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SYMMETRIC BI-SKEW MAPS AND SYMMETRIZED MOTION PLANNING IN PROJECTIVE SPACES

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Abstract This work is motivated by the question of whether there are spaces X for which the Farber–Grant symmetric topological complexity $\mathrm{TC}^S(X)$ differs from the Basabe–González–Rudyak–Tamaki symmetric topological complexity $\mathrm{TC}^\Sigma(X)$. For a projective space $\mathbb{R}\mathrm{P}^m$, it is known that $\mathrm{TC}^S(\mathbb{R}\mathrm{P}^m)$ captures, with a few potential exceptional cases, the Euclidean embedding dimension of $\mathbb{R}\mathrm{P}^m$. We now show that, for all $m \geq 1$, $\mathrm{TC}^\Sigma(\mathbb{R}\mathrm{P}^m)$ is characterized as the smallest positive integer n for which there is a symmetric $\mathbb{Z}_2$-biequivariant map $S^m \times S^m \to S^n$ with a 'monoidal' behaviour on the diagonal. This result thus lies at the core of the efforts in the 1970s to characterize the embedding dimension of real projective spaces in terms of the existence of symmetric axial maps. Together with Nakaoka's description of the cohomology ring of symmetric squares, this allows us to compute both TC numbers in the case of $\mathbb{R}\mathrm{P}^{2^e}$ for $e \geq 1$. In particular, this leaves the torus $S^1 \times S^1$ as the only closed surface whose symmetric (symmetrized) TC^S (TC^Σ) invariant is currently unknown.

Keywords: topological complexity; symmetric motion planning; axial maps with further structure; equivariant partition of unity; symmetric square of a space

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1. Introduction

Farber's topological complexity of a space X, $\mathrm{TC}(X)$, can be defined as the sectional category* of the double evaluation map $e_{0,1}\colon \mathrm{P}(X) \to X \times X$, i.e. the fibration which sends a path $\gamma\colon [0,1] \to X$ into the ordered pair $e_{0,1}(\gamma) = (\gamma(0), \gamma(1))$. This concept, originally motivated by the motion planning problem in robotics [11], has been found to have interesting connections with classical problems in differential topology and homotopy theory. This paper develops on one such connection.

A number of variants of Farber's TC concept have been proposed as models of the motion planning problem in the presence of symmetries. One such line of investigation was initiated by Farber and Grant in [12], who considered the pullback (restricted) fibration

* We follow the standard normalization convention for the sectional category: a fibration with a global section has zero sectional category.

$\epsilon_{0,1}\colon \mathrm{P}^{\mathrm{op}}(X) \to X \times X - \Delta_X$ of $e_{0,1}$ under the inclusion $X \times X - \Delta_X \hookrightarrow X \times X$, where $\Delta_X = \{(x,x)\colon x \in X\}$ is the diagonal. Both $X \times X$ and $\mathrm{P}(X)$ come equipped with a natural switching involution τ, namely $\tau(x_1,x_2) = (x_2,x_1)$ and $(\tau \cdot \gamma)(t) = \gamma(1-t)$. The restricted involutions on $X \times X - \Delta_X$ and on the space of open paths $\mathrm{P}^{\mathrm{op}}(X)$ are fix-point free, and $\epsilon_{0,1}$ becomes a τ-fibration.

Definition 1.1. The symmetric topological complexity of a space X, $\mathrm{TC}^S(X)$, is one more than the τ-equivariant sectional category of $\epsilon_{0,1}\colon \mathrm{P}^{\mathrm{op}}(X) \to X \times X - \Delta_X$.

Thus, in the TC^S-view, one considers motion planners, i.e. local sections for $\epsilon_{0,1}$, for which the movement from an initial point A to a final point B (with $A \neq B$) is the time-reversed motion from B to A. 'One more than' in Definition 1.1 can be thought of as taking into account (a neighbourhood of) the diagonal when describing actual symmetric motion planners on X.

The fact that $\mathrm{TC}^S(X)$ is not a homotopy invariant of X is one of the motivations for introducing in [**2**] the following variant of Farber and Grant's TC^S.

Definition 1.2. The symmetrized topological complexity of a space X, $\mathrm{TC}^\Sigma(X)$, is the smallest positive integer n for which $X \times X$ can be covered by $n+1$ open sets U, each of which is closed under the switching involution τ on $X \times X$, and admits a continuous τ-equivariant section $U \to \mathrm{P}(X)$ of the (τ-equivariant) double evaluation map $e_{0,1}\colon \mathrm{P}(X) \to X \times X$.

As noted in [**16**, Example 2.6], $e_{0,1}$ is a τ-fibration, so $\mathrm{TC}^\Sigma(X)$ can equivalently be defined as $\mathrm{secat}_\tau(e_{0,1})$, the τ-equivariant sectional category of $e_{0,1}\colon \mathrm{P}(X) \to X \times X$.

Much of the interest in $\mathrm{TC}^\Sigma(X)$ comes from the fact that it is a homotopy invariant of X [**2**, Proposition 4.7]. Further, $\mathrm{TC}^\Sigma(X)$ differs from $\mathrm{TC}^S(X)$ by at most a unit. In fact, the inequalities

$$\mathrm{TC}^S(X) - 1 \leq \mathrm{TC}^\Sigma(X) \leq \mathrm{TC}^S(X) \tag{1.1}$$

hold for any reasonable space X (see [**2**, Proposition 4.2]).

The equality $\mathrm{TC}^\Sigma(X) = \mathrm{TC}^S(X)$ is known to hold for a number of spaces: spheres (see [**2**, Example 4.5] for even dimensional spheres and [**16**] for odd dimensional spheres), simply connected closed symplectic manifolds (as follows from [**12**, proof of Proposition 10] and [**13**, Theorem 1]; see [**15**, Theorem 6.1] for the case of complex projective spaces) and all closed surfaces with the potential exceptional case of the torus $S^1 \times S^1$ (see Remark 4.2 below). Surprisingly, except for homotopically uninteresting situations [**2**, Example 4.4], *no example of a space* X *with* $\mathrm{TC}^\Sigma(X) \neq \mathrm{TC}^S(X)$ *is known.* This paper explores the differences between the two invariants in the case of a real projective space $\mathbb{R}\mathrm{P}^m$ – one of the central benchmarks in homotopy theory.

Remark 1.3. For later reference, we note here that the possibility of numerically distinguishing $\mathrm{TC}^S(\mathbb{R}\mathrm{P}^m)$ from $\mathrm{TC}^\Sigma(\mathbb{R}\mathrm{P}^m)$ turns out to be amazingly subtle, complicated and closely related to a challenge with roots in Hopf's work [**17**]: a (hoped-for) characterization of the Euclidean embedding dimension of real projective spaces in terms of *symmetric* axial maps. Such a question was intensively studied in the 1970s, and it has still not been successfully answered.

