Pulse interaction in nonlinear vacuum electrodynamics

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Abstract

The energy-momentum conservation law is used to investigate the interaction of pulses in the framework of nonlinear electrodynamics with Lorentz-invariant constitutive relations. It is shown that for the pulses of the arbitrary shape, the interaction results in phase shift only.

Keywords: Nonlinear electrodynamics; Solitons

1. INTRODUCTION

Although classical electromagnetic theory deals with linear Maxwell equations, there have been numerous attempts to bring the nonlinear phenomena into the stage. All relativistic and gauge invariant versions of electromagnetism are based on the Lagrangian density, L, which depends on the invariants of the field tensor. Generally, in terms of the electric ($\mathbf{E}$) and magnetic ($\mathbf{B}$) fields, the Maxwell equations in absence of external charges may be written in a standard form:

$$\mathbf{D}_t - \nabla \times \mathbf{H} = 0, \qquad \nabla \mathbf{D} = 0,$$
$$\mathbf{B}_t + \nabla \times \mathbf{E} = 0, \qquad \nabla \mathbf{B} = 0, \tag{1}$$

where we put $c = 1$ and $\mathbf{D} = \partial L/\partial \mathbf{E}$, $\mathbf{H} = -\partial L/\partial \mathbf{B}$. The Lagrangian $L(I, J^2)$ depends on the Poincaré invariants $I = \mathbf{E}^2 - \mathbf{B}^2$ and $J = \mathbf{EB}$ only. The distinctive feature of Eqs. (1) is that since the Poincaré invariants are identically zero for the plane electromagnetic wave, the latter is insensitive to vacuum nonlinearity and propagates without distortion.

Of particular interest are the nonlinear corrections to the linear electrodynamics arising due to vacuum polarization in the strong electromagnetic field. In the ultimate case of slowly varying fields, this results in Heisenberg–Euler electrodynamics, which is discussed in many textbooks (e.g., Akhiezer & Berestetskii, 1964).

Recently a number of studies have been performed aimed to investigation of the characteristic surfaces in the framework of nonlinear electrodynamics, that is, the propagation of infinitesimal discontinuities or "photons" (e.g., Muñoz, 1996; Dittrich & Gies, 1998; De Lorenci *et al.*, 2000; Novello *et al.*, 2000; Rikken & Rizzo, 2000; Novello & Salim, 2001). However, very little is known about the finite amplitude waves.

The main point of this paper is to describe the simplest, in a sense, nonlinear vacuum process: the interaction of two electromagnetic waveforms propagating in opposite directions. A similar particular solution of the Einstein–Born–Infeld equations was found by Bretón (1996). Here we demonstrate that the pulses of the arbitrary shape behave like solitons, that is, the scattering does not alter the shape of the pulse but result in a phase shift only.

2. MAXWELL EQUATIONS

We consider a linearly polarized wave propagating in the z direction of the form $E_x = E(z, t)$, $B_y = B(z, t)$ with all other components being zero. In this situation, the second Poincaré invariant vanishes, $J \equiv 0$, so the Maxwell equations are written as

$$(EL(I)_I)_t + (BL(I)_I)_z = 0,$$
$$(B)_t + (E)_z = 0, \tag{2}$$

where the subscript denotes the derivative with respect to the corresponding variable and $I = E^2 - B^2$. The Lagrangian in Eq. (2) is expanded in powers of I. Keeping the lowest order nonlinear corrections we have

$$L(I) = I + \frac{1}{2} \sigma I^2 + \ldots. \tag{3}$$

With the help of the appropriate scale transform, the coefficient σ may be reduced to ± 1. For the particular case of the Heisenberg–Euler electrodynamics, $\sigma = 1$ (Akhiezer & Berestetskii, 1964). Of interest also is to keep in mind the Born–Infeld electrodynamics (e.g., Tonnelat, 1959) with the Lagrangian

