eisenstein series $E(\phi', z)$ for $\phi' \in \mathscr{A}_{P_a}^{\rho}(\mathbf{G})$.

Proposition 5.3. *With the same notation as above, let σ be an irreducible cuspidal automorphic representation of* $\mathrm{GL}_a(\mathbb{A})$ *isomorphic to one of the isobaric summands of* $\mathrm{BC}_\psi(\pi')$. *Put $\rho = \sigma_{\psi^{-1}} \boxtimes \pi'$. Then there exists $\phi' \in \mathscr{A}_{P_a}^{\rho}(\mathbf{G})$ such that $E(\phi', z)$ has a pole at $z = 1$.*

Proof. By the Langlands theory of Eisenstein series, the analytic properties of the family $E(\phi', z)$ are controlled by those of the family

$$E_{P_a}(g, \phi', z) = \phi'(g)|a(g)|^z + [M(z)\phi'](g)|a(g)|^{-z},$$

where $M(z)$ is the relevant intertwining operator. That $E(\phi', z)$ has a pole at $z = 1$ is equivalent to saying that $M(z)\phi'$ has a pole at $z = 1$. If $\phi' = \otimes_v \phi'_v$ is factorizable, then $M(z)\phi'$ can be expressed as an infinite product

$$[M(z)\phi'](g) = \prod_v [M_v(z)\phi'_v](g_v),$$

where $M_v(z)$ is the local intertwining operator. Let S be a finite set of places of F containing all the archimedean places such that if $v \notin S$, then ψ_v, σ_v, π'_v and ϕ'_v are unramified and $g_v \in K_v$. We know

$$[M_v(z)\phi'_v](g_v) = \frac{L(z, \sigma_v \times \pi'_v)L(2z, \sigma_v, \mathrm{sym}^2)}{L(z+1, \sigma_v \times \pi'_v)L(2z+1, \sigma_v, \mathrm{sym}^2)}$$

for $v \notin S$ (cf. [25]). It follows that

$$[M(z)\phi'](g) = \frac{L^S(z, \sigma \times \pi')L^S(2z, \sigma, \mathrm{sym}^2)}{L^S(z+1, \sigma \times \pi')L^S(2z+1, \sigma, \mathrm{sym}^2)} \prod_{v \in S}[M_v(z)\phi'_v](g_v).$$

One can always choose the local sections ϕ'_v so that $M_v(z)\phi'_v$ are nonzero at $z = 1$. The L-function

$$L^S(s, \sigma \times \pi') := L^S(s, \sigma \times \mathrm{BC}_\psi(\pi')) = \prod_{i=1}^{k} L^S(s, \sigma \times \sigma_i)$$

has a pole at $s = 1$ if and only if there is i such that $\sigma \simeq \sigma_i^\vee (\simeq \sigma_i)$. The automorphic L-function $L^S(s, \sigma \times \pi')$ converges absolutely and does not vanish for $\Re s > 1$ by [16, Theorem 5.3]. The infinite product $L^S(s, \sigma, \mathrm{sym}^2)$ converges absolutely and does not vanish for $\Re s \geqslant 2$ by the bounds of Luo *et al.* [21] towards the generalized Ramanujan conjecture for general linear groups. □

When $E(\phi', z)$ has a pole of order $l \geqslant 1$ at $z = 1$ as $\phi' \in \mathscr{A}^\rho_{P_a}(\mathbf{G})$ varies, we write

$$\mathcal{E}(\phi') = \lim_{z \to 1}(z-1)^l E(\phi', z), \quad M(\phi') = \lim_{z \to 1}(z-1)^l M(z)\phi'.$$

We write $\mathcal{E}(\sigma, \pi')$ for the automorphic representation of $\mathbf{G}$ generated by the residues $\mathcal{E}(\phi')$.