A convenient way to approach the task set forward in Remark 1.3 is to start with a non-symmetric version of the problem.

For a real projective space $\mathbb{R}P^m$ which is not parallelizable, i.e. with $m \notin \{1, 3, 7\}$, the invariant $\mathrm{TC}(\mathbb{R}P^m)$ is known to agree with the number $\mathrm{Imm}(\mathbb{R}P^m)$ defined as the minimal dimension of Euclidean spaces where $\mathbb{R}P^m$ admits an immersion. In unrestricted terms, $\mathrm{TC}(\mathbb{R}P^m)$ agrees with the smallest positive integer n for which there is an axial map $a\colon \mathbb{R}P^m \times \mathbb{R}P^m \to \mathbb{R}P^n$, i.e. a map whose restriction to either of the axes is essential. Passing to universal covers, the above fact can be rephrased by saying that $\mathrm{TC}(\mathbb{R}P^m)$ is the smallest positive integer n for which there is a map $b\colon S^m \times S^m \to S^n$ which is $\mathbb{Z}_2$-biequivariant,* i.e. which satisfies $b(-x, y) = -b(x, y) = b(x, -y)$ for all $x, y \in S^m$ (see [**1**, **13**]).

The TC-Imm-axial phenomenon just described has a symmetric counterpart, summarized in (1.2) and (1.3) below. Let $\mathrm{Emb}(\mathbb{R}P^m)$ stand for the smallest dimension of Euclidean spaces where $\mathbb{R}P^m$ admits an embedding. Let $\mathrm{sb}(m)$ stand for the smallest positive integer n for which there exists a *symmetric* axial map $\mathbb{R}P^m \times \mathbb{R}P^m \to \mathbb{R}P^n$, i.e. an axial map which is $\mathbb{Z}_2$-equivariant with respect to the switching-axes involution τ on $\mathbb{R}P^m \times \mathbb{R}P^m$ and the trivial involution on $\mathbb{R}P^n$. Equivalently, $\mathrm{sb}(m)$ denotes the smallest positive integer n for which there exists a $\mathbb{Z}_2$-biequivariant $b\colon S^m \times S^m \to S^n$ which is symmetric, i.e. so that $b(x, y) = b(y, x)$ for all $x, y \in S^m$.

Remark 1.4. At the level of projective spaces, the difference $\mathrm{sb}(m) - m$ can be thought of as giving a measure of the failure of $\mathbb{R}P^m$ to be a (strictly) commutative H-space. We work with $\mathbb{Z}_2$-biequivariant maps, rather than with their axial-map counterparts, as the characterization of $\mathrm{TC}^\Sigma(\mathbb{R}P^m)$ (in Theorem 1.6 below) is naturally given in terms of a slight specialization of the former maps.

It is known that $\mathrm{TC}^S(\mathbb{R}P^m) \leq \mathrm{Emb}(\mathbb{R}P^m)$ for any m, and in fact

$$\mathrm{TC}^S(\mathbb{R}P^m) = \mathrm{Emb}(\mathbb{R}P^m), \tag{1.2}$$

except *possibly* for $m \in \{6, 7, 11, 12, 14, 15\}$ (see [**7**, **14**, **15**]). In addition, the main result in [**4**] asserts that $\mathrm{Emb}(\mathbb{R}P^m)$ agrees, up to 1, with $\mathrm{sb}(m)$. Explicitly,

$$\mathrm{Emb}(\mathbb{R}P^m) - 1 \leq \mathrm{sb}(m) \leq \mathrm{Emb}(\mathbb{R}P^m) \tag{1.3}$$

where the first inequality is asserted only if the 'metastable range' condition $2\,\mathrm{sb}(m) > 3m$ holds (e.g. for $m > 15$).

To the best of our knowledge, the gap in (1.3) has not been solved in either direction for general m. In fact, despite the fact that $\mathrm{Emb}(\mathbb{R}P^m)$ has been studied extensively, no explicit projective space $\mathbb{R}P^m$ with $m > 1$ and

$$\mathrm{sb}(m) < \mathrm{Emb}(\mathbb{R}P^m) \tag{1.4}$$

seems to have been singled out in the literature (although the slightly related Example 2 in [**3**, p. 415] should be noted). The problem can be approached via $\mathrm{TC}^\Sigma(\mathbb{R}P^m)$, which sits in a subtle way in between the two terms in (1.4). In fact, our main results (Theorems 1.6

* The term 'bi-skew' has been used in the literature as an alternative for '$\mathbb{Z}_2$-biequivariant'.

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and 1.7 below) are motivated by comparing (1.1)–(1.3). Specifically, we show that the two middle terms in the inequalities

$$\mathrm{TC}^S(\mathbb{R}\mathrm{P}^m) - 1 \leq \mathrm{TC}^\Sigma(\mathbb{R}\mathrm{P}^m) \leq \mathrm{TC}^S(\mathbb{R}\mathrm{P}^m),$$
$$\mathrm{TC}^S(\mathbb{R}\mathrm{P}^m) - 1 \leq \mathrm{sb}(m) \leq \mathrm{TC}^S(\mathbb{R}\mathrm{P}^m)$$

(the second chain of inequalities holding, say, for $m > 15$) can be characterized geometrically in *almost* the same way. In order to make precise the last assertion we need the following definition.

Definition 1.5. $\overline{\mathrm{sb}}(m)$ stands for the smallest positive integer n for which there is a symmetric $\mathbb{Z}_2$-biequivariant map $b\colon S^m \times S^m \to S^n$ with the property that the image under b of the diagonal $\Delta_{S^m} = \{(x,x)\colon x \in S^m\} \subset S^m \times S^m$ does not intersect some n-dimensional subspace of $\mathbb{R}^{n+1}$.

Of course, after a suitable rotation of S^n, we can assume that the first Euclidean coordinate $b_0\colon S^m \times S^m \to \mathbb{R}$ of the map $b = (b_0, b_1, \ldots, b_n)$ in Definition 1.5 satisfies $b_0(x,x) > 0$ for all $x \in S^m$.

Theorem 1.6. *For $m \geq 1$,*

$$\mathrm{sb}(m) \leq \overline{\mathrm{sb}}(m) = \mathrm{TC}^\Sigma(\mathbb{R}\mathrm{P}^m) \leq \mathrm{TC}^S(\mathbb{R}\mathrm{P}^m) \leq \mathrm{Emb}(\mathbb{R}\mathrm{P}^m).$$

Before specializing the above result, we highlight three interesting points in connection with the content and scope of Theorem 1.6.

