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AN INVERSE RESULT OF APPROXIMATION BY SAMPLING KANTOROVICH SERIES

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Abstract In the present paper, an inverse result of approximation, i.e. a saturation theorem for the sampling Kantorovich operators, is derived in the case of uniform approximation for uniformly continuous and bounded functions on the whole real line. In particular, we prove that the best possible order of approximation that can be achieved by the above sampling series is the order one, otherwise the function being approximated turns out to be a constant. The above result is proved by exploiting a suitable representation formula which relates the sampling Kantorovich series with the well-known generalized sampling operators introduced by Butzer. At the end, some other applications of such representation formulas are presented, together with a discussion concerning the kernels of the above operators for which such an inverse result occurs.

Keywords: inverse results; sampling Kantorovich series; order of approximation; central B-splines; generalized sampling operators; saturation theorem

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1. Introduction

The theory of sampling-type operators has been widely studied since 1980s, when, in order to provide an approximate version of the classical Whittaker–Kotel'nikov–Shannon sampling theorem (see e.g. [**30**, **31**]), Butzer introduced the generalized sampling operators G_w (see (4) of § 2) and studied their main properties (see e.g. [**12**, **38**]). The operators G_w allow us to reconstruct (in some sense) a given continuous signal f by a sequence of its sample values, which are of the form $f(k/w)$, $k \in \mathbb{Z}$, $w > 0$. Subsequently, such operators have been widely studied by many authors (see e.g. [**7**, **15**–**17**, **27**, **32**, **40**, **41**]).

In 2007, an L^1-version of the above operators was introduced, with the definition of the sampling Kantorovich series S_w (see (3) of § 2; [**6**]) obtained by replacing the sample values in G_w with the mean values $w \int_{k/w}^{(k+1)/w} f(u)\, du$, for any locally integrable signal f. The main advantage of the operators S_w compared with G_w is that a not necessarily

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continuous signal can also be approximated. Multivariate extensions of the above theory are given in [19, 20]. In the latter case, applications to digital images for earthquake engineering have been studied in [2, 13, 14]. The operators G_w and S_w are both based upon suitable kernel functions satisfying certain assumptions.

Recently, the sampling Kantorovich operators have been studied with respect to various aspects, e.g. the convergence in suitable spaces of function (see e.g. [43]), the order of approximation (see e.g. [21, 22, 37]), their behaviour at the discontinuity points of a given signal (see [26]), and so on. However, the following inverse problem is still open: whether there exists a positive non-increasing function $\varphi(w)$, $w \in \mathbb{R}^+$, with $\lim_{w\to+\infty} \varphi(w) = 0$, and a class of functions $\mathcal{K} \subseteq C(\mathbb{R})$ (the space of uniformly continuous and bounded functions) such that

$$\text{(I)} \quad \|S_w f - f\|_\infty = o(\varphi(w)) \text{ as } w \to +\infty, \text{ implies } f = \text{constant}.$$

The main result shown in the present paper consists of proving (I) with $\mathcal{K} = C(\mathbb{R})$ and $\varphi(w) = 1/w$, i.e. we prove that the best possible order of approximation that can be achieved by the sampling Kantorovich operators is 'one'. The main steps required in order to prove the above result are the following. First, we prove a representation formula for the sampling Kantorovich series in terms of the generalized sampling operators of f and its derivatives until order r, provided that they exist and are all uniformly continuous and bounded, namely, f belongs to the class $C^{(r)}(\mathbb{R})$, $r \in \mathbb{N}^+$. Subsequently, we obtain a saturation result for the subspace $C^{(2)}(\mathbb{R})$. Finally, we consider functions in $C(\mathbb{R})$, and by the regularization provided by the convolution with suitable test functions, we are able to prove a version of the desired result (I) by exploiting the inverse results for $C^{(2)}$-functions (see § 3).

The solution of problem (I) can open the way to obtaining a characterization of the saturation (Favard) classes of the approximation process defined by the sampling Kantorovich operators.

Note that the inverse result just discussed is quite different with respect to what happens in the case of operators G_w, where, in order to obtain a similar result, we would require that $f \in C^{(r)}(\mathbb{R})$, $r \in \mathbb{N}^+$; therefore, our problem cannot be solved using the result for G_w.

In the conclusion of § 3, we prove a further consequence of the above representation formula, by showing that under suitable assumptions on the kernels, the sampling Kantorovich operators map algebraic polynomials into other polynomials with the same degree. Examples of kernels for which the above results hold are provided in § 4.

2. Preliminaries

We first introduce some notation. In what follows, for any arbitrary finite or infinite interval $I \subseteq \mathbb{R}$, we denote by $C(I)$ the space of all uniformly continuous and bounded functions $f : I \to \mathbb{R}$, endowed with the supremum norm $\|f\|_\infty := \sup_{x\in I} |f(x)|$. Further, we denote by $C^{(r)}(I)$, $r \in \mathbb{N}^+$ the subspace of $C(I)$ for which the derivatives $f^{(s)}$ exist, for every $s \leq r$, $s \in \mathbb{N}^+$, and each $f^{(s)} \in C(I)$. Moreover, we define by $C_c(I)$ the subspace of $C(I)$ of functions having compact support, and similarly we can define $C_c^{(r)}(I)$, $r \in \mathbb{N}^+$. Finally, by $C_c^\infty(I)$ we denote the space of test functions, i.e. the space of functions with

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compact support which have continuous derivatives of any order, each one belonging to $C_c(I)$.

