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# BLOCK AND BASU'S BIVARIATE EXPONENTIAL DISTRIBUTION WITH APPLICATION TO DROUGHT DATA

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Motivated by hydrological applications, the exact distributions of $R = X + Y$, $P = XY$, and $W = X/(X + Y)$ and the corresponding moment properties are derived when $X$ and $Y$ follow Block and Basu's bivariate exponential distribution. An application of the results is provided to drought data from Nebraska.

## 1. INTRODUCTION

The aim of this article is to derive the exact distributions of $R = X + Y$, $P = XY$, and $W = X/(X + Y)$ when $(X, Y)$ follows Block and Basu's [1] bivariate exponential distribution given by the joint probability density function (p.d.f.)

$$f(x,y) = \begin{cases} \dfrac{\lambda_1 \lambda(\lambda_2 + \lambda_{12})}{\lambda_1 + \lambda_2} \exp\{-\lambda_1 x - (\lambda_2 + \lambda_{12}) y\} & \text{if } 0 < x < y \\ \dfrac{\lambda_2 \lambda(\lambda_1 + \lambda_{12})}{\lambda_1 + \lambda_2} \exp\{-\lambda_2 y - (\lambda_1 + \lambda_{12}) x\} & \text{if } 0 < y \le x \end{cases} \tag{1}$$

 

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for $x > 0$, $y > 0$, $\alpha > 0$, $\beta > 0$, $\alpha' > 0$, and $\beta' > 0$, where $\lambda = \lambda_1 + \lambda_2 + \lambda_{12}$. This is one of the most flexible bivariate exponential distributions in the literature: It was derived by Block and Basu by omitting the singular part of Marshall and Olkin's [6] distribution.

Since the pioneering work of Gumbel [4], bivariate exponential distributions have attracted many applications in hydrological sciences. The above model due to Block and Basu [1] would be an ideal model for these applications. There is clear reason to believe that distributions of $R = X + Y$, $P = XY$, and $W = X/(X + Y)$ will be of interest in hydrological applications. For example, if $X$ and $Y$ denote the drought intensity and the drought duration, respectively, then $P = XY$ will represent the magnitude of the drought. If $X$ and $Y$ denote the drought duration and the successive nondrought duration, respectively, then $R = X + Y$ and $W = X/(X + Y)$ will represent the interarrival time of drought events and the proportion of drought events, respectively (see Section 4).

This article is organized as follows. In Sections 2 and 3 explicit expressions for the p.d.f.s and moments of $R = X + Y$, $P = XY$, and $W = X/(X + Y)$ are derived. In Section 4 an application of the results to drought data from Nebraska is provided. The calculations in this article involve the complementary incomplete gamma function defined by

$$\Gamma(a, x) = \int_x^\infty t^{a-1} \exp(-t)\, dt.$$

The properties of this special function can be found in Prudnikov, Brychkov, and Marichev [7] and Gradshteyn and Ryzhik [3].

## 2. PROBABILITY DENSITY FUNCTIONS

Theorems 1–3 derive the p.d.f.s of $R = X + Y$, $P = XY$, and $W = X/(X + Y)$ when $X$ and $Y$ are distributed according to (1).

THEOREM 1: *If X and Y are jointly distributed according to (1), then*

$$\begin{aligned} f_R(r) = {} & \frac{\lambda\lambda_1(\lambda_2 + \lambda_{12})\exp\{-(\lambda_2 + \lambda_{12})r\}}{(\lambda_1 + \lambda_2)(\lambda_2 - \lambda_1 + \lambda_{12})}\left[\exp\left\{\left(\frac{1}{2}\right)(\lambda_2 - \lambda_1 + \lambda_{12})r\right\} - 1\right] \\ & + \frac{\lambda\lambda_2(\lambda_1 + \lambda_{12})\exp\{-(\lambda_1 + \lambda_{12})r\}}{(\lambda_1 + \lambda_2)(\lambda_2 - \lambda_1 - \lambda_{12})} \\ & \times \left[1 - \exp\left\{-\left(\frac{1}{2}\right)(\lambda_2 - \lambda_1 - \lambda_{12})r\right\}\right] \qquad (2) \end{aligned}$$

*for* $0 < r < \infty$.