$$L_{BI}(I) = 1 - \sqrt{1 - I}. \tag{4}$$

3. ENERGY-MOMENTUM TENSOR

The conservation laws for Eqs. (2) are given by

$$W_t + N_z = 0, \qquad N_t + P_z = 0, \tag{5}$$

where the components of the energy-momentum tensor, namely, the energy density, W, the momentum density, N, and the stress, P, may be obtained using standard variation procedure (e.g., Landau & Lifshitz, 1971). Explicitly,

$$W = 2E^2 L_I - L$$

$$N = 2EB L_I \tag{6}$$

$$P = 2B^2 L_I + L.$$

Usually Eqs. (5) and (6) are thought of as a consequence of the Maxwell equations (2). However, we may consider the relations (6) as a constraint implied upon the components of the momentum-energy tensor, so there are two independent variables in Eqs. (5), for example, W and N. One can easily check that for the nontrivial solutions of Eqs. (2), that is, for $I \neq 0$, the Jacobian of the transform $E, B \to W, N$ is nonzero. Thus, instead of looking for the solutions of Eqs. (2), we can solve Eqs. (5) and (6) excluding the Poincaré invariant I from Eqs. (6).

4. SOLUTION

To exclude I, it is convenient to introduce the invariants of the energy-momentum tensor, that is, its trace, $S = P - W$, and the determinant $T = WP - N^2$. As it follows from Eqs. (6)

$$S = 2(L - IL_I) \tag{7}$$

$$T = I(L^2)_I - L^2.$$

The latter relations implicitly define the dependence $T = T(S)$. Substituting the Lagrangian (3) into Eqs. (7), we find that the first nonvanishing term of the expansion of T in powers of S is linear and it is provided by the quadratic term of the expansion (3): $T(S) = -\sigma S + \ldots$. It is noteworthy that the Born–Infeld Lagrangian (4) yields exactly the linear dependence $T(S) = -S$.

The relations (5) are resolved introducing the potential ψ: $W = \psi_{zz}$, $N = -\psi_{zt}$, $P = \psi_{tt}$. Restricting ourselves with the linear relation between T and S, we obtain the Ampere–Monge type equation for ψ:

$$\psi_{zz}\psi_{tt} - \psi_{zt}^2 = \sigma(\psi_{zz} - \psi_{tt}). \tag{8}$$

There are trivial solutions to this equation $\psi(z, t) = F(z \pm t)$ with an arbitrary function F, which correspond to the plane electromagnetic waveforms described by Eqs. (2) with $I = 0$. Besides these, implementing the Legendre transform (Courant, 1962), one can easily obtain the general integral of Eq. (8) valid for $T \neq 0$ and, consequently, for $I \neq 0$. As a result, we get the components of the energy-momentum tensor in a parametric form:

$$W = \sigma(F_1'(\xi) + F_2'(\eta) + 2F_1'(\xi)F_2'(\eta))/\Delta(\xi, \eta),$$

$$P = \sigma(F_1'(\xi) + F_2'(\eta) - 2F_1'(\xi)F_2'(\eta))/\Delta(\xi, \eta),$$

$$N = \sigma(F_2'(\eta) - F_1'(\xi))/\Delta(\xi, \eta), \tag{9}$$

$$z = \tfrac{1}{2}(\xi + \eta - F_1(\xi) - F_2(\eta)),$$

$$t = \tfrac{1}{2}(\xi - \eta + F_1(\xi) - F_2(\eta)),$$

where $F_{1,2}$ are arbitrary functions and $\Delta(\xi, \eta) = 1 - F_1'(\xi)F_2'(\eta)$.

Consider, for example, two localized pulses of the arbitrary shape propagating in opposite directions. This corresponds to the following initial conditions:

$$W(z, t)|_{t \to -\infty} = W_1(z + t) + W_2(z - t), \tag{10}$$

where $W_{1,2}(\xi)|_{\xi \to \pm\infty} \to 0$. This initial condition is provided by the following choice of $F_{1,2}$ in Eqs. (9): $F_{1,2}'(\xi) = \sigma W_{1,2}(\xi)$ and $F_1(\xi)|_{\xi \to -\infty} \to 0$, $F_2(\xi)|_{\xi \to \infty} \to 0$. The asymptotic of the solution (9) at $t \to \infty$ is then given by

$$W(z, t) = W_1(z + t - \sigma K_2) + W_2(z - t + \sigma K_1), \tag{11}$$

where

$$K_{1,2} = \int_{-\infty}^{\infty} d\xi \, W_{1,2}(\xi) \tag{12}$$

is the net energy carried by the corresponding pulse.

