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ON THE ORNSTEIN–ZERNIKE EQUATION FOR STATIONARY CLUSTER PROCESSES AND THE RANDOM CONNECTION MODEL

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Abstract

In the first part of this paper we consider a general stationary subcritical cluster model in $\mathbb{R}^d$. The associated pair-connectedness function can be defined in terms of two-point Palm probabilities of the underlying point process. Using Palm calculus and Fourier theory we solve the Ornstein–Zernike equation (OZE) under quite general distributional assumptions. In the second part of the paper we discuss the analytic and combinatorial properties of the OZE solution in the special case of a Poisson-driven random connection model.

Keywords: Ornstein–Zernike equation; Gilbert graph; random connection model; pair-connectedness function; Poisson process; percolation; analyticity

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1. Introduction

In their seminal paper, Ornstein and Zernike [11] proposed to split the interaction between molecules in a liquid into a direct and an indirect part. While the resulting spatial convolution equation is of great importance in physics, it seems to be hardly known among mathematicians. The aim of this paper is to bridge this gap and to lay a rigorous mathematical foundation for further studies.

We start with a simple example of a stationary cluster process, which is also a special case of the random connection model studied later. Let η_t be a stationary Poisson process on $\mathbb{R}^d$ with intensity $t \geq 0$. Let $B \subset \mathbb{R}^d$ be a *gauge body*, that is, a compact set containing the origin $0 \in \mathbb{R}^d$ in its interior. We define a *random geometric graph* $\mathcal{G}(\eta_t)$ (or *Gilbert graph*) with vertex set η_t as follows. Two distinct points $x, y \in \eta_t$ are adjacent in $\mathcal{G}(\eta_t)$ whenever $(B+x) \cap (B+y) \neq \varnothing$, where $B + x := \{x + z : z \in B\}$; see [14]. For $x \in \eta_t$, let $C(x, \eta_t) \subset \eta_t$ denote the *cluster* of x, that is, the connected component of $\mathcal{G}(\eta_t)$ containing x. These definitions apply to any point process η and, in particular, to deterministic locally finite subsets of $\mathbb{R}^d$. For each point process η on $\mathbb{R}^d$ and $x, y \in \mathbb{R}^d$, we write $\eta^x := \eta \cup \{x\}$ and $\eta^{x,y} := \eta \cup \{x, y\}$; see also Appendix A. We wish to study the *pair-connectedness function* (see [17])

$$P_t(x, y) := \mathbb{P}(y \in C(x, \eta_t^{x,y})), \quad x, y \in \mathbb{R}^d.$$

By Corollary 4.15 of [4], there is a *percolation threshold* $t_c \in (0, \infty)$ such that $\mathbb{P}(|C(0, \eta_t^0)| = \infty) > 0$ for $t > t_c$ and $\mathbb{P}(|C(0, \eta_t^0)| = \infty) = 0$ for $t < t_c$. We seek a function $Q_t(x, y)$ solving

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the *Ornstein–Zernike equation* (OZE)

$$P_t(x, y) = Q_t(x, y) + t \int_{\mathbb{R}^d} Q_t(x, z) P_t(z, y)\, \mathrm{d}z, \qquad x, y \in \mathbb{R}^d,\ t < t_c. \tag{1.1}$$

We shall formulate and solve (1.1) in the following much more general setting. Let η be a stationary point process on $\mathbb{R}^d$ with finite intensity γ_η. The points are partitioned into clusters (sets of points of η) according to a translation-invariant rule. This rule might be very general and can incorporate additional randomness (e.g. in the random connection model). The point process η is assumed to be jointly stationary with the cluster process. The pair-connectedness function $P(x, y)$ is then informally defined as the conditional probability that $x, y \in \mathbb{R}^d$ belong to the same cluster, given that x and y are points of η (suitably weighted by the pair-correlation function). Then the OZE (1.1) takes the form

$$P(x, y) = Q(x, y) + \gamma_\eta \int_{\mathbb{R}^d} Q(x, z) P(z, y)\, \mathrm{d}z, \qquad x, y \in \mathbb{R}^d. \tag{1.2}$$

Our Theorem 3.1 shows under rather weak assumptions that (1.2) has a unique solution. The proof of this result is based on Palm calculus for stationary point processes (see Appendix A) and a classical theorem by Wiener on the inversion of Fourier transforms.

