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On infinitely cohomologous to zero observables

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Abstract. We show that for a large class of piecewise expanding maps T, the bounded p-variation observables u_0 that admit an infinite sequence of bounded p-variation observables u_i satisfying

$$u_i = u_{i+1} \circ T - u_{i+1}$$

are constant. The method of the proof consists of finding a suitable Hilbert basis for $L^2(hm)$, where hm is the unique absolutely continuous invariant probability of T. On this basis, the action of the Perron–Frobenius and the Koopman operator on $L^2(hm)$ can be easily understood. This result generalizes earlier results by Bamón, Kiwi, Rivera-Letelier and Urzúa for the case $T(x) = \ell x \bmod 1$, $\ell \in \mathbb{N}\backslash\{0, 1\}$ and Lipschitzian observables u_0.

1. *Introduction*

Let $T : I \to I$ be a dynamical system. Consider the *cohomological operator* defined by

$$\mathcal{L}(\psi) = \psi \circ T - \psi.$$

Given an observable, that is, a function $u_0 : I \to \mathbb{R}$, one can ask if there exists a solution u_1 to the *Livsic cohomologous equation*

$$\mathcal{L}(u_1) = u_0.$$

This equation was intensively studied after its introduction in the seminal work of Livsic. These studies mainly concern the existence and regularity of the solution u_1.

Let μ be an invariant probability measure of T. We say that a function $u : I \to \mathbb{R}$ in $L^1(\mu)$ is cohomologous to zero if there is a function $w : I \to \mathbb{R}$ in $L^1(\mu)$ such that

$$u = \mathcal{L}(w).$$

An observable u_0 is *infinitely cohomologous to zero* if there exists a sequence of functions $u_n \in L^1(\mu)$, $n \in \mathbb{N}$, such that $\mathcal{L}^n u_n = u_0$, for all $n \in \mathbb{N}$.

Bamón *et al* [**4**] consider the expanding maps defined by

$$T_\ell(x) = \ell x \mod 1,$$

where $\ell \geq 2$ is an integer. The Lebesgue measure on [0, 1] is invariant by T_ℓ. They show that every non-constant Lipschitzian function $u : I \to \mathbb{R}$ is not infinitely cohomologous to zero. In this work, we generalize this result to a much larger class of observables and piecewise expanding maps.

In [**4**], the study of this problem is motivated by the following observation. Let $\lambda \in (-1, 1)$, $u_0 \colon I \to \mathbb{R}$ be a Lipschitz function and define

$$A \colon I \times \mathbb{R} \to I \times \mathbb{R}$$

by

$$A_{\lambda,u_0}(x, y) = (T_\ell(x), \lambda y + u_0(x)).$$

In [**4**], they note that:

(i) if $\mathcal{L}(u_1) = u_0$, then $A_{\lambda,u_1} \circ H = H \circ A_{\lambda,u_0}$, where H is the homeomorphism

$$H(x, y) = \left(x, \frac{y + u_1(x)}{1 - \lambda}\right); \textit{ and}$$

(ii) it turns out that the analysis of topological structure of the attractor of $A_{\lambda,u}$ is easier if u is *not* cohomologous to zero.

So, if u_0 is not infinitely cohomologous to zero, by (i), we can reduce the analysis of the topological dynamics of A_{λ,u_0} to the analysis of A_{λ,u_n}, where $\mathcal{L}^n(u_n) = u_0$ and u_n is not cohomologous to zero. Using our results, a similar analysis of attractors could potentially be achieved to far more general skew-products.

1.1. *Statement of results.* Let I be an interval. We say that $T \colon I \to I$ is a *piecewise monotonic* map if there exists a partition by intervals $\{I_1, \ldots, I_m\}$ of I such that for each $i \le m$, the map T is continuous and strictly monotonic in I_i. A piecewise monotonic map is *onto* if, furthermore, $T(I_i) = I$ for every i. A piecewise monotonic map is called *expanding* if T is differentiable on each I_i and

$$\inf_i \inf_{x \in I_i} |T'(x)| > 1.$$

In this work, we will consider mainly maps $T \colon I \to I$ satisfying the following conditions.

(D1) T is piecewise monotonic, Lipschitz on each interval of the partition I_i, $i \le m$. In particular, T' is defined almost everywhere and is an essentially bounded function. We also assume

$$\operatorname{ess\,inf}_m |T'| > 0. \tag{1}$$

Here $\operatorname{ess\,inf}_m$ denotes the essential infimum with respect to the Lebesgue measure m.

(D2) We have $T(I) = I$ and, moreover, for every interval $H \subset I$, there is a finite collection of pairwise disjoint open subintervals $H_1, \ldots, H_k \subset H$ and n such that T^n is a homeomorphism on H_i and

$$\operatorname{int} I \subset \bigcup_i T^n(H_i).$$

(D3) T has a horseshoe, that is, there are three open intervals $J_1, J_2 \subset J \subset I$, with $J_1 \cap J_2 = \emptyset$, such that T is a homeomorphism on each J_i and $T(J_i) = J$, $i = 1, 2$.

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(D4) T has an invariant probability μ that is absolutely continuous with respect to the Lebesgue measure m, so

$$\mu(A) = \int_A h\, dm$$

for some $h \in L^1(m)$. We will denote $\mu = hm$, where $h \in L^1(m)$ and m is the Lebesgue measure on I. Moreover, μ is exact and there exist a, b such that

$$0 < a \leq h(x) \leq b < \infty \tag{2}$$

for hm-almost every x and the support of μ is I.

Our main result is the following theorem.

THEOREM 1. *Let T be a transformation satisfying (D1)–(D4) and let $u_0 \colon I \to \mathbb{R}$ be an observable with bounded p-variation. Then either u_0 is constant in I up to a countable set or there exist $M \geq 0$ and bounded p-variation functions $u_i : I \to \mathbb{R}$, with $i \leq M$, which are unique (in $L^1(hm)$ and $BV_{p,I}$) up to an addition by a constant, such that:*

- *we have*
$$\mathcal{L}^i u_i = u_0,$$
in I up to a countable set, for every $i \leq M$; and
- *for every function ρ with bounded p-variation and every $c \in \mathbb{R}$, we have $\mathcal{L}\rho \neq u_M + c$ in a non-empty open set in I.*

With somewhat distinct, but related, assumptions on T and u_0, which are satisfied in many interesting situations, we can improve this result in such a way that $\mathcal{L}\rho \neq u_M + c$ for every $\rho \in L^1(hm)$. Avila [**2**] contributed in this area by improving the results in the original version of this work, and we are grateful to him for allowing us to include them here. Avila's contribution is the following.

THEOREM 2. [**2**] *Let $u_0 \in L^1(hm)$ be such that*

$$\int u_0 h\, dm = 0$$

and such that for every $v \in L^\infty(hm)$, there exist $C > 0$ and $\lambda \in [0, 1)$ such that

$$\left| \int u_0 \cdot v \circ T^i \cdot h\, dm \right| \leq C\lambda^i.$$

Then either u_0 is constant hm-almost everywhere or there exist a unique $M \geq 0$ and functions $u_i : I \to \mathbb{R}$, with $i \leq M$, $u_i \in L^1(hm)$, which are unique in $L^1(hm)$, up to an addition by a constant, such that:

- *we have*
$$\mathcal{L}^i u_i = u_0 \quad \text{in } L^1(hm)$$
for every $i \leq M$; and
- *for every function $\rho \in L^1(hm)$ and every $c \in \mathbb{R}$, we have $\mathcal{L}\rho \neq u_M + c$ on $L^1(hm)$.*

Let $(\mathbb{B}, |\cdot|_{\mathbb{B}})$ be a Banach space of real-valued, Lebesgue-measurable functions defined on I such that:

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(D5) (i) T is a piecewise expanding map satisfying (D1) and (D4);

(ii) there exist C such that

$$|f|_{L^1(hm)} \leq C|f|_{\mathbb{B}}$$

for every $f \in \mathbb{B}$;

(iii) the Perron–Frobenius operator Φ_T of T is a bounded operator on $\mathbb{B}$ and there exists $h \in \mathbb{B}$, $h > 0$, with $\int h\, dm = 1$, $\lambda \in [0, 1)$ and a linear operator $\Psi : \mathbb{B} \to \mathbb{B}$ such that

$$\Phi_T(f) = \int f\, dm \cdot h + \Psi(f),$$

with

$$|\Psi^n(f)|_{\mathbb{B}} \leq C\lambda^n |f|_{\mathbb{B}},$$

for every $f \in \mathbb{B}$ and $n \in \mathbb{N}$. Moreover, $\Psi(h) = 0$;

(iv) $1/h \in \mathbb{B}$;

(v) the multiplication

$$(f, g) \to f \cdot g$$

is a bounded bilinear transformation on $\mathbb{B}$; and

(vi) the set $\mathbb{B}$ is dense in $L^1(hm)$.

THEOREM 3. *Let T be a transformation satisfying (D1) and (D4) and suppose that the Banach space of functions $\mathbb{B}$ and T satisfy (D5). Let $u_0 \in \mathbb{B}$ be an observable. Then either u_0 is constant hm-almost everywhere or there exist a unique $M \geq 0$ and functions $u_i : I \to \mathbb{R}$, with $i \leq M$, $u_i \in L^1(hm)$, which are unique in $L^1(hm)$, up to an addition by a constant, such that:*

- *we have*

$$\mathcal{L}^i u_i = u_0 \text{ in } L^1(hm)$$

  for every $i \leq M$; and
- *for every function $\rho \in L^1(hm)$ and every $c \in \mathbb{R}$, we have $\mathcal{L}\rho \neq u_M + c$ on $L^1(hm)$.*

Moreover, u_i belongs to $\mathbb{B}$, for $i \leq M$.