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PROOF: From (1), the joint p.d.f. of $(R, W) = (X + Y, X/R)$ becomes

$$f(r,w) = \begin{cases} \dfrac{\lambda_1 \lambda (\lambda_2 + \lambda_{12}) r}{\lambda_1 + \lambda_2} \exp\{-\lambda_1 rw - (\lambda_2 + \lambda_{12}) r(1-w)\} & \text{if } w < \dfrac{1}{2} \\ \dfrac{\lambda_2 \lambda (\lambda_1 + \lambda_{12}) r}{\lambda_1 + \lambda_2} \exp\{-\lambda_2 r(1-w) - (\lambda_1 + \lambda_{12}) rw\} & \text{if } w \geq \dfrac{1}{2}. \end{cases} \quad \textbf{(3)}$$

Thus, the p.d.f. of $R$ can be written as

$$f_R(r) = \frac{\lambda_1 \lambda (\lambda_2 + \lambda_{12}) r}{\lambda_1 + \lambda_2} \exp\{-(\lambda_2 + \lambda_{12}) r\} \int_0^{1/2} \exp\{(\lambda_2 - \lambda_1 + \lambda_{12}) rw\}\, dw$$

$$+ \frac{\lambda_2 \lambda (\lambda_1 + \lambda_{12}) r}{\lambda_1 + \lambda_2} \exp(-\lambda_2 r) \int_{1/2}^{1} \exp\{(\lambda_2 - \lambda_1 - \lambda_{12}) rw\}\, dw.$$

The result of the theorem follows by elementary integration of the above two integrals. ■

THEOREM 2: *If X and Y are jointly distributed according to (1), then*

$$f_P(p) = \frac{\lambda \lambda_1 (\lambda_2 + \lambda_{12})}{\lambda_1 + \lambda_2} \sum_{k=0}^{\infty} \frac{(-\lambda_1)^k}{k!} \{(\lambda_2 + \lambda_{12}) p\}^k \Gamma(-k, (\lambda_2 + \lambda_{12})\sqrt{p})$$

$$+ \frac{\lambda \lambda_2 (\lambda_1 + \lambda_{12})}{\lambda_1 + \lambda_2} \sum_{k=0}^{\infty} \frac{(-\lambda_2)^k}{k!} \{(\lambda_1 + \lambda_{12}) p\}^k \Gamma(-k, (\lambda_1 + \lambda_{12})\sqrt{p}) \quad \textbf{(4)}$$

*for* $0 < p < \infty$.

PROOF: From (1), the joint p.d.f. of $(X, P) = (X, XY)$ becomes

$$f(x,p) = \begin{cases} \dfrac{\lambda_1 \lambda (\lambda_2 + \lambda_{12})}{(\lambda_1 + \lambda_2) x} \exp\left\{-\lambda_1 x - (\lambda_2 + \lambda_{12}) \dfrac{p}{x}\right\} & \text{if } x < \sqrt{p} \\ \dfrac{\lambda_2 \lambda (\lambda_1 + \lambda_{12})}{(\lambda_1 + \lambda_2) x} \exp\left\{-\lambda_2 \dfrac{p}{x} - (\lambda_1 + \lambda_{12}) x\right\} & \text{if } x > \sqrt{p}. \end{cases}$$

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Thus, the p.d.f. of $P$ can be written as