Start by recalling that the relations in (1.2) and (1.3) are asserted for slightly restricted values of m. More explicitly, the inequality $\mathrm{Emb}(\mathbb{R}\mathrm{P}^m) - 1 \leq \mathrm{sb}(m)$ is currently known to hold for all values of m, except possibly $m \in \{5, 6, 7, 9, 11, 12, 15\}$. Actually, if attention is restricted to the TC^S–sb relationship, then the inequality $\mathrm{TC}^S(\mathbb{R}\mathrm{P}^m) - 2 \leq \mathrm{sb}(m)$ is currently known to hold for all values of m, except possibly $m = 12$. Yet, the characterization $\mathrm{TC}^\Sigma(\mathbb{R}\mathrm{P}^m) = \overline{\mathrm{sb}}(m)$ in Theorem 1.6 holds *without restrictions* on m. Loosely speaking, Theorem 1.6 implies that the TC^Σ-analogue of (1.4) can be ruled out effectively by strengthening slightly the definition of symmetric $\mathbb{Z}_2$-biequivariant maps.

Additionally, we have noted that the equality $\mathrm{TC}^S(\mathbb{R}^m) = \mathrm{Emb}(\mathbb{R}^m)$ and the inequality $\mathrm{Emb}(\mathbb{R}\mathrm{P}^m) - 1 \leq \mathrm{sb}(m)$ both hold for virtually all values of m. In those cases, at most one of the first two inequalities in the conclusion of Theorem 1.6 could fail to be an equality. The subtle (and interesting) point is the possibility that the potential failing inequality (if it really existed) could depend on m.

Last, the characterization $\mathrm{TC}^\Sigma(\mathbb{R}\mathrm{P}^m) = \overline{\mathrm{sb}}(m)$ in Theorem 1.6 can be thought of as the first step in the hoped-for characterization of the embedding dimension of real projective spaces in terms of symmetric axial maps (Remark 1.3). What remains open, of course, is a discussion of the potential difference (if any) between $\mathrm{TC}^\Sigma(\mathbb{R}\mathrm{P}^m)$ and $\mathrm{TC}^S(\mathbb{R}\mathrm{P}^m)$, as well as between $\mathrm{sb}(m)$ and $\overline{\mathrm{sb}}(m)$.

We also prove the following.

Theorem 1.7. *All three inequalities in the conclusion of Theorem 1.6 are sharp, provided that $m = 2^e$ with $e \geq 1$:* $\mathrm{Emb}(\mathbb{R}\mathrm{P}^{2^e}) = \mathrm{TC}^S(\mathbb{R}\mathrm{P}^{2^e}) = \mathrm{TC}^\Sigma(\mathbb{R}\mathrm{P}^{2^e}) = \overline{\mathrm{sb}}(2^e) = \mathrm{sb}(2^e) = 2^{e+1}$.

Theorem 1.7 should be compared with the fact that $\mathrm{TC}(\mathbb{R}\mathrm{P}^{2^e}) = \mathrm{Imm}(\mathbb{R}\mathrm{P}^{2^e}) = 2^{e+1} - 1$, for $e \geq 1$.

Remark 1.8. The case $e = 0$ in Theorem 1.7 is indeed exceptional in that, while $\mathrm{sb}(1) = 1$ is obvious (owing to the multiplication of complex numbers of norm 1), the equality $\mathrm{TC}^{\Sigma}(S^1) = 2$ is asserted in [**6**, **16**] after subtle considerations. In the final section of this paper, we offer a streamlined proof of the equality $\mathrm{TC}^{\Sigma}(S^1) = 2$.

It is tempting to think of the agreement between TC^{Σ} and its monoidal version (asserted in [**16**, Theorem 5.2]) as indirect evidence for the possibility that $\overline{\mathrm{sb}}(m) = \mathrm{sb}(m)$. However, this would have to be done with care in view of Remark 1.8.

We do not expect the equality $\mathrm{TC}^{S} = \mathrm{TC}^{\Sigma}$ in Theorem 1.7 to be generic; we believe that the equality $\mathrm{TC}^{S}(X) = \mathrm{TC}^{\Sigma}(X)$ would have to fail even for reasonably well-behaved spaces X. In other words, it is hard to think that considering a neighbourhood of the diagonal on its own would have to lead to the most efficient way to symmetrically motion plan. It would be interesting if the equality $\mathrm{TC}^{S} = \mathrm{TC}^{\Sigma}$ actually failed for some $\mathbb{R}\mathrm{P}^m$, as then the inequality $\mathrm{Emb}(\mathbb{R}\mathrm{P}^m) \neq \mathrm{sb}(m)$ would be forced (as would be the equality $\mathrm{sb}(m) = \overline{\mathrm{sb}}(m)$, at least if $m \geq 16$).

2. TC^{S}, TC^{Σ} and equivariant partitions of unity

Although the inequality $\overline{\mathrm{sb}}(m) \leq \mathrm{TC}^{\Sigma}(\mathbb{R}\mathrm{P}^m)$ in Theorem 1.6 follows easily from the asserted characterization $\overline{\mathrm{sb}}(m) = \mathrm{TC}^{\Sigma}(\mathbb{R}\mathrm{P}^m)$, it is convenient to start with the following.

Proof of the inequality $\overline{\mathrm{sb}}(m) \leq \mathrm{TC}^{\Sigma}(\mathbb{R}\mathrm{P}^m)$ in Theorem 1.6. Let $U_0, \ldots, U_n$ be a covering of $\mathbb{R}\mathrm{P}^m \times \mathbb{R}\mathrm{P}^m$ (say $n = \mathrm{TC}^{\Sigma}(\mathbb{R}\mathrm{P}^m)$) by open sets, each of which:

- is closed under the swapping involution $\tau((L_1, L_2)) = (L_2, L_1)$ of $\mathbb{R}\mathrm{P}^m \times \mathbb{R}\mathrm{P}^m$; and
- admits a τ-equivariant section $s_i \colon U_i \to \mathrm{P}(\mathbb{R}\mathrm{P}^m)$ for the double evaluation map $e_{0,1} \colon \mathrm{P}(\mathbb{R}\mathrm{P}^m) \to \mathbb{R}\mathrm{P}^m \times \mathbb{R}\mathrm{P}^m$. (Recall from the introduction that τ acts on the path space $\mathrm{P}(\mathbb{R}\mathrm{P}^m)$ by $\tau(\gamma)(t) = \gamma(1-t)$.)

