POINT-LIKE, SIMPLICIAL MAPPINGS OF A 3-SPHERE

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1. Introduction. A *decomposition* of a topological space X is a partitioning of X into non-empty, disjoint sets called *elements* of the decomposition. An element of a decomposition is *non-degenerate* if it contains more than one point. Associated with each decomposition D of X is a topological space D^*, called the *hyperspace* of the decomposition. A classical problem on decompositions of topological spaces is to find conditions under which D^* is homeomorphic to X. Often decompositions arise from mappings: if g is a mapping of a space X onto a space Y, then $D = \{g^{-1}(y) \mid y \in Y\}$ is a decomposition of X. Moreover, if X is compact and if Y is a Hausdorff space, then D^* is homeomorphic to Y and we may solve the problem by finding conditions under which Y is homeomorphic to X. (By *mapping* we shall always mean *continuous function*, and by *space* we shall always mean T_1-*space*.)

In 1925, R. L. Moore showed that if X is a 2-sphere, then Y is homeomorphic to X if and only if for each point y in Y the inverse image $g^{-1}(y)$ is a continuum which does not separate X **(3)**.

In 1938, J. H. Roberts and N. E. Stennrod showed that if X is a compact, connected 2-manifold, then Y is homeomorphic to X if and only if Y contains more than one point and the 1-dimensional Betti number (mod 2) of each of the sets $g^{-1}(y)$, where y is a point of Y, is zero **(4)**.

In 1936, G. T. Whyburn posed a question embodying a generalization of Moore's theorem **(5)**, asking whether Y is homeomorphic to X whenever X is a 3-sphere and g is point-like. A subset A of an n-sphere S^n is *point-like* if the space $(S^n - A)$ is homeomorphic to Euclidean n-space E^n. A mapping g of an n-sphere onto a space Y is *point-like* if the set $g^{-1}(y)$ is point-like for each point y in Y. Whyburn's question concerns a possible generalization of Moore's theorem because the point-like subsets of the 2-sphere are exactly those subsets which are continua that do not separate the sphere.

In 1957, R. H. Bing published an example of a point-like mapping of S^3 onto a space topologically different from S^3 **(1)**.

In 1958, O. G. Harrold, Jr. published sufficient conditions for a monotone image of a 3-sphere to be a 3-sphere. The conditions require that the closure of the set of points in the image-space which have non-degenerate inverse-images be totally disconnected, but are stronger and more interesting **(2)**.

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Going in a different direction from Harrold's result, we prove the following theorem.

THEOREM. *Let M be a triangulated 3-sphere and let T be a triangulated topological space. If there exists a point-like, simplicial mapping of M onto T, then T is homeomorphic to M.*

Here a triangulated 3-sphere is a finite, simplicial complex whose geometric realization is homeomorphic to a 3-sphere. Throughout this paper, complex will mean finite, simplicial complex and a single symbol will denote both a complex and its associated topological space.

2. The proof in outline. Let f be the given point-like simplicial mapping of M onto T, and let $G = \{f^{-1}(t) \mid t \in T\}$. We see that the union of the non-degenerate elements of G is a subcomplex K of M. In fact, K is the union of all faces of all 3-simplices on which f fails to be one-to-one. If K is empty, then f is the required homeomorphism. We suppose henceforth that K is non-empty. Since f is one-to-one on at least one 3-simplex of M, the frontier of K is non-empty. We can find a 3-simplex σ in K which has the properties:

(1) σ has a 2-face in $\mathrm{Fr}(K)$;

(2) if g is a non-degenerate element of G, then $\mathrm{Cl}(g - \sigma)$ is empty or still point-like; and

(3) $\mathrm{Cl}(K - \sigma)$ is a subcomplex of M with fewer simplices than K.

Using σ we define a new decomposition G_1 of M as follows: g_1 is an element of G_1 if either g_1 is a point of $(M - \mathrm{Cl}(K - \sigma))$ or g_1 is one of those sets $\mathrm{Cl}(g - \sigma)$ which is non-empty, g being an element of G. The elements of G_1 are point-like, the hyperspace G_1^* is homeomorphic to G^* and hence to T, and the union of the non-degenerate elements of G_1 is a proper subcomplex K_1 of K.

In K_1 there will be a simplex σ_1 with properties analogous to those stated for σ. Just as we used σ to construct G_1, we use σ_1 to construct a decomposition G_2 of M into point-like sets. As before, G_2^* is homeomorphic to G_1^* and hence to T, and the union K_2 of the non-degenerate elements of G_2 is a proper subcomplex of K_1.

We continue to construct new decompositions (although σ and σ_1 must be 3-simplices, eventually 2-simplices may be used) until at last we construct a subcomplex K_n of K which can be reduced no further. While G_n is still a decomposition of M into point-like sets, and while G_n^* is still homeomorphic to T, the decomposition G_n now has only finitely many non-degenerate elements. A decomposition of the 3-sphere into point-like sets, which has only a finite number of non-degenerate elements, has a hyperspace homeomorphic to the 3-sphere. Hence G_n^* is homeomorphic to M as well as to T, and the theorem is proved.

The technique of constructing a new decomposition by deleting a simplex is an unpublished technique of E. E. Moise. The technique is formalized for our purposes as the construction of D_1^* from D^* in Lemma 4.5. Once Lemma 4.5 has been proved, the burden of proof rests on showing that the construction may be applied repeatedly to G and K until G_n is reached.

3. Preliminary lemmas. If A is a subset of a topological space X, then $D(A)$ is the subset of X that is the union of those elements of the decomposition D which meet A. If $D(A)$ is closed whenever A is closed, D is *upper semi-continuous*. Thus each element d of an upper semi-continuous decomposition D of a T_1-space is closed, for if x is a point in d, then $D(x) = d$ is closed.