6. Periods of residues

Lemma 6.1. *Put $m = n - a$. Let σ be an irreducible cuspidal automorphic representation of $\mathrm{GL}_a(\mathbb{A})$, π an irreducible globally generic cuspidal automorphic representation of $H_m(\mathbb{A})$ and π' an irreducible genuine cuspidal automorphic representation of $\mathbf{H}_m$. Suppose that π' admits a weak ψ-functorial lift to an automorphic representation of $\mathrm{GL}_{2m}(\mathbb{A})$ of the form $\sigma_1 \times \cdots \times \sigma_k$, where σ_i are irreducible cuspidal automorphic representations of general linear groups such that $L^S(s, \sigma_i, \Lambda^2)$ has a pole at $s = 1$. If $\sigma \simeq \sigma_i$ for some i, then $I(\varphi', \Theta^\psi(f), E(\phi, z))$ is identically zero for all $\phi \in \mathscr{A}^{\sigma \boxtimes \pi}_{P_a}(G)$, $\varphi' \in \mathcal{E}(\sigma, \pi')$ and $f \in \omega_{W,\psi}$.*

Proof. Since the cuspidal support of the residues in $\mathcal{E}(\sigma, \pi')$ consists only of $\sigma_{\psi^{-1}}\nu^{-1} \boxtimes \pi'$ on $\mathbf{P}_a$, the residues are square-integrable by [22, Lemma I.4.11]. Thus $\mathcal{E}(\sigma, \pi')$ is a unitary quotient of $\mathrm{Ind}^{\mathbf{G}}_{\mathbf{P}_a}(\sigma_{\psi^{-1}}\nu \boxtimes \pi')$. Since the Langlands quotient of $\mathrm{Ind}^{\mathbf{G}}_{\mathbf{P}_a}(\sigma_{\psi^{-1}}\nu \boxtimes \pi')$ is its unique semisimple quotient, it is isomorphic to $\mathcal{E}(\sigma, \pi')$.

The proof is based on the structure of its local unramified components. Fix a finite place v such that the local v-components of ψ, σ, π, π' and $\mathcal{E}(\sigma, \pi')$ are unramified. We drop v and the field F_v from the notation. From what we have seen, $\mathcal{E}(\sigma, \pi')$ is the unique irreducible unramified quotient of $\mathrm{Ind}^{\mathbf{G}}_{\mathbf{P}_a}(\sigma_{\psi^{-1}}\nu \boxtimes \pi')$.

Recall $m = n - a$. Let B'_a and B_m denote the standard Borel subgroups of GL_a and H_m, respectively. On account of Remark 5.2, a is even, σ is isomorphic to the irreducible unramified constituent of

$$\mathrm{Ind}^{\mathrm{GL}_a}_{B'_a}(\chi_1 \boxtimes \cdots \boxtimes \chi_b \boxtimes \chi_b^{-1} \boxtimes \cdots \boxtimes \chi_1^{-1})$$

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and π' is isomorphic to the irreducible unramified constituent of

$$\mathrm{Ind}_{\mathbf{B}_m}^{\mathbf{H}_m}(\chi_{1,\psi^{-1}}\boxtimes\cdots\boxtimes\chi_{b,\psi^{-1}}\boxtimes\chi_{b+1,\psi^{-1}}\boxtimes\cdots\boxtimes\chi_{m,\psi^{-1}})$$

for some characters χ_i of $F^\times$, where $a=2b$. It is not difficult to see that $\mathcal{E}(\sigma,\pi')$ is the irreducible unramified constituent of

$$\mathrm{Ind}_{\mathbf{Q}}^{\mathbf{G}}(\chi_{1,\psi^{-1}}\circ\det_{\mathrm{GL}_3}\boxtimes\cdots\boxtimes\chi_{b,\psi^{-1}}\circ\det_{\mathrm{GL}_3}\boxtimes\chi_{b+1,\psi^{-1}}\boxtimes\cdots\boxtimes\chi_{m,\psi^{-1}})$$