In Sections 4–6 we shall consider the (Poisson driven) random connection model (RCM) (see [9]), a significant generalization of the Gilbert graph introduced above. The RCM with parameters $t \geq 0$ and $\varphi\colon \mathbb{R}^d \to [0, 1]$ is a random graph $\mathscr{G}_t$ where the set of vertices is a Poisson process η_t with intensity t. Any two distinct vertices $x, y \in \eta_t$ are adjacent with probability $\varphi(x - y)$ independently of all other pairs and of η_t. We call φ the *connection function* of the RCM. The clusters in this model are just the connected components of $\mathscr{G}_t$. In Section 4 we shall give a detailed description of this model along with formulas on degree distributions (that are basically well known) and a Margulis–Russo-type formula. The latter result might be of some independent interest. In Section 5 we shall first show that the RCM satisfies the assumptions of Theorem 3.1, so that a solution $Q_t \equiv Q$ of (1.1) (with $P_t \equiv P$ denoting the pair-connectedness function) exists in the whole subcritical regime. Then we prove that P_t is an analytic function of t on the interval $[0, t_*)$, where t_* is the smallest number such that for $t < t_*$ the typical cluster has an exponentially decreasing tail. In the Gilbert graph with fixed gauge body B (mentioned above), the arguments from [14] can be extended to show that t_* is equal to the percolation threshold t_c. In fact, Theorem 5.2 shows that this result holds for general integrable functions of the typical cluster. We then proceed with deriving similar properties for Q_t; see Proposition 5.2. We are not aware of a direct probabilistic interpretation of Q_t. However, for small intensities t, there is a simple combinatorial relationship between the coefficients in the expansions of P_t and Q_t; see Theorem 6.1.

In writing this paper we strongly benefited from the large physics literature on the topic. In particular, the combinatorial formulas in the final section are well known, although not in a mathematically rigorous form. Two key references are [1] and [2]. However, we have not been able to find a justification of the existence of a solution of the OZE, not even in the very special case of a Poisson-driven Gilbert graph. Moreover, the analytic properties of P_t and Q_t (often taken as granted) have not been proved either. In our opinion it is one of the main contributions of the present paper to apply modern point process methods (Palm calculus and Margulis–Russo-type formulas for Poisson driven systems) to the OZE. The original motivation for our work came from [18], where the author used the OZE to derive putative lower bounds for the percolation threshold in the Gilbert graph. This is one of the potential applications of the present paper.

2. Preliminaries on stationary point processes

In this paper all random elements are defined on a measurable space $(\Omega, \mathcal{A})$ equipped with a *measurable flow* $\theta_x\colon \Omega \to \Omega$, $x \in \mathbb{R}^d$. This is a family of measurable mappings such that $(\omega, x) \mapsto \theta_x \omega$ is measurable, θ_0 is the identity on Ω, and

$$\theta_x \circ \theta_y = \theta_{x+y}, \qquad x, y \in \mathbb{R}^d,$$

where '$\circ$' denotes composition. We may think of $\theta_x \omega$ as ω *shifted* by the vector $-x$. We fix a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{A})$ and assume that it is *stationary*, that is,

$$\mathbb{P} \circ \theta_x = \mathbb{P}, \qquad x \in \mathbb{R}^d,$$

where θ_x is interpreted as a mapping from $\mathcal{A}$ to $\mathcal{A}$ in the usual way:

$$\theta_x A := \{\theta_x \omega\colon \omega \in A\}, \qquad A \in \mathcal{A},\ x \in \mathbb{R}^d.$$

Let $\boldsymbol{N}(\mathbb{R}^d)$ denote the space of all locally finite subsets μ of $\mathbb{R}^d$. Hence, $\mu \in \boldsymbol{N}(\mathbb{R}^d)$ if and only if $\mu \cap B$ is finite for each bounded set. For each $\mu \in \boldsymbol{N}(\mathbb{R}^d)$ and each $B \subset \mathbb{R}^d$, we write $\mu(B) := |\mu \cap B|$ for the number of points of μ lying in B. As usual, we equip $\boldsymbol{N}(\mathbb{R}^d)$ with the smallest σ-field $\mathcal{N}$ making the mappings $\mu \mapsto \mu(B)$ measurable for all B in the Borel σ-field $\mathcal{B}^d$ on $\mathbb{R}^d$.

A *point process* on $\mathbb{R}^d$ is a measurable mapping $\eta\colon \Omega \to \boldsymbol{N}(\mathbb{R}^d)$. It is called *invariant* (or *stationary*) if

$$\eta(\omega, B + x) = \eta(\theta_x \omega, B), \qquad \omega \in \Omega,\ x \in \mathbb{R}^d,\ B \in \mathcal{B}^d.$$

Let η be an invariant point process. The *intensity* of η is the number $\gamma_\eta := \mathbb{E}\eta([0,1]^d)$. If the latter is positive and finite, we can define the probability measure

$$\mathbb{P}^0_\eta(A) := \gamma_\eta^{-1} \int \sum_{x \in \eta(\omega)} \mathbf{1}\{\theta_x \omega \in A,\ x \in [0,1]^d\} \mathbb{P}(\mathrm{d}\omega), \qquad A \in \mathcal{A}.$$