$$
\begin{aligned}
f_P(p) &= \frac{\lambda_1 \lambda (\lambda_2 + \lambda_{12})}{\lambda_1 + \lambda_2} \int_0^{\sqrt{p}} \frac{1}{x} \exp\left\{ -\lambda_1 x - (\lambda_2 + \lambda_{12}) \frac{p}{x} \right\} dx \\
&\quad + \frac{\lambda_2 \lambda (\lambda_1 + \lambda_{12})}{\lambda_1 + \lambda_2} \int_{\sqrt{p}}^{\infty} \frac{1}{x} \exp\left\{ -\lambda_2 \frac{p}{x} - (\lambda_1 + \lambda_{12}) x \right\} dx \\
&= \frac{\lambda_1 \lambda (\lambda_2 + \lambda_{12})}{\lambda_1 + \lambda_2} \int_{1/\sqrt{p}}^{\infty} \frac{1}{y} \exp\left\{ -\frac{\lambda_1}{y} - (\lambda_2 + \lambda_{12}) p y \right\} dy \\
&\quad + \frac{\lambda_2 \lambda (\lambda_1 + \lambda_{12})}{\lambda_1 + \lambda_2} \int_{\sqrt{p}}^{\infty} \frac{1}{x} \exp\left\{ -\lambda_2 \frac{p}{x} - (\lambda_1 + \lambda_{12}) x \right\} dx \\
&= \frac{\lambda_1 \lambda (\lambda_2 + \lambda_{12})}{\lambda_1 + \lambda_2} \int_{1/\sqrt{p}}^{\infty} \sum_{k=0}^{\infty} \frac{(-\lambda_1)^k}{k! y^{k+1}} \exp\{ -(\lambda_2 + \lambda_{12}) p y \} dy \\
&\quad + \frac{\lambda_2 \lambda (\lambda_1 + \lambda_{12})}{\lambda_1 + \lambda_2} \int_{\sqrt{p}}^{\infty} \sum_{k=0}^{\infty} \frac{(-\lambda_2 p)^k}{k! x^{k+1}} \exp\{ -(\lambda_1 + \lambda_{12}) x \} dx \\
&= \frac{\lambda_1 \lambda (\lambda_2 + \lambda_{12})}{\lambda_1 + \lambda_2} \sum_{k=0}^{\infty} \frac{(-\lambda_1)^k}{k!} \int_{1/\sqrt{p}}^{\infty} y^{-(k+1)} \exp\{ -(\lambda_2 + \lambda_{12}) p y \} dy \\
&\quad + \frac{\lambda_2 \lambda (\lambda_1 + \lambda_{12})}{\lambda_1 + \lambda_2} \int_{\sqrt{p}}^{\infty} \sum_{k=0}^{\infty} \frac{(-\lambda_2 p)^k}{k!} \int_{\sqrt{p}}^{\infty} x^{-(k+1)} \exp\{ -(\lambda_1 + \lambda_{12}) x \} dx,
\end{aligned} \tag{5}
$$

where we have set $y = 1/x$ and used the series expansion

$$\exp(-x) = \sum_{k=0}^{\infty} \frac{(-x)^k}{k!}.$$

By the definition of the complementary incomplete gamma function,

$$\int_{1/\sqrt{p}}^{\infty} y^{-(k+1)} \exp\{ -(\lambda_2 + \lambda_{12}) p y \} dy = (\lambda_2 + \lambda_{12})^k p^k \Gamma(-k, (\lambda_2 + \lambda_{12}) \sqrt{p}) \tag{6}$$

and

$$\int_{\sqrt{p}}^{\infty} x^{-(k+1)} \exp\{ -(\lambda_1 + \lambda_{12}) x \} dx = (\lambda_1 + \lambda_{12})^k \Gamma(-k, (\lambda_1 + \lambda_{12}) \sqrt{p}). \tag{7}$$

The result of the theorem follows by substituting (6) and (7) into (5). ■

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THEOREM 3: *If X and Y are jointly distributed according to (1), then*

$$f_W(w) = \begin{cases} \dfrac{\lambda\lambda_1(\lambda_2+\lambda_{12})}{(\lambda_1+\lambda_2)\{\lambda_1 w + (\lambda_2+\lambda_{12})(1-w)\}^2} & \text{if } w < \dfrac{1}{2} \\ \dfrac{\lambda\lambda_2(\lambda_1+\lambda_{12})}{(\lambda_1+\lambda_2)\{\lambda_2(1-w) + (\lambda_1+\lambda_{12})w\}^2} & \text{if } w \geq \dfrac{1}{2} \end{cases} \tag{8}$$

*for* $0 < w < 1$.

PROOF: Using (3), one can write

$$f_W(w) = \begin{cases} \dfrac{\lambda_1\lambda(\lambda_2+\lambda_{12})}{\lambda_1+\lambda_2} \displaystyle\int_0^\infty r\exp\{-\lambda_1 rw - (\lambda_2+\lambda_{12})r(1-w)\}\,dr & \text{if } w < \dfrac{1}{2} \\ \dfrac{\lambda_2\lambda(\lambda_1+\lambda_{12})}{\lambda_1+\lambda_2} \displaystyle\int_0^\infty r\exp\{-\lambda_2 r(1-w) - (\lambda_1+\lambda_{12})rw\}\,dr & \text{if } w \geq \dfrac{1}{2}. \end{cases}$$

The result of the theorem follows by elementary integration of the above integrals. ■

## 3. MOMENTS

Here we derive the moments of $R = X + Y$, $P = XY$, and $W = X/(X+Y)$ when $X$ and $Y$ are distributed according to (1). We need the following lemma.