(cf. [13, Lemma 3.2]). In view of Proposition 3.4(3) and Lemma 4.1, it suffices to show that if χ is a character of $F^\times$ and τ is a genuine irreducible smooth representation of $\mathbf{H}_{n-3}$, then

$$\mathrm{Hom}_{H_m}(J_{\psi_{a-1}}(\varrho\otimes\Omega_{W_m,\psi}),\pi^\vee)=0$$

for any irreducible subquotient ϱ of $\mathrm{Ind}_{\mathbf{P}_3}^{\mathbf{G}}(\chi_{\psi^{-1}}\circ\det_{\mathrm{GL}_3}\boxtimes\tau)$. We use [12, Theorem 6.1] with $j=3$ and $\ell=a-1$. Since the Bernstein–Zelevinsky derivative of $\chi\circ\det_{\mathrm{GL}_3}$ is given by

$$(\chi\circ\det_{\mathrm{GL}_3})^{(t)}=\begin{cases}\chi\circ\det_{\mathrm{GL}_{3-t}} & \text{if } t\leqslant 1,\\ 0 & \text{if } t\geqslant 2,\end{cases}$$

we have $t=0,1$ in the first sum of [12, (6.9)]. Since $(\chi\circ\det_{\mathrm{GL}_3})_{(t)}=0$ for $t\geqslant 1$, the second term does not contribute. Proposition 6.6 of [12] allows us to write $J_{\psi_{a-1}}(\mathrm{Ind}_{\mathbf{P}_3}^{\mathbf{G}}(\chi_{\psi^{-1}}\circ\det_{\mathrm{GL}_3}\boxtimes\tau)\otimes\Omega_{W_m,\psi})$ as the sum

$$\oplus_{t=0,1}\mathrm{Ind}_{Q_{3-t}}^{H_m}(\chi\nu^{(1-t)/2})\circ\det_{\mathrm{GL}_{3-t}}\boxtimes J_{\psi_{a-1-t}}(\tau\otimes\Omega_{W_{m+t-3},\psi})) \tag{6.1}$$

up to semisimplification, where Q_{3-t} denotes the standard parabolic subgroup of H_m preserving the standard totally isotropic subspace of W_m of dimension $3-t$. Since the functor $J_{\psi_{a-1}}$ is exact, we conclude that irreducible subquotients of $J_{\psi_{a-1}}(\varrho\otimes\Omega_{W_m,\psi})$ are those of (6.1). Theorem 2.9 of [2] and Corollary 10.1 of [5] imply that (6.1) and $\pi^\vee$ have no common composition factor, which completes our proof. □

The group $\mathbf{G}$ acts on the Schwartz space $\mathcal{S}(Y_n(\mathbb{A}))$ on $Y_n(\mathbb{A})$ via the Schrödinger model of the global Weil representation $\omega_{W,\psi}$. The action of $\mathbf{P}_n$ on $\mathcal{S}(Y_n(\mathbb{A}))$ is described as follows:

$$(\omega_{W,\psi}((m,\zeta))f)(y)=\gamma_\psi((\det m,\zeta))|\det m|^{1/2}f(m^{-1}y), \tag{6.2}$$

$$(\omega_{W,\psi}(u)f)(y)=\psi(\langle uy,y\rangle/2)f(y) \tag{6.3}$$

for $m\in\mathrm{GL}(X_n,\mathbb{A})$, $u\in U_n(\mathbb{A})$ and $y\in Y_n(\mathbb{A})$. Recall that $\Theta(f)=\sum_{y\in Y_n(F)}f(y)$. We write Y_a' for the subspace of Y_n spanned by $f_{a+1},f_{a+2},\dots,f_n$.