This *Palm probability measure* of η satisfies the *refined Campbell formula*

$$\int \sum_{x \in \eta(\omega)} f(\theta_x \omega, x)\, \mathbb{P}(\mathrm{d}\omega) = \gamma_\eta \iint f(\omega, x)\, \mathrm{d}x \mathbb{P}^0_\eta(\mathrm{d}\omega) \tag{2.1}$$

for all measurable $f\colon \Omega \times \mathbb{R}^d \to [0, \infty)$, where $\mathrm{d}x$ refers to integration with respect to the Lebesgue measure on $\mathbb{R}^d$. Using standard conventions we write this as

$$\mathbb{E} \sum_{x \in \eta} f(\theta_x, x) = \gamma_\eta \mathbb{E}^0_\eta \int f(\theta_0, x)\, \mathrm{d}x,$$

where $\mathbb{E}^0_\eta$ denotes integration with respect to $\mathbb{P}^0_\eta$. The measure $\mathbb{P}^0_\eta$ is concentrated on the measurable set Ω_0 of all $\omega \in \Omega$ such that the origin 0 is in $\eta(\omega)$. The *Palm distribution* of η is the distribution $\mathbb{P}^0_\eta(\eta \in \cdot)$ of η under $\mathbb{P}^0_\eta$. It is concentrated on the measurable set of all $\mu \in \boldsymbol{N}(\mathbb{R}^d)$ such that $0 \in \mu$. The number $\mathbb{P}^0_\eta(A)$ can be interpreted as the conditional probability of $A \in \mathcal{A}$ given that η has a point at the origin.

Let η be a point process on $\mathbb{R}^d$ and $n \in \mathbb{N}$. The nth *factorial moment measure* $\alpha^{(n)}$ of η is the measure on $(\mathbb{R}^d)^n$ defined by

$$\alpha^{(n)} := \mathbb{E} \sum_{x_1,\dots,x_n \in \eta}^{\neq} \mathbf{1}\{(x_1, \dots, x_n) \in \cdot\}, \tag{2.2}$$

where the superscript '$\neq$' indicates summation over all ordered n-tuples of distinct elements of η (a notation that is also used for multiindices). Assume now that η is an invariant point process with a positive and finite intensity. Assume also that $\alpha^{(2)}$ is locally finite (finite on bounded Borel sets) and absolutely continuous with respect to the Lebesgue measure, that is (using also stationarity),

$$\alpha^{(2)}(\mathrm{d}(x, y)) = \gamma_\eta^2 \rho(y - x)\, \mathrm{d}(x, y)$$

for a locally integrable measurable $\rho\colon \mathbb{R}^d \to \mathbb{R}$. The latter is the *pair correlation function* of η. The *two-point Palm probability measures* of η is a family $\{\mathbb{P}_\eta^{x,y} \colon x, y \in \mathbb{R}^d\}$ of probability measures on $(\Omega, \mathcal{A})$ such that $(x, y) \mapsto \mathbb{P}_\eta^{x,y}(A)$ is measurable for all $A \in \mathcal{A}$ and

$$\mathbb{E} \sum_{x,y \in \eta}^{\neq} f(\theta_0, x, y) = \gamma_\eta^2 \int \mathbb{E}_\eta^{x,y} f(\theta_0, x, y) \rho(y - x)\, \mathrm{d}(x, y) \tag{2.3}$$

for all measurable $f\colon \Omega \times \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty)$, where $\mathbb{E}_\eta^{x,y}$ denotes expectation with respect to $\mathbb{P}_\eta^{x,y}$. The number $\mathbb{P}_\eta^{x,y}(A)$ can be interpreted as the probability of $A \in \mathcal{A}$ given that η has points at x and y. Under a Borel assumption on $(\Omega, \mathcal{A})$, we will show, in Appendix A, that the two-point Palm probability measures can be chosen so as to satisfy

$$\mathbb{E}_\eta^{x,y} F = \mathbb{E}_\eta^{0,y-x} F \circ \theta_{-x}, \qquad x, y \in \mathbb{R}^d, \tag{2.4}$$

for all measurable $F\colon \Omega \to [0, \infty)$. More details on Palm calculus for stationary point processes can, e.g. be found in [6], [16], and in Appendix A.

3. The OZE

In this section we establish (1.2) for general stationary cluster processes defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. We let η be an invariant (and, therefore, stationary) point process on $\mathbb{R}^d$ with finite intensity $\gamma_\eta := \mathbb{E}\eta([0, 1]^d)$. We also assume that $\mathbb{P}(\eta \neq \varnothing) = 1$. To describe the clusters, we consider a measurable mapping $(\omega, x) \mapsto \tau(\omega, x)$ from $\Omega \times \mathbb{R}^d$ into $\mathbb{R}^d$ with the covariance property

$$\tau(\theta_y \omega, x - y) = \tau(\omega, x) - y, \qquad \omega \in \Omega,\ x, y \in \mathbb{R}^d. \tag{3.1}$$

(For convenience we also assume that $\tau(x) = x,\ x \in \mathbb{R}^d$, whenever $\eta(\mathbb{R}^d) = 0$.) The points of the random set

$$\xi := \{\tau(x) \colon x \in \eta\}$$

are interpreted as locations (or centers) of the clusters of η. Note that τ need not be a deterministic function of η like in the Gilbert graph, but might incorporate additional randomness; see Section 4. The refined Campbell formula (2.1) and the covariance property (3.1) imply that

$$\mathbb{E}|\xi \cap B| = \gamma_\eta \lambda_d(B)\, \mathbb{P}_\eta^0(\tau(0) \in B), \qquad B \in \mathcal{B}^d,$$