The usefulness of upper semi-continuous decompositions is illustrated by the following assertions. If D is an upper semi-continuous decomposition of a metric space X into compact sets, then the convergence of points in D^* corresponds to the convergence of the elements of D, as subsets in X (Lemma 3.4). If D is upper semi-continuous and if X is normal, then D^* is a Hausdorff space (Lemma 3.2). In particular, if D is an upper semi-continuous decomposition of a compact Hausdorff space, then D^* is a (compact) Hausdorff space.

The lemmas are presented without proof.

LEMMA 3.1. *Let X be a compact topological space, and let g be a one-to-one mapping of X onto a topological space Y. If Y is a Hausdorff space, then g is a homeomorphism.*

LEMMA 3.2. *Let X be a topological space, and let D be an upper semi-continuous decomposition of X. If X is normal, then D^* is a Hausdorff space.*

LEMMA 3.3. *If D is a decomposition of a compact topological space, then D^* is compact.*

LEMMA 3.4. *Let D be an upper semi-continuous decomposition of a metric space X into compact sets. Let $\{s_i\}$ be a sequence of points of D^*. If A and B are compact subsets of X, then let*

$$z(A, B) = \inf_{\substack{a \in A \\ b \in B}} m(a, b),$$

where m is the given metric on X. Let s be a point of D^*. The sequence $\{s_i\}$ converges to s if and only if

$$\lim z(p^{-1}(s_i), p^{-1}(s)) = 0.$$

LEMMA 3.5. *Let X be a compact topological space, let Y be a Hausdorff space, let g be a mapping of X onto Y, and let $D = \{g^{-1}(y) \mid y \in Y\}$. Then D^* is homeomorphic to Y under the correspondence $p(d) \leftrightarrow f(d)$, where d is an element of D and p is the projection mapping of X onto D^*. Also, D is an upper semi-continuous decomposition of X.*

4. The structure of K. Suppose that L is a subcomplex of K and that each non-degenerate element of G meets L in either the empty set or in a non-degenerate, point-like subset of M. Let D be the decomposition of M that consists of (1) each of the points of $(M - L)$, and (2) those of the sets $(g \cap L)$ which are non-empty. We note that L is the union of the non-degenerate elements of D, and that D is an upper semi-continuous decomposition of M.

When we reduce K according to the scheme mentioned before, the union K_i of the non-degenerate elements at the ith stage will have properties identical with those now hypothesized for L. What we deduce now about the structure of L will be used to reduce K_i to obtain K_{i+1}. Lemma 4.5 is a statement of the feasibility of the inductive step: if the subcomplex L of K contains a certain kind of simplex, then L and the decomposition D associated with L may be reduced to give a decomposition D_1 of M into point-like sets of which the non-degenerate elements form a proper subcomplex H of L and of which the hyperspace D_1^* is homeomorphic to D^*. In the section following this one we show how to use Lemma 4.5 to prove the theorem. Proofs of some of the simpler lemmas will be omitted.

Whenever σ^n is an n-simplex, $\text{Int}\,\sigma^n$ will denote the topological interior of σ^n, and $\text{Bdy}\,\sigma^n$ will denote $(\sigma^n - \text{Int}\,\sigma^n)$.

DEFINITION. *If L is a complex, then the 2-frontier* $\text{Fr}_2(L)$ *of L is the collection of those 2-simplices of L each of which lies in exactly one 3-simplex of L.*

LEMMA 4.1. *If σ^2 is a 2-simplex of L, and if $f(\sigma^2)$ is a 2-simplex, then σ^2 lies in a 3-simplex of L.*

Proof. Let σ_1^3 and σ_2^3 be the two 3-simplices of M that contain σ^2, and let x be a point of $\text{Int}(\sigma^2)$. Since x lies in L, the set $f^{-1}(f(x))$ is a non-degenerate element of G. If $f(\sigma^2)$ is a 2-simplex, then $[f^{-1}f(x) \cap \sigma^2] = x$. If neither σ_1^3 nor σ_2^3 lies in L, then either $[f^{-1}(f(x)) \cap L] = x$, or $[f^{-1}(f(x)) \cap L]$ is not connected, and hence not point-like. Neither alternative is possible. Hence at least one of the two 3-simplices σ_1^3, σ_2^3 lies in L.

LEMMA 4.2. *If σ^3 is a 3-simplex of L one face of which is mapped by f onto a 2-simplex τ^2 of T, then $f(\sigma^3) = \tau^2$ and σ^3 has two 2-faces which are mapped by f onto τ^2.*

LEMMA 4.3. *If σ^2 is a 2-simplex of L, and if $f(\sigma^2) = \tau^2$ is a 2-simplex of T, then every element of D which meets* $\text{Int}(\sigma^2)$ *is a polygonal arc. If σ^3 is a 3-simplex of L which f maps onto τ^2, then* $\text{Int}(\sigma^3)$ *is decomposed into line-segments each of which*

(1) *is parallel to whichever edge of σ^3 it is that is mapped onto a vertex of τ^2, and*

(2) *joins together the 2-faces of σ^3 that f maps onto τ^2.*

LEMMA 4.4. *Let σ^2 be a 2-simplex of L which f maps onto a 2-simplex. Suppose that f maps n 3-simplices of L onto $f(\sigma^2)$; $n > 0$. The 3-simplices may be indexed to form a sequence $\{\sigma_i^3\}_{i=1}^n$ such that $(\sigma_i^3 \cap \sigma_{i+1}^3)$ is a 2-simplex which is mapped* *onto* $f(\sigma^2)$. *The set of the 2-simplices of L that are mapped onto* $f(\sigma^2)$ *can be indexed to form a sequence* $\{\sigma_i^2\}_{i=0}^n$ *such that, for* $1 \leqslant i < n$, $\sigma_i^2 = (\sigma_i^3 \cap \sigma_{i+1}^3)$, $\sigma_0^2 \in \sigma_1^3$, $\sigma_n^2 \in \sigma_n^3$, *and that both* σ_0^2 *and* σ_n^2 *lie in* $\mathrm{Fr}_2(L)$.

