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Ergodic properties of β-adic Halton sequences

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Abstract. We investigate a parametric extension of the classical s-dimensional Halton sequence where the bases are special Pisot numbers. In a one-dimensional setting the properties of such sequences have already been investigated by several authors. We use methods from ergodic theory in order to investigate the distribution behavior of multidimensional versions of such sequences. As a consequence it is shown that the Kakutani–Fibonacci transformation is uniquely ergodic.

1. *Introduction*

In this article we consider the distribution properties of deterministic point sequences in $[0, 1)^s$. We use the following notation: for two points $\mathbf{a}, \mathbf{b} \in [0, 1)^s$ put $\mathbf{a} \leq \mathbf{b}$ and $\mathbf{a} < \mathbf{b}$ if the corresponding inequalities hold in each coordinate; furthermore, we write $[\mathbf{a}, \mathbf{b})$ for the set $\{\mathbf{x} \in [0, 1)^s : \mathbf{a} \leq \mathbf{x} < \mathbf{b}\}$, and such a set is called an s-dimensional interval. Moreover, we denote by $\mathbf{1}_I$ the indicator function of the set $I \subseteq [0, 1)^s$ and by λ_s the s-dimensional Lebesgue measure; we use λ instead of λ_1. Vectors are written in bold face and $\mathbf{0}$ denotes the s-dimensional vector $(0, \ldots, 0)$.

A sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$ of points in $[0, 1)^s$ is defined to be uniformly distributed modulo 1 (u.d.) if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mathbf{1}_{[\mathbf{a},\mathbf{b})}(\mathbf{x}_n) = \lambda_s([\mathbf{a}, \mathbf{b}))$$

for all s-dimensional intervals $[\mathbf{a}, \mathbf{b}) \subseteq [0, 1)^s$. A characterization of uniform distribution is due to Weyl [**29**]: a sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$ of points in $[0, 1)^s$ is u.d. if and only if for every real-valued continuous function f the relation

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(\mathbf{x}_n) = \int_{[0,1)^s} f(\mathbf{x})\, d\mathbf{x}$$

holds. Weyl's criterion suggests a numerical integration technique which is usually called quasi Monte Carlo (QMC) integration. However, this characterization gives no information on the quality of the estimator.

The Koksma–Hlawka inequality [**18**] states that the error term of QMC integration can be bounded by the product of the variation of f (in the sense of Hardy and Krause), denoted by $V(f)$, and the so-called star-discrepancy D_N^* of the point sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$, that is,

$$\left|\frac{1}{N}\sum_{n=1}^{N} f(\mathbf{x}_n) - \int_{[0,1]^s} f(\mathbf{x})\,d\mathbf{x}\right| \leq V(f)D_N^*(\mathbf{x}_n),$$

where D_N^* is defined by

$$D_N^*(\mathbf{x}_n) = D_N^*(\mathbf{x}_1, \ldots, \mathbf{x}_N) = \sup_{\mathbf{a}\in[0,1)^s}\left|\frac{1}{N}\sum_{n=1}^{N} \mathbf{1}_{[\mathbf{0},\mathbf{a})}(\mathbf{x}_n) - \lambda_s([\mathbf{0}, \mathbf{a}))\right|.$$

Thus in order to minimize the integration error we have to use point sequences with small discrepancy. There are several constructions for sequences which achieve a star-discrepancy of order $\mathcal{O}(N^{-1}(\log N)^s)$, so-called low-discrepancy sequences. This convergence rate, which is best possible among known sequences, is for all $s \geq 1$ better than that of the probabilistic error of the standard Monte Carlo method, where a sequence of random instead of deterministic points is used. QMC integration is successfully used in several different areas of applied mathematics, for example in actuarial and financial mathematics, where high-dimensional numerical integrals frequently appear; see, for example, [**4, 23**]. For a detailed survey on low-discrepancy sequences and their applications we refer to [**10**].

In this article we construct point sequences by a combination of methods from uniform distribution theory and dynamical systems. In particular, we give answers to problems considered in [**1, 8**]. For the basic definitions we refer to these articles and to standard textbooks such as [**28**].

An important tool for analyzing the properties of a dynamical system is the construction of an isometry to a known ergodic or uniquely ergodic system. A classical example of a uniquely ergodic system is $(\mathbb{Z}_b, \tau)$ (see, for instance, [**28**]), where $\mathbb{Z}_b$ is the compact group of b-adic integers and $\tau : \mathbb{Z}_b \longrightarrow \mathbb{Z}_b$ the addition-by-one map (or odometer). In the following we briefly recall the connection between $(\mathbb{Z}_b, \tau)$ and low-discrepancy sequences on $[0, 1)$. For an integer $b \geq 2$, every $z \in \mathbb{Z}_b$ has a unique expansion of the form

$$z = \sum_{j\geq 0} z_j b^j$$

with digits $z_j \in \{0, 1, \ldots, b-1\}$. For $z \in \mathbb{Z}_b$ we define the b-adic Monna map $\varphi_b : \mathbb{Z}_b \longrightarrow [0, 1)$ (see also [**13**]) by

$$\varphi_b\left(\sum_{j\geq 0} z_j b^j\right) = \sum_{j\geq 0} z_j b^{-j-1}.$$

The restriction of φ_b to $\mathbb{N}_0$ is called the radical-inverse function in base b and the sequence

$$(\varphi_b(n))_{n\in\mathbb{N}}$$

is the so-called van der Corput sequence in base b which is a low-discrepancy sequence.

The Monna map is continuous and surjective but not injective. In order to make it an isomorphism we only consider the so-called regular representations, that is, representations with infinitely many digits z_j different from $b-1$. The Monna map restricted to regular representations admits an inverse (called a pseudo-inverse) $\varphi_b^{-1} : [0, 1) \longrightarrow \mathbb{Z}_b$, defined by

$$\varphi_b^{-1}\bigg(\sum_{j\geq 0} z_j b^{-j-1}\bigg) = \sum_{j\geq 0} z_j b^j,$$

where $\sum_{j\geq 0} z_j b^{-j-1}$ is a b-adic rational in $[0, 1)$.