Lemma 6.2. *For all $f\in\mathcal{S}(Y_n(\mathbb{A}))$ and $1\leqslant a\leqslant n$, the constant term of $\Theta^\psi(f)$ along P_a is given by*

$$\Theta_{P_a}^\psi(g,f)=\sum_{y\in Y_a'(F)}(\omega_{W,\psi}(g)f)(y).$$

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Proof. The centre Z_a of U_a is the subgroup leaving the subspace W_{n-a} pointwise fixed. Note that $U_a \simeq \mathrm{Hom}(W_{n-a}, X_a) \ltimes Z_a$. For given $y \in Y_n(F)$, the character on $Z_a(\mathbb{A})$ defined by $u \mapsto \psi(\langle uy, y\rangle/2)$ is trivial if and only if $y \in Y'_a(F)$. The stated identity follows from (6.3). □

Observe that

$$\Theta^{\psi}_{P_a}((m,\zeta)g, f) = \gamma_{\psi}((\det m, \zeta))|\det m|^{1/2}\Theta^{\psi}_{P_a}(g, f) \tag{6.4}$$

by (6.2) for $m \in \mathrm{GL}(X_a, \mathbb{A})$ and $g \in \mathbf{G}$.

Proposition 6.3. *With the same notation as in Lemma 6.1, we assume that $\sigma \simeq \sigma_i$ for some i. If $\phi \in \mathscr{A}^{\sigma\boxtimes\pi}_{P_a}(G)$, $\varphi' \in \mathcal{E}(\sigma, \pi')$ and $f \in \mathcal{S}(Y_n(\mathbb{A}))$, then*

$$I(\varphi', \Theta^{\psi}(f), \mathcal{E}(\phi)) = \int_K \int_{M_a(F)\backslash M_a(\mathbb{A})^1} \phi(mk)\varphi'_{P_a}(mk)\Theta^{\psi}_{P_a}(mk, f)\, dm\, dk.$$

Remark 6.4. It is not difficult to show that the integral

$$\int_{G(F)\backslash G(\mathbb{A})} \mathcal{E}(g,\phi)\varphi'(g)\Theta^{\psi}(g, f)\, dg$$

is absolutely convergent, and so by Proposition 3.4(4), it is equal to $I(\varphi', \Theta^{\psi}(f), \mathcal{E}(\phi))$.

Proof. Note that

$$\int_{G(F)\backslash G(\mathbb{A})} \mathcal{E}(g,\phi)\Lambda^T_m(\varphi'\otimes\Theta^{\psi}(f))(g)\, dg$$
$$= \lim_{z\to 1/2}\left(z-\frac{1}{2}\right)\int_{G(F)\backslash G(\mathbb{A})} E(g,\phi,z)\Lambda^T_m(\varphi'\otimes\Theta^{\psi}(f))(g)\, dg.$$

The zero coefficient of the left-hand side is equal to the regularized period $I(\varphi', \Theta^{\psi}(f), \mathcal{E}(\phi))$ by Definition 3.3. We compute the zero coefficient of the right-hand side. Proposition 3.5 and Lemma 6.1 give

$$\int_{G(F)\backslash G(\mathbb{A})} E(g,\phi,z)\Lambda^T_m(\varphi'\otimes\Theta^{\psi}(f))(g)\, dg$$
$$= -\int^{*}_{P_a(F)\backslash G(\mathbb{A})} E_{P_a}(g,\phi,z)\varphi'_{P_a}(g)\Theta^{\psi}_{P_a}(g, f)\hat{\tau}_{P_a}(H(g)-T)\, dg$$
$$= -\int^{*}_{P_a(F)\backslash G(\mathbb{A})} (\phi(g)|a(g)|^{z} + [M(z)\phi](g)|a(g)|^{-z})\varphi'_{P_a}(g)\Theta^{\psi}_{P_a}(g, f)\hat{\tau}_{P_a}(H(g)-T)\, dg.$$