Take a τ-equivariant partition of unity $\{h_i\}$ subordinate to the cover $\{U_i\}_i$, i.e. a family of continuous functions $h_i \colon \mathbb{R}\mathrm{P}^m \times \mathbb{R}\mathrm{P}^m \to [0,1]$, $0 \leq i \leq n$, satisfying:

(i) $h_i(L_1, L_2) = h_i(L_2, L_1)$, for $L_1, L_2 \in \mathbb{R}\mathrm{P}^m$;

(ii) $\mathrm{supp}(h_i) \subseteq U_i$;

(iii) $\max\{h_i(L_1, L_2) \colon 0 \leq i \leq n\} = 1$ for each $(L_1, L_2) \in \mathbb{R}\mathrm{P}^m \times \mathbb{R}\mathrm{P}^m$.

(We will make use of Schwarz's cone model for the joining of spaces, which explains our use of the form (2) for the alternative condition $\sum_i h_i = 1$ in the definition of a partition of a unit.) For the existence of such a partition, see, for instance, [**16**, Lemma 3.2], [**20**, p. 321] or, more generally, [**21**, Theorem 5.2.5].

Recall the factorization $\mathrm{P}(\mathbb{R}\mathrm{P}^m) \xrightarrow{f} S^m \times_{\mathbb{Z}_2} S^m \xrightarrow{\pi} \mathbb{R}\mathrm{P}^m \times \mathbb{R}\mathrm{P}^m$ of $e_{0,1}$, where

- the middle space is the Borel construction $S^m \times S^m \,/(x,y) \sim (-x,-y)$;

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- $f(\gamma)$ is the class of the pair $(\widetilde{\gamma}(0), \widetilde{\gamma}(1))$, where $\widetilde{\gamma}$ is a lifting of $\gamma\colon [0,1] \to \mathbb{R}\mathrm{P}^m$ through the usual double covering $S^m \to \mathbb{R}\mathrm{P}^m$;
- $\pi([(x_1, x_2)]) = (L_{x_1}, L_{x_2})$, where L_{x_i} is the line determined by x_i.

The maps $\sigma_i := f \circ s_i$ are τ-equivariant local sections of π, where τ acts on $S^m \times_{\mathbb{Z}_2} S^m$ by $\tau \cdot ([(x,y)]) = [(y,x)]$. Since π is a $\mathbb{Z}_2$-principal fibration, where the generator g of $\mathbb{Z}_2$ acts on $S^m \times_{\mathbb{Z}_2} S^m$ via the formula

$$g \cdot [(x,y)] = [(-x,y)] = [(x,-y)], \tag{2.1}$$

σ_i yields a trivialization of the restriction of π to U_i, i.e. a $\mathbb{Z}_2$-equivariant homeomorphism $\lambda_i\colon \pi^{-1}(U_i) \to \mathbb{Z}_2 \times U_i$ characterized by the condition

$$\lambda_i(x) = (g^\epsilon, \pi(x)) \quad \text{where } \epsilon \in \{0,1\} \text{ and } x = g^\epsilon \cdot \sigma_i(\pi(x)). \tag{2.2}$$

Note that $\mathbb{Z}_2 \times U_i$ inherits a τ-involution via λ_i; in fact, since the action (2.1) commutes with that of τ, we see that this inherited τ-involution on $\mathbb{Z}_2 \times U_i$ takes the form

$$\tau \cdot (g^\epsilon, (L_1, L_2)) = (g^\epsilon, \tau \cdot (L_1, L_2)) = (g^\epsilon, (L_2, L_1)). \tag{2.3}$$

Let $C\mathbb{Z}_2$ stand for the cone $\mathbb{Z}_2 \times [0,1]/(g,0) \sim (g^2,0)$ – an interval $[0,1]$ in disguise. As observed by Schwarz [**22**, p. 87], the composition of λ_i with the map $\mu_i\colon \mathbb{Z}_2 \times U_i \to C\mathbb{Z}_2$ given by $\mu_i(g^\epsilon, u) = [(g^\epsilon, h_i(u))]$ extends to a (continuous) map $\Lambda_i\colon S^m \times_{\mathbb{Z}_2} S^m \to C\mathbb{Z}_2$ which is $\mathbb{Z}_2$-equivariant (g acts 'horizontally' on the cone: $g \cdot [(g^\epsilon, t)] = [(g^{1+\epsilon}, t)]$). Further, in the present situation,

(iv) Λ_i is τ-invariant, i.e. $\Lambda_i([x,y]) = \Lambda_i([y,x])$ for $x, y \in S^m$, in view of (i) and (2.3).

Of course, Schwarz's goal is to obtain that, in view of (iii), the product $\prod_i \Lambda_i$ yields a $\mathbb{Z}_2$-equivariant map $\Lambda\colon S^m \times_{\mathbb{Z}_2} S^m \to (\mathbb{Z}_2)^{*(n+1)} = S^n$, which is τ-invariant in view of (iv). Consequently, the composition of the canonical projection $S^m \times S^m \to S^m \times_{\mathbb{Z}_2} S^m$ with Λ yields a symmetric $\mathbb{Z}_2$-biequivariant map, completing the proof of Theorem 1.6. □

The proof above can be used to show the strengthened inequality $\overline{\mathrm{sb}}(m) \leq \mathrm{TC}^\Sigma(\mathbb{R}\mathrm{P}^m)$ in Theorem 1.6.

Proof of the inequality $\overline{\mathrm{sb}}(m) \leq \mathrm{TC}^\Sigma(\mathbb{R}\mathrm{P}^m)$ in Theorem 1.6. In view of [**16**, Theorem 5.2], we can start with an open covering $U_0, \ldots, U_n$ of $\mathbb{R}\mathrm{P}^m \times \mathbb{R}\mathrm{P}^m$ (say $n = \mathrm{TC}^\Sigma(\mathbb{R}\mathrm{P}^m)$) so that each U_i:

- contains the diagonal $\Delta_{\mathbb{R}\mathrm{P}^m}$;
- is closed under the swapping involution $\tau((L_1, L_2)) = (L_2, L_1)$ of $\mathbb{R}\mathrm{P}^m \times \mathbb{R}\mathrm{P}^m$; and
- admits a τ-equivariant section $s_i\colon U_i \to \mathrm{P}(\mathbb{R}\mathrm{P}^m)$ for the double evaluation map $e_{0,1}\colon \mathrm{P}(\mathbb{R}\mathrm{P}^m) \to \mathbb{R}\mathrm{P}^m \times \mathbb{R}\mathrm{P}^m$ such that, for all $L \in \mathbb{R}\mathrm{P}^m$, $s_i(L,L)$ is the constant path at L.