For any $f : \mathbb{R} \to \mathbb{R}$, we can define the discrete moment of order $\beta \in \mathbb{N}$, at point $u \in \mathbb{R}$, by:

$$m_\beta(f, u) := \sum_{k\in\mathbb{Z}} f(u-k)(u-k)^\beta, \tag{1}$$

and the discrete absolute moment of order $\beta \geq 0$, by:

$$M_\beta(f) := \sup_{u\in\mathbb{R}} \sum_{k\in\mathbb{Z}} |f(u-k)| \cdot |u-k|^\beta. \tag{2}$$

Clearly, if the function f belongs to $C_c(\mathbb{R})$, it turns out that both $m_\beta(f, u)$ and $M_\beta(f)$ are finite for every $\beta \in \mathbb{N}$, with $u \in \mathbb{R}$ and $\beta \geq 0$.

Now, we are able to recall the definition of the sampling Kantorovich operators, introduced in [6]:

$$(S_w f)(x) := \sum_{k\in\mathbb{Z}} \chi(wx-k)\left[w \int_{k/w}^{(k+1)/w} f(u)\, \mathrm{d}u\right], \quad x \in \mathbb{R}, \tag{3}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a locally integrable function, such that the above series is convergent for every $x \in \mathbb{R}$, and $\chi : \mathbb{R} \to \mathbb{R}$ is a kernel, i.e. a function which satisfies the following assumptions:

(χ1) χ belongs to $L^1(\mathbb{R})$ and is locally bounded at the origin;

(χ2) the series $\sum_{k\in\mathbb{Z}} \chi(u-k) = 1$, for every $u \in \mathbb{R}$;

(χ3) there exists $\beta > 0$ for which $M_\beta(\chi)$ is finite.

Note that, in general, it is possible to prove that (χ3) implies that $M_\nu(\chi)$ is finite, for every $0 \leq \nu \leq \beta$, see e.g. [6, 21, 22].

For instance, if we assume that $f \in L^\infty(\mathbb{R})$, it turns out that $S_w f \in L^\infty(\mathbb{R})$, i.e. S_w maps $L^\infty(\mathbb{R})$ to $L^\infty(\mathbb{R})$, see [6].

Moreover, under the assumptions (χi), $i = 1, 2, 3$, the family of the sampling Kantorovich series $S_w f$ converges to f pointwise at $x \in \mathbb{R}$, as $w \to +\infty$, provided that f is bounded and continuous at x; the convergence is uniform on $\mathbb{R}$ if f belongs to $C(\mathbb{R})$, see [6] again.

We recall that the sampling Kantorovich operators have been introduced in order to provide an L^1-version of the classical generalized sampling operators, which are defined by:

$$(G_w f)(x) := \sum_{k\in\mathbb{Z}} \chi(wx-k) f\left(\frac{k}{w}\right), \quad x \in \mathbb{R}, \tag{4}$$

with $w > 0$, and where χ is a kernel satisfying assumptions (χi), $i = 1, 2, 3$. Both the operators S_w and G_w are instances of 'quasi-interpolation' operators, see e.g. [29, 33, 39, 44].

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Pointwise and uniform convergence results analogous to those proved for the sampling Kantorovich series can be proved also for $(G_w f)_{w>0}$, see e.g. [12]. Now, we recall also the following high-order convergence result, which will be useful in the present paper.

Theorem 2.1 (Butzer and Stens [12]). *Let χ be a kernel that satisfies the following condition:*

$$m_j(\chi, u) := \begin{cases} 0, & j = 1, 2, \dots, r-1, \\ 1, & j = r, \end{cases} \tag{5}$$

for every $u \in \mathbb{R}$ and some $r \in \mathbb{N}^+$.
Then, for any $f \in C^{(r)}(\mathbb{R})$, it holds that:

$$\|G_w f - f\|_\infty \le \|f^{(r)}\|_\infty \frac{M_r(\chi)}{r!} w^{-r},$$

for every $w > 0$.
Moreover, the following property occurs:

$$(G_w p_{r-1})(x) = p_{r-1}(x), \quad x \in \mathbb{R},$$

for every $w > 0$, where $p_{r-1}(x)$ denotes any algebraic polynomial of degree $r-1$.

Conditions such as (5) are to be found in connection with finite element approximation, see e.g. [28].

In general, it can be difficult to check whether a given kernel χ satisfies Assumption (5. For this reason, the following lemma can be useful.

Lemma 2.2 (Butzer and Stens [12]). *Let χ be a continuous kernel. Condition (5) is equivalent to the following:*

$$(\widehat{\chi})^{(j)}(2\pi k) = \begin{cases} 1, & k = j = 0, \\ 0, & k \in \mathbb{Z} \setminus \{0\}, \quad j = 0, \\ 0, & k \in \mathbb{Z}, \quad j = 1, 2, \dots, r-1, \end{cases}$$

where $\widehat{\chi}(v) := \int_{\mathbb{R}} \chi(u) e^{-iuv}\, \mathrm{d}u$, $v \in \mathbb{R}$, denotes the Fourier transform of χ.