By (6.4) the integral of ϕ is given by

$$-\int_K \int_{M_a(F)\backslash M_a(\mathbb{A})^1} \int_T^{\infty} e^{(z-1/2)X}\phi(mk)\varphi'_{P_a}(mk)\Theta^{\psi}_{P_a}(mk, f)\, dX\, dm\, dk$$
$$= \frac{e^{(z-1/2)T}}{z-1/2}\int_K \int_{M_a(F)\backslash M_a(\mathbb{A})^1} \phi(mk)\varphi'_{P_a}(mk)\Theta^{\psi}_{P_a}(mk, f)\, dm\, dk.$$

The integral of $M(z)\phi$ does not contribute to the zero coefficient at $z=\frac{1}{2}$. □

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Lemma 6.5. *With the same notation as in Lemma 6.1, we assume that $\sigma \simeq \sigma_i$ for some i. If there are $\xi \in \pi$, $\xi' \in \pi'$ and $\xi'' \in \omega_{W_m,\psi}$ such that*

$$\int_{H_m(F)\backslash H_m(\mathbb{A})} \xi(h)\xi'(h)\Theta^{\psi}(h, \xi'')\, dh \neq 0,$$

then there are $\phi \in \mathscr{A}_{P_a}^{\sigma\boxtimes\pi}(G)$, $\varphi' \in \mathcal{E}(\sigma, \pi')$ and $f \in \omega_{W,\psi}$ such that

$$\int_K \int_{M_a(F)\backslash M_a(\mathbb{A})^1} \phi(mk)\varphi'_{P_a}(mk)\Theta^{\psi}_{P_a}(mk, f)\, dm\, dk \neq 0.$$

Proof. Since $L^S(s, \sigma_i, \Lambda^2)$ has a pole at $s = 1$ by assumption, it follows from Remark 5.2 that $\sigma \simeq \sigma^\vee$. Put

$$\Pi = \sigma\nu^{1/2} \boxtimes \pi, \quad \Pi' = \sigma_{\psi^{-1}}\nu^{-1} \boxtimes \pi', \quad \Pi'' = \nu_{\psi}^{1/2} \boxtimes \omega_{W_m,\psi}.$$

We define a functional on $\Pi \boxtimes \Pi' \boxtimes \Pi''$ by

$$\ell(\eta \boxtimes \eta' \boxtimes \eta'') = \int_{M_a(F)\backslash M_a(\mathbb{A})^1} \eta(m)\eta'(m)\eta''(m)\, dm.$$

Since cusp forms are bounded, this functional can uniquely be extended to a continuous functional $\ell : (\Pi \boxtimes \Pi' \boxtimes \Pi'')^\infty \to \mathbb{C}$, where $(\Pi \boxtimes \Pi' \boxtimes \Pi'')^\infty$ is the canonical Casselman–Wallach globalization of $\Pi \boxtimes \Pi' \boxtimes \Pi''$ (cf. [4], [26, Chapter 11]), realized in the space of smooth automorphic forms without the $K_{M_a} \times \mathbf{K}_{M_a} \times \mathbf{K}_{M_a}$-finiteness condition, where $K_{M_a} = K \cap M_a(\mathbb{A})$. Choose $\eta \in \Pi$, $\eta' \in \Pi'$ and $\eta'' \in \Pi''$ so that $\ell(\eta \boxtimes \eta' \boxtimes \eta'') \neq 0$. We can assume that η, η' and η'' are pure tensors. The functional ℓ is a product of local functionals $\ell_v \in \mathrm{Hom}_{M_{a,v}}((\Pi_v \boxtimes \Pi'_v \boxtimes \Pi''_v)^\infty, \mathbb{C})$ by [20, 24], where we have set $(\Pi_v \boxtimes \Pi'_v \boxtimes \Pi''_v)^\infty = \Pi_v \boxtimes \Pi'_v \boxtimes \Pi''_v$ if v is finite. We have $\ell_v(\eta_v \boxtimes \eta'_v \boxtimes \eta''_v) \neq 0$.