Proof. Let x be a point of $\mathrm{Int}(\sigma^2)$, and let $\{s_i\}_{i=1}^n$ be the sequence formed by indexing the segments of the arc $\alpha = (f^{-1}f(x) \cap L)$ in one of their two possible linear orderings. Let σ_i^3 be the 3-simplex that contains the segment s_i. The sequence $\{\sigma_i^3\}_{i=1}^n$ includes each 3-simplex that is mapped by f onto $f(\sigma^2)$, because each 3-simplex which f maps onto $f(\sigma^2)$ contributes a segment to α (see Lemma 4.3). The ordering is independent of the choice of x. For if x' is another point of $\mathrm{Int}(\sigma^2)$, then one of the two linear orderings $\{s_i'\}_{i=1}^{n'}$ of the segments of the arc $(f^{-1}(f(x')) \cap L)$ has the property that s_i' belongs to σ_i^3, and $n' = n$. Since σ_i^3 and σ_{i+1}^3 have a common point that lies in the interior of a 2-face of each, namely $(s_i \cap s_{i+1})$, they must have a 2-face in common. Let σ_i^2 be this face. Since f maps a point of $\mathrm{Int}(\sigma_i^2)$ into the interior of $f(\sigma^2)$, $f(\sigma_i^2) = f(\sigma^2)$. Let σ_0^2, σ_n^2 be the face of σ_1^3, σ_n^3 that is mapped onto $f(\sigma^2)$ but does not lie in σ_2^3, σ_{n-1}^3. (Here we assume $n \geqslant 2$; if $n = 1$, the proof of the lemma is trivial.) The sequence $\{\sigma_i^2\}_{i=0}^n$ contains all of the 2-simplices of L that are mapped onto $f(\sigma^2)$; for by Lemma 4.1, each 2-simplex of L that is mapped onto $f(\sigma^2)$ lies in a 3-simplex of L which is mapped onto $f(\sigma^2)$, and the sequence $\{\sigma_i^3\}_{i=1}^n$ contains all such 3-simplices.

Let ρ be the other 3-simplex of M of which σ_0^2 is a face, and let y be a point of $\mathrm{Int}(\sigma_0^2)$. If ρ belongs to L, then $f(\rho) = f(\sigma^2)$, by Lemma 4.2. If $f(\rho) = f(\sigma^2)$, then y lies in the interior of the arc $(f^{-1}f(y) \cap L)$. But y is an end-point of the arc. Hence ρ does not lie in L, and σ_0^2 lies in $\mathrm{Fr}_2(L)$. A similar argument shows that σ_n^2 lies in $\mathrm{Fr}_2(L)$.

DEFINITION. *A simplex* σ *of L has property R if it lies in* $\mathrm{Fr}(L)$, *if it is a proper face of exactly one simplex* τ *of L, and further, if* σ *and* τ *satisfy one of the following two sets of conditions*:

(1) σ *is a 2-simplex*, τ *is a 3-simplex, and* $f(\sigma) = f(\tau)$;

(2) σ *is a 1-simplex*, τ *is a 2-simplex, and* $f(\sigma) = f(\tau)$.

LEMMA 4.5. *Let* σ *be a simplex of L which has property R, and let* τ *be that simplex of L of which* σ *is a proper face. Let* $E = \mathrm{Int}(\sigma) \cup \mathrm{Int}(\tau)$, *and let H be the complex* $(L - E)$. *Let* D_1 *be the collection of subsets of M that consists of* (1) *each of the points of* $(M - H)$ *and* (2) *each of the sets* $(d - E)$, *where d is an element of D which lies in L. The collection* D_1 *is an upper semi-continuous decomposition of M into point-like sets. The union* L_1 *of the non-degenerate elements of* D_1 *is a subcomplex of H. Moreover,* D_1^* *is homeomorphic to* D^*.

Proof. We shall furnish first a detailed proof of a special case of the lemma: σ is a 2-simplex σ^2, τ is a 3-simplex τ^3, and $f(\sigma^2) = f(\tau^3)$ is a 2-simplex. We shall then suggest analogous proofs for the other cases.

We know from Lemmas 4.3 and 4.4 that $E = \mathrm{Int}(\sigma^2) \cup \mathrm{Int}(\tau^3)$ is decomposed in D into parallel line-segments each of which is a half-open end line-

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segment of a polygonal arc of D. It follows that every set in D_1 is either an element of D, a point, or a polygonal arc. Hence D_1 is a decomposition of M into point-like sets. That D_1 is an upper semi-continuous decomposition follows from the fact that $D_1(A) = \mathrm{Cl}(A - H) \cup (D(A \cap H) \cap H)$ for each closed subset A of M.