Note, for the sake of completeness, that no high-order approximation theorem for the sampling Kantorovich operators, analogous to the above, can be proved, see e.g. [3–5]. Moreover, the rate of convergence for the family $(S_w f)_{w>0}$ has also been studied in [21–23] in $C(\mathbb{R})$, and in the Orlicz spaces $L^\varphi(\mathbb{R})$, by considering functions in suitable Lipschitz classes.

3. Inverse result

In order to prove an inverse result for the sampling Kantorovich series, we need the following representation formula, which allows us to state the relation between the operators $S_w f$ and $G_w f$ when functions belonging to $C^{(r)}(\mathbb{R})$, $r \in \mathbb{N}^+$, are considered.

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Theorem 3.1. *For any $f \in C^{(r)}(\mathbb{R})$, $r \in \mathbb{N}^+$, it holds that:*

$$(S_w f)(x) = \sum_{j=0}^{r-1} \frac{w^{-j}}{(j+1)!} (G_w f^{(j)})(x) + \mathcal{R}_r^w(x), \quad x \in \mathbb{R},$$

where the remainder of order r is the following absolutely convergent series:

$$\mathcal{R}_r^w(x) := \frac{1}{r!} \sum_{k \in \mathbb{Z}} w \left[\int_{k/w}^{(k+1)/w} f^{(r)}(\theta_{k,w}(u)) \cdot (u - k/w)^r \, \mathrm{d}u \right] \chi(wx - k),$$

where $\theta_{k,w}(u)$ are measurable functions, such that $k/w < \theta_{k,w}(u) < (k+1)/w$, $k \in \mathbb{Z}$, for every $u \in [k/w, (k+1)/w]$, $w > 0$.

Proof. By considering the Taylor formula with the Lagrange remainder, applied to f, we have:

$$f(u) = \sum_{j=0}^{r-1} \frac{f^{(j)}(x)}{j!} (u - x)^j + \frac{f^{(r)}(\theta_{u,x})}{r!} (u - x)^r,$$

for x, $u \in \mathbb{R}$ and $\theta_{u,x} \in (x, u)$. Now, if we set $x = k/w$, $k \in \mathbb{Z}$ and $w > 0$ in the above formula, for every $u \in (k/w, (k+1)/w]$ it turns out that $k/w < \theta_{u,k/w} =: \theta_{k,w}(u) < (k+1)/w$. Then, replacing the above expansion with $x = k/w$ in the integrals $w \int_{k/w}^{(k+1)/w} f(u)\, \mathrm{d}u$, we can write the following:

$$\begin{aligned} w &\int_{k/w}^{(k+1)/w} f(u)\, \mathrm{d}u \\ &= w \int_{k/w}^{(k+1)/w} \left[\sum_{j=0}^{r-1} \frac{f^{(j)}(k/w)}{j!} (u - k/w)^j + \frac{f^{(r)}(\theta_{k,w}(u))}{r!} (u - k/w)^r \right] \mathrm{d}u \\ &= w \sum_{j=0}^{r-1} \frac{f^{(j)}(k/w)}{j!} \int_{k/w}^{(k+1)/w} (u - k/w)^j \, \mathrm{d}u + \frac{w}{r!} \int_{k/w}^{(k+1)/w} f^{(r)}(\theta_{k,w}(u))(u - k/w)^r \, \mathrm{d}u \\ &= \sum_{j=0}^{r-1} \frac{f^{(j)}(k/w)}{(j+1)!} w^{-j} + \frac{w}{r!} \int_{k/w}^{(k+1)/w} f^{(r)}(\theta_{k,w}(u))\, (u - k/w)^r \, \mathrm{d}u. \end{aligned} \tag{6}$$

Now, by exploiting (6) in the definition of $(S_w f)(x)$, $x \in \mathbb{R}$, we obtain:

$$(S_w f)(x) = \sum_{k \in \mathbb{Z}} \chi(wx - k) \left[\sum_{j=0}^{r-1} \frac{f^{(j)}(k/w)}{(j+1)!} w^{-j} + \frac{w}{r!} \int_{k/w}^{(k+1)/w} f^{(r)}(\theta_{k,w}(u))(u - k/w)^r \, \mathrm{d}u \right]$$

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$$= \sum_{j=0}^{r-1} \frac{w^{-j}}{(j+1)!} \sum_{k\in\mathbb{Z}} \chi(wx-k) f^{(j)}\left(\frac{k}{w}\right)$$

$$+ \frac{1}{r!} \sum_{k\in\mathbb{Z}} \chi(wx-k)\left[w \int_{k/w}^{(k+1)/w} f^{(r)}(\theta_{k,w}(u))(u-k/w)^r \,\mathrm{d}u\right]$$

$$= \sum_{j=0}^{r-1} \frac{w^{-j}}{(j+1)!} (G_w f^{(j)})(x) + \mathcal{R}_r^w(x),$$

for every $w > 0$, where:

$$\mathcal{R}_r^w(x) := \frac{1}{r!} \sum_{k\in\mathbb{Z}} \chi(wx-k)\left[w \int_{k/w}^{(k+1)/w} f^{(r)}(\theta_{k,w}(u))(u-k/w)^r \,\mathrm{d}u\right].$$