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We then proceed as in the previous proof, to find that $\sigma_i(L, L) = [(x, x)]$ whenever $x \in L$, so that $\lambda_i([(x, x)]) = (g^0, (L, L))$ for all $x \in S^m$. This immediately implies that the resulting symmetric $\mathbb{Z}_2$-biequivariant map

$$S^m \times S^m \to S^m \times_{\mathbb{Z}_2} S^m \xrightarrow{\Lambda} S^n = (\mathbb{Z}_2)^{*(n+1)} \subset \prod_{i=0}^{n} C\mathbb{Z}_2$$

sends the diagonal Δ_{S^m} into the simplex generated by the various neutral elements g^0 of each factor $C\mathbb{Z}_2$. □

Proof of the inequality $\overline{\mathrm{sb}}(m) \leq \mathrm{TC}^S(\mathbb{R}\mathrm{P}^m)$ in Theorem 1.6. Let $n = \mathrm{TC}^s(\mathbb{R}\mathrm{P}^m)$ and pick a covering $U_1, \ldots, U_n$ of $\mathbb{R}\mathrm{P}^m \times \mathbb{R}\mathrm{P}^m - \Delta_{\mathbb{R}\mathrm{P}^m}$ by open sets which are closed under the switching-axes involution τ, each with a τ-equivariant section $s_i \colon U_i \to \mathrm{P}(\mathbb{R}\mathrm{P}^m)$ for the double evaluation map $e_{0,1}$. As noted in the proof of [**12**, Corollary 9], we can also pick an open neighbourhood U_0 of $\Delta_{\mathbb{R}\mathrm{P}^m}$ in $\mathbb{R}\mathrm{P}^m \times \mathbb{R}\mathrm{P}^m$, which is closed under the action of τ, together with a τ-equivariant section $s_0 \colon U_0 \to \mathrm{P}(\mathbb{R}\mathrm{P}^m)$ of $e_{0,1}$ with the property that, for each line $L \in \mathrm{P}(\mathbb{R}\mathrm{P}^m)$, $s_0(L, L)$ is the constant path at L. Then we are in the situation at the start of the proof of the inequality $\mathrm{sb}(m) \leq \mathrm{TC}^\Sigma(\mathbb{R}\mathrm{P}^m)$, so we apply the same constructions (using the same notation), except that this time we can assume the additional property that the τ-equivariant partition of unity satisfies $h_0(L, L) = 1$ for all $L \in \mathbb{R}\mathrm{P}^m$. In such a setting, it follows that $\Lambda_0([x, x]) = (g^0, 1)$ and $\Lambda_i([x, x]) = (g^0, 0) = (g, 0)$ for all $x \in S^m$ and all $i > 0$. Therefore, the resulting $\Lambda \colon S^m \times_{\mathbb{Z}_2} S^m \to S^n$ is now constant on points of the form $[x, x]$, and the corresponding symmetric $\mathbb{Z}_2$-biequivariant map $S^m \times S^m \to S^n$ is constant on the diagonal Δ_{S^m}. □

The proof of Theorem 1.6 will be complete once we show (in the next section) the inequality $\mathrm{TC}^\Sigma(\mathbb{R}\mathrm{P}^m) \leq \overline{\mathrm{sb}}(m)$. (In view of (1.1), the proof we have just given for the inequality $\overline{\mathrm{sb}}(m) \leq \mathrm{TC}^S(\mathbb{R}\mathrm{P}^m)$ can be waived; we included the additional idea in support of Remark 3.1 below.)

3. Symmetrized motion rules

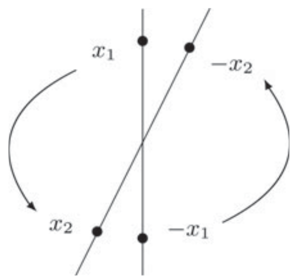
Definition 1.5 allows us to apply, word for word, the proof of [**13**, Proposition 6.3] in order to complete the proof of Theorem 1.6. This short section includes the easy details for completeness.

Proof of the inequality $\mathrm{TC}^\Sigma(\mathbb{R}\mathrm{P}^m) \leq \overline{\mathrm{sb}}(m)$ in Theorem 1.6. Let $n = \overline{\mathrm{sb}}(m)$ and pick a symmetric $\mathbb{Z}_2$-biequivariant map $b = (b_0, \ldots, b_n) \colon S^m \times S^m \to S^n$ such that

$$b_0(x, x) > 0 \quad \text{for all } x \in S^m. \tag{3.1}$$

For $0 \leq i \leq n$, set $V_i' = V_i - \Delta_{\mathbb{R}\mathrm{P}^m}$, where V_i is the image under the projection $\pi \colon S^m \times S^m \to \mathbb{R}\mathrm{P}^m \times \mathbb{R}\mathrm{P}^m$ of the set $U_i = \{(x, y) \in S^m \times S^m \colon b_i(x, y) \neq 0\}$. All sets U_i, V_i and V_i' are open, and are closed under the action of the corresponding switching-axes involutions τ. Furthermore, τ-equivariant (continuous) sections $s_i \colon V_i' \to \mathrm{P}(\mathbb{R}\mathrm{P}^m)$ for the double evaluation map $e_{0,1} \colon \mathrm{P}(\mathbb{R}\mathrm{P}^m) \to \mathbb{R}\mathrm{P}^m \times \mathbb{R}\mathrm{P}^m$ are defined as follows.

For $(L_1, L_2) \in V_i'$ there are four pairs $(\pm x_1, \pm x_2) \in U_i \cap \pi^{-1}(L_1, L_2)$. Only two of these, say (x_1, x_2) and $(-x_1, -x_2)$, have positive images under b_i. We then set $s_i(L_1, L_2)$ to be the path in $\mathrm{P}(\mathbb{R}\mathrm{P}^m)$ corresponding to the rotation from L_1 to L_2, through the plane these lines generate, so that x_1 rotates toward x_2 through an angle less than 180°. As illustrated below, the resulting path $s_i(L_1, L_2)$ does not depend on whether (x_1, x_2) or $(-x_1, -x_2)$ is used.



Because of (3.1), s_0 extends to a continuous τ-equivariant section of $e_{0,1}$ on V_0, so that $s_0(L, L)$ is the constant path (with constant value L). The proof is complete since V_0, V_1, $\ldots$, V_n cover $\mathbb{R}\mathrm{P}^m \times \mathbb{R}\mathrm{P}^m$. □

In view of (1.3), Theorem 1.6 implies that instances with

$$\mathrm{TC}^{\Sigma}(\mathbb{R}\mathrm{P}^m) < \mathrm{TC}^{S}(\mathbb{R}\mathrm{P}^m) \tag{3.2}$$

could only happen when optimal embeddings of $\mathbb{R}\mathrm{P}^m$ are not realizable by symmetric axial maps – a possibility that, to the best of our knowledge, cannot be currently overruled for $m > 1$. Furthermore, the equalities $\mathrm{TC}^{\Sigma}(\mathbb{R}\mathrm{P}^m) = \overline{\mathrm{sb}}(m) = \mathrm{sb}(m)$ would be forced whenever (3.2) holds (here we are implicitly assuming that m lies in the range where the first inequality in (1.3) holds).