Let ρ be the other 3-simplex of M of which σ^2 is a face. Since σ^2 has property R, the simplex ρ does not lie in L. Let v_1, v_2, and v_3 be the vertices of σ^2. Let v_0 be the remaining vertex of τ^3, and let v_4 be the remaining vertex of ρ. Let $'\sigma^2 = (v_0v_2v_3)$. We may suppose with no loss of generality that the line-segments into which $\mathrm{Int}(\sigma^3)$ is decomposed in D are parallel to (v_0v_1). Thus $'\sigma^2$ is decomposed in D into the individual points of $'\sigma^2$. Let $C = \mathrm{Int}(\rho)$, and let $C_1 = \mathrm{Int}(\rho \cup \tau^3)$. Note, for later reference, that $C = (C_1 - E)$. (See Fig. 1.) Let p be the projection mapping of M onto D^*, and let p_1 be the projection mapping of M onto D_1^*.

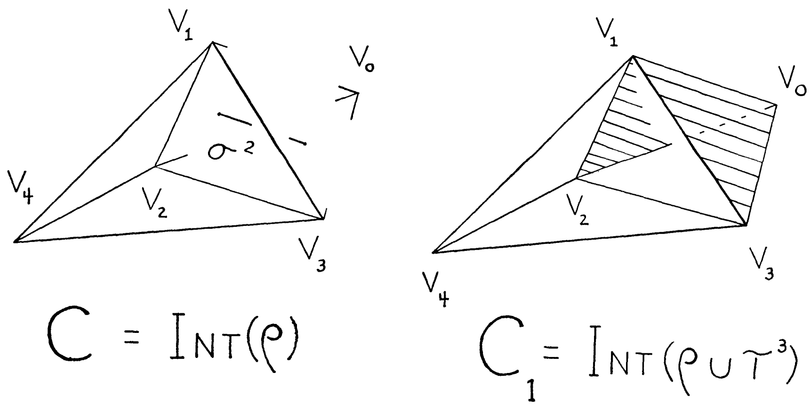

FIGURE 1

To show that D_1^* is homeomorphic to D^* we shall first construct a homeomorphism h of $(D^* - p(C))$ onto $(D_1^* - p_1(C_1))$ which maps $\mathrm{Bdy}(p(\mathrm{Cl}(C)))$ onto $\mathrm{Bdy}(p_1(\mathrm{Cl}(C_1)))$. We then show that $p_1(\mathrm{Cl}(C_1))$ is a 3-cell. Since $p(\mathrm{Cl}(C))$ is also a 3-cell, the homeomorphism $h|\mathrm{Bdy}(p(\mathrm{Cl}(C))$ can be extended to a homeomorphism h' of $p(\mathrm{Cl}(C))$ onto $p_1(\mathrm{Cl}(C_1))$. The mapping

$$g = \begin{cases} h & \text{on } D^* - p(C), \\ h' & \text{on } p(\mathrm{Cl}(C)) \end{cases}$$

will then be the required homeomorphism of D^* onto D_1^*.

Let $D \mid (M - C) = \{d \mid d \in D;\ d \subset (M - C)\}$.

The collection $D \mid (M - C)$ is an upper semi-continuous decomposition of $(M - C)$. Certainly $D|(M - C)$ is a decomposition of $(M - C)$, because each point of C is an element of D. Let F be a closed subset of $(M - C)$. The set F is also closed in M. Hence $D(F)$ is closed in M. Since $D(F)$ and C have no point in common, $D(F)$ is closed in $(M - C)$. For the same reason, $D(F)$ is the union of those elements of $D|(M - C)$ which meet F. Therefore, $D|(M-C)$ is an upper semi-continuous decomposition.

The space $(D^* - p(C))$, with the relative topology, is the hyperspace of $D|(M - C)$. The mapping $\pi = p|(M - C)$ throws $(M - C)$ onto $D|(M - C)$. Let S be a subset of $D|(M - C)$. If $\pi^{-1}(S)$ is open in $(M - C)$, then $(\pi^{-1}(S) \cup C)$ is open in M. Since $(\pi^{-1}(S) \cup C) = p^{-1}(S \cup p(C))$, the set $(S \cup p(C))$ is open in D^*, and S is open in the relative topology of $(D^* - p(C))$. Now let S be an open subset of $(D^* - p(C))$. Then the set $(S \cup p(C))$ is open in D^*, and the set $p^{-1}(S \cup p(C))$ is open in M. Moreover, $p^{-1}(S \cup p(C)) = (p^{-1}(S) \cup C)$. Therefore, $p^{-1}(S)$ is open in $(M - C)$. Since $\pi^{-1}(S) = p^{-1}(S)$, the set $\pi^{-1}(S)$ is open in $(M - C)$. Thus the relative topology for $(D^* - p(C))$ is exactly the topology induced on $D|(M - C)$ by π, and π is the projection mapping of $D|(M - C)$ onto $(D^* - p(C))$.

In a similar manner it can be proved that

$$D_1|(M - C_1) = \{d_1 \mid d_1 \in D_1, d_1 \subset (M - C_1)\}$$

is an upper semi-continuous decomposition of $(M - C_1)$, and that

$$\pi_1 = p_1 \mid (M - C_1)$$

is the projection mapping of $D_1|(M - C_1)$ onto $(D_1^* - p_1(C_1))$.

For each point x in $(D^* - p(C))$, let $h(x) = \pi_1(\pi^{-1}(x) - E)$.

For each x in $(D^* - p(C))$, $h(x)$ is a well-defined point. Let x_1 and x_2 be points of $(D^* - p(C))$. If $\pi_1(\pi^{-1}(x_1) - E)$ is not equal to $\pi_1(\pi^{-1}(x_2) - E)$, then

$$(\pi^{-1}(x_1) - E) \neq (\pi^{-1}(x_2) - E),$$
$$\pi^{-1}(x_1) \neq \pi^{-1}(x_2),$$

and $x_1 \neq x_2$.

The function h throws $(D^* - p(C))$ into $(D_1^* - p_1(C_1))$. For if x is a point of $(D^* - p(C))$, then $(\pi^{-1}(x) - E)$ is an element of $D_1|(M - C_1)$.