Note that the series $\mathcal{R}_r^w(x)$ is absolutely convergent for every $x \in \mathbb{R}$, for every $w > 0$. Indeed,

$$\frac{1}{r!} \sum_{k\in\mathbb{Z}} |\chi(wx-k)| \left| w \int_{k/w}^{(k+1)/w} f^{(r)}(\theta_{k,w}(u))(u-k/w)^r \,\mathrm{d}u \right|$$

$$\leq \frac{\|f^{(r)}\|_\infty}{(r+1)!} w^{-r} \sum_{k\in\mathbb{Z}} |\chi(wx-k)| \leq \frac{\|f^{(r)}\|_\infty}{(r+1)!} w^{-r} M_0(\chi) < +\infty. \tag{7}$$

This completes the proof. □

Remark 3.2. Note that it easily follows from (7) that the remainder $\mathcal{R}_r^w(x)$ in the representation formula of Theorem 3.1 is such that:

$$\mathcal{R}_r^w(x) = \mathcal{O}(w^{-r}) \quad \text{as } w \to +\infty,$$

for every $x \in \mathbb{R}$.

Now, we can state the main result of this section.

Theorem 3.3. *Let χ be a kernel that satisfies the moment condition (5) for every $u \in \mathbb{R}$, with $r = 2$. Now, let $f \in C(\mathbb{R})$, and suppose in addition that:*

$$\|S_w^\pi f - f\|_\infty = o(w^{-1}) \quad \text{as } w \to +\infty, \tag{8}$$

uniformly with respect to every sequence $\pi = (t_k)_{k\in\mathbb{Z}} \subset \mathbb{R}$, such that $\lim_{k\to\pm\infty} t_k = \pm\infty$, with $t_{k+1} - t_k = 1$, $k \in \mathbb{Z}$, and where:

$$(S_w^\pi f)(x) := \sum_{k\in\mathbb{Z}} \left[w \int_{t_k/w}^{t_{k+1}/w} f(u)\,\mathrm{d}u\right] \chi(wx - t_k), \quad x \in \mathbb{R}.$$

Then, f is constant over $\mathbb{R}$.

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Note that assumption (8), which involves the operators (3) for a general sampling scheme $\pi = (t_k)_{k\in\mathbb{Z}} \subseteq \mathbb{R}$, is meaningful and not restrictive, in view of the results concerning the order of approximation proved in [**21**], for the series S_w^π.

Moreover, we also point out that to prove the above theorem it is sufficient that the assumptions on χ are satisfied for the sequence $t_k = k$, $k \in \mathbb{Z}$ (as in the form given in § 2).

In order to obtain the proof of Theorem 3.3, we first prove the above result for functions belonging to $C^{(2)}(\mathbb{R})$. We have the following.

Theorem 3.4. *Let χ be a kernel that satisfies the moment condition (5) for every $u \in \mathbb{R}$, with $r = 2$. Now, let $f \in C^{(2)}(\mathbb{R})$, and suppose that:*

$$\|S_w f - f\|_\infty = o(w^{-1}) \quad \text{as } w \to +\infty. \tag{9}$$

Then, it turns out that f is constant on $\mathbb{R}$.

Proof. Since f belongs to $C^{(2)}(\mathbb{R})$, the representation formula of Theorem 3.1 can be applied, e.g. until order $r = 1$, i.e. for every $x \in \mathbb{R}$ we can write:

$$(S_w f)(x) = (G_w f)(x) + \mathcal{R}_1^w(x),$$

for every $w > 0$. Then, assumption (9) can be rewritten as follows:

$$|(G_w f)(x) + \mathcal{R}_1^w(x) - f(x)| = o(w^{-1}) \quad \text{as } w \to +\infty,$$

i.e.

$$\lim_{w\to+\infty} w[(G_w f)(x) + \mathcal{R}_1^w(x) - f(x)] = 0,$$

for every $x \in \mathbb{R}$. Now, splitting the above limit (since, as we will show below, they exist and are both finite), we can write:

$$\lim_{w\to+\infty} w[(G_w f)(x) - f(x)] + \tfrac{1}{2} \lim_{w\to+\infty} 2\, w\, \mathcal{R}_1^w(x) = 0. \tag{10}$$

Now, since (5) is satisfied for $r = 2$, in view of Theorem 2.1 we know that $\|G_w f - f\| = \mathcal{O}(w^{-2})$, as $w \to +\infty$. Then it is easy to see that:

$$\lim_{w\to+\infty} w[(G_w f)(x) - f(x)] = 0,$$

and so we can deduce from (10) that:

$$\lim_{w\to+\infty} 2w\mathcal{R}_1^w(x) = 0. \tag{11}$$

Now, we claim that the family $(2\, w\, \mathcal{R}_1^w)_{w>0}$ converges uniformly (then also pointwise) to f' on $\mathbb{R}$. In order to prove the above statement, we proceed by estimating:

$$|2w\mathcal{R}_1^w(x) - f'(x)| \le |2w\mathcal{R}_1^w(x) - (G_w f')(x)| + |(G_w f')(x) - f'(x)| =: I_1 + I_2,$$

$w > 0$. Let $\varepsilon > 0$ be fixed. Since f' is uniformly continuous and bounded, by the well-known convergence results concerning the generalized sampling series, we immediately