Remark 1.1. In the first version of this work, Theorem 3 had additional assumptions. For instance, we assumed that $\mathbb{B}$ was contained in the space of functions with p-bounded variation. This is no longer necessary because of Avila's contribution (Theorem 2).

Remark 1.2. The finiteness result for the family of cohomological operators

$$\mathcal{L}_\lambda(v) = v \circ T - \lambda v,$$

with $\lambda \in (0, 1]$, $T(x) = \ell x \mod 1$, for integers $\ell \geq 2$ and Lipschitz observables, obtained in [**4**, Main Lemma, page 225], can also be generalized for maps described in Remarks 1.3, 1.4 and 1.5, replacing Lipschitz observables by bounded variation observables. The methods for achieving this generalization are quite similar to those in [**4**], so we will not give a full proof here. It is necessary to use Theorem 3, and to replace in their argument the usual Fourier basis by the basis obtained in §3, and the compactness of closed balls centered at zero of the space of Lipschitz functions as subsets of the space of continuous functions by Helly's theorem, that is, the compactness of closed balls centered at zero of the space of bounded variation functions as subsets of $L^1(hm)$.

Remark 1.3. There are plenty of examples of transformations $T: I \to I$ satisfying (D1)–(D4). Let T be a piecewise monotonic, expanding map C^2 on each I_i. Consider the $m \times m$ matrix $A_T = (a_{ij})$ defined by $a_{ij} = 1$ if

$$\overline{T(\text{int } I_i)} \subset \text{int } I_j,$$

and $a_{ij} = 0$ otherwise. Here the closure and interior are taken with respect to the topology of [0, 1]. Suppose that $A_T^k > 0$ for some k. Then T satisfies (D1), (D2) and (D4), and some iteration of T satisfies (D1)–(D4). If we add the assumption that T has a horseshoe, then T fulfills (D1)–(D4). The space of bounded variation functions $BV(I)$ and T satisfy (D5).

Remark 1.4. A class of examples satisfying (D1)–(D4) is β-transformations $T(x) = \beta x \bmod 1$, with $\beta \geq 2$, $\beta \in \mathbb{R}$, $I = [0, 1]$. The space of bounded variation functions $BV(I)$ and T satisfy (D5).

Remark 1.5. Let $T: [-1, 1] \to [-1, 1]$ be a continuous map with $T(-1) = T(1) = -1$, C^2 on the intervals $[-1, 0]$ and $[0, 1]$, with $T' > 0$ in $[-1, 0]$ and $T' < 0$ in $[0, 1]$ and $T(-x) = T(x)$ for every $x \in [-1, 1]$. Define

$$\theta = \inf_x |T'(x)|.$$

If $\theta > 1$, then there exists a unique fixed point $p \in [0, 1]$. Define $J = [-p, p]$. If $\theta > \sqrt{2}$, then T^2 has a horseshoe in J and satisfies (D1)–(D4) with $I = [T^2(0), T(0)]$. The space of bounded variation functions $BV(I)$ and T satisfies (D5).

Remark 1.6. Let $T: I \to I$ be a piecewise expanding and onto map, $C^{1+\alpha_0}$ in each I_i, $\alpha_0 \in (0, 1)$. Then T satisfies (D1)–(D4). The space of Hölder continuous functions $C^\alpha(I)$, for $\alpha \leq \alpha_0$, and T satisfy (D5).

Remark 1.7. Let $T: I \to I$ be a piecewise expanding map, linear in each I_i. Suppose that T has a horseshoe and satisfies the conditions on the matrix A_T as in Remark 1.3. One can prove, using Wong's results [**10**], that T satisfies (D1)–(D4). The space of bounded p-variation functions $BV_p(I)$, with $p \geq 1$, and T satisfy (D5).

Remark 1.8. The mixing assumptions on the invariant measure μ are necessary, as is shown by the following example. Consider a piecewise C^2 expanding map $T: I \to I$, unimodal (continuous and only one turning point), and with a cycle of intervals, that is, there are open intervals $J_j \subset I$, $j < p$ pairwise disjoint, such that $f(\overline{J}_j) \subset \overline{J}_{j+1 \bmod p}$ and $f(\partial J_j) \subset \partial J_{j+1 \bmod p}$. Then T has an absolutely continuous invariant probability μ and its support is contained in $\bigcup_j \overline{J}_j$. Let $\delta \in \mathbb{C} \backslash \{1\}$ be a p-root of unity, $\delta^p = 1$. Define $u_i: I \to \mathbb{C}$, $i \geq 0$, as

$$u_i(x) = \frac{\delta^j}{(\delta - 1)^i},$$

for $x \in J_j$. Define u_i arbitrarily elsewhere. It is easy to see that $u_i = u_{i+1} \circ T - u_{i+1}$ on $L^1(hm)$. To obtain real-valued functions, we can consider the real and imaginary parts of u_i.

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1.2. *Topological results.* Replacing Lipschitzian by bounded p-variation observables has the advantage of allowing us to obtain results similar to Theorems 1 and 3 for maps which are just *topologically conjugate* with maps satisfying the assumptions of those theorems.

We will say that two functions $f, g\colon W \to \mathbb{R}$ are equal except in a countable set, $f = g$ on W (e.c.s.), if $\{x \in W\colon f(x) \neq g(x)\}$ is countable.

THEOREM 4. *Let $H\colon I \to I$ be a homeomorphism, let T be a piecewise monotonic map and $\tilde{T}$ satisfy (D1)–(D4). Suppose that*

$$H \circ \tilde{T} = T \circ H$$

in I (e.c.s.). Let $u_0\colon H(I) \to \mathbb{R}$ be an observable with bounded p-variation. Then either u_0 is constant in $H(I)$ (e.c.s.) or there exist a unique $M \geq 0$ and bounded p-variation functions $u_i : H(I) \to \mathbb{R}$, with $i \leq M$, which are unique up to an addition by a constant (e.c.s.), such that:

- *we have*

$$\mathcal{L}^i u_i = u_0,$$

  on $H(I)$ (e.c.s.) for every $i \leq M$; and
- *for every function ρ with bounded p-variation and every $c \in \mathbb{R}$, we have $\mathcal{L}\rho \neq u_M + c$ in a non-empty open subset in $H(I)$.*

THEOREM 5. *Let $H\colon I \to I$ be a homeomorphism, let T be a piecewise monotonic map and $\tilde{T}$ satisfy (D1)–(D4). Suppose that*

$$H \circ \tilde{T} = T \circ H$$

in I (e.c.s.). Suppose that the space of functions with bounded p_0-variation $BV_{p_0,I}$ and $\tilde{T}$ satisfy (D5). Let $u_0\colon H(I) \to \mathbb{R}$ be an observable with bounded p_0-variation. Then either u_0 is constant in $H(I)$ (e.c.s.) or there exist a unique $M \geq 0$ and continuous (e.c.s.) bounded Borelian functions $u_i : H(I) \to \mathbb{R}$, with $i \leq M$, which are unique up to an addition by a constant (e.c.s.), such that:

- *we have*

$$\mathcal{L}^i u_i = u_0,$$

  on $H(I)$ (e.c.s.) for every $i \leq M$; and
- *we have $\mathcal{L}\rho \neq u_M + c$:*
  - *(A) in an uncountable subset of $H(I)$, if ρ is a Borel-measurable, bounded function and $c \in \mathbb{R}$; and*
  - *(B) in a non-empty open subset of $H(I)$, if ρ is a Borel-measurable, bounded function which is continuous in $H(I)$ (e.c.s.) and $c \in \mathbb{R}$.*

Moreover, u_i has bounded p_0-variation, $i \leq M$.

Remark 1.9. Let $T\colon [0, 2] \to [0, 2]$ be piecewise monotonic, C^1 in $[0, 1]$ and $[1, 2]$, $T[0, 1] = T[1, 2] = [0, 2]$, with $T(0) = 0$, $T' \geq \lambda > 1$ in $[1, 2]$ and $T'(x) > 1$ in $x \in (0, 1)$, and $T'(0) = 1$. Then T is conjugate with $\tilde{T}(x) = 2 \cdot x \mod 1$, so T satisfies the assumptions of Theorems 4 and 5, considering $p_0 = 1$ in Theorem 5.

Remark 1.10. Let $T: [-1, 1] \to [-1, 1]$, $T(-1) = T(1) = -1$, C^3 in $[-1, 1]$, $T'(0) = 0$, $T' > 0$ on $[-1, 0)$, $T' < 0$ on $(0, 1]$. If T has a negative Schwarzian derivative and is non-renormalizable, then T is conjugate to a tent map $\tilde{T}_\beta: [-1, 1] \to [-1, 1]$, defined as $\tilde{T}(x) = -\beta|x| + \beta - 1$, with $\beta = \exp(h_{\text{top}}(T))$. Here $h_{\text{top}}(T)$ denotes the topological entropy of T. If $h_{\text{top}}(T) \geq \ln(2)/2$, then $T^2: I \to I$, with $I = [T^2(0), T(0)]$, satisfies the assumptions of Theorems 4 and 5, considering $p_0 = 1$ in Theorem 5.