Moreover, φ_b is measure preserving from $\mathbb{Z}_b$ onto $[0, 1)$ and it transports the normalized Haar measure on $\mathbb{Z}_b$ to the Lebesgue measure. Hence, by the unique ergodicity of τ it follows that the sequence $(\tau^n z)_{n\in\mathbb{N}}$ is uniformly distributed in $\mathbb{Z}_b$ for all $z \in \mathbb{Z}_b$, in particular for $z = 0$. Thus the van der Corput sequence $(\varphi_b(\tau^n 0))_{n\in\mathbb{N}}$ in base b is uniformly distributed modulo 1.

In order to construct multidimensional sequences we need a criterion to ensure that the Cartesian product of ergodic systems is again ergodic (see, for example, [**13**]).

THEOREM 1. *Let* $\mathcal{T}_i = (X_i, T_i)$, $i = 1, \ldots, s$, *be ergodic dynamical systems. Then the dynamical system* $\mathcal{T}_1 \times \cdots \times \mathcal{T}_s$ *is ergodic if and only if for all* $i, j \in \{1, \ldots, s\}$, $i \neq j$, *the discrete parts of the spectra of* T_i *and* T_j *intersect only at* 1.

In [**13**], the spectrum of $(\mathbb{Z}_b, \tau)$ is given explicitly. Furthermore, the authors show that if b_1, b_2 denote positive integers and τ_{b_i} denotes the addition-by-one on $\mathbb{Z}_{b_i}$, then the dynamical systems $(\mathbb{Z}_{b_1}, \tau_{b_1})$ and $(\mathbb{Z}_{b_2}, \tau_{b_2})$ are spectrally disjoint if and only if b_1 and b_2 are coprime. This is exactly the condition proved by Halton in [**17**] in order to obtain a low-discrepancy sequence in $[0, 1)^s$ by combining coordinatewise van der Corput sequences. The resulting sequence $(\phi_{\mathbf{b}}(n))_{n\in\mathbb{N}} = (\phi_{b_1}(n), \ldots, \phi_{b_d}(n))_{n\in\mathbb{N}}$ is called **b**-adic Halton sequence, where **b** is an s-dimensional vector of pairwise coprime integers b_i, $i = 1, \ldots, s$.

The aim of the present article is to extend the above idea to point sequences with irrational bases. In a one-dimensional setting such sequences have been investigated by several authors. For instance, Barat and Grabner [**5**] consider the so-called β-adic van der Corput sequence $(\phi_\beta(n))_{n\in\mathbb{N}}$ on $[0, 1)$ and similar constructions. They prove that $(\phi_\beta(n))_{n\in\mathbb{N}}$ is low-discrepancy, if β is the characteristic root of a special linear recurrence. Ninomiya [**22**] considers the discrepancy of point sequences on $[0, 1)$ for a slightly greater class of irrational bases β. Furthermore, Steiner [**26**] investigates so-called bounded remainder sets in a similar setting. In a second article Steiner [**27**] considers van der Corput sequences on abstract numeration systems and gives conditions under which they are low-discrepancy. Discrepancy bounds for a multidimensional extension of [**22**] are formulated by Mori and Mori [**21**] by using algebraic methods. Note that the construction in [**21**] is different from that in the present article.

Carbone [**7**] and Drmota and Infusino [**9**] investigate the discrepancy of point sequences generated by the so-called Kakutani splitting procedure. Carbone completely characterizes the growth order of the discrepancy for a two parametric subfamily, so-called LS-sequences. Moreover, Aistleitner *et al* [**1**] give conditions under which an s-dimensional vector of LS-sequences is not u.d. in $[0, 1)^s$.

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The remainder of this article is structured as follows: in the next section we formulate a characterization of uniquely ergodic systems which are constructed as Cartesian products of odometers on numeration systems related to linear recurrences. In the third section we give conditions under which the s-dimensional $\boldsymbol{\beta}$-adic Halton sequence is uniformly distributed in $[0, 1)^s$. Furthermore, we present a parametric class of sequences which satisfies these conditions. Finally, we prove that the ergodic Kakutani–Fibonacci transformation, presented in [**8**], is in fact uniquely ergodic.

2. *General G-odometers*

In this section we consider odometers on numeration systems with respect to a linear recurring base sequence. For a detailed discussion of such number systems we refer to [**11, 14–16**].

Definition 1. Let $(G_n)_{n\geq 0}$ be an increasing sequence of positive integers with $G_0 = 1$. Then every positive integer can be expanded as

$$n = \sum_{k=0}^{\infty} \varepsilon_k(n) G_k \quad \text{for all } n \in \mathbb{N},$$

where $\varepsilon_k(n) \in \{0, \ldots, \lfloor G_{k+1}/G_k \rfloor\}$ and $\lfloor x \rfloor$ denotes the integer part of x, that is, the greatest integer less than or equal to $x \in \mathbb{R}$. This expansion (called the G-expansion) is uniquely determined and finite, provided that for every K,

$$\sum_{k=0}^{K-1} \varepsilon_k(n) G_k < G_K. \tag{1}$$

We write ε_k for the kth digit of the G-expansion.

The digits ε_k can be computed by the greedy algorithm (see, for instance, [**11**]) and $G = (G_n)_{n\geq 0}$ is called a numeration system.