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have that $I_2 < \varepsilon$ for sufficiently large $w > 0$, see e.g. [**7**, **12**]. Now, we estimate I_1. We can write the following:

$$I_1 \leq \sum_{k\in\mathbb{Z}} \left| 2w^2 \int_{k/w}^{(k+1)/w} f'(\theta_{k,w}(u))(u-k/w)\,\mathrm{d}u - f'(k/w) \right| |\chi(wx-k)|.$$

For any $k \in \mathbb{Z}$ and sufficiently large $w > 0$, we have:

$$\begin{aligned} &\left| 2w^2 \int_{k/w}^{(k+1)/w} f'(\theta_{k,w}(u))(u-k/w)\,\mathrm{d}u - f'(k/w) \right| \\ &\quad = \left| 2w^2 \int_{k/w}^{(k+1)/w} f'(\theta_{k,w}(u))(u-k/w)\,\mathrm{d}u - 2w^2 f'(k/w) \int_{k/w}^{(k+1)/w} (u-k/w)\,\mathrm{d}u \right| \\ &\quad \leq\ 2\,w^2 \int_{k/w}^{(k+1)/w} |f'(\theta_{k,w}(u)) - f'(k/w)|\,(u-k/w)\,\mathrm{d}u, \qquad (12) \end{aligned}$$

where $k/w < \theta_{k,w}(u) < (k+1)/w$. Now, since f' is uniformly continuous, and $\theta_{k,w}(u) - k/w \leq 1/w$, we have, for $\varepsilon > 0$, that

$$|f'(\theta_{k,w}(u)) - f'(k/w)| < \varepsilon, \qquad (13)$$

for sufficiently large $w > 0$. Now, replacing (13) in (12) we finally obtain:

$$\left| 2w^2 \int_{k/w}^{(k+1)/w} f'(\theta_{k,w}(u))(u-k/w)\,\mathrm{d}u - f'(k/w) \right| < \varepsilon.$$

In conclusion, we have:

$$I_1 \leq \varepsilon \sum_{k\in\mathbb{Z}} |\chi(wx-k)| \leq \varepsilon M_0(\chi),$$

for $w > 0$ sufficiently large, and thus the above claim is now proved, i.e.

$$\lim_{w\to+\infty} 2\,w\,\mathcal{R}_1^w(x) = f'(x) \qquad (14)$$

for every $x \in \mathbb{R}$. Then, in view of (11) and (14), we obtain that $f'(x) = 0$ for every $x \in \mathbb{R}$, i.e, f is constant on the whole $\mathbb{R}$. □

Now, we are able to provide the proof of Theorem 3.3.

Proof of Theorem 3.3. Let $f \in C(\mathbb{R})$ be fixed, such that (8) is satisfied. Moreover, let $\Phi \in C_c^\infty(\mathbb{R})$ be a test function. We denote:

$$F_\Phi(x) := (\Phi * f)(x) = \int_{\mathbb{R}} \Phi(x-t) f(t)\,\mathrm{d}t, \quad x \in \mathbb{R},$$

where '$*$' is the usual convolution product. Note that $F_\Phi(x)$ is well defined, since f is continuous then it belongs to $L^1_{\mathrm{Loc}}(\mathbb{R})$, and in view of the regularization properties of '$*$',

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it turns out that F_Φ belongs, e.g. to $C^{(2)}(\mathbb{R})$. Indeed, it is easy to see that both the first and the second derivatives of F_Φ are uniformly continuous, together with F_Φ itself, in view of the uniform continuity of f. Now, for every fixed $x \in \mathbb{R}$, by exploiting condition (χ2) and the Fubini–Tonelli theorem, we can write the following:

$$\begin{aligned}
&(S_w F_\Phi)(x) - F_\Phi(x)\\
&= \sum_{k\in\mathbb{Z}} \left\{ w \int_{k/w}^{(k+1)/w} [F_\Phi(u) - F_\Phi(x)]\,\mathrm{d}u \right\} \chi(wx-k)\\
&= \sum_{k\in\mathbb{Z}} \left\{ w \int_{k/w}^{(k+1)/w} \left[\int_{\mathbb{R}} \Phi(u-t) f(t)\,\mathrm{d}t - \int_{\mathbb{R}} \Phi(x-t) f(t)\,\mathrm{d}t \right] \mathrm{d}u \right\} \chi(wx-k)\\
&= \sum_{k\in\mathbb{Z}} \left\{ w \int_{k/w}^{(k+1)/w} \left[\int_{\mathbb{R}} \Phi(y) f(x-y)\,\mathrm{d}y - \int_{\mathbb{R}} \Phi(y) f(u-y)\,\mathrm{d}y \right] \mathrm{d}u \right\} \chi(wx-k)\\
&= \sum_{k\in\mathbb{Z}} \left\{ w \int_{k/w}^{(k+1)/w} \left(\int_{\mathbb{R}} \Phi(y)[f(x-y) - f(u-y)]\,\mathrm{d}y \right) \mathrm{d}u \right\} \chi(wx-k)\\
&= \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} \Phi(y) \left\{ w \left(\int_{k/w}^{(k+1)/w} [f(x-y) - f(u-y)]\,\mathrm{d}u \right) \chi(wx-k) \right\} \mathrm{d}y.
\end{aligned}$$