Remark 3.1. A close look at the techniques in this and the previous section reveals that, for any $m \geq 1$, $\mathrm{TC}^{\Sigma}(\mathbb{R}\mathrm{P}^m)$ agrees with the smallest positive integer n for which there is a symmetric $\mathbb{Z}_2$-biequivariant map $S^m \times S^m \to S^n$ which is constant on the diagonal. The latter fact is the right symmetrization of the corresponding property for $\mathrm{TC}(\mathbb{R}\mathrm{P}^m)$, although the proof in the non-symmetric case reduces to the simpler homotopy fact that an axial map $\mathbb{R}\mathrm{P}^m \times \mathbb{R}\mathrm{P}^m \to \mathbb{R}\mathrm{P}^n$, being nullhomotopic on the diagonal, is homotopic to a (necessarily axial) map $\mathbb{R}\mathrm{P}^m \times \mathbb{R}\mathrm{P}^m \to \mathbb{R}\mathrm{P}^n$ which is in fact constant on the diagonal.

4. Symmetric squares and TC^{Σ}

Recall that the symmetric square of a space X, $\mathrm{SP}^2(X)$, is the orbit space of $X \times X$ by the switching involution τ described at the beginning of the introduction. Let $\rho\colon X \times X \to \mathrm{SP}^2(X)$ denote the canonical projection. We think of X as being embedded diagonally both in $X \times X$ and (via ρ) in $\mathrm{SP}^2(X)$.

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Here is a useful observation when trying to estimate from below the number $\mathrm{sb}(m)$. Any symmetric axial map $\alpha\colon \mathbb{R}\mathrm{P}^m \times \mathbb{R}\mathrm{P}^m \to \mathbb{R}\mathrm{P}^n$ factors in the form

$$\mathbb{R}\mathrm{P}^m \times \mathbb{R}\mathrm{P}^m \xrightarrow{\rho} \mathrm{SP}^2(\mathbb{R}\mathrm{P}^m) \xrightarrow{\widetilde{\alpha}} \mathbb{R}\mathrm{P}^n. \tag{4.1}$$

Let $x = x_i$ stand for the one-dimensional generator of the mod 2 cohomology ring of $\mathbb{R}\mathrm{P}^i$, and set $\omega := \widetilde{\alpha}^*(x) \in H^1(\mathrm{SP}^2(\mathbb{R}\mathrm{P}^m); \mathbb{Z}_2)$, which is a non-zero element since the relation $\alpha^*(x) = x \otimes 1 + 1 \otimes x$ is well known. If some algebraic topology property (such as height or action of the Steenrod algebra) of the class $x \in H^1(\mathbb{R}\mathrm{P}^n; \mathbb{Z}_2)$ should fail to be compatible with the corresponding property for its $\widetilde{\alpha}^*$-image ω, we would infer that $\mathrm{sb}(m) > n$, as no such α could exist.

In this section we use the above idea to show the inequality $\mathrm{sb}(m) \geq 2m$ for $m = 2^e$ with $e \geq 1$. This, Theorem 1.6 and the well-known fact that $\mathrm{Emb}(\mathbb{R}\mathrm{P}^{2^e}) = 2^{e+1}$ yield Theorem 1.7.

It is worth isolating the particularly amenable situation for $m = 2$.

Example 4.1. A nice geometric fact shown by Massey in [**18**, Lemma 1] is that $\mathrm{SP}^2(\mathbb{R}\mathrm{P}^2)$ is homeomorphic to $\mathbb{R}\mathrm{P}^4$, with the diagonal inclusion $\mathbb{R}\mathrm{P}^2 \hookrightarrow \mathrm{SP}^2(\mathbb{R}\mathrm{P}^2) = \mathbb{R}\mathrm{P}^4$ being nullhomotopic. In particular, the generator $x \in H^1(\mathrm{SP}^2(\mathbb{R}\mathrm{P}^2); \mathbb{Z}_2) = \mathbb{Z}_2$ satisfies $x^4 \neq 0$, forbidding the existence of a map $\widetilde{\alpha}$ (with $m = 2$ and $n = 3$) as in (4.1), as $x \in H^*(\mathbb{R}\mathrm{P}^3; \mathbb{Z}_2) = \mathbb{Z}_2$ has $x^4 = 0$. We thus get $\mathrm{sb}(2) \geq 4$, the case $m = 2$ in Theorem 1.7.

Remark 4.2. Don Davis has observed that the assertions

(i) $\mathrm{TC}(X) \leq \mathrm{TC}^{\Sigma}(X) \leq \mathrm{TC}^{S}(X)$;

(ii) all closed surfaces Γ have $\mathrm{TC}^{S}(\Gamma) \leq 4$ [**12**, Proposition 10];

(iii) except for S^2, $S^1 \times S^1$ and $\mathbb{R}\mathrm{P}^2$, all closed surfaces Γ have $\mathrm{TC}(\Gamma) = 4$ (see [**5**, **8**–**10**]);

imply that both inequalities in (i) above are in fact equalities for all closed surfaces Γ, except perhaps for $\Gamma \in \{S^2, S^1 \times S^1, \mathbb{R}\mathrm{P}^2\}$. The corresponding equality $\mathrm{TC}^{\Sigma}(\mathbb{R}\mathrm{P}^2) = \mathrm{TC}^{S}(\mathbb{R}\mathrm{P}^2)$ is now accounted for by Example 4.1. As noted in the introduction of this paper, the equality $\mathrm{TC}^{S}(S^2) = \mathrm{TC}^{\Sigma}(S^2)$ is also known. (See Example 4.7 below for a discussion of what is currently known in the case of the torus.)

The topology of symmetric squares $\mathrm{SP}^2(\mathbb{R}\mathrm{P}^m)$ for $m > 2$ is much more subtle than that for $m = 2$. In order to deal with the general form of Theorem 1.7, we shall make use of the description in [**19**] of the mod 2 cohomology ring of $\mathrm{SP}^2(X)$. We give a short description of Nakaoka's results after stating the main goal in this section, Proposition 4.3, and observe that it yields Theorem 1.7. The proof of Proposition 4.3 will then be given.