1.3. *Continuous observables infinitely cohomologous to zero.*

Avila put forward a nice argument showing the existence of continuous and non-constant observables that are infinitely cohomologous to zero. He kindly agreed to the inclusion of this result here.

THEOREM 6. [**2**] *Let $T: \mathbb{S}^1 \to \mathbb{S}^1$ be a C^1 expanding map on a circle. Then there exists a non-constant continuous observable u that is infinitely cohomologous to zero.*

2. *Preliminaries*

In this section, we present some notations and definitions.

Definition 2.1. Given a function $f: I \to \mathbb{C}$ and $p \geq 1$, we define the p-variation of f by

$$v_{p,I}(f) = \sup \bigg(\sum_{i=1}^{n} |g(a_i) - g(a_{i-1})|^p \bigg)^{1/p},$$

where the supremum is taken over all finite sequences $a_0 < a_1 < \cdots < a_n$, $a_i \in I$.

We say that f has bounded p-variation if

$$v_{p,I}(f) < \infty.$$

Since the Perron–Frobenius operator is not properly defined at points which are images of points where DT is not defined, to define the Perron–Frobenius operator acting in the space of p-bounded variation functions it is convenient to identify functions u and v defined on I so that $u = v$ up to a countable subset of I. We write $u \sim v$. The set of equivalence classes $[f]$ with respect to the relation $\sim$ such that

$$v_{p,I}([f]) = \inf_{f \sim g} v_{p,I}(g) < \infty$$

will be called the space of the functions on I with bounded p-variation and denoted $BV_{p,I}$. The function $f \to v_{p,I}([f])$ is a pseudo-norm on $BV_{p,I}$. We can define a norm by

$$|[f]|_{BV_{p,I}} = \inf_{g \sim f} (\sup |g| + v_{p,I}(g)).$$

$(BV_{p,I}, |\cdot|_{BV_{p,I}})$ is a Banach space. As usual, from now on we will omit the brackets $[\cdot]$ in the notation of equivalence classes.

Note that $1/p$-Hölder continuous functions have bounded p-variation. When $p = 1$, we say that the function has bounded variation.

Remark 2.2. One of the greatest advantages of dealing with p-bounded variation observables, as opposed to either Hölder or Lipschitzian ones, for instance, is that the pseudo-norm $v_{p,I}$ is invariant by homeomorphisms, that is, if $h: J \to I$ is a homeomorphism and $f: I \to \mathbb{R}$ is an observable, then

$$v_{p,I}(f) = v_{p,J}(f \circ h).$$

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Definition 2.3. Given a piecewise monotonic, expanding map T, satisfying (D1), define the Perron–Frobenius operator associated to T by

$$\Phi_T f(x) = \sum_{j\in J} f(\sigma_j(x)) \frac{1}{|T'(\sigma_j x)|} \mathbb{1}_{T(I_j)}(x),$$

where $\sigma_j : T(I_j) \to I_j$ stands for the inverse branch of T restricted to I_j and $\mathbb{1}_J$ denotes the characteristic function of the set J.

The main properties of Φ_T are (see [**3, 5**], for instance):

(i) Φ_T is a continuous linear operator on $L^1(hm)$;
(ii) $\int_0^1 \Phi_T f \cdot g \, dm = \int_0^1 f \cdot g \circ T \, dm$, where $f \in L^1(m)$ and $g \in L^\infty(m)$; and
(iii) $\Phi_T f = f$ if and only if the measure $\mu = fm$ is invariant by T.

3. *A special basis of $L^2(hm)$*

In this section, we assume that T satisfies (D1) and (D4). Consider the Hilbert space $L^2(hm)$ with the inner product

$$\langle u, w\rangle_{hm} = \int uwh \, dm.$$

$\langle u, w\rangle_{hm}$ is well defined, even for $u \in L^k(hm)$ and $w \in L^b(hm)$, with $k, b \in [1, \infty) \cup \{+\infty\}$ satisfying

$$\frac{1}{k} + \frac{1}{b} = 1.$$

Since the measure hm is T-invariant, we have

$$\langle u \circ T, w \circ T\rangle_{hm} = \langle u, w\rangle_{hm}.$$

In this section, we will construct a special Hilbert basis for $L^2(hm)$. Consider the bounded linear operator $P : L^k(hm) \to L^k(hm)$, $k \geq 1$, defined by

$$P(u) = \frac{\Phi(uh)}{h}.$$

From equation (2), the operator P is well defined. Indeed,

$$\sum_{j\in J} \frac{h(\sigma_j(x))}{h(x)} \frac{1}{|T'(\sigma_j x)|} \mathbb{1}_{T(I_j)}(x) = 1$$

for every x, and z^k is convex, so we have

$$\begin{aligned}
\int |Pu|^k h \, dm &\leq \int \bigg(\sum_{j\in J} \frac{h(\sigma_j(x))}{h(x)} \frac{1}{|T'(\sigma_j x)|} |u|(\sigma_j(x)) \mathbb{1}_{T(I_j)}(x) \bigg)^k h(x) \, dm \\
&\leq \int \sum_{j\in J} \frac{h(\sigma_j(x))}{h(x)} \frac{1}{|T'(\sigma_j x)|} |u|^k(\sigma_j(x)) \mathbb{1}_{T(I_j)}(x) h(x) \, dm \\
&= \int P(|u|^k) h \, dm \\
&\leq \int \Phi(|u|^k) \, dm = \int |u|^k \, dm \leq \frac{1}{a} \int |u|^k h \, dm.
\end{aligned}$$

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Note that for $k = 1$, we have

$$\int |Pu|h\, dm \le \int \Phi(|u|h)\, dm = \int |u|h\, dm,$$

so $\|P\|_{L^1(hm)} \le 1$.

Let $\mathcal{B} = \{\varphi_i\}_{i\in\mathbb{N}}$ be an orthonormal basis for

$$\mathrm{Ker}(P) = \{u \in L^2(hm) \text{ such that } P(u) = 0\}.$$

Define

$$\mathcal{W} = \{\varphi_i \circ T^j : \varphi_i \in \mathcal{B} \text{ and } j \in \mathbb{N}\} \cup \{\mathbb{1}_I\}.$$

Recall that $\mathbb{1}_A$ denotes the indicator function of a set A. The main result of this section is the following proposition.

PROPOSITION 3.1. *Suppose that T satisfies (D1) and (D4). Then $\mathcal{W}$ is a Hilbert basis for $L^2(hm)$. Indeed, we can choose $\mathcal{B}$ such that $\mathcal{W} \subset L^\infty(hm)$.*

Remark 3.2. A very interesting example of this theorem is given by the function $T\colon [0, 1] \to [0, 1]$, defined by $T(x) = \ell x \mod 1$, with $\ell \in \mathbb{N}\backslash\{0, 1\}$. In this case, the Ruelle–Perron–Frobenius operator is just

$$(\Phi_T\psi)(x) = \frac{1}{\ell}\sum_{i=0}^{\ell-1}\psi\left(\frac{x+i}{\ell}\right).$$

The Lebesgue measure m is an invariant probability, so $P = \Phi_T$. Moreover,

$$\mathcal{B} = \{\sin(2\pi nx), \cos(2\pi nx) : \ell \text{ does not divide } n\}$$

is a basis for Ker P. Note that

$$\sin(2\pi nT^j(x)) = \sin(2\pi n\ell^j x) \quad \text{and} \quad \cos(2\pi nT^j(x)) = \cos(2\pi n\ell^j x),$$

so the corresponding set $\mathcal{W}$ is just the classical Fourier basis of $L^2([0, 1])$.

By property (ii) of the Perron–Frobenius operator, it is easy to see that the Koopman operator $U : L^k(hm) \to L^k(hm)$, $k \ge 1$, defined by

$$U(w) = w \circ T,$$

is the adjoint operator of P, that is,

$$\langle P(u), w\rangle_{hm} = \langle u, U(w)\rangle_{hm} \tag{3}$$

for every $u \in L^k(hm)$ and $w \in L^b(hm)$. Note that U preserves $L^k(hm)$ because hm is invariant. Moreover,

$$P \circ U(f) = f$$

for every $f \in L^1(hm)$.

LEMMA 3.3. *$\mathcal{W}$ is an orthonormal set.*

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Proof. Indeed,

$$|\mathbb{1}_I|_{L^2(hm)} = 1,$$
$$|\varphi_i \circ T^j|^2_{L^2(hm)} = |\varphi_i|^2_{L^2(hm)} = 1.$$

Furthermore, if

$$(i_1, j_1) \neq (i_2, j_2),$$

then either $j_1 = j_2$, so we have

$$\langle \varphi_{i_1} \circ T^{j_1}, \varphi_{i_2} \circ T^{j_2} \rangle_{hm} = \langle \varphi_{i_1}, \varphi_{i_2} \rangle_{hm} = 0,$$

or, without loss of generality, we can assume $j_1 < j_2$ and

$$\langle \varphi_{i_1} \circ T^{j_1}, \varphi_{i_2} \circ T^{j_2} \rangle_{hm} = \langle \varphi_{i_1}, \varphi_{i_2} \circ T^{j_2 - j_1} \rangle_{hm} = \langle P^{j_2 - j_1}(\varphi_{i_1}), \varphi_{i_2} \rangle_{hm} = 0,$$

and

$$\langle \varphi_{i_1} \circ T^{j_1}, \mathbb{1}_I \rangle_{hm} = \int \varphi_i \circ T^j h \, dm = \int \varphi_i h \, dm = \int P(\varphi_i) h \, dm = 0. \qquad \square$$

LEMMA 3.4. *There exists a countable set of functions $\Lambda \subset L^\infty(hm) \cap \mathrm{Ker}(P)$ with the following property. Let $w \in L^k(hm)$, with $k \geq 1$. If for all $\varphi \in \Lambda$ we have*

$$\int w \varphi h \, dm = 0,$$

then there exists $\beta \in L^k(hm)$ such that

$$w = \beta \circ T$$

hm-almost everywhere. Moreover, $\mathrm{Ker}(P)^\perp = U(L^2(hm))$.