LEMMA 1: *If X and Y are jointly distributed according to (1), then*

$$E(X^m Y^n) = \frac{\lambda\lambda_1}{(\lambda_1+\lambda_2)(\lambda_2+\lambda_{12})^n}\sum_{k=0}^{n}\frac{(m+k)!(\lambda_2+\lambda_{12})^k}{k!(\lambda_1+\lambda_2+\lambda_{12})^{m+k+1}}$$

$$+ \frac{\lambda\lambda_2}{(\lambda_1+\lambda_2)(\lambda_1+\lambda_{12})^m}\sum_{k=0}^{m}\frac{(n+k)!(\lambda_1+\lambda_{12})^k}{k!(\lambda_1+\lambda_2+\lambda_{12})^{n+k+1}}$$

*for* $m \geq 1$ *and* $n \geq 1$.

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PROOF: One can express

$$
\begin{aligned}
E(X^m Y^n) &= \frac{\lambda_1 \lambda(\lambda_2+\lambda_{12})}{\lambda_1+\lambda_2} \int_0^\infty \int_x^\infty x^m y^n \exp\{-\lambda_1 x - (\lambda_2+\lambda_{12})y\}\, dy\, dx \\
&\quad + \frac{\lambda_2 \lambda(\lambda_1+\lambda_{12})}{\lambda_1+\lambda_2} \int_0^\infty \int_y^\infty x^m y^n \exp\{-\lambda_2 y - (\lambda_1+\lambda_{12})x\}\, dx\, dy \\
&= \frac{\lambda_1 \lambda}{(\lambda_1+\lambda_2)(\lambda_2+\lambda_{12})^n} \int_0^\infty x^m \exp(-\lambda_1 x)\Gamma(n+1,(\lambda_2+\lambda_{12})x)\, dx \\
&\quad + \frac{\lambda_2 \lambda}{(\lambda_1+\lambda_2)(\lambda_1+\lambda_{12})^m} \int_0^\infty y^n \exp(-\lambda_2 y)\Gamma(m+1,(\lambda_1+\lambda_{12})y)\, dy \\
&= \frac{\lambda_1 \lambda}{(\lambda_1+\lambda_2)(\lambda_2+\lambda_{12})^n} \\
&\quad \times \int_0^\infty x^m \exp\{-(\lambda_1+\lambda_2+\lambda_{12})x\} \sum_{k=0}^{n} \frac{(\lambda_2+\lambda_{12})^k x^k}{k!}\, dx \\
&\quad + \frac{\lambda_2 \lambda}{(\lambda_1+\lambda_2)(\lambda_1+\lambda_{12})^m} \\
&\quad \times \int_0^\infty y^n \exp\{-(\lambda_1+\lambda_2+\lambda_{12})y\} \sum_{k=0}^{m} \frac{(\lambda_1+\lambda_{12})^k y^k}{k!}\, dy \\
&= \frac{\lambda_1 \lambda}{(\lambda_1+\lambda_2)(\lambda_2+\lambda_{12})^n} \sum_{k=0}^{n} \frac{(\lambda_2+\lambda_{12})^k}{k!} \\
&\quad \times \int_0^\infty x^{m+k} \exp\{-(\lambda_1+\lambda_2+\lambda_{12})x\}\, dx \\
&\quad + \frac{\lambda_2 \lambda}{(\lambda_1+\lambda_2)(\lambda_1+\lambda_{12})^m} \sum_{k=0}^{m} \frac{(\lambda_1+\lambda_{12})^k}{k!} \\
&\quad \times \int_0^\infty y^{n+k} \exp\{-(\lambda_1+\lambda_2+\lambda_{12})y\}\, dy,
\end{aligned}
\tag{9}
$$

where we have used the definition of the complementary incomplete gamma function and the identity

$$
\Gamma(n+1,x) = n! \exp(-x) \sum_{k=0}^{n} \frac{x^k}{k!}.
$$

The result of the theorem follows by elementary integration of the two integrals in (9). ■

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The moments of $R = X + Y$ and $P = XY$ are now simple consequences of this lemma, as illustrated in Theorems 4 and 5. The moments of $W = X/(X+Y)$ require a separate treatment, as shown by Theorem 6.