We denote by $\mathcal{K}_G$ the subset of sequences that satisfy (1), and the elements of $\mathcal{K}_G$ are called G-admissible. In order to extend the addition-by-one map defined on $\mathbb{N}$ to $\mathcal{K}_G$ we introduce $\mathcal{K}_G^0 \subseteq \mathcal{K}_G$:

$$\mathcal{K}_G^0 = \left\{ x \in \mathcal{K}_G : \exists M_x, \forall j \geq M_x, \sum_{k=0}^{j} \varepsilon_k G_k < G_{j+1} - 1 \right\}. \tag{2}$$

Put $x(j) = \sum_{k=0}^{j} \varepsilon_k G_k$, and set

$$\tau(x) = (\varepsilon_0(x(j) + 1) \cdots \varepsilon_j(x(j) + 1))\varepsilon_{j+1}(x)\varepsilon_{j+2}(x) \cdots , \tag{3}$$

for every $x \in \mathcal{K}_G^0$ and $j \geq M_x$. This definition does not depend on the choice of $j \geq M_x$ and can be easily extended to sequences x in $\mathcal{K}_G \backslash \mathcal{K}_G^0$ by $\tau(x) = 0 = (0^\infty)$. In this way the transformation τ is defined on $\mathcal{K}_G$ and it is called a G-odometer. We refer to [**14**] for a complete survey on odometers related to general numeration systems.

In this article we consider only numeration systems where the base sequence is a linear recurrence. Let $G_0 = 1$ and $G_k = a_0 G_{k-1} + \cdots + a_{k-1} G_0 + 1$ for $k < d$. Then G_n for $n \geq d$ is determined by a recurrence of order $d \geq 1$, that is,

$$G_{n+d} = a_0 G_{n+d-1} + \cdots + a_{d-1} G_n, \quad n \geq 0. \tag{4}$$

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The solution of the characteristic equation of the numeration system G,

$$x^d = a_0 x^{d-1} + \cdots + a_{d-1}, \tag{5}$$

plays a crucial role. We are only interested in numeration systems where the solution of (5) is a Pisot number β. Note that β is always a Pisot number if

$$a_0 \geq \cdots \geq a_{d-1} \geq 1; \tag{6}$$

see [**12**, Theorem 2]. Parry [**24**] shows that in this case the so-called Parry's β-expansion of β is finite, that is,

$$\beta = a_0 + \frac{a_1}{\beta} + \cdots + \frac{a_{d-1}}{\beta^{d-1}}, \tag{7}$$

where $a_0 = \lfloor \beta \rfloor$. At the end of the last section we also consider numeration systems where (6) does not hold.

For numeration systems where the characteristic root β is a Pisot number which satisfies (7), a sum $\sum_{k=0}^{M} \varepsilon_k G_k$, $M < \infty$, is the expansion of an integer if and only if the digits ε_k of the G-expansion satisfy

$$(\varepsilon_k, \varepsilon_{k-1}, \ldots, \varepsilon_0, 0^\infty) < (a_0, a_1, \ldots, a_{d-1})^\infty, \tag{8}$$

for every k and $<$ denoting the lexicographic order (see [**24**]). Representations $(\varepsilon_k, \ldots, \varepsilon_0)$ satisfying this condition are called admissible representations and thus they belong to $\mathcal{K}_G$.

In [**14**, Theorem 5], the authors show that the odometer on an admissible numeration system G is uniquely ergodic and the corresponding invariant measure μ is given by

$$\mu(Z) = \frac{F_{K,0}\beta^{d-1} + (F_{K,1} - a_0 F_{K,0})\beta^{d-2} + \cdots + (F_{K,d-1} - a_0 F_{K,d-2} - \cdots - a_{d-2} F_{K,0})}{\beta^K (\beta^{d-1} + \beta^{d-2} + \cdots + 1)}, \tag{9}$$

where $F_{K,r} := \#\{n < G_{K+r} : (\varepsilon_0(n), \varepsilon_1(n), \ldots) \in Z\}$ and Z is a cylinder with length K. Note that the corresponding formula in [**14**, Theorem 5] contains a misprint which was corrected in [**5**].

In the following we want to apply Theorem 1, thus we need information on the spectrum of the G-odometer. We introduce the following two hypotheses.

HYPOTHESIS 1. (Grabner *et al* [**14**]) *There exists an integer $b > 0$ such that for all k and*

$$N = \sum_{i=0}^{k} \epsilon_i(N) G_i + \sum_{j=k+b+2}^{\infty} \epsilon_j(N) G_j,$$

the addition of G_m to N, where $m \geq k + b + 2$, does not change the first $k + 1$ digits in the greedy representation, that is,

$$N + G_m = \sum_{i=0}^{k} \epsilon_i(N) G_i + \sum_{j=k+1}^{\infty} \epsilon_j(N + G_m) G_j.$$

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HYPOTHESIS 2. (Frougny and Solomyak [**12**]) *The characteristic root β of the numeration system G is a Pisot number such that all elements of the set $\mathbb{Z}[\beta^{-1}]$ have finite β-expansions.*

There are only a few results concerning Hypothesis 1. In [**14**] the authors remark that the Multinacci sequence, $a_0 = \cdots = a_{d-1} = 1$, satisfies Hypothesis 1. Furthermore, Bruin *et al* [**6**] show that the numeration system with coefficients $(a_0, a_1, a_2) = (1, 0, 1)$ satisfies Hypothesis 1.

Several researchers have worked on algebraic characterizations of Pisot numbers β which satisfy Hypothesis 2. Frougny and Solomyak [**12**] show that (6) implies Hypothesis 2 and they give a full characterization of all Pisot numbers of degree two with this property. Furthermore, Hollander [**19**] states another sufficient condition for Hypothesis 2, and Akiyama [**2**] characterizes all Pisot units of degree three satisfying Hypothesis 2. Further progress was also made by Akiyama *et al* [**3**] who prove Hypothesis 2 for a large class of Pisot numbers of degree three by using the theory of shift radix systems. Nevertheless, there exists no complete algebraic characterization for Pisot numbers satisfying Hypothesis 2 of degree greater than two. Note that both hypotheses can be satisfied by the same numeration system but, to the best of the authors' knowledge, it is unknown if the two hypotheses are equivalent; see [**14**].