Typical plots $W(z, t)$ are depicted in Figures 1 and 2. The first picture shows the interaction of two identical pulses. The interaction of one pulse with a sequence of two pulses is shown in Figure 2.

5. DISCUSSION

Of interest is the geometrical sense of the obtained solution (9). The parameters ξ and η are, in fact, the light-cone
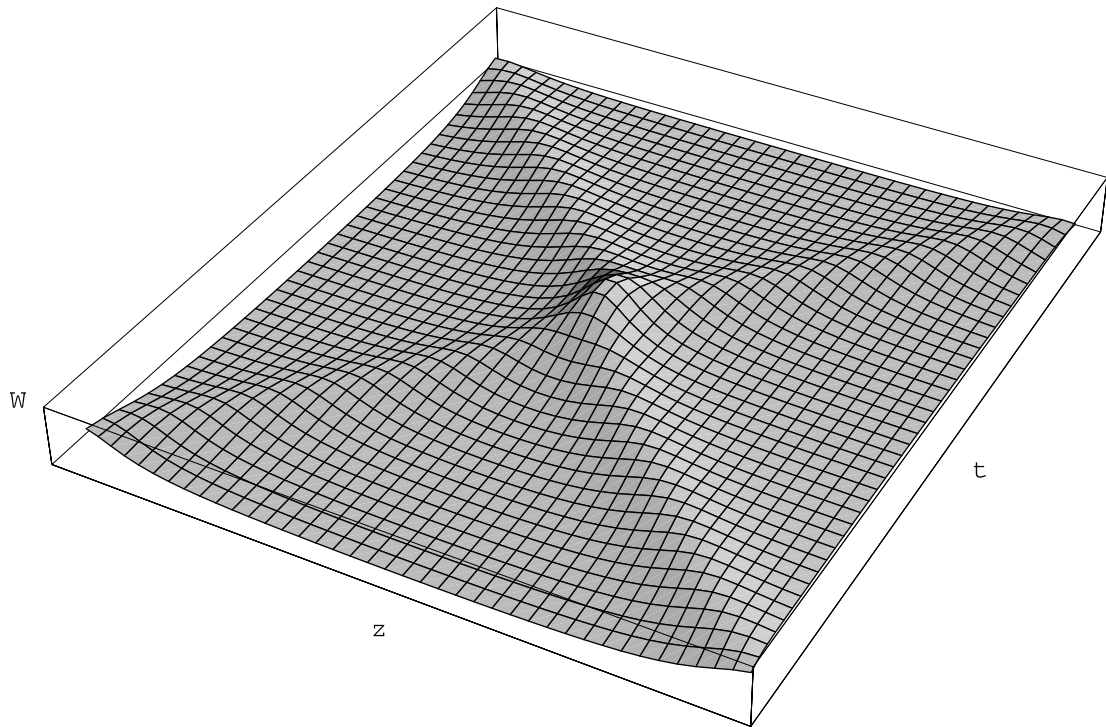
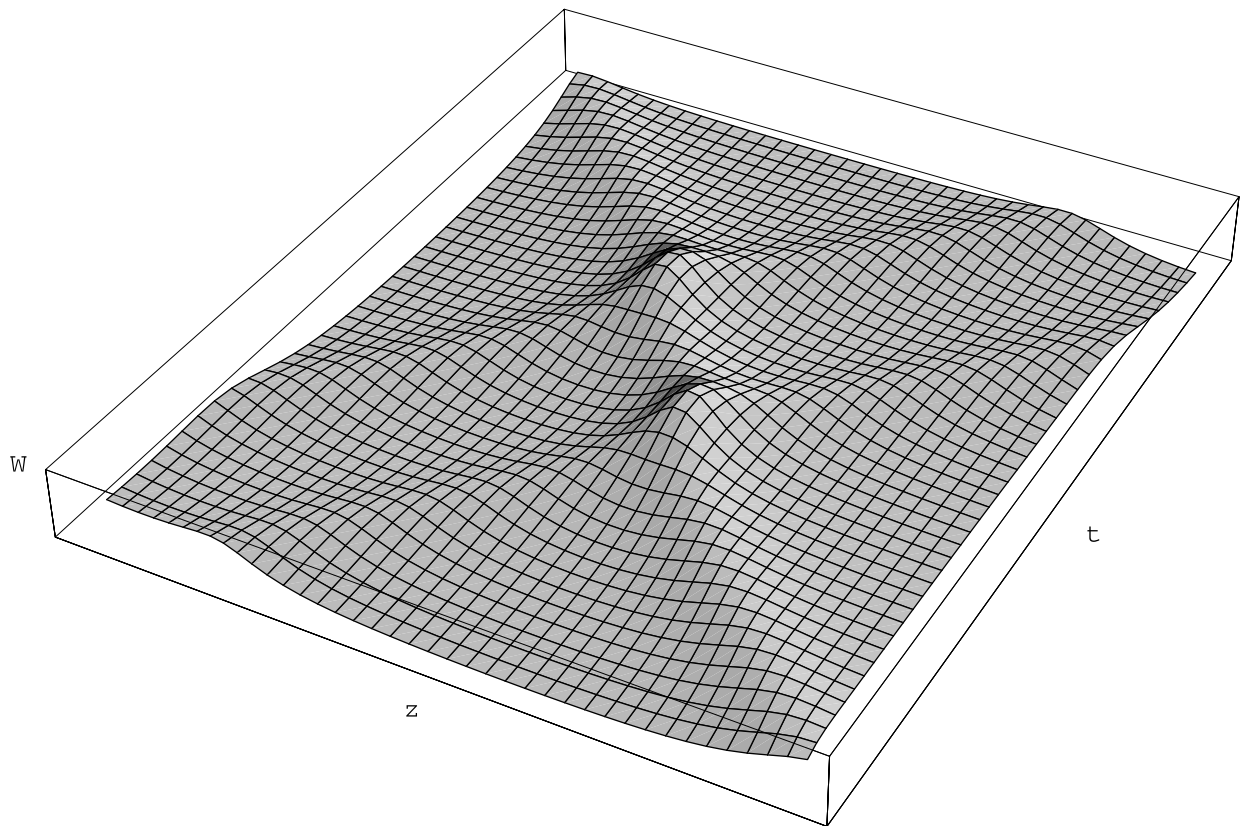**Fig. 1.** Interaction of two identical pulses.

Fig. 2. A single pulse interacting with a sequence of two pulses.

coordinates disturbed by the electromagnetic field. One may say that the electromagnetic field alters the space–time metric due to the dependence of the speed of light on the field strength. The same conclusion stems also from the analysis of the characteristic surfaces (Novello *et al.*, 2000). However, the effective geometries originating from linear analysis and our approach should not generally coincide.

Another interesting point is that, for $\sigma = 1$, the increase in the pulse amplitude results in delay in energy (and information) exchange between distant points, that is, the solution described by (11) is subluminal. This takes place for both the Heisenberg–Euler electrodynamics, which is currently the only one of physical sense, and for the elegant Born–Infeld theory, for which our results are exact. However, for $\sigma = -1$, the pulse propagation would be superluminal.

From the viewpoint of nonlinear physics, the electromagnetic pulses in vacuum exhibit the soliton-like behavior: The collision results in a phase shift but the form of a pulse remains unchanged. The main interesting point with this respect is that unlike solitons described by usual integrable nonlinear equations, the shape of the Maxwellian soliton is arbitrary.

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