Proof. We claim that for the existence of $\beta \in L^k(hm)$ such that $w = \beta \circ T$, it is necessary and sufficient that for hm-almost every $y \in I$ we have

$$\sharp\{w(x) \colon h(x) \neq 0 \text{ and } T(x) = y\} = 1. \tag{4}$$

Indeed, if equation (4) holds, then for every y satisfying (4), choosing x such that $T(x) = y$ and $h(x) \neq 0$, we can define

$$\beta(y) = w(x).$$

If y does not satisfy (4), define $\beta(y) = 0$. Of course, $w = \beta \circ T$ hm-almost everywhere and, since hm is an invariant measure of T, β belongs to $L^k(hm)$.

On the other hand, suppose that there exists $\beta \in L^k(hm)$ such that $w = \beta \circ T$. Then

$$K = \{x \colon w(x) = \beta(T(x))\}$$

has full hm-measure. Since the support of hm is I and $I \subset Im\ T$, it follows that for hm-almost every y, we have $\sharp A_y \geq 1$, where

$$A_y = \{w(x) \colon h(x) \neq 0 \text{ and } T(x) = y\}.$$

Suppose there is Ω, with $hm(\Omega) > 0$, such that $\sharp A_y \geq 2$ for every $y \in \Omega$. Note that (D1) implies that f and its inverse branches are absolutely continuous functions, so it is easy to see that there are X_1 X_2 such that $m(X_1)$, $m(X_2) > 0$, $T(X_i) = \Omega$ and for each $y \in \Omega$ and $i = 1, 2$ there exists only one $x_i \in X_i$ such that $T(x_i) = y$. Furthermore, $w(x_1) \neq w(x_2)$, $h(x_i) \neq 0$. The absolute continuity of T and its inverses branches implies that

$$\tilde{\Omega} = T(X_1 \cap K) \cap T(X_2 \cap K) \subset \Omega$$

has positive measure. Let $y \in \tilde{\Omega}$ and x_i be as above. Then $w(x_i) = \beta(T(x_i)) = \beta(y)$, which contradicts $w(x_1) \neq w(x_2)$. This concludes the proof of the claim.

Let C_i be the set of points $x_0 \in I$ such that the function

$$F_i(a) = \int_0^a w \circ \sigma_i(Tx) \cdot \mathbb{1}_{T(I_i)}(T(x)) \cdot h(x)\, dm(x) \tag{5}$$

has the derivative $w \circ \sigma_i(T(x_0))\mathbb{1}_{T(I_i)}(T(x_0))h(x_0)$ at $a = x_0$. The function in the above integral belongs to $L^1(m)$, so by the Lebesgue differentiation theorem, the set

$$C = \bigcap_i C_i \Big\backslash \bigcup_i \partial I_i$$

has full Lebesgue measure in I. Since T is piecewise Lipschitz, we obtain

$$m(T(I \backslash C)) = 0.$$

Suppose that equation (4) does not hold for hm-almost every $y \in I$. Then it is not true that equation (4) holds for hm-almost every $y \in I \backslash T(I \backslash C)$. Since hm-almost every point has at least one preimage x with $h(x) \neq 0$, we conclude that there exist $y_0 \in I \backslash T(I \backslash C)$ and two inverse branches of T, denoted by σ_1 and σ_2, such that y_0 belongs to the interior of $T(I_1) \cap T(I_2)$ and, furthermore,

$$w \circ \sigma_1(y_0) \neq w \circ \sigma_2(y_0), \quad h(\sigma_1(y_0)) \neq 0, \quad h(\sigma_2(y_0)) \neq 0.$$

We can assume

$$w \circ \sigma_1(y_0) > w \circ \sigma_2(y_0),$$

so

$$\begin{aligned} & w \circ \sigma_1 \circ T \circ \sigma_2(y_0)\mathbb{1}_{T(I_1)} \circ T \circ \sigma_2(y_0) h \circ \sigma_2(y_0) \\ & \quad > w \circ \sigma_2 \circ T \circ \sigma_2(y_0)\mathbb{1}_{T(I_2)} \circ T \circ \sigma_2(y_0) h \circ \sigma_2(y_0). \end{aligned} \tag{6}$$

Since $\sigma_2(y_0) \in C$, the derivatives of the functions F_1 and F_2 at $a = \sigma_2(y_0)$ are respectively the left- and right-hand sides of equation (6), so there exists $\varepsilon > 0$ such that for every closed non-degenerate interval $\tilde{I}_2$ satisfying

$$\sigma_2(y_0) \in \tilde{I}_2 \subset (\sigma_2(y_0) - \varepsilon, \sigma_2(y_0) + \varepsilon) \cap I_2, \tag{7}$$

we have

$$\int_{\tilde{I}_2} w \circ \sigma_1(Tx)\mathbb{1}_{T(I_1)} \circ T(x) \cdot h(x)\, dm(x) > \int_{\tilde{I}_2} w \circ \sigma_2(Tx)\mathbb{1}_{T(I_2)} \circ T(x) \cdot h(x)\, dm(x).$$

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Choose an interval $\tilde{I}_2$ satisfying equation (7) and small enough such that $T(\tilde{I}_2) \subset T(I_1)$. We can assume, without loss of generality, that $\partial \tilde{I}_2 \subset \mathbb{Q}$. Then

$$\int_{\tilde{I}_2} w \circ \sigma_1(Tx) \cdot h(x)\, dm(x) > \int_{\tilde{I}_2} w \circ \sigma_2(Tx) \cdot h(x)\, dm(x).$$

Let $\tilde{I}_1 := \sigma_1(T(\tilde{I}_2)) \subset I_1$. Define φ as

$$\varphi(x) = \begin{cases} -\dfrac{|T'(x)|}{|T'(\sigma_2(Tx))|} \cdot \dfrac{h(\sigma_2(Tx))}{h(x)} & \text{if } x \in \tilde{I}_1, \\ 1 & \text{if } x \in \tilde{I}_2, \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

Note that $\varphi \in L^\infty(hm)$ and $\Phi(\varphi h) = 0$.

Hence

$$\begin{aligned} \int w\varphi h\, dm &= \int_{\tilde{I}_1} w\varphi h\, dm + \int_{\tilde{I}_2} w\varphi h\, dm \\ &= \int_{\tilde{I}_1} w \cdot \left(-\frac{|T'|}{|T' \circ \sigma_2 \circ T|} \frac{h \circ \sigma_2 \circ T}{h} \right) h\, dm + \int_{\tilde{I}_2} wh\, dm. \end{aligned}$$

Since $\sigma_2 \circ T : \tilde{I}_1 \to \tilde{I}_2$ is Lipschitzian and monotonically increasing, we can change the variables to get

$$\begin{aligned} &-\int_{\tilde{I}_1} w \frac{|T'|}{|T' \circ \sigma_2 \circ T|} \frac{h \circ \sigma_2 \circ T}{h} h\, dm + \int_{\tilde{I}_2} wh\, dm \\ &= -\int_{\tilde{I}_2} w \circ \sigma_1 \circ T \cdot h\, dm + \int_{\tilde{I}_2} w \circ \sigma_2 \circ T \cdot h\, dm \\ &< -\int_{\tilde{I}_2} w \circ \sigma_2 \circ T \cdot h\, dm + \int_{\tilde{I}_2} w \circ \sigma_2 \circ T \cdot h\, dm = 0. \end{aligned}$$

Therefore

$$\int w\varphi h\, dm \neq 0.$$

Let Λ be the set of functions φ of the form in equation (8), with:

- the intervals $\tilde{I}_j \subset I_{i_j}$, $j = 1, 2$, and $\sigma_2 \colon T(I_{i_2}) \to I_{i_2}$ the inverse of $T \colon I_{i_2} \to T(I_{i_2})$;
- $T(\tilde{I}_2) = T(\tilde{I}_1)$; and
- $\partial \tilde{I}_2 \subset \mathbb{Q}$.

Then it is easy to see that Λ is countable and $\Lambda \subset L^\infty(hm) \cap \mathrm{Ker} P$ and, by the argument above, Λ has the desired property.