Grabner *et al* [**14**, Theorem 6] and Solomyak [**25**, Theorem 4.1] show that the odometer on the base system G has purely discrete spectrum provided that one of the above Hypotheses holds. Furthermore, we obtain in both cases that the set of eigenvalues of the transformation is given by

$$\Gamma := \left\{ z \in \mathbb{C} : \lim_{n\to\infty} z^{G_n} = 1 \right\}. \tag{10}$$

THEOREM 2. *Let $G^1, \ldots, G^s$ be numeration systems defined by*

$$G^i_{n+d_i} = a^i_0 G^i_{n+d_i-1} + \cdots + a^i_{d_i-1} G^i_n$$

and let the coefficients of the linear recurrences be given as $a^i_j = b_i$, $j = 0, \ldots, (d_i - 1)$, $i = 1, \ldots, s$, with pairwise coprime positive integers b_i, $i = 1, \ldots, s$. Furthermore, let $\beta_i^k/\beta_j^l \notin \mathbb{Q}$, for all $l, k \in \mathbb{N}$, where $\beta_1, \ldots, \beta_s$ denote the characteristic roots of the numeration systems. Then the dynamical system which is constructed as the s-dimensional Cartesian product of the odometers, $((\mathcal{K}_{G^1}, \tau_1) \times \cdots \times (\mathcal{K}_{G^s}, \tau_s))$, is uniquely ergodic.

Proof. It follows by [**25**, Main theorem] that each numeration system G^j, $j = 1, \ldots, s$, satisfies Hypothesis 2, thus the components of the s-dimensional dynamical system are uniquely ergodic. Furthermore, we obtain that their spectrum is given by (10). By Theorem 1, we derive that the Cartesian product is uniquely ergodic if and only if $\Gamma_j \cap \Gamma_k = \{1\}$ for all $1 \le j < k \le s$. As noted in [**14**], we have

$$\lim_{n\to\infty} \frac{G^j_n}{\beta^n_j} = C_j, \tag{11}$$

where the constant C_j can be computed by residue calculus. Using the standard notation $\sim$ for asymptotic equality (if $n \to \infty$) we obtain for fixed $l \in \mathbb{N}$,

$$\exp\left(2\pi i \frac{G_n^j}{\beta_j^l}\right) \sim \exp(2\pi i C_j \beta_i^{n-l}) \sim \exp(2\pi i G_{n-l}^j),$$

and thus

$$\lim_{n\to\infty} \exp\left(2\pi i \frac{G_n^j}{\beta_j^l}\right) = \lim_{n\to\infty} \exp(2\pi i G_{n-l}^j) = 1,$$

where C_j is given by (11). Furthermore, it is easy to see that for every $k \in \mathbb{N}$ there exists an n_0 with $b_j^k \mid G_n^j$ for all $n \geq n_0$ and there exist no $b', n_0' \in \mathbb{N}$ with $\gcd(b', b_j) = 1$ such that $b' \mid G_n^j$ for all $n \geq n_0'$. Then Γ_j can be written as

$$\Gamma_j = \left\{\exp\left(2\pi i \frac{c}{b_j^m \beta_j^l}\right) : m, l, c \in \mathbb{N} \cup \{0\}\right\}.$$

By Theorem 1 the s-dimensional dynamical system is ergodic. Since, by our assumption, the spectrum is purely discrete the system is isomorphic to a group rotation, and thus uniquely ergodic. □

3. *Uniform distribution of the β-adic Halton sequence*

First we extend the definition of the Monna map to irrational bases $\beta > 1$. Let

$$n = \sum_{j\geq 0} \epsilon_j(n) G_j$$

be the G-expansion of an integer n. We write ϵ_j for short and define the β-adic Monna map $\phi_\beta : \mathcal{K}_G \to \mathbb{R}^+$ as

$$\phi_\beta(n) = \phi_\beta\left(\sum_{j\geq 0} \epsilon_j(n) G_j\right) = \sum_{j\geq 0} \epsilon_j(n) \beta^{-j-1}.$$

Furthermore, we define the radical inverse function as the restriction of ϕ_β on $\mathcal{K}_G^0$ and the pseudo-inverse ϕ_β^{-1} similarly. In this context the $\boldsymbol{\beta}$-adic Halton sequence is given as $(\phi_{\boldsymbol{\beta}}(n))_{n\in\mathbb{N}} = (\phi_{\beta_1}(n), \ldots, \phi_{\beta_s}(n))_{n\in\mathbb{N}}$, where $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_s)$ and the β_i are characteristic roots of the numeration systems G^i, respectively.

Note that even if one of the Hypotheses 1 or 2 holds, this does not imply that the image of $\mathcal{K}_G^0$ under ϕ_β is contained and dense in $[0, 1)$.

LEMMA 1. *Let* $\mathbf{a} = (a_0, \ldots, a_{d-1})$, *where the integers* $a_0, \ldots, a_{d-1} \geq 0$ *are the coefficients defining the numeration system G, and assume that the corresponding characteristic root* β *satisfies (7). Furthermore, assume that there is no* $\mathbf{b} = (b_0, \ldots, b_{k-1})$ *with* $k < d$ *such that* β *is the characteristic root of the polynomial defined by* $\mathbf{b}$. *Then* $\phi_\beta(\mathbb{N}) \subset [0, 1)$ *and* $\phi_\beta(\mathbb{N}) \not\subset [0, x)$ *for all* $0 < x < 1$ *if and only if* $\mathbf{a}$ *can be written either as*

$$\mathbf{a} = (a_0, \ldots, a_0) \tag{12}$$

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or as

$$\mathbf{a} = (a_0, a_0 - 1, \ldots, a_0 - 1, a_0), \tag{13}$$

where $a_0 > 0$.

