ON THE "EDGE OF THE WEDGE" THEOREM

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Introduction. In the mathematical justification of the formal calculations of axiomatic quantum field theory and the theory of dispersion relations, a strategic role is played by a theorem on analytic functions of several complex variables which has been given the euphonious name of *the edge of the wedge theorem*. The statement of the theorem seems to be due originally to N. Bogoliubov (cf. **3**, Mathematical Appendix, pp. 654–673) but no complete proof which is fully satisfactory from the mathematical point of view has yet appeared in the literature. The main step in this direction was the treatment of a special case by Bremmerman, Oehme, and Taylor in (**3**). (The most recent discussion of analytic functions of several complex variables from the viewpoint of axiomatic field theory by Wightman (**7**) contains no explicit reference to the theorem. A companion paper by Omnes (**5**) gives an incorrect proof of related results by a mis-reading of Hartog's theorem, for example, the proof of Theorem 9 (**5**, p. 340). Several years ago, the writer was told by H. Grauert that L. Gårding had constructed a proof of the edge of the wedge theorem, though this proof has not yet been published, and less definite reports have it that such a proof is contained in an unpublished manuscript of Beurling and Gårding.)

It is the object of the present paper to show that the edge of the wedge theorem in its most general form may be obtained in a very direct fashion by combining simple arguments about distribution kernels with a theorem on analyticity of functions of several variables proved by the writer in (**4**). This theorem, which is Theorem 2 below, was established by the writer in connection with joint work on analyticity of distribution kernels with Barros-Neto (**1**). Theorem 2 has recently become very popular and been announced in the past several months by R. Kunze and E. M. Stein, who obtained it in connection with the study of bounded representation of the classical Lie groups on Hilbert space, and by R. H. Cameron, who proved it in connection with a study of Feynman integrals.

Theorem 2, the proof of which is both elementary and transparent, furnishes a very effective tool for a direct intuitive proof of the edge of the wedge theorem without any essential use of Fourier transform calculations. We emphasize the intuitive character of this proof since most discussions of most topics in this area proceed by extended formal calculations.

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1. Let $n \geqslant 1$. We consider the complex space C^n of n dimensions whose points will be denoted by $\zeta = (\zeta_1, \ldots, \zeta_n)$. Let R^n be the real n-dimensional euclidean space imbedded in the natural way in C^n so that $C^n = R^n + iR^n$, that is, $\zeta = \xi + i\eta$ with ξ, η in R^n. Let E be an open subset of R^n, C a cone in R^n with non-vacuous interior.

Definition. By the positive and negative wedges W^+ and W^- determined by E and C, we mean

$$W^+ = \{\zeta : \zeta = \xi + i\eta, \xi \in E, \eta \in C\}$$

$$W^- = \{\zeta : \zeta = \xi - i\eta, \xi \in E, \eta \in C\}.$$

THEOREM 1. (*The edge of the wedge theorem.*) *Let F^+ and F^- be two functions defined and holomorphic in the interior of the wedges W^+ and W^-, respectively. Suppose that $F^+(\xi + i\eta)$, considered as a distribution on E for fixed η in C, converges to $F_0^+ \in \mathfrak{D}'(E)$ in the distribution topology as $\eta \to 0$ in C. Suppose also that $F^-(\xi - i\eta)$, considered as a distribution on E for fixed η in C, converges to $F_0^- \in \mathfrak{D}'(E)$ in the distribution topology as $\eta \to 0$ in C. Suppose finally that $F_0^+ = F_0^-$. Then there exists a holomorphic function F on a neighbourhood of $E \times \{0\}$ in C^n which coincides with F^+ and F^- on W^+ and W^- respectively.*

In the statement of Theorem 1, as in its proof below, we have used the definitions of the theory of distributions of L. Schwartz **(6)** and we shall use some of the standard results of that theory in the proof.

We shall derive Theorem 1 from another more classical result which does not involve references to distributions and the distribution topology. This is the following:

THEOREM 1′. *Let F^+ and F^- be two functions defined and holomorphic in the interior of the wedges W^+ and W^-, respectively. Suppose that F^+ is continuous on the closure of W^+ and that $F^+(\xi + i\eta)$ converges uniformly to $F_0^+(\xi)$ for ξ in E as $\eta \to 0$ in C. Suppose also that F^- is continuous on the closure of W^- and that $F^-(\xi - i\eta)$ converges uniformly to $F_0^-(\xi)$ for ξ in E as $\eta \to 0$ in C. Suppose finally that $F_0^+(\xi) = F_0^-(\xi)$, $\xi \in E$. Then there exists a neighbourhood N of $E \times \{0\}$ in C^n, with N independent of the particular functions F^+ and F^-, and a holomorphic function F defined on N which coincides with F^+ and F^- on W^+ and W^-, respectively.*