In particular, for $k = 2$, we obtain $\mathrm{Ker}(P)^\perp \subset U(L^2(hm))$. The inclusion $U(L^2(hm)) \subset \mathrm{Ker}(P)^\perp$ follows from equation (3). □

PROPOSITION 3.5. *Let Λ be as in Lemma 3.4. Let $u : I \to \mathbb{R}$ be a non-constant function in $L^1(hm)$. Then there exist $\varphi \in \Lambda$ and an integer $p \geq 0$ such that*

$$\int u \cdot \varphi \circ T^j \cdot h\, dm = 0 \quad \textit{for all } 0 \leq j < p$$

and

$$\int u\varphi \circ T^p \cdot h\, dm \neq 0.$$

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Proof. Suppose that, for all $\varphi \in \Lambda$ and for all $k \geq 0$,

$$\int u\varphi \circ T^k \cdot h \, dm = 0. \tag{9}$$

We claim that for every n, there exists $\beta_n \in L^1(hm)$ such that

$$u = \beta_n \circ T^n. \tag{10}$$

Indeed, choosing $k = 0$ in equation (9), we obtain that for all $\varphi \in \Lambda$,

$$\int u\varphi h \, dm = 0.$$

By Lemma 3.4, there exists $\beta_1 \in L^1(hm)$ such that

$$u = \beta_1 \circ T.$$

Suppose, by induction, that $u = \beta_n \circ T^n$, with $\beta_n \in L^1(hm)$. By equation (9), when $k = n$, for all $\varphi \in \Lambda$, we have

$$\int \beta_n \varphi h \, dm = \int \beta_n \circ T^n \cdot \varphi \circ T^n \cdot h \, dm = \int u\varphi \circ T^n \cdot h \, dm = 0.$$

By Lemma 3.4, there exists $\beta_{n+1} \in L^1(hm)$ such that

$$\beta_n = \beta_{n+1} \circ T.$$

Hence one has $u = \beta_{n+1} \circ T^{n+1}$.

Since the measure hm is an exact measure, we can conclude that u is a constant function. So, $u = 0$. □

COROLLARY 3.6. *Let $u : I \to \mathbb{R}$ be a non-constant function in $L^2(hm)$. Then there exist $\varphi_i \in \mathcal{B}$ and an integer $p \geq 0$ such that*

$$\langle u, \varphi_i \circ T^j \rangle_{hm} = 0 \quad \textit{for all } 0 \leq j < p$$

and

$$\langle u, \varphi_i \circ T^p \rangle_{hm} \neq 0.$$

Proof. Suppose that for every $\varphi_i \in \mathcal{B}$ and every $j \in \mathbb{N}$,

$$\langle u, \varphi_i \circ T^j \rangle_{hm} = 0.$$

Since $\mathcal{B}$ is a base for $\mathrm{Ker}(P)$ and $U^j : L^2(hm) \to L^2(hm)$ is an isometry, it follows that

$$\int \varphi \circ T^j \cdot u \cdot h \, dm = 0$$

for every $\varphi \in \mathrm{Ker}(P)$ and $j \in \mathbb{N}$. This contradicts Proposition 3.5. □

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Proof of Proposition 3.1. It follows from Lemma 3.3 and Corollary 3.6 that $\mathcal{W}$ is a basis of $L^2(hm)$. To construct a basis $\hat{\mathcal{W}} \subset L^\infty(hm)$, consider an enumeration of the set $\Lambda = \{\psi_i\}$ defined in Lemma 3.4. Apply the Gram–Schmidt process in the sequence ψ_i to obtain a sequence $\tilde{\psi}_i$ of pairwise orthogonal functions. Discarding the null functions and normalizing the remaining functions, we obtain an orthonormal set of functions $\hat{\mathcal{B}}$. From Lemma 3.4,

$$\overline{\text{span}(\hat{\mathcal{B}})} = \overline{\text{span}(\Lambda)} = \text{Ker } P,$$

so $\hat{\mathcal{B}}$ is a basis of Ker P, and

$$\hat{\mathcal{W}} = \{\phi \circ T^j : \phi \in \hat{\mathcal{B}},\ j \in \mathbb{N}\} \cup \{\mathbb{1}_I\}$$

is a basis of $L^2(hm)$. □

COROLLARY 3.7. *Let $u : I \to \mathbb{R}$ be a non-constant function in $L^1(hm)$. Let $\hat{\mathcal{B}}$ be as in the proof of Proposition 3.1. Then there exist $\varphi_i \in \hat{\mathcal{B}}$ and an integer $p \geq 0$ such that*

$$\langle u, \varphi_i \circ T^j \rangle_{hm} = 0 \quad \textit{for all } 0 \leq j < p$$

and

$$\langle u, \varphi_i \circ T^p \rangle_{hm} \neq 0.$$

Proof. Suppose that for every $\varphi \in \hat{\mathcal{B}}$ and every $j \in \mathbb{N}$,

$$\langle u, \varphi \circ T^j \rangle_{hm} = 0. \tag{11}$$

Let Λ be as in Lemma 3.4. Since $\hat{\mathcal{B}}$ was obtained by applying the Gram–Schmidt process to Λ, it follows that equation (11) holds for every $\varphi \in \Lambda$. This contradicts Proposition 3.5. □

From now on, we assume $\mathcal{W} \subset L^\infty(hm)$. Let $u \in L^1(hm)$ and consider the Fourier coefficients of u with respect to the basis $\mathcal{W}$:

$$c_{i,j}(u) = \langle u, U^j(\varphi_i) \rangle_{hm} = \int u \cdot \varphi_i \circ T^j \cdot h\, dm.$$

PROPOSITION 3.8. *The functionals $c_{i,j}$ have the following properties.*

(1) $c_{i,j}$ *is linear on* $L^1(hm)$.
(2) $c_{i,j}(U(u)) = c_{i,j-1}(u)$ *for* $j \geq 1$.
(3) $c_{i,0}(U(u)) = 0$.
(4) $c_{i,j}(P(u)) = c_{i,j+1}(u)$.

Proof. We have:

(1) the proof is straightforward;
(2) $c_{i,j}(u \circ T) = \langle u \circ T, \varphi_i \circ T^j \rangle_{hm} = \langle u, \varphi_i \circ T^{j-1} \rangle_{hm} = c_{i,j-1}(u)$;
(3) $c_{i,0}(u \circ T) = \langle U(u), \varphi_i \rangle_{hm} = \langle u, P(\varphi_i) \rangle_{hm} = \langle u, 0 \rangle_{hm} = 0$; and
(4) $c_{i,j}(Pu) = \langle P(u), U^j(\varphi_i) \rangle_{hm} = \langle u, U^{j+1}(\varphi_i) \rangle_{hm} = c_{i,j+1}(u)$. □

PROPOSITION 3.9. *For every $u \in L^1(hm)$ and $\varphi_i \in \hat{\mathcal{B}}$, we have*

$$\lim_j c_{i,j} = 0.$$

Proof. Since hm is exact, it is mixing, so

$$\lim_j c_{i,j} = \lim_j \int u \cdot \varphi_i \circ T^j \cdot h\, dm = 0. \qquad \square$$

Remark 3.10. Baladi drew our attention to the method used by Pollicott [**7**] to build eigenvectors of transfer operators for eigenvalues inside the essential spectral radius in certain function spaces. In our setting, the method is the following. Pick $\varphi \in \operatorname{Ker}(P)$ and $|\lambda| < 1$. Then

$$v = \sum_{j=0}^{\infty} \lambda^j \varphi \circ T^j$$

is a λ-eigenvector of P in $L^2(hm)$. Using Propositions 3.1 and 3.8, one can easily show that *all* λ-eigenvalues of P in $L^2(hm)$, for every $|\lambda| < 1$, can be built in this way.

4. *Proof of Theorem 1*

In this section, we will study the linear operator

$$\mathcal{L}u = u \circ T - u$$

acting on functions with bounded p-variation $u : I \to \mathbb{R}$.

First, we will present some properties and then, at the end of this section, we will prove the theorems announced in the introduction. The following results are well known.

LEMMA 4.1. *Let $\mathcal{L}$ be the linear operator defined above acting on $L^1(hm)$. Then:*
(1) *if $f \in \operatorname{Im}(\mathcal{L})$, then $\int fh\, dm = 0$; and*
(2) *$\operatorname{Ker}(\mathcal{L}) = \{f \in L^1(hm) : f$ is constant hm-almost everywhere$\}$.*

COROLLARY 4.2. *Let $u \in L^1(hm)$ and suppose that there exist functions $v, w \in L^1(hm)$ such that*

$$\mathcal{L}^n v = u = \mathcal{L}^n w.$$

Then $v = w + c$ on $L^1(hm)$ for some $c \in \mathbb{R}$. Moreover, if v, w have bounded p-variation then $v = w + c$ on I (e.c.s.).

Proof. Define $v_i = \mathcal{L}^{n-i} v$, $w_i = \mathcal{L}^{n-i} w$. We will prove by induction on i that $v_i = w_i$, if $i < n$ and $v_n = w_n + c$, for some $c \in \mathbb{R}$. Indeed, for $i = 0$, we have $w_0 = v_0 = u$. Suppose that $v_i = w_i$, $i < n$. Then

$$\mathcal{L}(v_{i+1} - w_{i+1}) = v_i - w_i = 0,$$

so $v_{i+1} - w_{i+1}$ is hm-almost everywhere constant. If $i + 1 = n$, we are done. If $i + 1 < n$, then $\mathcal{L}v_{i+2} = v_{i+1}$ and $\mathcal{L}w_{i+2} = w_{i+1}$, so

$$\int v_{i+1} h\, dm = \int w_{i+1} h\, dm = 0,$$

which implies $c = 0$. Now assume that u, v and w have bounded p-variation. Since the support of hm is I and $v = w + c$ hm-almost everywhere, we have $v = w + c$ on a set $\Lambda \subset I$ such that for every non-empty open subset O of I, we have that $O \cap \Lambda$ is a dense and uncountable subset of O. Since v and w have just a countable number of discontinuities in I, it follows that $v = w + c$ in I (e.c.s.). $\square$

LEMMA 4.3. *Let J be an open interval as in (D3) and $u \in BV_{p,I}$. Then*

$$v_{p,J}(\mathcal{L}u) \geq v_{p,J}(u)$$

for every $n \in \mathbb{N}$.