Proof. It follows by (7) that

$$\frac{a_0}{\beta} + \cdots + \frac{a_{d-1}}{\beta^d} = 1. \tag{14}$$

Furthermore, for all admissible representations $(\varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots)$ we have

$$(\varepsilon_k, \varepsilon_{k-1}, \ldots, \varepsilon_0, 0^\infty) < (a_0, a_1, \ldots, a_{d-1})^\infty \tag{15}$$

for every k and $<$ denoting the lexicographic order.

We consider a representation given by $\mathbf{c} = (c_0, \ldots, c_{k-1})^\infty$ for $k > 0$ and assume that there are no positive integers c'_i and $m < k$ such that $\mathbf{c} = (c'_0, \ldots, c'_{m-1})^\infty$. We obtain

$$\begin{aligned}
\phi_\beta(\mathbf{c}) &= \sum_{i=0}^{\infty} \left(\frac{c_0}{\beta} + \cdots + \frac{c_{k-1}}{\beta^k}\right)\left(\frac{1}{\beta^k}\right)^i \\
&= \left(\frac{c_0}{\beta} + \cdots + \frac{c_{k-1}}{\beta^k}\right)\frac{1}{1 - (1/\beta^k)} \\
&= \left(\frac{c_0}{\beta} + \cdots + \frac{c_{k-1}}{\beta^k}\right)\frac{\beta^k}{\beta^k - 1}.
\end{aligned}$$

To finish the proof of the lemma, it will be shown that the maximum of $\phi_\beta(\mathbf{c})$ (extended over all representations $\mathbf{c}$) is 1, provided that (12) or (13) are satisfied, that is,

$$\frac{c_0}{\beta} + \cdots + \frac{c_{k-2}}{\beta^{k-1}} + \frac{c_{k-1} + 1}{\beta^k} = 1. \tag{16}$$

We obtain $k \geq d$ since $\mathbf{a}$ is by assumption the minimal representation of β. In the case $k = d$ we get

$$\frac{c_0}{\beta} + \cdots + \frac{c_{d-2}}{\beta^{d-1}} + \frac{c_{d-1} + 1}{\beta^d} = \frac{a_0}{\beta} + \cdots + \frac{a_{d-1}}{\beta^d} = 1,$$

thus $c_0 = a_0, \ldots, c_{d-2} = a_{d-2}$ and $c_{d-1} + 1 = a_{d-1}$. Moreover, by (15), we have $\max_{i=0,\ldots,d-1} c_i \leq a_0$ and thus $a_0 = \max(\max_{i=0,\ldots,d-2} a_i, a_{d-1} - 1)$. Let $m \geq 0$ be the maximal integer such that $a_i = a_0$ for $i \leq m$. Then, if $a_{d-1} = a_0 + 1$ we obtain that $\mathbf{c} = (a_0, a_1, \ldots, a_{d-2}, a_0)^\infty$ contains $m + 1$ successive a_0s, thus $\mathbf{c}$ is not admissible and $a_0 = \max_{i=0,\ldots,d-1} a_i$. Similar arguments yield $k \leq d$ since by assumption $\mathbf{c} \neq (c'_0, \ldots, c'_{m-1})^\infty$ for $m < k$.

Assume now that $k = d$, $a_0 = \max_{i=0,\ldots,d-1} a_i$ and let $m \geq 0$ be defined as above. By (14) and (15) $\mathbf{c}^* = (a_0, \ldots, a_m, (a_0 - 1, a_0, \ldots, a_m)^\infty)$ is admissible and

$$\phi_\beta(\mathbf{c}^*) \geq 1,$$

where equality holds if m is either 0 or $d - 1$, which are the cases (12) and (13). This yields assertion (16), thus the proof of the lemma is complete. □

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If we drop the condition that d has to be minimal, we obtain the following additional cases for which the above lemma is satisfied:

$$\mathbf{a} = (a_0, \ldots, a_0, a_0 + 1) \tag{17}$$

and

$$\mathbf{a} = (\mathbf{a}', \ldots, \mathbf{a}', \mathbf{a}''), \tag{18}$$

where $a_0 > 0$, $\mathbf{a}'$, $\mathbf{a}''$ are of equal length and of the form

$$\mathbf{a}' = (a_0, \ldots, a_0, a_0 - 1), \quad \mathbf{a}'' = (a_0, \ldots, a_0)$$

or

$$\mathbf{a}' = (a_0, a_0 - 1, \ldots, a_0 - 1), \quad \mathbf{a}'' = (a_0, a_0 - 1, \ldots, a_0 - 1, a_0).$$

Note that (17) is another way to represent the $(a_0 + 1)$-adic number system, which is a special case of (12) and obviously satisfies the lemma. Furthermore, condition (18) is a reformulation of (12) and (13), thus in the following we only consider numeration systems which satisfy (12) or (13).

LEMMA 2. *Let G be a numeration system of the form (4), assume that the coefficients of the linear recurrence are given by $a_j = a$, $j = 0, \ldots, (d-1)$, for a positive integer a and let β denote the corresponding characteristic root. Then $\mu(Z) = \lambda(\phi_\beta(Z))$ for every cylinder set Z.*

Proof. Let the cylinder set Z be defined by the fixed digits $\epsilon_0, \ldots, \epsilon_{k-1}$. Assume first that $\epsilon_{k-1} < a$, then $F_{k,r} = (a+1)^r$ for $0 \le r < d$. Thus, by (9), we obtain that

$$\mu(Z) = \beta^{-k}.$$

Consider the β-adic Monna map of $n \in \mathbb{N}$, that is,

$$\phi_\beta(n) = \sum_{i=0}^{\infty} \frac{\epsilon_i}{\beta^{i+1}}.$$

If $\epsilon_{k-1} < a$ we easily see that $\phi_\beta(Z)$ is dense in

$$I = \left[\sum_{i=0}^{k-1} \frac{\epsilon_i}{\beta^{i+1}}, \sum_{i=0}^{k-2} \frac{\epsilon_i}{\beta^{i+1}} + \frac{(\epsilon_{k-1}+1)}{\beta^k}\right)$$

and that $\phi_\beta(x') \notin I$ if $x' \notin Z$. Thus $\phi_\beta(Z)$ is λ-measurable and $\lambda(\phi_\beta(Z)) = \lambda(I) = \beta^{-k}$.