We shall apply the following result established in **(4)** in the derivation of Theorem 1′:

THEOREM 2. *Consider $C^n \times C^m$ with co-ordinates $z = x + iy \in C^n$, $w = u + iv \in C^m$. Let $G(x, u)$ be a function defined for $|x| < 1$, $|u| < 1$. Suppose that $G(x, u)$ satisfies either of the following (equivalent) conditions:*

(a) *There exist constants $M_1, R_1, M_2, R_2 > 0$ such that for each fixed x in R^n with $|x| < 1$, $G(x, u)$ has a holomorphic extension $G(x, w)$ in w to the set $|w| < R_2$ satisfying the inequality $|G(x, w)| \leqslant M_2$, while for each fixed u in R^m with $|u| < 1$,*

$G(x, u)$ has a holomorphic extension $G(z, u)$ in z to set $|z| < R_1$ satisfying the inequality $|G(z, u)| \leqslant M_1$.

(b) *$G(x, u)$ is real analytic on the set $\{(x, u): |x| < 1, |u| < 1\}$ and there exist constants M_1, p_1, M_2, p_2 independent of x and u such that*

$$|D_x^\alpha G(x, u)| \leqslant M_1 p_1^{|\alpha|} |\alpha|!$$

$$|D_u^\alpha G(x, u)| \leqslant M_2 p_2^{|\alpha|} |\alpha|!$$

Then there exists an open neighbourhood N of the form

$$N = \{(z, w): |x| < 1 + \epsilon, |u| < 1 + \epsilon, |y| < \epsilon, |v| < \epsilon\}, \quad (\epsilon > 0)$$

such that $G(x, u)$ has a holomorphic extension to N. The constant ϵ depends on the R_j and the p_j but not on the M_j. (Note that $p_j = R_j^{-1}$.) The extension of G to N is bounded by a constant depending only on the M_j and R_j.

We remark that the existence of the domain N dependent only on the R_j is not explicitly remarked in **(4)** but follows without further argument from the proof given there.

If we apply the criterion (b) of Theorem 2 in an iterative argument, we obtain immediately the following variant of Theorem 2:

THEOREM 2′. *Let $G(x) = G(x_1, \ldots, x_n)$ be a function defined on a neighbourhood of a closed subset K of R^n which is real-analytic in each x_j with the others held fixed. Suppose further that there exists $M > 0, p > 0$, such that*

$$\left| \frac{\partial^r}{\partial x_j^r} G(x) \right| \leqslant M p^r r!$$

for $1 \leqslant j \leqslant n$ and all x in the fixed neighbourhood of K. Then G is real analytic on a neighbourhood of K and there exists a neighbourhood N of K in C^n, depending only on K and p but not on M, such that G admits a holomorphic extension to N.

A simple consequence of Theorem 2′ is the following:

THEOREM 2″. *Let $\{a_1, \ldots, a_n\}$ be a set of n linearly independent unit vectors in R^n. Let $G(x)$ be a function defined on a neighbourhood of a closed subset K of R^n with the property that for each x in the neighbourhood and each a_j, the function $h_j(\lambda) = G(x + \lambda a_j)$ of the real variable λ admits a holomorphic extension to the disk $|\lambda| < R$, R a fixed constant, which satisfies the inequality*

$$|h_j(\lambda)| \leqslant M \tag{1}$$

for $|\lambda| \leqslant R, \lambda \in C^1$.

Then G admits a holomorphic extension to a neighbourhood N of K in C^n (with N depending only on K and R but not on M).

Proof of Theorem 2″. Let S be the uniquely defined non-singular transformation of R^n carrying each unit vector e_j along the positive x_j-axis into a_j. Then

$G(Sx)$ is a function satisfying the hypotheses of Theorem 2′ but with K replaced by $S^{-1}K$. Indeed it follows from the Cauchy integral formula that

$$\left|\left(\frac{d}{d\lambda}\right)^r h_{j,x}(\lambda)\right|_{\lambda=0} \leqslant MR^{-r}.$$

However,

$$\left(\frac{d}{d\lambda}\right)^r G(x + \lambda a_j)|_{\lambda=0} = ((a_j \cdot \nabla)^r G)(x) = \left(\left(\frac{\partial}{\partial x_j}\right)^r G(Sx)\right).$$

Thus $G(Sx)$ admits an analytic extension $G_1(\zeta)$ to a neighbourhood N_1 of $S^{-1}K$ in C^n. Setting $G(\zeta) = G_1(S^{-1}\zeta)$, $N = SN_1$, we find that N is a neighbourhood of K in C^n and G is a holomorphic extension of $G(x)$ to N.