Proof. Let $J_1, J_2 \subset J$ be as in (D3). Since T is a homeomorphism on J_1 and J_2, by Remark 2.2,

$$v_{p,J}(u \circ T) \geq v_{p,J_1}(u \circ T) + v_{p,J_2}(u \circ T) = 2v_{p,J}(u),$$

so

$$v_{p,J}(u \circ T - u) \geq v_{p,J}(u \circ T) - v_{p,J}(u) \geq v_{p,J}(u). \qquad \square$$

LEMMA 4.4. *There exists C with the following property. Let $u_n : I \to \mathbb{R}$, $n \leq M+1$ be observables with bounded p-variation, $p \geq 1$, such that for every $n \leq M$,*

$$u_n = \mathcal{L}u_{n+1}.$$

Then

$$|u_n|_{L^\infty(hm)} \leq v_{p,I}(u_n) \leq C v_{p,I}(u_0)$$

for every $n \leq M$.

Proof. Let $J \subset I$ be one interval as in (D3). By Lemma 4.3,

$$v_{p,J}(u_n) \leq v_{p,J}(u_0) \tag{12}$$

for every $n \geq 0$. By (D2), there is a finite collection of pairwise disjoint open intervals $H_1, \ldots, H_k \subset J$ and j such that T^j is a homeomorphism on each H_i and

$$\text{int } I \subset \bigcup_{i=1}^{k} T^j(H_i). \tag{13}$$

We claim that for every $\ell \leq j$ and n,

$$v_{p,T^\ell(H_i)}(u_n) \leq 2^\ell v_{p,J}(u_0). \tag{14}$$

We will prove this by induction on ℓ. Of course, since $H_i \subset J$, equation (12) implies that for every $i = 1, \ldots, k$,

$$v_{p,H_i}(u_n) \leq v_{p,J}(u_0), \tag{15}$$

so equation (14) holds for $\ell = 0$. Suppose, by induction, that equation (14) holds for $\ell < j$ and every n. Since T is a homeomorphism on $T^\ell(H_i)$ and $u_{n-1} = u_n \circ T - u_n$, we have

$$\begin{aligned} v_{p,T^{\ell+1}(H_i)}(u_n) = v_{p,T^\ell(H_i)}(u_n \circ T) &\leq v_{p,T^\ell(H_i)}(u_n) + v_{p,T^\ell(H_i)}(u_{n-1}) \\ &\leq 2^{\ell+1} v_{p,J}(u_0). \end{aligned}$$

By equation (13),

$$v_{p,I}(u_n) = v_{p,\text{int } I}(u_n) \leq \sum_{i=1}^{k} v_{p,T^j(H_i)}(u_n) \leq k2^j v_{p,J}(u_0) \leq k2^j v_{p,I}(u_0).$$

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Note that since $u_n = u_{n+1} \circ T - u_{n+1}$, it follows that

$$\int u_n h \, dm = 0.$$

Suppose that

$$\operatorname*{ess\ sup}_{hm} u_n = |u_n|_{L^\infty(hm)}.$$

Then

$$\begin{aligned} 0 = \int u_n h \, dm &\geq \text{ ess inf } u_n = (\text{ess inf } u_n - \text{ess sup } u_n) + \text{ess sup } u_n \\ &\geq -v_{p,I}(u_n) + |u_n|_{L^\infty(hm)}, \end{aligned}$$

so $|u_n|_{L^\infty(hm)} \leq v_{p,I}(u_n)$. We can obtain the same conclusion for the case

$$-\text{ess inf}_{hm} \, u_n = |u_n|_{L^\infty(hm)},$$

replacing u_n by $-u_n$ in the argument above. □

Proof of Theorem 1. Define by induction the (either finite or infinite) sequence $u_n : I \to \mathbb{R}$ of functions in the following way: u_0 is given. If u_n is defined and there exists a function $v : I \to \mathbb{R}$ with bounded p-variation such that $\mathcal{L} v = u_n$ in $L^1(hm)$, then define

$$u_{n+1} = v - \int vh \, dm.$$

Otherwise, the sequence ends with u_n. Note that

$$\mathcal{L}^n u_n = u_0.$$

Define

$$M_0 = \sup\{n \in \mathbb{N} : u_n \text{ is defined}\} \in \mathbb{N} \cup \{\infty\}.$$

We will show that $M_0 < \infty$. Let $M \in \mathbb{N}$, $M \leq M_0$. Recall the basis $\mathcal{W}$ defined in §3. By Corollary 3.6, if u_0 is not constant almost everywhere there exist i and $q \geq 0$ such that

$$c_{i,j}(u_0) = \int u_0 \varphi_i \circ T^j \cdot h \, dm = 0 \quad \text{for all } 0 \leq j < q$$

and

$$c_{i,q}(u_0) = \int u_0 \varphi_i \circ T^q \cdot h \, dm \neq 0.$$

By Lemma 4.4, we have that $|u_n|_{L^2(hm)} \leq |u_n|_{L^\infty(hm)} \leq C v_{p,I}(u_0)$, so since

$$|\varphi_i \circ T^i|_{L^2(hm)} = 1,$$

we obtain

$$|c_{i,k}(u_n)| = \left| \int u_n \cdot \varphi_i \circ T^k \cdot h \, dm \right| \leq C v_{p,I}(u_0).$$

Using Proposition 3.8, we can now use an argument quite similar to [**4**]. Observe that

$$c_{i,l}(u_{n-1}) = c_{i,l}(u_n \circ T - u_n) = c_{i,l}(u_n \circ T) - c_{i,l}(u_n) = c_{i,l-1}(u_n) - c_{i,l}(u_n)$$

for $l \geq 1$.

For $l = 0$,

$$c_{i,0}(u_{n-1}) = c_{i,0}(u_n \circ T - u_n) = c_{i,0}(u_n \circ T) - c_{i,0}(u_n) = -c_{i,0}(u_n)$$

for $0 < n \leq M$.

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Therefore, for $0 < n \leq M$,

$$c_{i,l}(u_n) = c_{i,l-1}(u_n) - c_{i,l}(u_{n-1}) \quad \text{for } l \geq 1, \tag{16}$$

$$c_{i,0}(u_{n-1}) = -c_{i,0}(u_n). \tag{17}$$

Since $c_{i,j}(u_0) = 0$ for $0 \leq j < q$, by equations (16) and (17), we can conclude that

$$c_{i,j}(u_n) = 0 \quad \text{for } 0 \leq j < q \quad \text{and} \quad 0 \leq n \leq M. \tag{18}$$

Now, by equation (16), considering $l = q$, we have

$$c_{i,q}(u_{n-1}) = c_{i,q-1}(u_n) - c_{i,q}(u_n).$$

By equation (18), for every $n \leq M$,

$$c_{i,q}(u_{n-1}) = -c_{i,q}(u_n). \tag{19}$$

By equation (19), we conclude that for $n \leq M$,

$$c_{i,q}(u_n) = (-1)^n c_{i,q}(u_0).$$

Considering $l = q + 1$ in equation (16),

$$c_{i,q+1}(u_n) = (-1)^n c_{i,q}(u_0) - c_{i,q+1}(u_{n-1}) \Rightarrow$$

$$c_{i,q}(u_0) = (-1)^n c_{i,q+1}(u_n) + (-1)^n c_{i,q+1}(u_{n-1}). \tag{20}$$

Putting $n = 1, \ldots, M$ in equation (20) and adding the resulting equations, we obtain

$$M \cdot c_{i,q}(u_0) = (-1)^M c_{i,q+1}(u_M) - c_{i,q+1}(u_0). \tag{21}$$

Therefore,

$$\begin{aligned} M &= \frac{-c_{i,q+1}(u_0) + (-1)^M c_{i,q+1}(u_M)}{c_{i,q}(u_0)} \\ &\leq \frac{|c_{i,q+1}(u_0)| + |c_{i,q+1}(u_M)|}{|c_{i,q}(u_0)|} \\ &\leq \frac{|c_{i,q+1}(u_0)| + C v_{p,I}(u_0)}{|c_{i,q}(u_0)|}, \end{aligned}$$

so M_0 is bounded. Note that by Corollary 4.2, if $v_n \in L^1(hm)$ satisfies $\mathcal{L}^n v_n = u_0$, then $v_n = u_n + c$ in $L^1(hm)$ for some $c \in \mathbb{R}$. This proves the uniqueness statements of Theorem 1. □

5. *Proof of Theorem 2*

Fix $\lambda < 1$. Denote by S_λ the linear space of the real sequences $x = (x^j)_{j \in \mathbb{N}}$ such that there exists C satisfying

$$|x^j| \leq C \lambda^j.$$

Here we use x^j to denote the jth element of the sequence x. Consider the linear space $\ell_0(\mathbb{N})$ of real sequences $x = (x^j)_{j \in \mathbb{N}}$ such that

$$\lim_j x^j = 0.$$

We define the operator $U : \ell_0(\mathbb{N}) \to \ell_0(\mathbb{N})$ as

$$U(x) = y,$$

where $y^0 = 0$ and $y^{j+1} = x^j$ for $j \geq 0$.

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We say that $x \in \ell_0(\mathbb{N})$ is infinitely cohomologous to zero with respect to U in $\ell_0(\mathbb{N})$ if there exists an infinite sequence $x_i \in \ell_0(\mathbb{N})$, with $x = x_0$, such that

$$x_i = U(x_{i+1}) - x_{i+1}. \tag{22}$$

for every $i \geq 0$.