The function h throws $(D^* - p(C))$ onto $(D_1^* - p_1(C_1))$. Let y be a point of $(D_1^* - p_1(C_1))$. Then $\pi_1^{-1}(y)$ is an element of $D_1|(M - C_1)$. Moreover, $\pi_1^{-1}(y)$ lies in an element d of $D|(M - C)$ such that $(d - E) = \pi_1^{-1}(y)$. If $\pi_1^{-1}(y)$ does not meet $\text{Int}('\sigma^2)$, then the element d is the set $\pi_1^{-1}(y)$ itself. If $\pi_1^{-1}(y)$ meets $\text{Int}('\sigma^2)$, then $(\pi_1^{-1}(y) \cap \text{Int}('\sigma^2))$ is a point q, because $'\sigma^2$ is decomposed into individual points. Moreover, if $\pi_1^{-1}(y)$ meets $\text{Int}('\sigma^2)$, then d is the polygonal arc $(f^{-1}(f(q)) \cap L)$ which has an end line-segment joining σ^2 to $\text{Int}('\sigma^2)$ and lies parallel to (v_0v_1). Thus if $\pi_1^{-1}(y)$ meets $\text{Int}('\sigma^2)$, then,

$$\pi_1^{-1}(y) = (f^{-1}(f(q)) \cap H) = (d - E).$$

In either case, there is a point x in $(D^* - p(C))$ such that $d = \pi^{-1}(x)$, and $y = h(x)$.

The function h is one-to-one. If x_1 and x_2 are points of $(D^* - p(C))$, then

$$\pi^{-1}(x_1) \cap \pi^{-1}(x_2) = \emptyset,$$

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and

$$(\pi^{-1}(x_1) - E) \cap (\pi^{-1}(x_2) - E) = \emptyset.$$

The sets $(\pi^{-1}(x_1) - E)$ and $(\pi^{-1}(x_2) - E)$ are distinct elements of $D_1|(M - C_1)$, and they have distinct images under π_1.

The function h^{-1} is continuous. Let $\{x_i\}_{i=1}^{\infty}$ be a sequence of points converging to a point x in $(D_1^* - p_1(C_1))$. By Lemma 3.4,

$$\lim z(\pi_1^{-1}(x_i),\ \pi_1^{-1}(x)) = 0.$$

The set $\pi_1^{-1}(x_i)$ lies in an element d_i of $D|(M - C)$, and $\pi_1^{-1}(x)$ lies in an element d of $D|(M - C)$. For each i,

$$z(\pi_1^{-1}(x_i),\ \pi_1^{-1}(x)) \geqslant z(d_i, d).$$

Therefore, $\lim z(d_i, d) = 0$. Therefore, by another application of Lemma 3.4, the sequence $\{\pi(d_i)\}_{i=1}^{\infty}$ converges to $\pi(d)$ in $(D^* - p(C))$. Since $\pi(d_i) = h^{-1}(x_i)$, and since $\pi(d) = h^{-1}(x)$, the sequence $\{h^{-1}(x_i)\}_{i=1}^{\infty}$ converges to the point $h^{-1}(x)$.

The function h is a homeomorphism. By Lemma 3.3, the space $(D_1^* - p_1(C_1))$ is compact. By Lemma 3.1, the mapping h^{-1} is a homeomorphism, for, by Lemma 3.2, $(D^* - p(C))$ is a Hausdorff space.

The next few paragraphs show that the set $p_1(\mathrm{Cl}(C_1))$ is a closed 3-cell.

In E^3, let $w_0 = (0, 0, 1)$, $w_1 = (0, 0, 0)$, $w_2 = (0, 1, 0)$, $w_3 = (1, 0, 0)$, and $w_4 = (0, 0, -1)$. Let $\rho' = (w_1w_2w_3w_4)$, $\tau' = (w_0w_1w_2w_3)$, $\sigma_1 = (w_1w_2w_3)$, and $\sigma_2 = (w_0w_2w_3)$. Map $\mathrm{Cl}(C_1)$ onto $(\rho' \cup \tau')$ with the simplicial homeomorphism l that sends v_i onto w_i. Line-segments parallel to (v_0v_1) are mapped onto line-segments parallel to (w_0w_1) in the z-axis.

Let b be the barycentre of σ_2, let Σ be the union of the affine 2-simplices (w_1w_3b) and (w_1w_2b), and let τ'' be the affine 3-simplex $(w_1w_2w_3b)$. (See Fig. 2.)

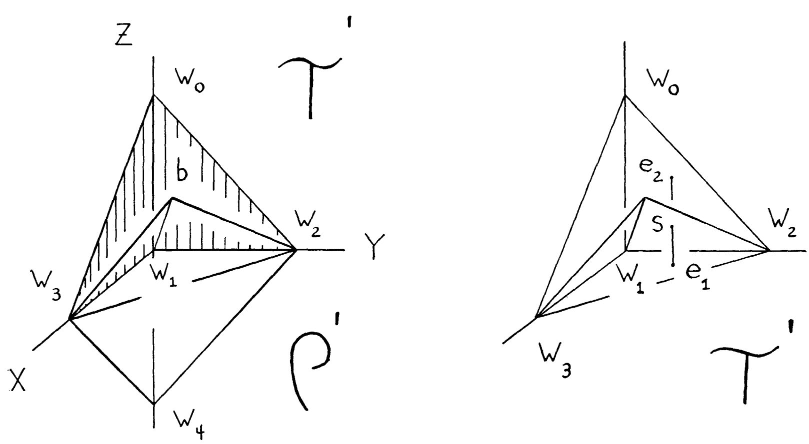

FIGURE 2

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Let S be a line-segment in τ' joining σ_1 to σ_2 and parallel to the z-axis. Let $(S \cap \sigma_i) = e_i$ $(i = 1, 2)$, and let $(S \cap \Sigma) = s$. Map S into itself by mapping e_1 onto e_1, by mapping e_2 onto s, and by mapping the linear interval $[e_2, e_1]$ linearly onto the linear interval $[s, e_1]$. Call this mapping π_s.