Proof of Theorem 1′. Let $\{a_1, \ldots, a_n\}$ be a linearly independent set of n unit vectors lying in the interior of C. (Such a set exists because C has a non-vacuous interior.) For each j, $1 \leqslant j \leqslant n$, we consider the two functions of the simple complex variable λ defined by

$$h_j^+(\lambda) = F^+(x + \lambda a_j),$$

$$h_j^-(\lambda) = F^-(x + \lambda a_j).$$

Since $\mathrm{Im}(x + \lambda a_j) = (\mathrm{Im}\,\lambda)\cdot a_j$, it follows that $\mathrm{Im}(x + \lambda a_j) \in C$ if $\mathrm{Im}(\lambda) \geqslant 0$, and $\mathrm{Im}(x + \lambda a_j) \in \mathrm{Int}(C)$ if $\mathrm{Im}(\lambda) > 0$. Similarly $\mathrm{Im}(x + \lambda a_j) \in (-C)$ if $\mathrm{Im}(\lambda) \leqslant 0$, and $\mathrm{Im}(x + \lambda a_j) \in \mathrm{Int}(-C)$ if $\mathrm{Im}(\lambda) > 0$. The two functions $h_j^+(\lambda)$ and $h_j^-(\lambda)$ are therefore defined and analytic in the upper and lower half-planes, respectively, for $|\lambda| < \mathrm{dist}(x, R^n - E)$. They are both continuous up to the real-axis moreover within this circle and are equal there. By the Schwartz reflection principle, they are both therefore restrictions of a single function $h_j(\lambda)$ analytic on the whole disk $|\lambda| < d_x$, and the function $h_j(\lambda)$ is bounded by the common bounds for h_j^+ and h_j^-. For any y in E such that $y = x + \lambda a_j$, $h_j(x + \lambda a_j) = F_0^+(y)$ is independent of j. We thus have a situation satisfying the hypotheses of Theorem 2″. There must therefore exist a common analytic extension $F(\zeta)$ of $F_0^+ = F_0^-$ on E which coincides with $h_j(x + \lambda a_j)$ wherever both are defined, that is, F coincides with F^+ and F^- on the intersections of the respective domains. The neighbourhood N on which $F(\zeta)$ is defined depends only on E by the fact that the radius of analyticity of $h_j(x + \lambda a_j)$ depends only on x and not on the choice of the functions F^+ and F^-.

2. We turn now to the proof of Theorem 1 itself, using Theorem 1′.

Proof of Theorem 1. To prove the existence of a common analytic extension of F^+ and F^-, it suffices to consider a complex neighbourhood of each point x of E. For simplicity, we may assume that $x = 0$ and choose a disk E_1 about 0 such that $4E_1 \subset E$.

Let $C_c^\infty(E)$ be the family of C^∞ functions with compact support in E. It follows from the hypotheses of Theorem 1 that if we denote by

$$\int f(y)\phi(y)dy$$

the pairing between a distribution f on E and a testing function ϕ in $C_c^\infty(E)$, then for each $\xi \in 2E_1$, the function

$$F_\phi^+(\xi + i\eta) = \int F^+(\xi - y + i\eta)\phi(y)dy,$$

is analytic for $\eta \in \mathrm{Int}(C)$, and continuous on the positive wedge $W_1^+ = 2E_1 + iC$. Similarly the function

$$F_\phi(\xi - i\eta) = \int F^-(\xi - y - i\eta)\phi(y)dy$$

is analytic in the interior of the negative wedge $W_1^- = 2E_1 - iC$ and continuous on the closed wedge. On $2E_1$, the common face of the two wedges, $F_\phi^+(\xi) = F_\phi^-(\xi)$, by the definition of equality of distributions. Therefore we may apply Theorem 1′ to this pair of functions and assert that there exists a fixed neighbourhood N_1 of $2E_1 \times \{0\}$ in C^n such that F_ϕ^+ and F_ϕ^- admit a common holomorphic extension F_ϕ to N_1 for each ϕ in $C_c^\infty(E)$.