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where λ_d is the Lebesgue measure on $\mathbb{R}^d$ and $\mathbb{P}_\eta^0$ is the Palm probability measure of η. In particular, $\mathbb{E}|\xi \cap B| < \infty$ for all bounded Borel sets B, so that it is no restriction of generality to assume that ξ is locally finite everywhere on Ω. It follows that ξ is an invariant point process with finite intensity $\gamma_\xi = \gamma_\eta \mathbb{P}_\eta^0(\tau(0) \in [0, 1]^d)$. The *clusters* can formally be defined as those points of η which have the same image under τ. Hence, $x, y \in \eta$ belong to the same cluster if and only if $\tau(x) = \tau(y)$ and the cluster of $x \in \eta(\omega)$ is given by $C(\omega, x) := \{y \in \eta(\omega) \colon \tau(\omega, y) = \tau(\omega, x)\}$ or, more succinctly,

$$C(x) := \{y \in \eta \colon \tau(y) = \tau(x)\}, \qquad x \in \eta. \tag{3.2}$$

(It is convenient to use this definition for all $x \in \mathbb{R}^d$.) In the random connection model, for instance, the mapping τ is defined so as to ensure that (3.2) is consistent with the definition of the clusters given in the introduction.

It follows from (3.1) that

$$C(\theta_y \omega, x) = C(\omega, x + y) - y, \qquad \omega \in \Omega,\ x, y \in \mathbb{R}^d. \tag{3.3}$$

The distribution of $C(0)$ under the Palm probability measure $\mathbb{P}_\eta^0$ can be interpreted as the distribution of the cluster containing the typical point of η. We make the crucial assumption that the size of this cluster has a finite expectation, that is,

$$\mathbb{E}_\eta^0|C(0)| < \infty. \tag{3.4}$$

The cluster with location $z \in \xi(\omega)$ is defined by $D(\omega, z) := \{x \in \eta(\omega) \colon \tau(\omega, x) = z\}$ or

$$D(z) := \{x \in \eta \colon \tau(x) = z\}, \qquad z \in \xi.$$

(Again we use this notation for all $z \in \mathbb{R}^d$.)

As we are interested in the second-order properties of η, we need to assume that the second-order factorial moment measure of η is locally finite and absolutely continuous. We then denote by ρ the pair correlation function and by $\mathbb{P}_\eta^{x,y}$, $x, y \in \mathbb{R}^d$, the two-point Palm distributions of η; see (2.3). Our interest in this paper focuses on the weighted *pair-connectedness function*

$$P(x, y) := \rho(x - y)\mathbb{P}_\eta^{x,y}(y \in C(x)), \qquad x, y \in \mathbb{R}^d.$$

In view of (2.4) and (3.3), we have $P(x, y) = P(y - x)$ and we define the (even) function $P \colon \mathbb{R}^d \to \mathbb{R}$ by $P(x) := P(0, x)$. Choosing $f := \mathbf{1}\{0 \in C(x)\}$ in (A.2) yields

$$\mathbb{E}_\eta^0|C(0)| = 1 + \gamma_\eta \int P(x)\,\mathrm{d}x. \tag{3.5}$$

Hence, (3.4) implies that P is in the space L^1 of all measurable functions $f \colon \mathbb{R}^d \to \mathbb{R}$ with $\|f\|_1 := \int |f(x)|\,\mathrm{d}x < \infty$.

The convolution of $f, g \in L^1$ is defined as

$$(f * g)(x) := \int f(x - y)g(y)\,\mathrm{d}y, \qquad x \in \mathbb{R}^d.$$

In the same way, we define the convolution for functions $f \in L^1$ and $g \in L^\infty$, where L^∞ is the space of bounded functions equipped with the supremum norm $\|\cdot\|_\infty$. Both definitions make sense due to the basic inequalities

$$\|f * g\|_\infty \le \|f\|_1 \|g\|_\infty, \qquad f \in L^1,\ g \in L^\infty, \tag{3.6}$$

$$\|f * g\|_1 \le \|f\|_1 \|g\|_1, \qquad f \in L^1,\ g \in L^\infty. \tag{3.7}$$

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We can now formulate and prove the OZE (1.1) in the present very general stationary setting. We need the regularity assumption

$$\mathbb{P}_\eta^0\Bigl(\sum_{x\in C(0)} \mathrm{e}^{\mathrm{i}wx} \neq 0\Bigr) > 0, \qquad w \in \mathbb{R}^d, \tag{3.8}$$

where wx is the Euclidean scalar product of $x, w \in \mathbb{R}^d$ and 'i' is the imaginary unit.