Take pure tensors $\delta_{P_a}^{-1/2} \cdot \varphi'_{P_a} = \boxtimes_v \varphi'_v$ and $\Theta^{\psi}_{P_a}(f) = \boxtimes_v f_v$ such that $\varphi'_v(e) = \eta'_v$ and $f_v(e) = \eta''_v$, where e denotes the identity element of $\mathbf{G}$. It is enough to choose a smooth function ϕ_v on G_v, with values in Π_v^∞, that satisfies

$$\phi_v(mug) = e^{\langle \rho_{P_a}, H(m)\rangle}\Pi_v^\infty(m)\phi_v(g), \quad m \in M_{a,v}, u \in U_{a,v}, g \in G_v$$

and such that the local integral

$$I(\phi_v) = \int_{K_v} \ell_v(\phi_v(k) \boxtimes \varphi'_v(k) \boxtimes f_v(k))\, dk$$

is not zero. Indeed, ϕ_v is a limit of K_v-finite sections and $I(\phi_v)$ is a limit of corresponding integrals.

We write U_a^- for the unipotent radical of the parabolic subgroup opposite to P_a. For a smooth function β_v of compact support on $U_{a,v}^-$ we can define a section ϕ_v by requiring

$$\phi_v(muu_-) = e^{\langle \rho_{P_a}, H(m)\rangle}\beta_v(u_-)\Pi_v^\infty(m)\eta_v, \quad m \in M_{a,v}, \quad u \in U_{a,v}, u_- \in U_{a,v}^-.$$

Since $P_{a,v} \cdot U_{a,v}^-$ is an open dense subset of G_v, we can rewrite the local integral as

$$I(\phi_v) = \int_{U_{a,v}^-} \beta_v(u_-)\ell_v(\eta_v \boxtimes \varphi'_v(u_-) \boxtimes f_v(u_-))\, du_-.$$

We have $I(\phi_v) \neq 0$ by choosing β_v to be supported in a small neighbourhood. $\square$

We are now ready to prove Theorem 1.1. For each $1 \leqslant i \leqslant k$, Proposition 6.3 and Lemma 6.5 applied to $\sigma = \sigma_i$, $a = a_i$ and $n = m + a_i$ imply that $I(\varphi', \Theta^{\psi}(f), \varphi) \neq 0$ for some $\varphi \in \mathcal{E}(\sigma_i, \pi)$, $\varphi' \in \mathcal{E}(\sigma_i, \pi')$ and $f \in \omega_{W,\psi}$. Then $\mathcal{E}(\sigma_i, \pi)$ cannot be zero. Therefore $L(1/2, \sigma_i \times \mathrm{BC}(\pi)) \neq 0$ by Proposition 5.1. We finally arrive at

$$L_{\psi}\left(\frac{1}{2}, \pi \times \pi'\right) = \prod_{i=1}^{k} L\left(\frac{1}{2}, \sigma_i \times \mathrm{BC}(\pi)\right) \neq 0.$$

Remark 6.6. From [6, Conjecture 25.1], any irreducible genuine tempered cuspidal automorphic representation π' of $\mathbf{H}$ is expected to satisfy the assumption of Theorem 1.1. The assumption that π is globally generic arises as follows:

(1) the weak functorial lift $\mathrm{BC}(\pi)$ exists;

(2) the normalized local intertwining operator is holomorphic and nonzero for $\Re z \geqslant \frac{1}{2}$;

(3) a weak bound towards the Ramanujan conjecture is valid.

Note that (2) (resp. (3)) plays an essential role in the proof of Proposition 3.2 of [9] (resp. Lemma 6.1). If π and π' lift to irreducible tempered automorphic representations of the general linear groups, then these hold.

Therefore, if we assumed Arthur's conjecture for irreducible tempered cuspidal automorphic representations of symplectic groups and its metaplectic analogue [6, Conjecture 25.1], then we would be able to prove Theorem 1.1 for all irreducible tempered cuspidal automorphic representations π and π'.

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