Assume now that Z is defined by the fixed digits $\epsilon_0, \ldots, \epsilon_{k-2}$ and $\epsilon_{k-1} = a$. By the above argument we derive that a cylinder with fixed digits $\epsilon_0, \ldots, \epsilon_{k-2}$, with $\epsilon_{k-2} < a$, has measure $\beta^{-(k-1)}$.

Now compute the measure of Z:

$$\mu(Z) = \beta^{-(k-1)} - (a-1)\beta^{-k}.$$

Next we consider $\phi_\beta(Z)$, hence

$$\phi_\beta(Z) = \left[\sum_{i=0}^{k-1} \frac{\epsilon_i}{\beta^{i+1}}, \sum_{i=0}^{k-3} \frac{\epsilon_i}{\beta^{i+1}} + \frac{(\epsilon_{k-2}+1)}{\beta^k}\right)$$

and thus $\lambda(\phi_\beta(Z)) = \mu(Z)$.

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Let $2 \le h \le \min(k, d-1)$ and consider a cylinder set Z with fixed digits $\epsilon_0, \ldots, \epsilon_{k-h-1} < a$ and $\epsilon_{k-l} = a$ for $l = 1, \ldots, h$. Then, the cylinder with fixed digits $\epsilon_0, \ldots, \epsilon_{k-h-1}$ has measure $\beta^{-(k-h)}$ and every cylinder with digits $\epsilon_0, \ldots, \epsilon_{k-h+1}$ has measure $\beta^{-(k-h+2)}$. Thus it follows that

$$\mu(Z) = \beta^{-(k-h+1)} - (a-1)\beta^{-(k-h+2)}.$$

Considering $\phi_\beta(Z)$, we have

$$\phi_\beta(Z) = \left[\sum_{i=0}^{k-h+1} \frac{\epsilon_i}{\beta^{i+1}}, \sum_{i=0}^{k-h} \frac{\epsilon_i}{\beta^{i+1}} + \frac{(\epsilon_{k-h-1}+1)}{\beta^{k-h+2}}\right),$$

and thus $\lambda(\phi_\beta(Z)) = \mu(Z)$. □

As mentioned in the previous section, a result of Frougny and Solomyak [**12**, Lemma 3] implies that the dominant root of

$$x^2 - a_0 x - a_1, \quad a_0, a_1 \ge 1,$$

is a Pisot number if and only if $a_0 \ge a_1$. Lemma 1 shows that the image of $\mathcal{K}_G^0$ under ϕ_β is not a subset of $[0, 1)$, when $a_0 > a_1$. Thus Lemma 2 characterizes all van der Corput-type constructions for $d = 2$.

THEOREM 3. *Let $G^1, \ldots, G^s$ be numeration systems as in Theorem 2 and let $\beta_1, \ldots, \beta_s$ denote the roots of the corresponding characteristic equations. Then the s-dimensional $\boldsymbol{\beta}$-adic Halton sequence $(\phi_{\boldsymbol{\beta}}(n))_{n\in\mathbb{N}}$ is u.d. in* $[0, 1)^s$.

Proof. By Lemma 2 and the definition of the Monna map we obtain an isometry between the dynamical systems $((\mathcal{K}_{G^1}, \tau_1) \times \cdots \times (\mathcal{K}_{G^s}, \tau_s))$ and $(([0, 1), T_1) \times \cdots \times ([0, 1), T_s))$, where

$$T_i : [0, 1) \to [0, 1), \quad T_i(x) := \phi_{\beta_i} \circ \tau_i \circ \phi_{\beta_i}^{-1}(x).$$

Let $\mathbf{T}\mathbf{x} = (T_1 x_1, \ldots, T_s x_s)$ for $\mathbf{x} = (x_1, \ldots, x_s) \in [0, 1)^s$. Hence by Birkhoff's ergodic theorem, $(\mathbf{T}^n \mathbf{x})_{n\in\mathbb{N}}$ is u.d. in $[0, 1)^s$ for all $\mathbf{x} \in [0, 1)^s$. In particular, $(\phi_{\boldsymbol{\beta}}(n))_{n\in\mathbb{N}} = (\mathbf{T}^n \mathbf{0})_{n\in\mathbb{N}}$ is u.d. □

Note that the classical **b**-adic Halton sequence with pairwise coprime integer bases $b_1, \ldots, b_s \ge 2$, is covered by Theorem 3.

THEOREM 4. *Let the numeration system G be defined by the coefficients $(a_0, a_1, a_2) = (1, 0, 1)$, β be its characteristic root and τ the odometer on G. Then $\mu(Z) = \lambda(\phi_\beta(Z))$ for all cylinder sets Z. Thus $T(x) = \phi_\beta \circ \tau \circ \phi_\beta^{-1}(x)$ is uniquely ergodic and $(T^n x)_{n\in\mathbb{N}}$ is u.d. for all x in $[0, 1)$. Furthermore, the spectrum of T is given by*

$$\Gamma = \left\{\exp\left(2\pi i \frac{c}{\beta^l}\right) : l, c \in \mathbb{N} \cup \{0\}\right\}. \tag{19}$$

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Proof. It is well known that β is a Pisot number and equation (7) holds since $\lfloor\beta\rfloor = 1 = a_0$. Hypothesis 1 was proved for this case in [**6**, Theorem 4]. The proof that Hypothesis 2 is satisfied can be found in [**2**, Theorem 3]. Equation (19) follows from the proof of Theorem 2.