Theorem 3.1. *Assume that (3.4) and (3.8) hold. Then there is a unique $Q \in L^1 \cap L^\infty$ such that*

$$P = Q + \gamma_\eta Q * P. \tag{3.9}$$

Remark 3.1. Assumption (3.8) is rather weak. It holds, for instance, if $\mathbb{P}_\eta^0(|C(0)| = 1) > 0$. Indeed, if $|C(0)| = 1$ then the sum in (3.8) reduces to the single term 1. Another sufficient condition can be formulated in terms of the factorial moment measures $\alpha^{(n)}$, $n \in \mathbb{N}$, of η defined by (2.2). If these measures are locally finite and absolutely continuous then

$$\mathbb{P}_\eta^0\Bigl(\sum_{x\in C(0)} \mathrm{e}^{\mathrm{i}wx} = 0\Bigr) = 0, \qquad w \in \mathbb{R}^d,$$

so that (3.8) holds. To see this, we note that

$$\begin{aligned}\mathbb{P}_\eta^0\Bigl(\sum_{x\in C(0)} \mathrm{e}^{\mathrm{i}wx} = 0\Bigr) &\le \mathbb{E}\sum_{n=2}^{\infty}\frac{1}{n!}\sum_{x_1,\dots,x_n\in\eta}^{\neq} \mathbf{1}\{\mathrm{e}^{-\mathrm{i}wx_1} f_n(x_2,\dots,x_n) = 1\}\\ &= \sum_{n=2}^{\infty}\frac{1}{n!}\mathbb{E}\int \mathbf{1}\{f_n(x_2,\dots,x_n) = \mathrm{e}^{\mathrm{i}wx_1}\}\alpha^{(n)}(\mathrm{d}(x_1,\dots,x_n)),\end{aligned}$$

where $f_n(x_2,\dots,x_n) := \sum_{k=2}^{n}\mathrm{e}^{\mathrm{i}wx_k}$.

We prepare the proof of Theorem 3.1 with some results of independent interest. We start with providing the connection between the typical cluster and the cluster of a typical point. This is an example of size biasing; see, e.g. [16, Chapter 9] and [6, Section 3.9] for the case of volume biasing and debiasing. In the following, we interpret $C(x)$ and $D(x)$ as point processes on $\mathbb{R}^d$, that is, as measurable mappings from Ω to $\boldsymbol{N}(\mathbb{R}^d)$.

Proposition 3.1. *For any measurable $f\colon \boldsymbol{N}(\mathbb{R}^d) \to [0,\infty)$,*

$$\gamma_\eta\, \mathbb{E}_\eta^0 f(C(0) - \tau(0)) = \gamma_\xi\, \mathbb{E}_\xi^0 |D(0)| f(D(0)). \tag{3.10}$$

Proof. Using Proposition A.1 with $(\omega, x) \mapsto f(D(\omega,0))\mathbf{1}\{\tau(\omega,x) = 0\}$, we obtain

$$\begin{aligned}\gamma_\xi \mathbb{E}_\xi^0 |D(0)| f(D(0)) &= \gamma_\xi\, \mathbb{E}_\xi^0 \sum_{x\in\eta} f(D(0))\mathbf{1}\{\tau(x) = 0\}\\ &= \gamma_\eta \mathbb{E}_\eta^0 \sum_{x\in\xi} f(D(\theta_x, 0))\mathbf{1}\{\tau(\theta_x, -x) = 0\}\\ &= \gamma_\eta \mathbb{E}_\eta^0 \sum_{x\in\xi} f(D(\theta_x, 0))\mathbf{1}\{\tau(\theta_0, 0) = x\},\end{aligned}$$

where we have used (3.1) to obtain the third identity. Using (3.1) again, it can be easily checked that $D(\theta_x\omega, 0) = C(\omega, 0) - x$ whenever $x \in \xi(\omega)$ and $\tau(\omega, 0) = x$. □

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Proposition 3.1 implies, in particular, that

$$\gamma_\xi \mathbb{E}^0_\xi |D(0)|^2 = \gamma_\eta \mathbb{E}^0_\eta |C(0)|, \tag{3.11}$$

which is finite by (3.4). Another consequence of Proposition 3.1 is

$$\gamma_\xi = \gamma_\eta \mathbb{E}^0_\eta |C(0)|^{-1}.$$

The number $\mathbb{E}^0_\eta |C(0)|^{-1}$ might be called the number of clusters per vertex in percolation theory; see, e.g. [3].

We also need the following consequence of Proposition 3.1.

Lemma 3.1. *The relationship (3.8) is equivalent to*

$$\mathbb{P}^0_\xi\Big(\sum_{x\in D(0)} \mathrm{e}^{\mathrm{i}wx} \neq 0\Big) > 0, \qquad w \in \mathbb{R}^d. \tag{3.12}$$

Proof. The relationship (3.12) holds if and only if

$$\mathbb{E}^0_\xi \Big| \sum_{x\in D(0)} \mathrm{e}^{\mathrm{i}wx} \Big| \neq 0.$$

By (3.10), this is equivalent to

$$0 \neq \mathbb{E}^0_\eta \Big| \sum_{x\in C(0)-\tau(0)} \mathrm{e}^{\mathrm{i}wx} \Big| = \mathbb{E}^0_\eta \Big| \mathrm{e}^{-\mathrm{i}w\tau(0)} \sum_{x\in C(0)} \mathrm{e}^{\mathrm{i}wx} \Big| = \mathbb{E}^0_\eta \Big| \sum_{x\in C(0)} \mathrm{e}^{\mathrm{i}wx} \Big|.$$

This implies the assertion. □

The *Fourier transform* of P is the function $\hat{P}\colon \mathbb{R}^d \to \mathbb{C}$ given by

$$\hat{P}(w) := \int P(x) \mathrm{e}^{\mathrm{i}wx}\, \mathrm{d}x, \qquad w \in \mathbb{R}^d.$$

This transform can be expressed in terms of the typical cluster as follows.