Now, if we set:

$$\sum_{k\in\mathbb{Z}} \Phi(y) \left\{ w \left(\int_{k/w}^{(k+1)/w} [f(x-y) - f(u-y)]\,\mathrm{d}u \right) \chi(wx-k) \right\} =: \sum_{k\in\mathbb{Z}} h_k(y),$$

we have that the above series is absolutely convergent (hence also convergent) for every $y \in \mathbb{R}$, since:

$$\sum_{k\in\mathbb{Z}} |h_k(y)| \leq 2\|\Phi\|_\infty \|f\|_\infty M_0(\chi) < +\infty,$$

and, moreover, for every $n \in \mathbb{N}^+$:

$$\left| \sum_{k=-n}^{n} h_k(y) \right| \leq 2\|f\|_\infty M_0(\chi) |\Phi(y)| =: H(y), \quad y \in \mathbb{R},$$

with $H \in L^1(\mathbb{R})$. Then, by the Lebesgue dominated convergence theorem, we can write:

$$\begin{aligned}
&(S_w F_\Phi)(x) - F_\Phi(x)\\
&\quad = \int_{\mathbb{R}} \Phi(y) \left(\sum_{k\in\mathbb{Z}} \left\{ w \int_{k/w}^{(k+1)/w} [f(x-y) - f(u-y)]\,\mathrm{d}u \right\} \chi(wx-k) \right) \mathrm{d}y.
\end{aligned}$$

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Now, by setting $g_y(x) := f(x-y)$ for every $x \in \mathbb{R}$ and $y \in \mathbb{R}$, and using Hölder's inequality, we obtain:

$$
\begin{aligned}
|(S_w F_\Phi)(x) - F_\Phi(x)| &= \left| \int_{\mathbb{R}} \Phi(y)[g_y(x) - (S_w g_y)(x)]\,\mathrm{d}y \right| \\
&\leq \int_{\mathbb{R}} |\Phi(y)|\,|(S_w g_y)(x) - g_y(x)|\,\mathrm{d}y \leq \|\Phi\|_1\, \|S_w g_{(\cdot)} - g_{(\cdot)}\|_\infty, \quad (15)
\end{aligned}
$$

for every $x \in \mathbb{R}$, where:

$$
\|(S_w g_{(\cdot)})(x) - g_{(\cdot)}(x)\|_\infty = \sup_{y\in\mathbb{R}} |(S_w g_y)(x) - g_y(x)|, \tag{16}
$$

for fixed x and w. Now, using respectively the changes of variables $u - y = t$ and $k - yw =: t_k^{(y,w)}$, $k \in \mathbb{Z}$, we obtain:

$$
\begin{aligned}
(S_w g_y)(x) &= \sum_{k\in\mathbb{Z}} \left[w \int_{k/w}^{(k+1)/w} g_y(u)\,\mathrm{d}u \right] \chi(wx-k) \\
&= \sum_{k\in\mathbb{Z}} \left[w \int_{k/w}^{(k+1)/w} f(u-y)\,\mathrm{d}u \right] \chi(wx-k) \\
&= \sum_{k\in\mathbb{Z}} \left[w \int_{(k/w)-y}^{((k+1)/w)-y} f(t)\,\mathrm{d}t \right] \chi(wx-k) \\
&= \sum_{k\in\mathbb{Z}} \left[w \int_{(k-yw)/w}^{(k+1-yw)/w} f(t)\,\mathrm{d}t \right] \chi(wx-k) \\
&= \sum_{k\in\mathbb{Z}} \left[w \int_{t_k^{(y,w)}/w}^{(t_k^{(y,w)}+1)/w} f(t)\,\mathrm{d}t \right] \chi(wx - (t_k^{(y,w)} + yw)) \\
&= \sum_{k\in\mathbb{Z}} \left[w \int_{t_k^{(y,w)}/w}^{(t_k^{(y,w)}+1)/w} f(t)\,\mathrm{d}t \right] \chi(w(x-y) - t_k^{(y,w)}) = (S_w^{\pi_y^w} f)(x-y),
\end{aligned}
$$

where $\pi_y^w = (t_k^{(y,w)})_{k\in\mathbb{Z}}$ for every $y \in \mathbb{R}$. Now, it is easy to observe that $\lim_{k\to\pm\infty} t_k^{(y,w)} = \pm\infty$, and:

$$
t_{k+1}^{(y,w)} - t_k^{(y,w)} = k + 1 - yw - k + yw = 1,
$$

for every $k \in \mathbb{Z}$. Hence, (16) becomes:

$$
\sup_{y\in\mathbb{R}} |(S_w g_y)(x) - g_y(x)| = \sup_{y\in\mathbb{R}} |(S_w^{\pi_y^w} f)(x-y) - f(x-y)|.
$$

In view of the above equality, since all the sequences of the form π_y^w, $y \in \mathbb{R}$, $w > 0$, satisfy the conditions required in assumption (8), and $\|\Phi\|_1 < +\infty$, using (15) we finally have:

$$
\|S_w F_\Phi - F_\Phi\|_\infty = o(w^{-1}) \quad \text{as } w \to +\infty,
$$

for every test function $\Phi \in C_c^\infty(\mathbb{R})$. We have proved that any F_Φ satisfies the assumptions of Theorem 3.4, and it turns out that $F_\Phi(x) = k$ for every $x \in \mathbb{R}$, for a suitable constant