Now we have to prove that ϕ_β transports the measure μ to the Lebesgue measure on [0, 1). First we assume $k \geq 3$. Let the cylinder Z be defined by the fixed digits $\epsilon_0, \ldots, \epsilon_{k-1}$. We consider four different cases, the first of which is $\epsilon_{k-3} = \epsilon_{k-2} = \epsilon_{k-1} = 0$. Then $F_{k,0} = 1$, $F_{k,1} = 2$, $F_{k,2} = 3$ and we get by (9) that

$$\mu(Z) = \beta^{-k}.$$

Furthermore, by the same argument as in the first part of the proof of Theorem 3 we obtain

$$\phi_\beta(Z) = \left[\sum_{i=0}^{k-1} \frac{\epsilon_i}{\beta^{i+1}}, \sum_{i=0}^{k-2} \frac{\epsilon_i}{\beta^{i+1}} + \frac{(\epsilon_{k-1}+1)}{\beta^k}\right),$$

and thus $\lambda(\phi_\beta(Z)) = \beta^{-k}$.

Now let $\epsilon_{k-3} = 1$, $\epsilon_{k-2} = \epsilon_{k-1} = 0$. Hence $F_{k,0} = 1$, $F_{k,1} = 2$, $F_{k,2} = 3$ and $\mu(Z) = \beta^{-k}$. We have

$$\begin{aligned}\phi_\beta(Z) &= \left[\sum_{i=0}^{k-1} \frac{\epsilon_i}{\beta^{i+1}}, \sum_{i=0}^{k-1} \frac{\epsilon_i}{\beta^{i+1}} + \beta^{-k}\sum_{i=0}^{\infty}\beta^{-(3i+1)}\right) \\ &= \left[\sum_{i=0}^{k-1} \frac{\epsilon_i}{\beta^{i+1}}, \sum_{i=0}^{k-1} \frac{\epsilon_i}{\beta^{i+1}} + \beta^{-k}\right),\end{aligned}$$

thus again $\lambda(\phi_\beta(Z)) = \beta^{-k}$. Now assume that $\epsilon_{k-2} = 1$, $\epsilon_{k-1} = 0$. Hence $F_{k,0} = 1$, $F_{k,1} = 1$, $F_{k,2} = 2$ and

$$\mu(Z) = \beta^{-k}\frac{\beta^{-2}+1}{\beta^{-2}+\beta^{-1}+1}.$$

Similarly as above we obtain

$$\begin{aligned}\phi_\beta(Z) &= \left[\sum_{i=0}^{k-1} \frac{\epsilon_i}{\beta^{i+1}}, \sum_{i=0}^{k-1} \frac{\epsilon_i}{\beta^{i+1}} + \beta^{-(k+1)}\sum_{i=0}^{\infty}\beta^{-(3i+1)}\right) \\ &= \left[\sum_{i=0}^{k-1} \frac{\epsilon_i}{\beta^{i+1}}, \sum_{i=0}^{k-1} \frac{\epsilon_i}{\beta^{i+1}} + \beta^{-(k+1)}\right),\end{aligned}$$

thus $\lambda(\phi_\beta(Z)) = \beta^{-(k+1)}$. Now we have

$$\beta^{-(k+1)} = \beta^{-k}\frac{\beta^{-2}+1}{\beta^{-2}+\beta^{-1}+1},$$

which is equivalent to $\beta^{-3} + \beta^{-2} + \beta^{-1} = \beta^{-2} + 1$ and holds by (5). In the last case we assume $\epsilon_{k-1} = 1$, thus $F_{k,0} = F_{k,1} = F_{k,2} = 1$ and

$$\mu(Z) = \beta^{-k}\frac{1}{\beta^{-2}+\beta^{-1}+1}.$$

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As above we get $\lambda(\phi_\beta(Z)) = \beta^{-(k+2)}$ and the result follows since

$$\beta^{-(k+2)} = \beta^{-k} \frac{1}{\beta^{-2} + \beta^{-1} + 1}$$

is equivalent to $\beta^{-3} + \beta^{-1} = 1$. The cases where $k < 3$ follow by the same arguments. □

As a consequence of Theorem 1 we can construct uniformly distributed two-dimensional sequences $(\phi_{\beta_1}(n), \phi_{\beta_2}(n))_{n\in\mathbb{N}}$, where β_1 is the characteristic root in Theorem 4, β_2 is the characteristic root of a numeration system in Theorem 2 and $\beta_1^k/\beta_2^l \notin \mathbb{Q}$ for all integers $k, l > 0$.

Theorem 4 extends the examples given in [**5**, Proposition 13,14], where the authors consider G-additive functions which lead to u.d. point sequences in the unit interval. Furthermore, it is possible to show that the one-dimensional point sequence in the previous theorem is a low-discrepancy sequence by mimicking the proof for the b-adic van der Corput sequence; see, for example, [**5, 7, 20**].

In [**8**], the authors present the so-called Kakutani–Fibonacci transformation, proving ergodicity on the unit interval and showing that the orbit of 0 is precisely the LS-sequence with parameters $L = S = 1$ defined in [**7**]. With our different approach we can show that this transformation is uniquely ergodic, thus the orbit of x under the transformation is u.d. for every $x \in [0, 1)$.

The Kakutani–Fibonacci transformation was first introduced and discussed in detail in [**8**] (for a related transformation see [**13**]). In the following we give an informal description.