Proposition 3.2. *For any* $w \in \mathbb{R}^d$,

$$\gamma_\eta + \gamma_\eta^2 \hat{P}(w) = \gamma_\xi\, \mathbb{E}^0_\xi \Big| \sum_{x\in D(0)} \mathrm{e}^{\mathrm{i}wx} \Big|^2. \tag{3.13}$$

Proof. First we apply (A.2) with $f(\omega, x) := \mathbf{1}\{x \in C(\omega, 0)\}\mathrm{e}^{\mathrm{i}wy}$ to obtain

$$\gamma_\eta + \gamma_\eta^2 \hat{P}(w) = \gamma_\eta + \gamma_\eta^2 \int \mathbb{P}^{0,x}(x \in C(0))\mathrm{e}^{\mathrm{i}wx} \rho(x)\, \mathrm{d}x = \gamma_\eta \mathbb{E}^0_\eta \sum_{x\in C(0)} \mathrm{e}^{\mathrm{i}wx},$$

where we recall the integrability assumption (3.4).

Since the clusters exhaust the points of η, we obtain

$$\gamma_\eta + \gamma_\eta^2 \hat{P}(w) = \gamma_\eta \mathbb{E}^0_\eta \sum_{x\in\xi} \sum_{y\in C(0)} \mathrm{e}^{\mathrm{i}wy} \mathbf{1}\{\tau(y) = x\}.$$

Using the exchange formula (A.1) (to be justified below), we have

$$\begin{aligned}\gamma_\eta + \gamma_\eta^2 \hat{P}(w) &= \gamma_\xi \mathbb{E}_\xi^0 \sum_{x\in\eta} \sum_{y\in C(\theta_x,0)} \mathrm{e}^{\mathrm{i}wy} \mathbf{1}\{\tau(\theta_x, y) = -x\} \\ &= \gamma_\xi \mathbb{E}_\xi^0 \sum_{x\in\eta} \sum_{y\in\eta} \mathbf{1}\{y - x \in C(\theta_x, 0)\} \mathrm{e}^{\mathrm{i}w(y-x)} \mathbf{1}\{\tau(\theta_x, y - x) = -x\} \\ &= \gamma_\xi \mathbb{E}_\xi^0 \sum_{x\in\eta} \sum_{y\in C(x)} \mathrm{e}^{-\mathrm{i}wx} \mathrm{e}^{\mathrm{i}wy} \mathbf{1}\{\tau(y) = 0\},\end{aligned}$$

where we have used the invariance properties (3.1) and (3.3). For $0 \in \xi$ and $x, y \in \eta$, the relations $y \in C(x)$ and $\tau(y) = 0$ are equivalent to $x, y \in D(0)$. Hence,

$$\gamma_\eta + \gamma_\eta^2 \hat{P}(w) = \gamma_\xi \mathbb{E}_\xi^0 \sum_{x\in D(0)} \mathrm{e}^{-\mathrm{i}wx} \sum_{y\in D(0)} \mathrm{e}^{\mathrm{i}wy},$$

implying (by Fubini's theorem) the asserted formula (3.13). The use of both the exchange formula and Fubini's theorem is justified by $\mathbb{E}_\xi^0 |D(0)|^2 < \infty$, a consequence of (3.11) and assumption (3.4). □

Proof of Theorem 3.1. We shall use a classical theorem by Wiener on the inversion of Fourier transforms. Recall that a *finite signed measure* μ on $\mathbb{R}^d$ is the difference of two finite measures. The Fourier transform of such a μ is defined by

$$\hat{\mu}(w) := \int \mathrm{e}^{\mathrm{i}wx} \mu(\mathrm{d}x), \qquad w \in \mathbb{R}^d.$$

The convolution $\mu * \nu$ of two finite signed measures μ and ν is the finite signed measure defined by

$$\mu * \nu(B) := \iint \mathbf{1}\{x + y \in B\} \mu(\mathrm{d}x)\nu(\mathrm{d}y), \qquad B \in \mathcal{B}^d.$$

Note that $\mu * \delta_0 = \mu$, where $\delta_0(B) := \mathbf{1}\{0 \in B\}$, $B \in \mathcal{B}^d$. Also note that $\widehat{\mu * \nu} = \hat{\mu}\hat{\nu}$. Each $f \in L^1$ defines a finite signed measure $\mu_f := \int \mathbf{1}\{x \in \cdot\} f(x)\,\mathrm{d}x$. (Later we will abuse notation and write f instead of μ_f.) For $f, g \in L^1$ we have $\mu_f * \mu_g = \mu_{f*g}$.