THEOREM 4: *If X and Y are jointly distributed according to (1), then*

$$E(R^n) = \sum_{k=0}^{n} \binom{n}{k} \left[ \frac{\lambda\lambda_1}{(\lambda_1+\lambda_2)(\lambda_2+\lambda_{12})^k} \sum_{l=0}^{k} \frac{(n-k+l)!(\lambda_2+\lambda_{12})^l}{l!(\lambda_1+\lambda_2+\lambda_{12})^{n-k+l+1}} + \frac{\lambda\lambda_2}{(\lambda_1+\lambda_2)(\lambda_1+\lambda_{12})^{n-k}} \sum_{l=0}^{n-k} \frac{(k+l)!(\lambda_1+\lambda_{12})^l}{l!(\lambda_1+\lambda_2+\lambda_{12})^{k+l+1}} \right] \quad \mathbf{(10)}$$

*for* $n \geq 1$.

PROOF: The result in (10) follows by writing

$$E((X+Y)^n) = \sum_{k=0}^{n} \binom{n}{k} E(X^{n-k}Y^k)$$

and applying Lemma 1 to each expectation in the sum. ■

THEOREM 5: *If X and Y are jointly distributed according to (1), then*

$$E(P^n) = \frac{\lambda\lambda_1}{(\lambda_1+\lambda_2)(\lambda_2+\lambda_{12})^n} \sum_{k=0}^{n} \frac{(n+k)!(\lambda_2+\lambda_{12})^k}{k!(\lambda_1+\lambda_2+\lambda_{12})^{n+k+1}} + \frac{\lambda\lambda_2}{(\lambda_1+\lambda_2)(\lambda_1+\lambda_{12})^n} \sum_{k=0}^{n} \frac{(n+k)!(\lambda_1+\lambda_{12})^k}{k!(\lambda_1+\lambda_2+\lambda_{12})^{n+k+1}} \quad \mathbf{(11)}$$

*for* $n \geq 1$.

PROOF: The proof follows by writing $E(P^n) = E(X^nY^n)$ and applying Lemma 1 with $m = n$. ■

THEOREM 6: *If X and Y are jointly distributed according to (1), then*

$$E(W^n) = \frac{\lambda\lambda_1(\lambda_2+\lambda_{12})}{(\lambda_1+\lambda_2)(\lambda_1-\lambda_2-\lambda_{12})^{n+1}} \sum_{l=0}^{n} \binom{n}{l} (-\lambda_2-\lambda_{12})^{n-l}\delta_1(l-2) + \frac{\lambda\lambda_2(\lambda_1+\lambda_{12})}{(\lambda_1+\lambda_2)(\lambda_1-\lambda_2+\lambda_{12})^{n+1}} \sum_{l=0}^{n} \binom{n}{l} (-\lambda_2)^{n-l}\delta_2(l-2) \quad \mathbf{(12)}$$

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*for* $n \geq 1$, *where*

$$\delta_1(k) = \begin{cases} \dfrac{1}{k+1}\left[\left(\dfrac{\lambda}{2}\right)^{k+1} - (\lambda_2 + \lambda_{12})^{k+1}\right] & \text{if } k \neq -1 \\ \log \dfrac{\lambda}{2(\lambda_2 + \lambda_{12})} & \text{if } k = -1 \end{cases}$$

*and*

$$\delta_2(k) = \begin{cases} \dfrac{1}{k+1}\left[(\lambda_1 + \lambda_{12})^{k+1} - \left(\dfrac{\lambda}{2}\right)^{k+1}\right] & \text{if } k \neq -1 \\ \log \dfrac{2(\lambda_1 + \lambda_{12})}{\lambda} & \text{if } k = -1. \end{cases}$$