LEMMA 5.1. [**2**] *Let $x \in S_\lambda$. Suppose that there exists a finite sequence $x = x_0$, $x_1, \ldots, x_k \in \ell_0(\mathbb{N})$ such that $x_i = U(x_{i+1}) - x_{i+1}$ for every $i < k$. Then $x_i \in S_\lambda$ for every $i \leq k$. If x is infinitely cohomologous to zero with respect to U in $\ell_0(\mathbb{N})$, then $x = 0 = (0, 0, \ldots)$.*

Proof. Let $x_i \in \ell_0(\mathbb{N})$, $i \leq k$, with $x_0 = x$, satisfying equation (22) for $i < k$. One can see that

$$x_{i+1}^j = -\sum_{p \leq j} x_i^p.$$

Since $\lim_j x_{i+1}^j = 0$, it follows that

$$\sum_p x_i^p = 0;$$

since $x_0 \in S_\lambda$, we can prove by induction on i that

$$|x_{i+1}^j| = \left|\sum_{p>j} x_i^p\right| \leq C_i \lambda^j$$

for some C_i. We concluded that $x_i \in S_\lambda$ for every $i \leq k$. With each $i \leq k$, we can associate the power series

$$f_i(z) = \sum_{j=0}^{\infty} x_i^j z^j.$$

Since $x_i = (x_i^j)_j \in S_\lambda$, the power series f_i converges to a complex analytic function on a disk with center at 0 and radius $1/\lambda > 1$. Note that the sequence $U(x_i)$ is the sequence of coefficients of the Taylor series (centered at 0) of the function $zf_i(z)$. So, equation (22) yields

$$f_i(z) = zf_{i+1}(z) - f_{i+1}(z) = (z-1)f_{i+1}(z).$$

So, if x_0 is infinitely cohomologous to zero, we conclude that

$$f_0(z) = (z-1)^k f_k(z)$$

for every k, where f_k is defined in a disk strictly larger than the unit disk. It follows that $f_0^{(k)}(1) = 0$ for every k, so $f_0(z) = 0$ everywhere. So, $x = x_0 = 0 = (0, 0, \ldots)$. □

Proof of Theorem 2. Corollary 4.2 gives the uniqueness of the sequence u_i. Now suppose that u_0 is infinitely cohomologous to zero. So, there exists a sequence $u_i \in L^1(hm)$ such that

$$u_i = u_{i+1} \circ T - u_{i+1}. \tag{23}$$

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Consider $\hat{\mathcal{B}}$ as in Corollary 3.7. Fix $\varphi \in \hat{\mathcal{B}}$. Define the sequence $x_i = (x_i^j)_j$ as

$$x_i^j = \int u_i \cdot \varphi \circ T^j \cdot h \, dm.$$

Since x_i^j are Fourier coefficients of $u_i \in L^1(hm)$ with respect to the Hilbert basis $\mathcal{W}$, by Proposition 3.9, we have that $\lim_j x_i^j = 0$. By equation (23) and Proposition 3.8 we have

$$x_i = U(x_{i+1}) - x_i,$$

so x_0 is infinitely cohomologous to zero in $\ell_0(\mathbb{N})$. Note that

$$|x_0^j| = \left| \int u_0 \cdot \varphi \circ T^j \cdot h \, dm \right| \le C\lambda^j,$$

so $x_0 \in S_\lambda$. By Lemma 5.1, we have that $x_0 = 0$; this holds for every $\varphi \in \hat{\mathcal{B}}$, so by Corollary 3.7, the function u_0 is zero. □

6. *Proof of Theorem 3*

We first make a couple of remarks on condition (D5).

Remark 6.1. Suppose that T and $\mathbb{B}$ satisfy (D5). Let $\tilde{h} \in L^1(m)$ be a function satisfying $\Phi_T(\tilde{h}) = \tilde{h}$. Then

$$\tilde{h} = \int \tilde{h} \, dm \cdot h, \tag{24}$$

where h is as in (D5)(iii) Indeed, by (D5)(vi), there exists a sequence $h_n \in \mathbb{B}$ such that $h_n \to_n \tilde{h}$ in $L^1(hm)$. Furthermore, since h, $1/h \in \mathbb{B}$, from (D5)(ii), there exist $a, b > 0$ such that

$$0 < a \le h(x) \le b < \infty \tag{25}$$

on I. So,

$$\begin{aligned}
\left| \int \tilde{h} \, dm \cdot h - \tilde{h} \right|_{L^1(m)} &\le |\tilde{h} - h_n|_{L^1(m)} + \left| \int h_n \, dm \cdot h - \Phi_T^k(h_n) \right|_{L^1(m)} \\
&\quad + |\Phi_T^k(h_n) - \Phi_T^k(\tilde{h})|_{L^1(m)} \\
&\le 2|\tilde{h} - h_n|_{L^1(m)} + \left| \int h_n \, dm \cdot h - \Phi_T^k(h_n) \right|_{\mathbb{B}} \\
&\le 2|\tilde{h} - h_n|_{L^1(m)} + C\lambda^k |h_n|_{\mathbb{B}}.
\end{aligned}$$

Given $\epsilon > 0$, choose n_0 such that

$$|\tilde{h} - h_{n_0}|_{L^1(m)} \le \frac{1}{a} |\tilde{h} - h_{n_0}|_{L^1(hm)} < \frac{\epsilon}{4},$$

and k_0 such that

$$C\lambda^{k_0} |h_{n_0}|_{\mathbb{B}} < \frac{\epsilon}{2}.$$

Then

$$\left| \int \tilde{h} \, dm \cdot h - \tilde{h} \right|_{L^1(m)} < \epsilon$$

for every $\epsilon > 0$, so equation (24) holds. In particular, if T and $\mathbb{B}$ satisfy (D1), (D4) and (D5), we have that functions h in (D4) and (D5) coincide.

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Remark 6.2. Note that (D5)(ii) implies that

$$\mathbb{B} \subset L^1(hm).$$

Moreover, (D5)(iii)–(v) imply that

$$\frac{1}{h}\Psi^j(vh)$$

converges exponentially to zero in $L^1(hm)$ and $\mathbb{B}$.

LEMMA 6.3. *Let T be a transformation satisfying (D1) and (D4), and suppose that $\mathbb{B}$ and T satisfy (D5). Let $u \in \mathbb{B}$, and suppose that there exists $v \in L^1(hm)$ such that*

$$u = \mathcal{L}v$$

on I. Then v coincides hm-almost everywhere with a function $v_1 \in \mathbb{B}$.

Proof. The method we are going to use here is very well known for specific kinds of dynamical systems and observables; for instance, see [**5**] for the case of C^2 piecewise smooth expanding maps and bounded variation observables. Replacing v by

$$v - \int vh\, dm\, \mathbb{1}_I,$$

we may assume, without loss of generality, that

$$\int vh\, dm = 0$$

since

$$u = v \circ T - v.$$

Applying P^j, $j \geq 1$, we get

$$P^j u = P^{j-1} v - P^j v. \tag{26}$$

Putting $j = 1, \ldots, n$ in equation (26) and adding the resulting equations, we obtain

$$v = P^n v + \sum_{j=1}^{n} P^j u.$$

We claim that $|P^j v|_{L^1(hm)} \to_j 0$. Indeed, from (D5)(vi), for every $\epsilon > 0$, there exists $w \in \mathbb{B}$ such that $\int wh\, dm = 0$ and $|v - w|_{L^1(hm)} < \epsilon$. Since $\|P\|_{L^1(hm)} \leq 1$, for every j,

$$|P^j v - P^j w|_{L^1(hm)} < \epsilon.$$

From (D5), for every $w \in \mathbb{B}$,

$$P^j(w) = \frac{1}{h}\Psi^j(wh),$$

and

$$|\Psi^j(wh)|_{L^1(hm)} \leq C|\Psi^j(wh)|_{\mathbb{B}} \leq C\lambda^j |wh|_{\mathbb{B}},$$

and we have that for j large enough,

$$|P^j v|_{L^1(hm)} \leq |P^j v - P^j w|_{L^1(hm)} + |P^j w|_{L^1(hm)} < 2\epsilon.$$

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This proves our claim. In particular,

$$v = \sum_{j=1}^{\infty} P^j u,$$

where the convergence of the series is in $L^1(hm)$. On the other hand, by Remark 6.2, this series converges in $L^1(hm)$ and $\mathbb{B}$ to a function $v_1 \in \mathbb{B}$. So, $v = v_1$ hm-almost everywhere. □

Proof of Theorem 3. Since $u_0 \in \mathbb{B}$, by (D5), for every $v \in L^\infty(hm)$, we have

$$\left| \int u_0 \cdot v \circ T^j \cdot h\, dm \right| = \left| \int P^j(u_0) \cdot v \cdot h\, dm \right| \le C\lambda^j |u_0|_{\mathbb{B}} |v|_{L^\infty(hm)}.$$

By Theorem 2, we have that u_0 is not infinitely cohomologous to zero in $L^1(hm)$. Now suppose $\mathcal{L} u_i = u_0$. The uniqueness (up to a constant) of u_i follows from Corollary 4.2. By Lemma 6.3, we have $u_i \in \mathbb{B}$. □