Map $(\rho' \cup \tau')$ onto $(\rho' \cup \tau'')$ by keeping the points of ρ' fixed and mapping x onto $\pi_s(x)$ if x does not lie in ρ'. Call this mapping k.

The mapping $p_1 l^{-1} k^{-1}$ is a homeomorphism of the 3-cell $(\rho' \cup \tau'')$ onto $p_1(\mathrm{Cl}(C_1))$.

Now that we know that D is upper semi-continuous and that D_1^* is homeomorphic to D^*, it remains to show that L_1 is a subcomplex of H. Let $'\rho^3$ be the other 3-simplex of M that contains $'\sigma^2$ (see page 596). We divide the proof into two parts, according to whether $'\rho^3$ lies in L.

(1) Suppose that $'\rho^3$ does not belong to L. Then by Lemma 4.4, no other 3-simplex of L maps onto $f(\sigma^2)$, and by Lemma 4.3, if an element d of D has a non-empty intersection with $\mathrm{Int}(\sigma^2)$ then d is a line-segment joining σ^2 to $'\sigma^2$ and lying parallel to that edge of τ^3 which does not meet $(\sigma^2 \cap {}'\sigma^2)$. Thus, if d meets $\mathrm{Int}(\sigma^2)$, then $(d - E)$ is a point of $\mathrm{Int}('\sigma^2)$ and if $(d \cap \mathrm{Int}(\sigma^2))$ is empty, then $(d - E) = d$. Hence $L_1 = (H - \mathrm{Int}('\sigma^2))$.

(2) If $'\rho^3$ lies in L, then, by Lemma 3.3, $f('\rho^3) = f('\sigma^2)$, and $L_1 = H$.

In either case, L_1 is a subcomplex of H, and we have proved Lemma 4.5 for the case $\sigma = \sigma^2$, $\tau = \tau^3$, and $f(\sigma) = f(\tau)$ is a 2-simplex.

With a very few alterations, the preceding arguments establish Lemma 4.5 for the (less complicated) cases in which σ is a 2-simplex, τ is a 3-simplex, and $f(\sigma) = f(\tau)$ is either a 1-simplex or a vertex. Define

$$\rho,\ C = \mathrm{Int}(\rho),\ C_1 = \mathrm{Int}(\rho \cup \tau),\ p,\ p_1,\ \pi,\ \pi_1, \text{ etc.},$$

as before. Then $h(x) = \pi_1(\pi^{-1}(x) - E)$ is a homeomorphism of $(D^* - p(C))$ onto $(D_1^* - p_1(C_1))$, the sets $p(\mathrm{Cl}(C))$ and $p_1(\mathrm{Cl}(C_1))$ are closed 3-cells, and the homeomorphism $h|\mathrm{Bdy}(\mathrm{Cl}(C))$ may be extended to a homeomorphism h' of $p(\mathrm{Cl}(C))$ onto $p_1(\mathrm{Cl}(C_1))$ in order to give a homeomorphism g of D^* onto D_1^*.

If σ is a 1-simplex, and if τ is a 2-simplex, however, the cells C and C_1 must be constructed in some other way. A suitable construction is the following. (See Fig. 3.)

Let v_1 and v_2 be the vertices of σ, and let v_0 be the other vertex of τ. Let A be the complex consisting of the faces of all the simplices of M of which σ is an edge. The complex A is a 3-cell, and $(A \cap L)$ lies in $(\mathrm{Bdy}(A) \cup \tau)$. The 3-simplices of A may be ordered cyclically around σ to form a sequence $\sigma_1^3, \ldots, \sigma_n^3$ such that σ_1^3 and σ_n^3 are the two 3-simplices of M of which τ is a face. For each i $(1 \leqslant i < n)$ let $\sigma_i^2 = (\sigma_i^3 \cap \sigma_{i+1}^3)$, and let b_i be the barycentre of σ_i^2. Let $\rho_1^3 = (v_0 v_1 v_2 b_1)$. For each i $(1 < i < n)$ let $\rho_i^3 = (b_i b_{i-1} v_1 v_2)$. Let $\rho_n^3 = (b_{n-1} v_0 v_1 v_2)$. The set $Z = \cup \rho_i^3$ is a 3-cell such that $(Z \cap L) = \tau$. (See Fig. 4.)

Recall that $E = \mathrm{Int}(\sigma) \cup \mathrm{Int}(\tau)$. The argument for the preceding cases can now be carried out for σ and τ by defining $C = (\mathrm{Int}(Z) - E)$ and