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$k \in \mathbb{R}$. Thus, for every $x \in \mathbb{R}$ we have:

$$\begin{aligned} 0 = F_\Phi(x) - F_\Phi(0) &= \int_{\mathbb{R}} \Phi(x-t) f(t)\,\mathrm{d}t - \int_{\mathbb{R}} \Phi(-t) f(t)\,\mathrm{d}t \\ &= \int_{\mathbb{R}} \Phi(y) f(-y)\,\mathrm{d}y - \int_{\mathbb{R}} \Phi(y) f(x-y)\,\mathrm{d}y = \int_{\mathbb{R}} \Phi(y)[f(-y) - f(x-y)]\,\mathrm{d}y, \end{aligned}$$

where the equality:

$$\int_{\mathbb{R}} \Phi(y)[f(-y) - f(x-y)]\,\mathrm{d}y = 0$$

holds for every test function $\Phi \in C_c^\infty(\mathbb{R})$.

Now, in order to conclude the proof, we suppose by contradiction that f is not constant on $\mathbb{R}$, i.e. that there exists $x_0 < y_0$ such that $f(x_0) \neq f(y_0)$. Now let $\widetilde{x} \in \mathbb{R}$ such that $\widetilde{x} + y_0 = x_0$, and let $n \in \mathbb{N}^+$ be sufficiently large, such that $y_0 \in I_n := (-n, n)$ (then also $-y_0 \in I_n$). Then, for every $\Phi \in C_c^\infty(I_n)$, we have:

$$\int_{-n}^{n} \Phi(y)[f(-y) - f(\widetilde{x} - y)]\,\mathrm{d}y = \int_{\mathbb{R}} \widetilde{\Phi}(y)[f(-y) - f(\widetilde{x} - y)]\,\mathrm{d}y = 0,$$

where $\widetilde{\Phi}$ denotes the zero-extension of Φ to the whole $\mathbb{R}$. Since the above equality holds for every $\Phi \in C_c^\infty(I_n)$, and f is continuous on $\mathbb{R}$, it turns out that (see [**10**]):

$$f(-y) - f(\widetilde{x} - y) = 0, \quad y \in (-n, n).$$

Now, setting $y = -y_0$ in the above equality, we finally obtain:

$$f(y_0) = f(\widetilde{x} + y_0) = f(x_0),$$

which is a contradiction. This completes the proof. □

In the conclusion of this section, we prove further nice properties of the sampling Kantorovich operators, which can be deduced from the representation formula achieved in Theorem 3.1.

Theorem 3.5. *Let χ be a kernel satisfying assumption (5) with $r \in \mathbb{N}^+$. Then:*

$$(S_w p_{r-1})(x) = \sum_{j=0}^{r-1} \frac{w^{-j}}{(j+1)!} p_{r-1}^{(j)}(x),$$

for every $w > 0$, and for any algebraic polynomials of degree at most $r-1$, i.e. S_w maps algebraic polynomials of degree at most $r-1$ into algebraic polynomials of the same degree.

Proof. The proof follows immediately from the representation formula of Theorem 3.1, the applications of Theorem 2.1 and, finally, observing that $p_{r-1}^{(r)}(x) = 0$ for every $x \in \mathbb{R}$. □

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4. The construction of the kernels

In § 2, the definition of a kernel for the sampling Kantorovich operators S_w (and also for G_w) was provided. Several examples of well-known functions χ which satisfy assumptions $(\chi 1)$, $(\chi 2)$ and $(\chi 3)$ are given e.g. in [**6**, **11**, **19**, **24**, **25**].

For instance, we can choose as kernels the following one-dimensional band-limited functions:

$$F(x) := \frac{1}{2}\left(\frac{\sin(\pi x/2)}{\pi x/2}\right)^2 \qquad \text{(Fejér's kernel)},$$

$$V(x) := \frac{3}{2\pi}\frac{\sin(x/2)\sin(3x/2)}{3x^2/4} \qquad \text{(de la Vallée Poussin's kernel)},$$

$$\chi(x) := \frac{\sin(\pi x/2)\sin(\pi x)}{\pi^2 x^2/2},$$

$$b^\alpha(x) := 2^\alpha \Gamma(\alpha+1)|x|^{-(n/2)+\alpha}\mathcal{B}_{(n/2)+\alpha}(|x|) \quad \text{(Bochner–Riesz kernels)},$$

where $\alpha > (n-1)/2$, $\mathcal{B}_\lambda$ is the Bessel function of order λ and Γ is the Euler function, and, finally,

$$J_k(x) = c_k \operatorname{sinc}^{2k}\left(\frac{x}{2k\pi\alpha}\right) \qquad \text{(Jackson-type kernels)}$$

with $k \in \mathbb{N}$, $\alpha \geq 1$, where the normalization coefficients c_k are given by

$$c_k := \left[\int_{\mathbb{R}} \operatorname{sinc}^{2k}\left(\frac{u}{2k\pi\alpha}\right) du\right]^{-1}.$$

Actually, the above examples of kernels can be used to show the convergence of the operators S_w and G_w, but they do not satisfy the moment condition (5), which we showed to be crucial in order to prove the inverse results of § 3.