PROOF: Using (8), one can write

$$\begin{aligned}
E(W^n) &= \frac{\lambda\lambda_1(\lambda_2+\lambda_{12})}{(\lambda_1+\lambda_2)} \int_0^{1/2} \frac{w^n}{\{\lambda_1 w + (\lambda_2+\lambda_{12})(1-w)\}^2}\, dw \\
&\quad + \frac{\lambda\lambda_2(\lambda_1+\lambda_{12})}{(\lambda_1+\lambda_2)} \int_{1/2}^{1} \frac{w^n}{\{\lambda_2(1-w) + (\lambda_1+\lambda_{12})w\}^2}\, dw \\
&= \frac{\lambda\lambda_1(\lambda_2+\lambda_{12})}{(\lambda_1+\lambda_2)(\lambda_1-\lambda_2-\lambda_{12})^{n+1}} \int_{\lambda_2+\lambda_{12}}^{\lambda/2} u^{-2}(u-\lambda_2-\lambda_{12})^n\, du \\
&\quad + \frac{\lambda\lambda_2(\lambda_1+\lambda_{12})}{(\lambda_1+\lambda_2)(\lambda_1-\lambda_2+\lambda_{12})^{n+1}} \int_{\lambda/2}^{\lambda_1+\lambda_{12}} v^{-2}(v-\lambda_2)^n\, dv \\
&= \frac{\lambda\lambda_1(\lambda_2+\lambda_{12})}{(\lambda_1+\lambda_2)(\lambda_1-\lambda_2-\lambda_{12})^{n+1}} \int_{\lambda_2+\lambda_{12}}^{\lambda/2} \sum_{l=0}^{n} \binom{n}{l} (-\lambda_2-\lambda_{12})^{n-l} u^{l-2}\, du \\
&\quad + \frac{\lambda\lambda_2(\lambda_1+\lambda_{12})}{(\lambda_1+\lambda_2)(\lambda_1-\lambda_2+\lambda_{12})^{n+1}} \int_{\lambda/2}^{\lambda_1+\lambda_{12}} \sum_{l=0}^{n} \binom{n}{l} (-\lambda_2)^{n-l} v^{l-2}\, dv \\
&= \frac{\lambda\lambda_1(\lambda_2+\lambda_{12})}{(\lambda_1+\lambda_2)(\lambda_1-\lambda_2-\lambda_{12})^{n+1}} \sum_{l=0}^{n} \binom{n}{l} (-\lambda_2-\lambda_{12})^{n-l} \int_{\lambda_2+\lambda_{12}}^{\lambda/2} u^{l-2}\, du \\
&\quad + \frac{\lambda\lambda_2(\lambda_1+\lambda_{12})}{(\lambda_1+\lambda_2)(\lambda_1-\lambda_2+\lambda_{12})^{n+1}} \sum_{l=0}^{n} \binom{n}{l} (-\lambda_2)^{n-l} \int_{\lambda/2}^{\lambda_1+\lambda_{12}} v^{l-2}\, dv, \qquad \textbf{(13)}
\end{aligned}$$

where we have set $u = \lambda_1 w + (\lambda_2 + \lambda_{12})(1-w)$ and $v = \lambda_2(1-w) + (\lambda_1 + \lambda_{12})w$. The result of the theorem follows by noting that the two integrals in (13) reduce to $\delta_1(l-2)$ and $\delta_2(l-2)$ after elementary integration. ■

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**Table 1.** Basic Drought Statistics for Nebraska PDSI Data

| Climate division | No. of droughts | Drought frequency (No./year) | Mean drought duration (months) |
|---|---|---|---|
| 1 | 83 | 0.75 | 6.0 |
| 2 | 66 | 0.60 | 8.6 |
| 3 | 89 | 0.81 | 6.3 |
| 5 | 81 | 0.74 | 6.3 |
| 6 | 90 | 0.82 | 6.3 |
| 7 | 81 | 0.74 | 6.1 |
| 8 | 76 | 0.69 | 6.5 |
| 9 | 74 | 0.67 | 7.5 |

## 4. APPLICATION

Here we return to the drought problem discussed in Section 1 and provide an application of the model given by (1). We use the drought data from the state of Nebraska. The data comprises the monthly modified Palmer Drought Severity Index (PDSI) from the period from January 1895 to December 2004. A drought is said to have happened when the PDSI is below zero. Defined by the theory of runs [9], some statistics of the observed drought for the eight climatic divisions of Nebraska are summarized in Table 1. The state of Nebraska is divided into eight climate divisions numbered 1, 2, 3, 5, 6, 7, 8, and 9; there is no climate division 4 for Nebraska.

**Table 2.** Expected Values of the Sum, Product, and Ratio

| Climate division | $E(R)$ (95% CI) | $E(P)$ (95% CI) | $E(W)$ (95% CI) |
|---|---|---|---|
| 1 | 15.740 (1.621, 52.640) | 91.4 (0.2, 572.7) | 0.492 (0.011, 0.977) |
| 2 | 19.716 (2.228, 59.144) | 201.2 (0.3, 1256.7) | 0.494 (0.014, 0.978) |
| 3 | 14.709 (1.704, 43.145) | 107.4 (0.2, 688.9) | 0.487 (0.015, 0.976) |
| 5 | 16.129 (1.813, 48.742) | 113.1 (0.2, 710.4) | 0.454 (0.011, 0.978) |
| 6 | 14.672 (1.724, 42.186) | 106.5 (0.2, 682.7) | 0.473 (0.017, 0.971) |
| 7 | 16.137 (1.741, 50.933) | 98.4 (0.2, 617.5) | 0.463 (0.009, 0.979) |
| 8 | 17.376 (1.995, 51.073) | 124.2 (0.2, 776.6) | 0.417 (0.010, 0.977) |
| 9 | 17.875 (1.949, 55.255) | 161.3 (0.3, 1015.7) | 0.507 (0.018, 0.968) |