Let M^1 denote the vector space of all finite signed measures of the form $r\delta_0 + \mu_f$, where $r \in \mathbb{R}$ and $f \in L^1$. The OZE (3.9) can be written as

$$\mu_P = t\mu_Q * \nu, \tag{3.14}$$

where $t := \gamma_\eta$ and $\nu := t^{-1}\delta_0 + \mu_P \in M^1$. Proposition 3.2, (3.8), and Lemma 3.1 imply that $\hat{\nu}(w) \neq 0$ for all $w \in \mathbb{R}^d$. A theorem of Wiener (see Theorem 13.2 of [5]) says that ν can be inverted within the convolution algebra M^1. This means that there is an $f \in L^1$ such that

$$\nu * (t\delta_0 + t^2\mu_f) = \delta_0.$$

The function $Q := P + tP * f$ is in L^1. Moreover,

$$t\mu_Q * \nu = t\nu * (\mu_P + t\mu_{P*f}) = \nu * (t\mu_P + t^2\mu_P * \mu_f) = \nu * \mu_P * (t\delta_0 + t^2\mu_f) = \mu_P * \delta_0 = \mu_P,$$

as required by (3.14).

To show that Q is bounded, we apply (3.6) to obtain

$$\|Q\|_\infty = \|P\|_\infty + \gamma_\eta\|Q * P\|_\infty \le \|P\|_\infty + \gamma_\eta\|Q\|_1\|P\|_\infty < \infty. \qquad □$$

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The solution Q of the OZE (3.9) has good integrability properties and can be used to express the mean size of the cluster containing a typical point.

Proposition 3.3. *Under the assumption of Theorem 3.1, we have*

$$0 \le \int Q(x)\,\mathrm{d}x < \gamma_\eta^{-1}$$

and

$$\mathbb{E}_\eta^0|C(0)| = \Big(1 - \gamma_\eta \int Q(x)\,\mathrm{d}x\Big)^{-1}. \tag{3.15}$$

Proof. Equation (3.5) and the OZE (3.9) imply that

$$\begin{aligned}\mathbb{E}_\eta^0|C(0)| &= 1 + \gamma_\eta \int Q(x)\,\mathrm{d}x + \gamma_\eta^2 \int P(x)\,\mathrm{d}x \int Q(x)\,\mathrm{d}x \\ &= 1 + \gamma_\eta \int Q(x)\,\mathrm{d}x + \gamma_\eta(\mathbb{E}_\eta^0|C(0)| - 1) \int Q(x)\,\mathrm{d}x.\end{aligned}$$

It follows that

$$\mathbb{E}_\eta^0|C(0)| = 1 + \gamma_\eta \mathbb{E}_\eta^0|C(0)| \int Q(x)\,\mathrm{d}x.$$

Since $\mathbb{E}_\eta^0|C(0)| \ge 1$ we conclude that $\int Q(x)\,\mathrm{d}x \ge 0$. Moreover, since $\mathbb{E}_\eta^0|C(0)| < \infty$ we have $\gamma_\eta \int Q(x)\,\mathrm{d}x < 1$ and, hence, (3.15). □

4. The random connection model

In this section we consider a stationary Poisson process η_t on $\mathbb{R}^d$ with intensity $t \ge 0$ together with a measurable function $\varphi\colon \mathbb{R}^d \to [0, 1]$ satisfying

$$\varphi(x) = \varphi(-x), \qquad x \in \mathbb{R}^d, \tag{4.1}$$

and

$$\int \varphi(x)\,\mathrm{d}x < \infty.$$

Suppose that any two distinct points $x, y \in \eta_t$ are adjacent with probability $\varphi(y - x)$ independently of all other pairs and independently of η_t. This yields the RCM, an undirected *random graph* $\mathcal{G}$ with vertex set η_t. Each $x \in \eta_t$ belongs to a uniquely defined connected component $C'(x)$. The mapping τ from Section 3 is defined as follows. If $x \in \eta_t$ and $|C'(x)| < \infty$ then we let $\tau(x)$ be the lexicographic minimum of $C'(x)$. (For all other $x \in \mathbb{R}^d$ we let $\tau(x) := x$.) Hence, if all connected components of $\mathcal{G}$ are finite, the set of clusters consists exactly of these connected components.

The Gilbert graph (briefly discussed in the introduction) based on η_t and a *gauge body* $B \subset \mathbb{R}^d$, that is, a compact and connected set containing the origin $0 \in \mathbb{R}^d$, is a special case of the RCM. It is obtained by choosing

$$\varphi(x) = \mathbf{1}\{(B + x) \cap B \ne \varnothing\}.$$

In contrast to the RCM, the Gilbert graph contains no additional randomness. Two points $x, y \in \eta_t$ are adjacent if the shifted gauge bodies $B + x$ and $B + y$ overlap.