Let us consider the unit interval $[0, 1)$ and split it into two consecutive subintervals $[0, \alpha)$ and $[\alpha, 1)$ of lengths α and α^2, respectively, where α is the reciprocal golden ratio $(\sqrt{5} - 1)/2$. So we obtain two systems of half-open intervals, in the first step $L_1 = \{[0, \alpha)\}$ ('long') and $S_1 = \{[\alpha, 1)\}$ ('short'). One can easily iterate this procedure in the following way. Let us split all intervals contained in L_1, proportionally to α^2 and α^3, respectively. Thus we obtain two systems of half-open intervals L'_2 and S'_2, the first one containing intervals of length α^2 and the second of length α^3, respectively.

This yields two new systems of half-open intervals, $L_2 = \{L'_2 \cup S_1\}$, $S_2 = S'_2$. Note that the ordering of the intervals is essential and follows from the illustration in the picture below.

In general, at the kth step, we split only the intervals in L_{k-1}, proportionally to α^k and α^{k+1} forming two systems of half-open intervals L'_k and S'_k, the first one containing intervals of length α^k and the second of length α^{k+1}, respectively. Again this yields a system of half-open intervals $L_k = \{L'_k \cup S_{k-1}\}$ and $S_k = S'_k$. This construction procedure is well known as the cutting-stacking method and the transformation associated to it is defined as the translation from the first interval of L_k onto the second, the second onto the third and so on, till the last interval of the column L_k. Then the transformation is given as a contraction with factor β from the top of L_k to the bottom of S_k, and then again as the translation from the first interval of S_k onto the same and so on.

In the following we give an explicit expression for the Kakutani–Fibonacci transformation T, which is the limit transformation of the above construction; see [**8**].

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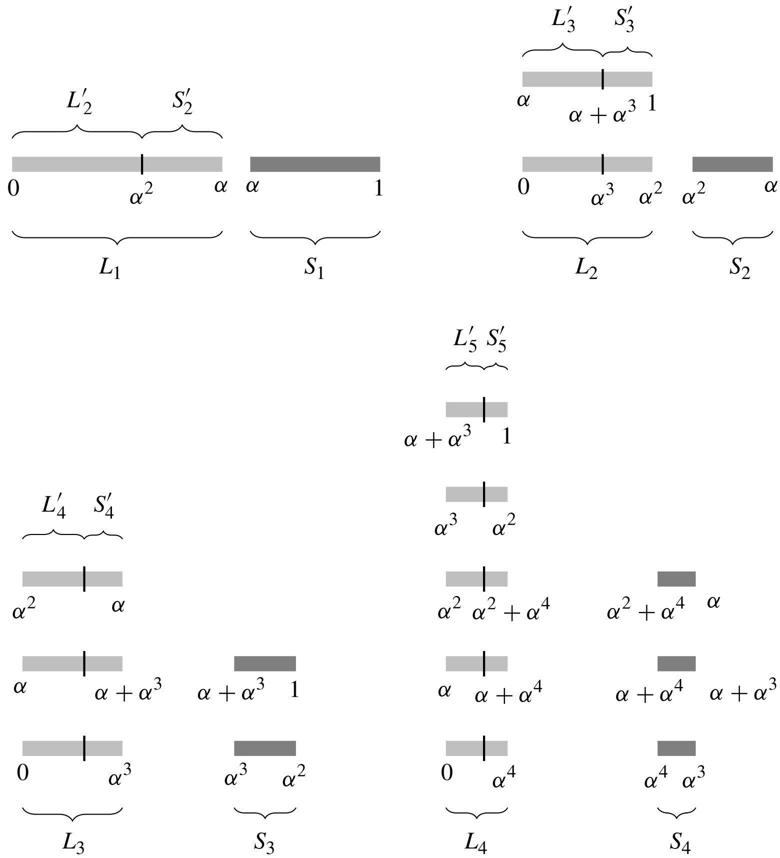

Definition 2. The Kakutani–Fibonacci transformation is a piecewise linear map $T : [0, 1) \to [0, 1)$, whose restriction to I_k is T_k, where

$$T_1 x = x + \alpha \quad \text{if } x \in I_1 = [0, \alpha^2)$$

and, for every $k \geq 1$,

$$T_{2k} x = x + \alpha^{2k} - \sum_{j=0}^{k-1} \alpha^{2j+1} \quad \text{if } x \in I_{2k} = \left[\sum_{j=0}^{k-1} \alpha^{2j+1}, \sum_{j=0}^{k} \alpha^{2j+1}\right)$$

and

$$T_{2k+1} x = x + \alpha^{2k+1} - \sum_{j=0}^{k-1} \alpha^{2(j+1)} \quad \text{if } x \in I_{2k+1} = \left[\sum_{j=0}^{k-1} \alpha^{2(j+1)}, \sum_{j=0}^{k} \alpha^{2(j+1)}\right).$$

To prove unique ergodicity of the Kakutani–Fibonacci transformation we need the following lemma.

LEMMA 3. *Let β be the golden ratio $(\sqrt{5}+1)/2$. Then we have $Tx = \phi_\beta \circ \tau \circ \phi_\beta^{-1} x$ for all β-adic rationals*

$$x = \sum_{i=1}^{k} \frac{\epsilon_i}{\beta^i}$$

with coefficients $\epsilon_i \in \{0, 1\}$.

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Proof. This follows from [**1**, Lemma 3] specializing $L = S = 1$. Applying this lemma combined with [**1**, Equation (2), §2] yields that the β-adic Monna map is bijective on the set of all β-adic rationals. Combining this with [**8**, Theorem 16] completes the proof of Lemma 3. □

THEOREM 5. *The Kakutani–Fibonacci transformation is uniquely ergodic.*

Proof. By Theorem 3, $\phi_\beta \circ \tau \circ \phi_\beta^{-1}$ is uniquely ergodic, thus by Lemma 3 and the denseness of the β-adic rationals T is uniquely ergodic. □

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