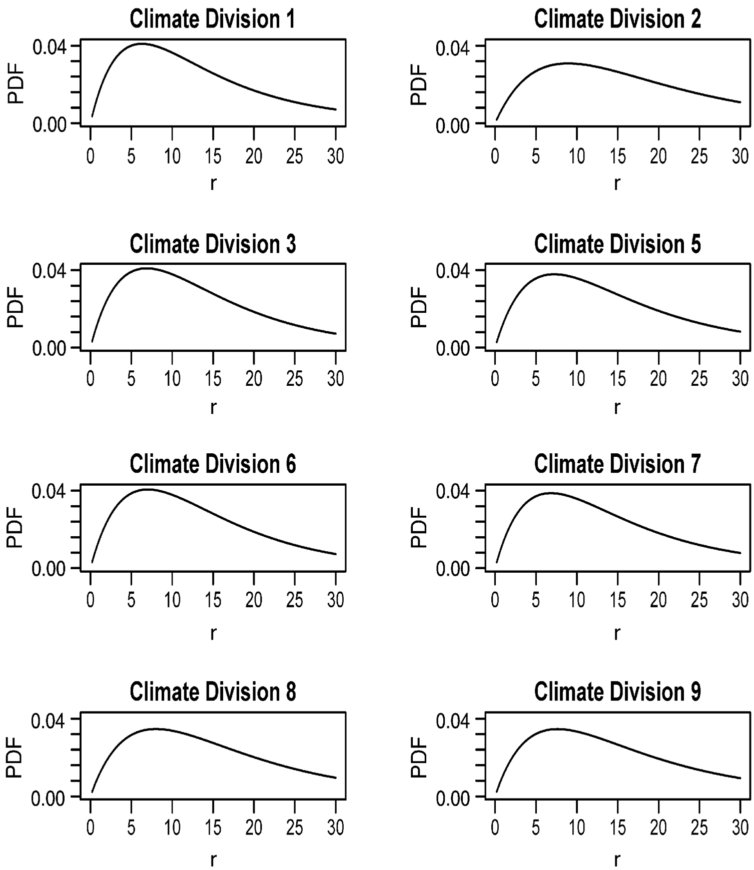


**FIGURE 1.** Fitted values of the p.d.f. (2) for the eight climate divisions of Nebraska ($X$ = drought duration and $Y$ = successive nondrought duration).

Using the PDSI data, data on drought duration, nondrought duration, and drought intensity were obtained for each climate division. The interest is in determining the distributions of the following:

1. The interarrival time of droughts ($R$) = Drought duration + Nondrought duration.
2. The magnitude of droughts ($P$) = Drought intensity × Drought duration.
3. The proportion of droughts ($W$) = Drought duration/(Drought duration + Nondrought duration).

The distribution of $R$ was determined by fitting the model given by (1) to the observed values of (drought duration ($X$), nondrought duration ($Y$)) and using (2) to com-

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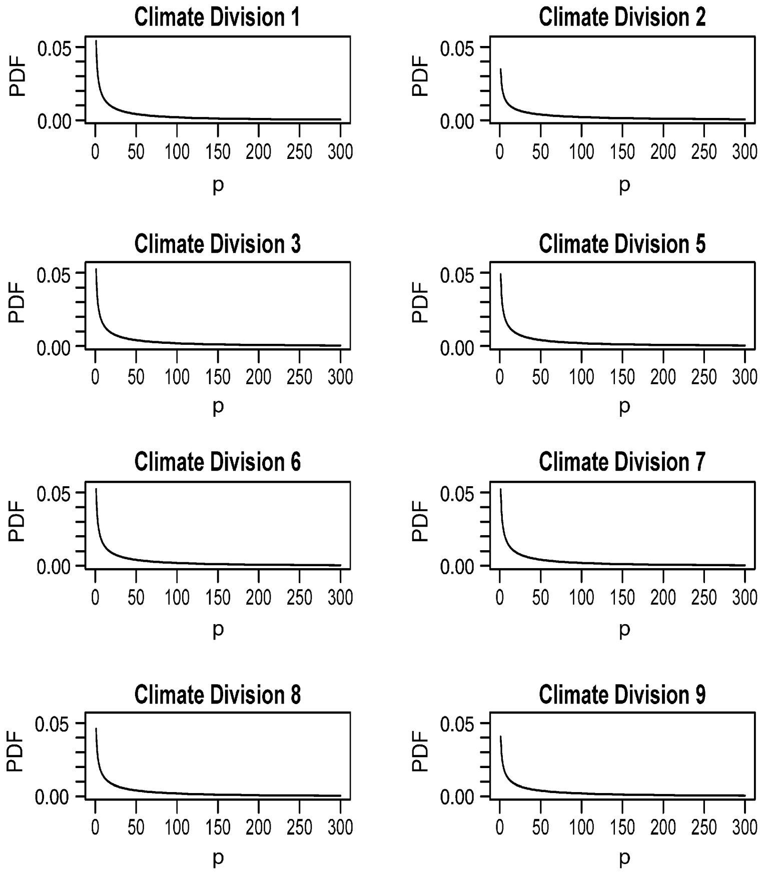