We thus have a mapping $\phi \to F_\phi$ from $C_c^\infty(E)$ to $H(N_1)$, where we let $H(N_1)$ denote the Frechet space of holomorphic functions on N_1. Call this mapping T. T is obviously linear. Moreover, T is a closed mapping since if $\phi_k \to \phi$ in $C_c^\infty(K)$, K compact in E, $F_{\phi_n} \to h$ in $H(N_1)$. Then it follows that $F_{\phi_k}{}^+ \to F_\phi{}^+$ in $\mathrm{Int}(W_1^+)$, F_ϕ^- in $\mathrm{Int}(W_1^-)$ so that h is the uniquely determined analytic extension of F_ϕ^+ and F_ϕ^- to N_1, that is, $h = F_\phi = T(\phi)$. By the closed graph theorem, therefore, T is a continuous linear map. Moreover, the imbedding map of $H(N_1)$ into $C^\infty(N_1)$ (that is, $\mathfrak{E}(N_1)$ in the notation of **(6)**) is continuous. By the Schwartz Kernel theorem, there exists a distribution $k_{\zeta,y}$ on $N_1 \times E$ such that

(2) $$F_\phi(\zeta) = \int k_\zeta(y)\phi(y)dy, \ (k_\zeta(y) = k_{\zeta,y}),$$

for all $\phi \in C_c^\infty(E)$, $\zeta \in N_1$. (Actually $k_\zeta(y)$ may be identified with an analytic map from N_1 to $\mathfrak{D}'(E)$.)

We now remark that if $u \in E_1$, $\phi \in C_c^\infty(E_1)$, then $\phi_u(y) = \phi(u + y)$ defines a function in $C_c^\infty(E)$. Moreover, if $\xi \in E_1$, $\eta \in \mathrm{Int}(C)$,

$$\begin{aligned} F_{\phi_u}^+(\xi + i\eta) &= \int F^+(\xi - y + i\eta)\phi(u + y)dy \\ &= \int F^+(\xi + u - y + i\eta)\phi(y)dy \\ &= F_\phi^+(\xi + u + i\eta). \end{aligned}$$

Similarly,

$$F_{\phi_u}^-(\xi - i\eta) = F_\phi(\xi + u - i\eta).$$

By the uniqueness of the analytic continuation,

$$F_{\phi_u}(\xi + i\eta) = F_\phi(\xi + u + i\eta).$$

In other words, T commutes locally with translations.

Applying the kernel representation (2), we obtain on the one hand

$$F_{\phi_u}(\zeta) = F_\phi(\zeta + u) = \int k_{\zeta+u}(y)\phi(y)dy,$$

and, on the other hand,

$$F_{\phi_u}(\zeta) = \int k_\zeta(y)\phi(u + y)dy = \int k_\zeta(y - u)\phi(y)dy.$$

Hence by the uniqueness of the kernel representation, for y in $2E_1$, $u \in 2E_1$, $\zeta \in N_1$, we have

$$k_{\zeta+u,y} = k_{\zeta,y-u}.$$

If $\zeta = \xi + i\eta$, $\xi \in E_1$, $y \in E_1$, and if we set $u = -\xi$, $y_1 = y - \xi$

$$k_{\zeta+u,y_1} = k_{\zeta,y_1-u}$$

becomes

$$k_{\zeta,y_1} = k_{i\eta,y_1-\xi}.$$

(Indeed, we have just repeated the standard argument that kernels which commute with translations must be of convolution type, that is, depend on the difference of the arguments.)

As a mapping from $C_c^\infty(E_1)$ to $H(N)$, where N is the intersection of N_1 with the tube above E_1, T is a "convolution" with the kernel $k_{\zeta,y} = k_{i\eta,y-\xi}$. Let $k(\zeta) = k_{i\eta-\xi}$. It follows that if a sequence $\{\phi_k\}$ from $C_c^\infty(E_1)$ converges to the Dirac delta at 0 in $\mathfrak{E}'(E_1)$, then $F_{\phi_k} = T(\phi_k)$ will converge to $k(\zeta)$ in $\mathfrak{D}'(N)$. However, all the functions F_{ϕ_k} are holomorphic on N so that their convergence in $\mathfrak{D}'(N)$ implies their uniform convergence on compact subsets on N and the holomorphic character of their limit $k(\zeta)$ on N. On the interiors of the wedges W_1^+ and W_1^-, F_{ϕ_k} will converge to $F^+(\zeta)$ and $F^-(\zeta)$, respectively. Therefore $k(\zeta)$ is a holomorphic extension of F^+ and F^- on a neighbourhood of x, and the proof of Theorem 1 is complete.

Added in proof. In a latter to the writer, Dr. R. Stora has pointed out that a mathematically complete proof of the edge of the wedge theorem (along lines different from the present one) was given by H. Epstein, J. Math. Phys., *1* (1960), 524–531. Epstein refers to the unpublished proof of Beurling and Garding based upon the ideas of Dyson, Phys. Rev., *110* (1958).

Another proof of Theorem 2 has been given by T. Kotake in his study of analyticity of fundamental solutions for parabolic equations.

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