Proposition 4.3. *Let $m \geq 2$ with $2^e \leq m < 2^{e+1}$. Then $H^1(\mathrm{SP}^2(\mathbb{R}\mathrm{P}^m); \mathbb{Z}_2) = \mathbb{Z}_2$, and the generator ϕ_1 of this group satisfies $\phi_1^{2^{e+1}} \neq 0 = \phi_1^{2^{e+1}+1}$.*

Since $\mathrm{Emb}(\mathbb{R}\mathrm{P}^{2^e}) = 2^{e+1}$ is well known, it is clear that Proposition 4.3 is all that is needed to have the argument in Example 4.1 prove the general case of Theorem 1.7.

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Here is a brief summary of Nakaoka's description of the mod 2 cohomology ring of $\mathrm{SP}^2(X)$ for a finite zero-connected polyhedron X [19]. Throughout the rest of the paper, cochain complexes and cohomology are taken with coefficients mod 2.

The identity and the involution τ induce maps at the cochain level $C^*(X \times X, X)$, and we let $\sigma\colon C^*(X \times X, X) \to C^*(X \times X, X)$ stand for the corresponding difference morphism. Note that the kernel and the image of σ agree; we let ${}^{\sigma}C^*(X \times X, X)$ stand for the resulting cochain subcomplex, writing ${}^{\sigma}H^*(X \times X, X)$ for its cohomology. The so-called Smith–Richardson short exact sequence

$$0 \to {}^{\sigma}C^*(X \times X, X) \to C^*(X \times X, X) \to {}^{\sigma}C^*(X \times X, X) \to 0$$

yields a connecting morphism

$$\partial\colon {}^{\sigma}H^*(X \times X, X) \to {}^{\sigma}H^{*+1}(X \times X, X).$$

Since the canonical projection $(X \times X, X) \to (\mathrm{SP}^2(X), X)$ identifies the cochain complexes $C^*(\mathrm{SP}^2(X), X)$ and ${}^{\sigma}C^*(X \times X, X)$, we get a morphism

$$\nu\colon H^*(\mathrm{SP}^2(X), X) \to H^{*+1}(\mathrm{SP}^2(X), X)$$

corresponding to ∂. Then, morphisms $E_s\colon H^*(X) \to H^{*+s}(\mathrm{SP}^2(X), X)$ are defined for $s \geq 1$ as the composition

$$E_s = (H^*(X) \xrightarrow{\delta} H^{*+1}(\mathrm{SP}^2(X), X) \xrightarrow{\nu^{s-1}} H^{*+s}(\mathrm{SP}^2(X), X)),$$

where δ is the usual connecting map associated with the pair $(\mathrm{SP}^2(X), X)$. On the other hand, note that the transfer map $C^*(X \times X) \to C^*(\mathrm{SP}^2(X))$ lands in the relative cochain subcomplex $C^*(\mathrm{SP}^2(X), X)$, thus defining a morphism $\phi\colon H^*(X \times X) \to H^*(\mathrm{SP}^2(X), X)$. Last, by restricting under the inclusion of pairs $(X, \varnothing) \hookrightarrow (\mathrm{SP}^2(X), X)$, we get corresponding maps $H^*(X) \to H^{*+s}(\mathrm{SP}^2(X))$ and $H^*(X \times X) \to H^*(\mathrm{SP}^2(X))$, which will also be denoted by E_s and ϕ, respectively (the context will clarify which map we refer to).

The results we need from Nakaoka's work [19] are packed in the following omnibus result.

Theorem 4.4. *Fix a homogeneous basis $\{b_0, b_1, \ldots, b_m\}$ of $H^*(X)$. Let R stand for either $\varnothing$ or X, and set*

$$\ell = \begin{cases} 1 & \text{if } R = X, \\ 2 & \text{if } R = \varnothing. \end{cases}$$

A basis for $H^*(\mathrm{SP}^2(X), R)$ consists of 1, the elements $E_s(b_i)$ with $\ell \leq s \leq \deg(b_i)$, and the elements $\phi(b_i \otimes b_j)$ with $i < j$. The ring structure is determined by the two relations:

(a) $\phi(b_i \otimes b_j) \cdot \phi(b_u \otimes b_v) = \phi((b_i \cdot b_u) \otimes (b_j \cdot b_v)) + \phi((b_i \cdot b_v) \otimes (b_j \cdot b_u))$;

(b) $E_s(b_i) \cdot \phi(b_u \otimes b_v) = E_s(b_i) \cdot E_t(b_j) = 0$.

The right-hand side in (a) can be expanded in terms of basis elements using the relations:

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(c) $\phi(b_j \otimes b_i) = \phi(b_i \otimes b_j)$;

(d) $\phi(b_i \otimes b_i) = \sum_{s=\ell}^{\deg(b_i)} E_s(\mathrm{Sq}^{\deg(b_i)-s}\, b_i)$.

The action of the Steenrod algebra is determined by the relations:

(e) $\mathrm{Sq}^k\, \phi(b_i \otimes b_j) = \phi\, \mathrm{Sq}^k(b_i \otimes b_j) + \sum_{s=\ell}^{k} E_s(\mathrm{Sq}^{k-s}(b_i \cdot b_j))$;

(f) $\mathrm{Sq}^k\, E_s(b_i) = \sum_{j=0}^{k} \binom{s-1}{k-j} E_{k+s-j}(\mathrm{Sq}^j\, b_i)$, *for* $\ell \le s \le \deg(b_i)$;

(g) $E_{\deg(b_i)+k}(b_i) = \sum_{s=\max(k,\ell)}^{\deg(b_i)+k-1} E_s(\mathrm{Sq}^{\deg(b_i)+k-s}\, b_i)$, *for* $k \ge 1$.

Of course, Theorem 4.4 is most useful when we actually know the structure of $H^*(X)$ as an algebra over the mod 2 Steenrod algebra $\mathcal{A}^*$; we then get a full description of $H^*(\mathrm{SP}^2(X))$ as an $\mathcal{A}^*$-algebra.