**FIGURE 2.** Fitted values of the p.d.f. (4) for the eight climate divisions of Nebraska ($X$ = drought intensity and $Y$ = drought duration).

pute the fitted p.d.f. The distribution of $W$ was determined in the same way by using (8). The distribution of $P$ was determined by fitting (1) to the observed values of (drought intensity ($X$), drought duration ($Y$)) and using (4) to compute the fitted p.d.f. The fitting of (1) was performed by the method of maximum likelihood. The quasi-Newton algorithm nlm in the R software package (Dennis and Schnabel [2], Schnabel, Koontz, and Weiss [8], Ihaka and Gentleman [5]) was used to maximize the likelihood.

The fitted p.d.f.s of $R$, $P$, and $W$ for the eight climate divisions are shown in Figures 1–3. The estimated values of the moments $E(R)$, $E(P)$, and $E(W)$ along with their 95% percentile-based confidence intervals (CI) are given in Table 2. These estimates were computed using (10)–(12). It is evident from both Figures 1–3 and

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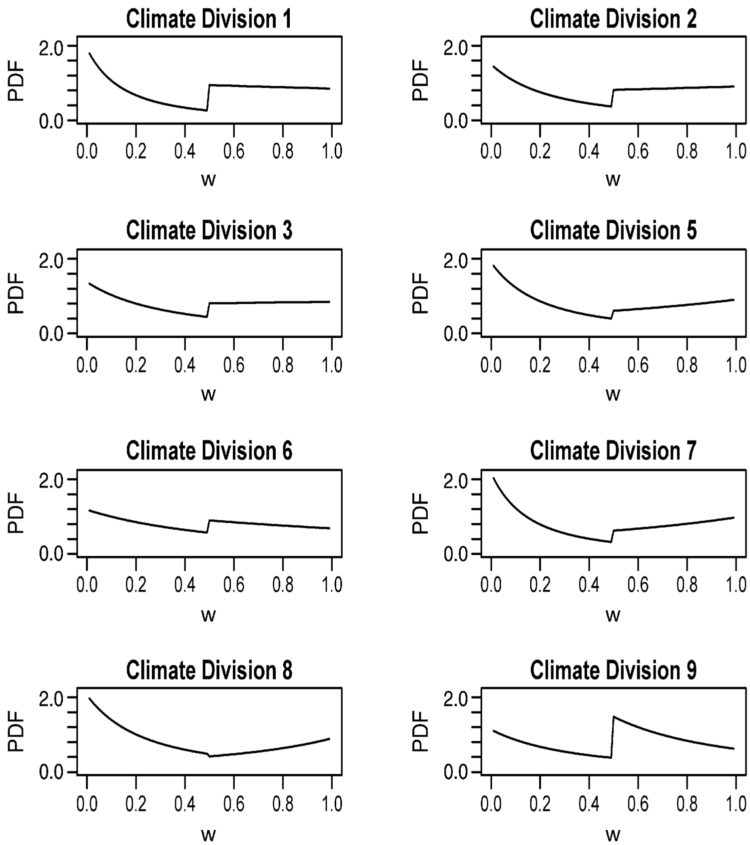


**FIGURE 3.** Fitted values of the p.d.f. (8) for the eight climate divisions of Nebraska ($X$ = drought duration and $Y$ = successive nondrought duration).

Table 2 that there is little difference among the climate divisions. This is what one would expect given the geography of the state of Nebraska.

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