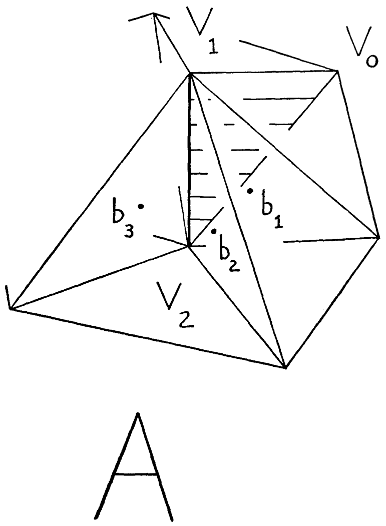

FIGURE 3

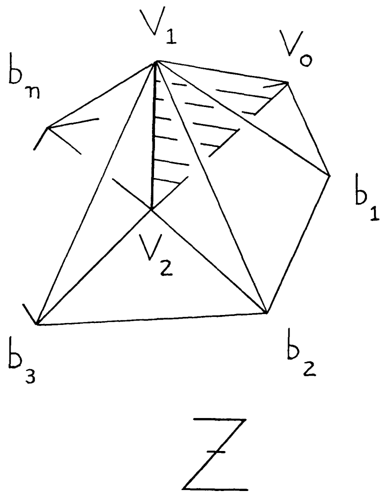

FIGURE 4

$C_1 = \text{Int}(Z)$, and by using the fact that, since $f(\tau)$ is a 1-simplex, τ is decomposed into line-segments parallel, say, to $(v_0 v_1)$. (Compare this last definition of C and of C_1 with the definition on page 596.)

5. Applying Lemma 4.5. Since we have assumed K to be non-empty, the following lemma shows that $\text{Fr}_2(K)$ is non-empty.

LEMMA 5.1. *The mapping f is one-to-one on some 3-simplex of M.*

Proof. The space T cannot consist of a single point, because then f would not be point-like. Nor can T consist entirely of 1-simplices, for then some pair of points would separate T and their inverses would separate M. Hence T contains a 2-simplex τ. If $\mathrm{Cl}(f^{-1}(\mathrm{Int}(\tau))) = C$ is a 2-simplex, then f is one-to-one on each of the two 3-simplices of M of which C is a face. If C is not a 2-simplex, then f maps at least one other 2-simplex of M onto τ, and by applying Lemma 4.4, with $L = K$, we find that C contains a 2-simplex σ in $\mathrm{Fr}(K)$. Then f is one-to-one on the other 3-simplex of M that contains σ.

Remark. Lemma 5.1 generalizes to n dimensions using the Vietoris mapping theorem.

LEMMA 5.2. *The construction of Lemma* 4.5 *may be applied to G and K.*

Proof. The decomposition G and the complex K satisfy the hypotheses of the preceding section for D and L, respectively. Also, $\mathrm{Fr}_2(K)$ is non-empty, and each 2-simplex of $\mathrm{Fr}_2(K)$ has property R. Let σ be a 2-simplex of $\mathrm{Fr}_2(K)$, let τ be the 3-simplex of K that contains σ, and let ρ be the other 3-simplex of M that contains σ. Since σ belongs to $\mathrm{Fr}_2(K)$, the simplex ρ does not lie in K. Hence $f(\rho)$ is a 3-simplex and $f(\sigma)$ is a 2-simplex. The simplices σ and τ satisfy the requirements in the definition of property R.

Let G_i be the decomposition obtained from the first $i > 0$ applications of the construction of Lemma 4.5, and let K_i be the union of the non-degenerate elements of G_i.

LEMMA 5.3. *The decomposition G_i is an upper semi-continuous decomposition of M into point-like sets, K_i is a proper subcomplex of K_{i-1}, and G_i^* is homeomorphic to G^*.*

COROLLARY. *If K_i contains a simplex with property R, then the construction of Lemma* 4.5 *may be applied to G_i and to K_i.*

Proof. The lemma follows easily by mathematical induction and from Lemma 4.5, the corollary from observing that G_i and K_i satisfy the hypotheses of Lemma 4.5 whenever K_i contains a simplex with property R.

DEFINITION. *If K_n contains no simplex with property R, then K_n is minimal.*

That a minimal complex K_n exists follows from the facts that (1) K is a finite complex, and (2) for each $i > 0$, the complex K_i is a proper subcomplex of K_{i-1}.

LEMMA 5.4. *If K_n is minimal, then K_n is the union of a finite number of disjoint, point-like sets.*

Proof. We first show that K_n contains no 3-simplex.

Suppose that C is a component of K_n which contains a 3-simplex. Since $(M - K) \neq \emptyset$ (see Lemma 5.1), the set $(M - C)$ is non-emtpy. Hence $\mathrm{Fr}_2(C)$ is non-empty. Let σ^2 be a 2-simplex lying in $\mathrm{Fr}_2(C)$, and let τ^3 be the 3-simplex of C that contains σ^2. If $f(\sigma^2)$ is a 2-simplex, then, by Lemma 4.2,

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$f(\sigma^2) = f(\tau^3)$, the simplex σ^2 has property R, and K_n is not minimal. Hence $f(\sigma^2)$ is either a 1-simplex or a vertex. If $f(\sigma^2)$ is a 1-simplex, then $f(\tau^3)$ must be a 2-simplex, for otherwise K_n again fails to be minimal. But if $f(\tau^3)$ is a 2-simplex, then, by Lemma 4.4, τ^3 is one of a sequence of 3-simplices mapped onto $f(\tau^3)$, one of these 3-simplices has a 2-face that both maps onto $f(\tau^3)$ and lies in $\mathrm{Fr}(K_n)$, this 2-face has property R, and K_n is not minimal. Thus every 2-simplex of $\mathrm{Fr}_2(C)$ is mapped onto a vertex by f.

Let B be the component of $\mathrm{Fr}_2(C)$ that contains σ^2. Since $f(\sigma^2)$ is a vertex, σ^2 lies in some element g of G_n. Since B is connected, and since each simplex of B is mapped by f onto a vertex, each simplex of B is mapped onto $f(\sigma^2)$, and B is a subset of g.