7. *Topological results*

Proof of Theorem 4. Define $\tilde{u}_0 = u_0 \circ H$. Then $\tilde{u}_0$ has bounded p-variation. By Theorem 1, there exist bounded p-variation functions $\tilde{u}_i$, $i \le M$, unique up to a constant, such that

$$\tilde{\mathcal{L}}^i \tilde{u}_i = \tilde{u}_0 \quad \text{on } L^1(hm),$$

and

$$\tilde{\mathcal{L}} \alpha \ne \tilde{u}_M + c \quad \text{on } L^1(hm), \tag{27}$$

for every bounded p-variation function α. Here $\tilde{\mathcal{L}} v = v \circ \tilde{T} - v$. Since the support of hm is I, it follows that $\tilde{\mathcal{L}}^i \tilde{u}_i = \tilde{u}_0$ in I (e.c.s.). Define $u_i = \tilde{u}_i \circ H^{-1}$. Then u_i has bounded p-variation and

$$\mathcal{L}^i u_i = u_0 \quad \text{on } H(I)\ (e.c.s.).$$

Suppose that there exists a function ρ with bounded p-variation such that $\mathcal{L}\rho = u_M + c$ (e.c.s.). Define $\tilde{\rho} = \rho \circ H$. Then $\tilde{\rho}$ has bounded p-variation and $\tilde{\mathcal{L}}\tilde{\rho} = \tilde{u}_M + c$ on $L^1(hm)$. This contradicts equation (27). So, $\mathcal{L}\rho \ne u_M + c$ in an uncountable subset of $H(I)$. Since the discontinuities of $\mathcal{L}\rho$ and $u_M + c$ are countable, it follows that there is a continuity point $x_0 \in H(I)$ of both functions such that $(\mathcal{L}\rho)(x_0) \ne u_M(x_0) + c$. So, there is a non-empty open subset of $H(I)$ such that $\mathcal{L}\rho \ne u_M + c$. □

Proof of Theorem 5. The proof of this theorem is quite similar to the proof of Theorem 4. Define $\tilde{u}_0 = u_0 \circ H$. Then $\tilde{u}_0$ has bounded p_0-variation. By Theorem 3, there exist bounded p_0-variation functions $\tilde{u}_i$, $i \le M$, unique up to a constant, such that

$$\tilde{\mathcal{L}}^i \tilde{u}_i = \tilde{u}_0 \quad \text{on } L^1(hm),$$

and

$$\tilde{\mathcal{L}} \alpha \ne \tilde{u}_M + c \quad \text{on } L^1(hm), \tag{28}$$

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for every $\alpha \in L^1(hm)$. Here $\tilde{\mathcal{L}}v = v \circ \tilde{T} - v$. Since the support of hm is I, it follows that $\tilde{\mathcal{L}}^i \tilde{u}_i = \tilde{u}_0$ in I (e.c.s.). Define $u_i = \tilde{u}_i \circ H^{-1}$. Then u_i has bounded p_0-variation and

$$\mathcal{L}^i u_i = u_0 \quad \text{on } H(I)\ (e.c.s.).$$

Now we show the uniqueness of u_i in the set of continuous (e.c.s.), bounded Borelian functions. If continuous (e.c.s.) bounded Borelian functions v_i satisfy $\mathcal{L}^i v_i = u_0$, then $\tilde{v}_i = v_i \circ H$ are also continuous (e.c.s.) and, moreover, they belong to $L^1(hm)$ and satisfy $\tilde{\mathcal{L}}^i \tilde{v}_i = \tilde{u}_0$. So, by Theorem 3, we have that $\tilde{v}_i = \tilde{u}_i + c_i$ for some $c_i \in \mathbb{R}$, where this equality holds in $L^1(hm)$. Since both functions $\tilde{v}_i$, $\tilde{u}_i$ are continuous (e.c.s.), it follows that $\tilde{v}_i = \tilde{u}_i + c_i\ (e.c.s.)$, so $v_i = u_i + c_i\ (e.c.s.)$.

To show conclusions (A) and (B), suppose that there exists a bounded Borelian function ρ such that $\mathcal{L}\rho = u_M + c$ (e.c.s.). Define $\tilde{\rho} = \rho \circ H$. Then $\tilde{\rho}$ is also a bounded Borelian function, so it belongs to $L^1(hm)$ and $\tilde{\mathcal{L}}\tilde{\rho} = \tilde{u}_M + c$ (e.c.s.). So, since hm has no atoms, it follows that this equality holds on $L^1(hm)$. This contradicts equation (28). So, $\mathcal{L}\rho \neq u_M + c$ in an uncountable subset of $H(I)$. If ρ is continuous (e.c.s.), we can now finish the proof exactly as in the proof of Theorem 4. □

Remark 7.1. One can ask why the conclusions of Theorem 5 are weaker than those of Theorem 3. The problem is that the conjugacy between one-dimensional maps can be singular with respect to the Lebesgue measure. Indeed, that is often the case, even when the two one-dimensional maps T and $\tilde{T}$ are very regular, as expanding maps on a circle (see [**8**]). In particular, the conjugacy H does not in general preserve either $L^1(hm)$, $L^1(m)$ or the space of Lebesgue-measurable functions (see [**6**]). So, note that if in the proof of Theorem 5 we pick ρ to be either in $L^1(m)$ or $L^1(hm)$, then it is not true in general that $\rho \circ H$ belongs to $L^1(hm)$. Moreover, since composition with H does not in general preserve Lebesgue-measurable functions, we need to assume that ρ is a Borel-measurable function, so $\rho \circ H$ is also Borel measurable. These are the reasons why we assume that ρ is bounded and Borelian in Theorem 5.

8. *Observables infinitely cohomologous to zero*

Consider the Banach space of summable sequences $\ell^1(\mathbb{N})$. For a sequence $x = (x^j)_{j \in \mathbb{N}}$, denote

$$|x|_{\ell^1(\mathbb{N})} = \sum_j |x^j|.$$

We define the operator $U : \ell^1(\mathbb{N}) \to \ell^1(\mathbb{N})$ as the norm-preserving map

$$U(x) = y,$$

where $y^0 = 0$ and $y^{j+1} = x^j$ for $j \geq 0$.

We say that $x \in \ell^1(\mathbb{N})$ is infinitely cohomologous to zero with respect to U if there exists an infinite sequence $x_i \in \ell^1(\mathbb{N})$, with $x = x_0$ such that

$$x_i = U(x_{i+1}) - x_{i+1}$$

for every $i \geq 0$.

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LEMMA 8.1. [**2**] *There is a non-vanishing sequence* $x \in \ell^1(\mathbb{N})$ *which is infinitely cohomologous to zero with respect to* U.

Proof. We claim that for every $k \in \mathbb{N}$, there exist

$$x_{0,k}, x_{1,k}, \ldots, x_{k,k} \in \ell^1(\mathbb{N}),$$

all of them with compact support, such that $x^0_{0,k} = 1$,

$$\begin{aligned} x_{i,k} &= U(x_{i+1,k}) - x_{i+1,k}, \\ |x_{i,k+1} - x_{i,k}|_{\ell^1(\mathbb{N})} &< 2^{-k-1}, \end{aligned} \tag{29}$$

for every $i < k$.

The proof is by induction on k. Choose $x_{0,0} = (1, 0, 0, 0, \ldots)$. Suppose that by induction we found a finite sequence $x_{i,k}$, $i \leq k$, with the above properties. Fix $N > 0$. Define $x_{k,k+1}$ as $x^0_{k,k+1} = x^0_{k,k}$, $x^j_{k,k+1} = x^j_{k,k} - \delta/N$, for $1 \leq j \leq N$, and $x^j_{k,k+1} = x^j_{k,k}$ for $j \geq N + 1$. Here $\delta = \sum_j x^j_{k,k}$. Defining

$$x^j_{k+1,k+1} = -\sum_{p \leq j} x^p_{k,k+1},$$

we have that $x_{k+1,k+1}$ has compact support and $x_{k,k+1} = U(x_{k+1,k+1}) - x_{k+1,k+1}$. Now define by induction

$$x_{i,k+1} = U(x_{i+1,k+1}) - x_{i+1,k+1}, \quad i < k.$$

In particular, $x^0_{i,k+1} = -x^0_{i+1,k+1}$ for $i \leq k$. Since $x^0_{i,k} = -x^0_{i+1,k}$ for $i < k$ and $x^0_{k,k+1} = x^0_{k,k}$, we have $x_{0,k+1} = 1$. Furthermore, it is not difficult to see that if N is large enough, then

$$|x_{i,k+1} - x_{i,k}|_{\ell^1(\mathbb{N})} < 2^{-k-1},$$

for every $i < k$. This completes the inductive step.

By equation (29), for every i, there exists $x_i \in \ell^1(\mathbb{N})$ such that $\lim_k x_{i,k} = x_i$ on $\ell^1(\mathbb{N})$. It is easy to check that $x_i = U(x_{i+1}) - x_{i+1}$ and $x^0_0 = 1$. Pick $x = x_0$. □

Proof of Theorem 6. Since T is topologically conjugate with $T_\ell = \ell x \mod 1$, $\ell \in \mathbb{Z} \backslash \{-1, 0, 1\}$, it is enough to show Theorem 6 for T_ℓ. Choose n such that ℓ does not divide n. Let $x = (x_j)_j \in \ell^1(\mathbb{N})$, as in Lemma 8.1. Define

$$u(x) = \sum_{j=0}^{\infty} x_j \sin(2\pi n \ell^j x).$$

The function u is continuous and non-constant. Using Remark 3.2 and Proposition 3.8, one can easily show that u is infinitely cohomologous to zero. □

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