Example 4.5. A basis for the mod 2 cohomology of the torus $T = S^1 \times S^1$ consists of the elements 1, x, y and xy (with trivial action of the Steenrod algebra), where x and y are one-dimensional classes. Then a basis for the mod 2 cohomology of $\mathrm{SP}^2(T)$ is given by 1, $\phi(1 \otimes x)$, $\phi(1 \otimes y)$, $\phi(1 \otimes xy)$, $\phi(x \otimes y)$, $\phi(x \otimes xy)$, $\phi(y \otimes xy)$ and $E_2(xy)$. Further, by straightforward calculation we can check that the only non-vanishing products are:

$$\phi(1 \otimes x)\phi(1 \otimes y) = \phi(1 \otimes xy) + \phi(x \otimes y);$$
$$\phi(1 \otimes xy)^2 = \phi(x \otimes y)^2 = \phi(xy \otimes xy) = E_2(xy);$$
$$\phi(1 \otimes x)\phi(1 \otimes y)\phi(x \otimes y) = \phi(xy \otimes xy) = E_2(xy);$$
$$\phi(1 \otimes x)\phi(1 \otimes y)\phi(1 \otimes xy) = \phi(xy \otimes xy) = E_2(xy).$$

Proof of Proposition 4.3. We use Theorem 4.4 with the basis $\{1, x, x^2, \ldots, x^m\}$ of $H^*(\mathbb{R}\mathrm{P}^m)$ and $R = \varnothing$ (so $\ell = 2$). Recall that $\mathrm{Sq}^s\, x^i = \binom{i}{s} x^{i+s}$. Then the only basis element of degree 1 in $H^*(\mathrm{SP}^2(X))$ is $\phi_1 = \phi(1 \otimes x)$, which by direct calculation has

$$\phi_1^2 = \phi(1 \otimes x^2) + \phi(x \otimes x) = \phi(1 \otimes x^2),$$
$$\phi_1^4 = \phi(1 \otimes x^4) + \phi(x^2 \otimes x^2) = \phi(1 \otimes x^4) + E_2(x^2).$$

Assuming $\phi_1^{2^i} = \phi(1 \otimes x^{2^i}) + E_{2^{i-1}}(x^{2^{i-1}})$, we get

$$\phi_1^{2^{i+1}} = (\phi(1 \otimes x^{2^i}) + E_{2^{i-1}}(x^{2^{i-1}}))^2 = \phi(1 \otimes x^{2^{i+1}}) + \phi(x^{2^i} \otimes x^{2^i}),$$

which by standard properties of mod 2 binomial coefficients (and, of course, Theorem 4.4) implies

$$\phi_1^{2^{i+1}} = \phi(1 \otimes x^{2^{e+1}}) + E_{2^i}(x^{2^i}).$$

The conclusion of Proposition 4.3 follows from the $i = e$ case of the last equality. □

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We close this section by observing that the arguments in this section suggest that a systematic analysis of the (rich but not yet fully explored) algebraic topology properties of the symmetric square $\mathrm{SP}^2(\mathbb{R}\mathrm{P}^m)$ could have implications for (and lead to a better understanding of) $\mathrm{TC}^{\Sigma}(\mathbb{R}\mathrm{P}^m)$ and $\mathrm{sb}(m)$. For instance, motivated by [**16**, Theorem 4.6], we observe below that $\mathrm{TC}^{\Sigma}(X)$ is bounded from below by the cup length of $H^*(\mathrm{SP}^2(X))$ (see Proposition 4.6 and Example 4.7). The latter fact can be used (with $X = S^1$, see Corollary 4.8 below) to re-prove, in a streamlined way, the equality $\mathrm{TC}^{\Sigma}(S^1) = 2$ (first noted in [**6**, **16**]).

We realized the following result after reading [**16**, Theorem 4.6].

Proposition 4.6. $\mathrm{TC}^{\Sigma}(X)$ *is bounded from below by the sectional category of the diagonal inclusion* $X \hookrightarrow \mathrm{SP}^2(X)$.

Proof. Consider the diagram

$$\begin{array}{ccccccc} X & \hookrightarrow & \mathrm{P}(X) & \longrightarrow & \mathrm{P}(X)/\tau & \longrightarrow & X \\ & \searrow & \downarrow{\scriptstyle e_{0,1}} & & {\scriptstyle e'_{0,1}}\downarrow & \swarrow & \\ & & X \times X & \longrightarrow & \mathrm{SP}^2(X) & & \end{array}$$

where both slanted maps are diagonal inclusions, $e'_{0,1}$ is induced by $e_{0,1}$, X is embedded in $\mathrm{P}(X)$ as the subspace of constant paths, and the right-most horizontal arrow sends the equivalence class of a path γ to $\gamma(1/2)$. The diagram is strictly commutative, except for the right-hand side triangle, which commutes only up to homotopy (by contraction of paths toward their middle point). Note that $\mathrm{TC}^{\Sigma}(X) = \mathrm{secat}_{\tau}(e_{0,1}) \geq \mathrm{secat}(e'_{0,1})$, the latter of which agrees with $\mathrm{secat}(X \hookrightarrow \mathrm{SP}^2(X))$, in view of the diagram. □

Example 4.7. For a finite polyhedron X, the diagonal inclusion $X \hookrightarrow \mathrm{SP}^2(X)$ induces the trivial map in mod 2 positive-dimensional cohomology (see [**19**, Theorems 11.2 and 11.4]). The usual nilker lower bound for $\mathrm{secat}(X \hookrightarrow \mathrm{SP}^2(X))$ then shows that $\mathrm{TC}^{\Sigma}(X)$ is bounded from below by the mod 2 cup length of $\mathrm{SP}^2(X)$. In particular, Example 4.5 implies $3 \leq \mathrm{TC}^{\Sigma}(S^1 \times S^1)$. On the other hand, $\mathrm{TC}^{\Sigma}(S^1 \times S^1) \leq \mathrm{TC}^{S}(S^1 \times S^1) \leq 4$, in view of [**12**, Proposition 10]. Note that the sharp estimate for both $\mathrm{TC}^{\Sigma}(S^1 \times S^1)$ and $\mathrm{TC}^{S}(S^1 \times S^1)$ is decidable by (primary) obstruction-theoretic methods (in the equivariant setting, for TC^{Σ}). It would be well worth taking a look at the actual computations needed, as this might lead to an example with $\mathrm{TC}^{\Sigma} \neq \mathrm{TC}^{S}$.

Corollary 4.8. $\mathrm{TC}^{\Sigma}(S^1) = 2$.

Proof. Recall $\mathrm{TC}^{\Sigma}(S^1) \leq \mathrm{TC}^{S}(S^1) = 2$. If $\mathrm{TC}^{\Sigma}(S^1) \leq 1$, the product inequality for TC^{Σ} (Lemma 4.9) would yield $\mathrm{TC}^{\Sigma}(S^1 \times S^1) \leq 2$, which is impossible in view of Example 4.7. □

The proof of [**10**, Theorem 11] can be used, word for word (using τ-equivariant partitions of units), to prove the following auxiliary lemma.

Lemma 4.9. *For paracompact spaces X and Y*, $\mathrm{TC}^{\Sigma}(X \times Y) \leq \mathrm{TC}^{\Sigma}(X) + \mathrm{TC}^{\Sigma}(Y)$.

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