Let x be a point of $(M - K)$. Since $g \subset K_n \subset K$, the point x does not lie in g. Let y be a point of $\mathrm{Int}(\tau^3)$. The component B separates x from y. Hence g separates x from y, unless y belongs to g. Because g is point-like, g cannot separate two points of M. Therefore, y does lie in g. Hence

$$f(\tau^3) = f(g) = f(\sigma^2),$$

and K_n is not minimal. Hence K_n contains no 3-simplex.

We show next that every 2-simplex of K_n is mapped by f onto a vertex.

Let σ^2 be a 2-simplex lying in K_n. The simplex $f(\sigma^2)$ cannot be a 2-simplex, for if it were, then by Lemma 4.1, σ^2 would belong to a 3-simplex of K_n. If $f(\sigma^2)$ is a 1-simplex σ', then $\mathrm{Fr}(K)$ contains a 1-simplex with property R. To see this, let y be a point of $\mathrm{Int}(\sigma')$, and let H be the set of all points of K_n that are mapped by f onto y. Since K_n contains no 3-simplex, H is the union of a finite number of line-segments, one from each 2-simplex of K_n that is mapped onto σ'. The line-segments meet only at their end-points, and H contains no 1-sphere because H, being an element of G_n, is point-like. We shall call an end-point of a line-segment of H a "free vertex" if it belongs to only that one line-segment. It is well known that a finite, 1-dimensional complex which contains no 1-sphere has at least two free vertices. Let σ'' be an edge of K_n which contains a free vertex of H. The simplex σ'' belongs to exactly one 2-simplex $'\sigma^2$ of K_n, and $f(\sigma'') = f('\sigma^2)$. Therefore, σ'' has property R. Since K_n contains no such simplex, f must map each 2-simplex of K_n onto a vertex.

We can now show that every 1-simplex of K_n is mapped by f onto a vertex. If σ is a 1-simplex of K_n, then $f(\sigma)$ is either a 1-simplex or a vertex. If σ is an edge of a 2-simplex τ^2 of K_n, then $f(\sigma) = f(\tau^2)$ is a vertex. If σ lies in no 2-simplex of K_n, then $f(\sigma)$ must still be a vertex, for if $f(\sigma)$ were a 1-simplex, each point of $\mathrm{Int}(\sigma)$ would be an element of G_n, contrary to the fact that K_n contains only non-degenerate elements of G_n.

Since every 1-simplex of K_n is mapped by f onto a vertex of Y, each component of K_n is mapped onto some vertex of Y and must therefore be an element of G_n. Elements of G_n are point-like, components of K_n are disjoint, and Lemma 5.4 is proved.

Remark. After seeing that a minimal K_n contains no 3-simplex, one might try to show that it also contains no 2-simplex. One could then continue the reduction of the one-dimensional K_n to a finite set of vertices, and there would be no need for Lemma 5.5. However, a minimal K_n may contain a 2-dimensional component each edge of which lies in more than one 2-simplex (e.g., a pleated disk). One cannot reduce such a component. Fortunately, all components of a minimal K_n are already point-like.

It remains to state one more lemma before proceeding to the proof of the theorem.

LEMMA 5.5. *If D is a decomposition of a 3-sphere into point-like sets, and if D has only finitely many non-degenerate elements, then D^* is homeomorphic to the sphere.*

Remark. This lemma is a special case of the following, which is easy to prove: If D is a decomposition of an n-manifold N into cellular sets and if D has only finitely many non-degenerate elements, then D^* is homeomorphic to N. A subset of an n-manifold is *cellular* if there exist closed n-cells C_i $(i = 1, 2, \ldots)$ in N such that $A = \bigcap C_i$, and such that, for each i, the cell C_{i+1} is contained in the interior of the cell C_i. The concepts *cellular* and *point-like* are equivalent for subsets of S^n.

6. The proof of the theorem.

THEOREM. *Let M be a triangulated 3-sphere and let T be a triangulated topological space. If there exists a point-like, simplicial mapping of M onto T, then T is homeomorphic to M.*

Apply the construction of Lemma 4.5 to G and K until a hyperspace G_n^* with a minimal complex K_n is obtained. The complex K_n will be the union of a finite number of point-like elements of G_n (Lemma 5.4) and hence G_n^* will be homeomorphic to M (Lemma 5.5). But G^* is homeomorphic to G_n^* (Lemma 5.3) and T is homeomorphic to G^* (Lemma 3.5). Thus T is homeomorphic to M.

Remark. A mapping g of an n-manifold N is called *cellular* if the set $g^{-1}(x)$ is cellular for each point x in $g(N)$. The arguments of this paper are easily modified to show that if there exists a cellular, simplicial mapping of a triangulated compact 3-manifold M onto a triangulated topological space T, then T is homeomorphic to M.

I should like to close with the following question. Can every orientation-preserving, point-like, simplicial mapping of the 3-sphere be factored into a product of simplicial mappings of the sphere onto itself each of which identifies exactly two vertices, those bounding a 1-simplex?

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References

1. R. H. Bing, *A decomposition of E^3 into points and tame arcs such that the decomposition space is topologically different from E^3*, Ann. of Math., *65* (1957), 484–500.
2. O. G. Harrold, Jr., *A sufficient condition that a monotone image of the three-sphere be a topological three-sphere*, Proc. Amer. Math. Soc., *9* (1958), 846–850.
3. R. L. Moore, *Concerning upper semi-continuous collections of continua*, Trans. Amer. Math. Soc., *27* (1925), 416–428.
4. J. H. Roberts and N. E. Steenrod, *Monotone transformations of two-dimensional manifolds*, Ann. of Math., *39* (1938), 851–862.
5. G. T. Whyburn, *On the structure of continua*, Bull. Amer. Math. Soc., *42* (1936), 49–73.

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