Hence, here we briefly describe how it is possible to construct examples of kernels satisfying condition (5). The most convenient instances can be constructed by using the so-called central B-splines.

First of all, we recall that a function $q : I \to \mathbb{R}$ is called a (polynomial) spline of order $n \in \mathbb{N}^+$ (degree $n-1$) with knots $a_1 < a_2 < \cdots < a_m$ belonging to I if it coincides with a polynomial of degree $n-1$ on each of the intervals (a_i, a_{i+1}), $i = 1, 2, \ldots, m-1$ (see e.g. [**1**, **34**, **35**]).

The central B-splines of order $n \in \mathbb{N}^+$ are defined by:

$$M_n(x) := \frac{1}{(n-1)!}\sum_{i=0}^{n}(-1)^i \binom{n}{i}\left(\frac{n}{2}+x-i\right)_+^{n-1}, \quad x \in \mathbb{R}, \tag{17}$$

where $(x)_+ := \max\{x, 0\}$ denotes the 'positive part' of $x \in \mathbb{R}$, see e.g. [**11**, **42**]. They have knots at the points $0, \pm 1, \pm 2, \ldots, \pm n/2$ in the case where n is even, and at $\pm 1/2$, $\pm 3/2, \ldots, \pm n/2$ in the case where n is odd, and their support is the compact interval

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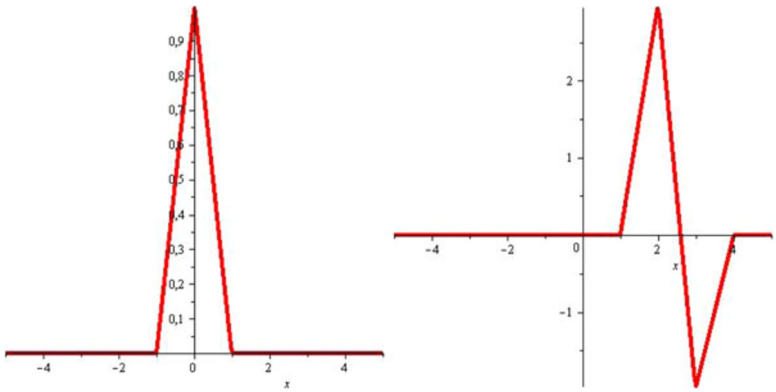

Figure 1. On the left: plot of the central B-spline of order 2 (roof-function). On the right: plot of the kernel χ_2 defined in (18).

$[-n/2, n/2]$. The Fourier transform of M_n (see e.g. [**36**]) is:

$$\widehat{M_n}(v) = \mathrm{sinc}(v/2)^n, \quad v \in \mathbb{R}.$$

The central B-splines M_n satisfy the assumptions ($\chi 1$), ($\chi 2$) and ($\chi 3$), i.e. M_n are kernels, see e.g. [**6**]. Now, we have the following classical theorem.

Theorem 4.1 (Butzer and Stens [12]). *For $r \in \mathbb{N}^+$, $r \geq 2$, let $\varepsilon_0 < \varepsilon_1 < \cdots < \varepsilon_{r-1}$ be any given real numbers, and let a_{μ_r}, $\mu = 0, 1, \ldots, r-1$, be the unique solutions of the linear system:*

$$\sum_{\mu=0}^{r-1} a_{\mu_r} (-i\varepsilon_\mu)^j = \left(\frac{1}{\widehat{M_r}} \right)^{(j)} (0),$$

for every $j = 0, 1, \ldots, r-1$, where i denotes the imaginary unit. Then:

$$\chi_r(x) := \sum_{\mu=0}^{r-1} a_{\mu_r} M_r(t - \varepsilon_\mu), \quad x \in \mathbb{R},$$

is a polynomial spline of order r, satisfying (5) and having support contained in $[\varepsilon_0 - r/2, \, \varepsilon_{r-1} + r/2]$.

For instance, an example of kernel generated as in Theorem 4.1 that satisfies (5) with $r = 2$ (see Figure 1) is the following:

$$\chi_2(x) = 3M_2(x-2) - 2M_2(x-3), \quad x \in \mathbb{R}. \tag{18}$$

By procedures similar to that described in Theorem 4.1, many other instances of kernels can be easily generated. For more details, and for other examples of kernels, see e.g. [**8**, **9**, **11**, **12**, **18**].

Moreover, for the reconstruction of signals in terms of splines using finite number of samples from the past, see [**12**, **26**].

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5. Conclusions

By using the representation formula proved in Theorem 3.1, we are able to obtain an inverse result for the sampling Kantorovich operators. In particular, we show that the best order of approximation that can be achieved for the aliasing error $\|S_w f - f\|_\infty$ is $\mathcal{O}(w^{-1})$, as $w \to +\infty$, for $f \in C^{(2)}(\mathbb{R})$ (Theorem 3.4). A similar result has been achieved on the space $C(\mathbb{R})$, as in Theorem 3.3.

Even if the above representation formula links the sampling Kantorovich operators with the generalized sampling operators of f and its derivatives, the proof of the above inverse result cannot be directly reconnected to the corresponding one for the generalized operators. Indeed, for the operators $S_w f$ it is not possible to establish an higher order of approximation theorem, which was revealed to be crucial for the proof of the aforementioned inverse result of [**12**] relative to G_w.

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