In the next sections we shall study the properties of the pair-connectedness function P_t of the RCM and the solution Q_t of the associated OZE. In particular, we shall show that P_t and Q_t are analytic and relate the coefficients of their series representation at 0. To do this properly we need to introduce the model in a more formal way. If the intensity t is positive then η_t can be (almost surely) represented as

$$\eta_t = \{X_i : i \in \mathbb{N}\},$$

where the X_i, $i \in \mathbb{N}$, are almost sure distinct random elements in $\mathbb{R}^d$. For $t = 0$, the Poisson process η_t has (almost surely) no points. Let $\mathbb{R}^{[2d]}$ denote the space of all sets $e \subset \mathbb{R}^d$ containing exactly two elements. Any $e \in \mathbb{R}^{[2d]}$ is a potential edge of the RCM. When equipped with the Hausdorff metric (see [15]) this space is a Borel subset of a complete separable metric space. Let '$<$' denote the strict lexicographic ordering on $\mathbb{R}^d$. Introduce independent random variables $U_{i,j}$, $i, j \in \mathbb{N}$, uniformly distributed on the unit interval $[0, 1]$ such that the double sequence $(U_{i,j})$ is independent of η_t. For $t > 0$,

$$\chi_t := \{(\{X_i, X_j\}, U_{i,j}) : X_i < X_j,\ i, j \in \mathbb{N}\} \tag{4.2}$$

is a point process on $\mathbb{R}^{[2d]} \times [0, 1]$. For $t = 0$, we let χ_t equal the zero measure. Note that η_t can be recovered from χ_t. For $t > 0$, we can define the RCM as a deterministic functional of χ_t by taking, for $i \neq j$ and $X_i < X_j$, the set $\{X_i, X_j\}$ as an edge of $\mathcal{G}$ if and only if $U_{i,j} \le \varphi(X_i - X_j)$.

Justified by assumption (4.1), we can introduce a measurable function $\varphi^* : \mathbb{R}^{[2d]} \to [0, 1]$ by

$$\varphi^*(e) := \varphi(y - x), \qquad e = \{x, y\} \in \mathbb{R}^{[2d]}.$$

If $\widetilde{\chi}$ is a point process on $\mathbb{R}^{[2d]} \times [0, 1]$, we can define a graph $\mathcal{G}(\widetilde{\chi}) := \mathcal{G}(\widetilde{\chi}) = (V(\widetilde{\chi}), E(\widetilde{\chi}))$ as follows. The vertex set is given by

$$V(\widetilde{\chi}) := \{e^-, e^+ : e \in \mathbb{R}^{[2d]}, \widetilde{\chi}(\{e\} \times [0, 1]) = 1\},$$

where e^- and e^+ are the points of $e \in \mathbb{R}^{[2d]}$. A set $e \in \mathbb{R}^{[2d]}$ belongs to the edge set $E(\widetilde{\chi})$ of this graph if and only if $\widetilde{\chi}(\{(e, u)\}) = 1$ for some $u \in [0, 1]$ with $u \le \varphi^*(e)$. In this notation our RCM is given as $\mathcal{G}(\chi_t)$. (For $t = 0$ this is the empty graph.) For $x \in V(\widetilde{\chi})$, we denote the cluster of x (the connected component of $\mathcal{G}(\widetilde{\chi})$) by $C(x, \widetilde{\chi})$. (For convenience we set $C(x, \widetilde{\chi}) := \{x\}$ for all other $x \in \mathbb{R}^d$.)

In the remaining part of this section we state a few fundamental results on the RCM that will be needed later but cannot be found in the literature. We extend the (double) sequence $(U_{i,j})_{i,j=1}^{\infty}$ featuring in (4.2) to a sequence $(U_{i,j})_{i,j=0}^{\infty}$ of independent random variables uniformly distributed on $[0, 1]$, independent of the Poisson process η_t. For $t > 0$, we then define a point process χ_t^0 on $\mathbb{R}^{[2d]} \times [0, 1]$ by

$$\chi_t^0 := \{(\{X_i, X_j\}, U_{i,j}) : X_i < X_j,\ i, j \in \mathbb{N}_0\}, \tag{4.3}$$

where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $X_0 := 0$. The graph $\mathcal{G}(\chi_t^0)$ can be interpreted as the RCM as seen from a *typical vertex* positioned at the origin. For $x \in \mathbb{R}^d$, we define

$$\chi_t^{0,x} := \{(\{X_i, X_j\}, U_{i,j}) : X_i < X_j,\ i, j \in \mathbb{N}_{-1}\}, \tag{4.4}$$

where $\mathbb{N}_{-1} := \mathbb{N}_0 \cup \{-1\}$, $X_{-1} := x$, and $(U_{i,j})_{i,j=-1}^{\infty}$ has similar properties as $(U_{i,j})_{i,j=0}^{\infty}$. In the $t = 0$ case, the point processes χ_t^0 and $\chi_t^{0,x}$ are defined to be the empty set.

For $k \in \mathbb{N}$, we let $[k] := \{1, 2, \dots, k\}$. For any $x_1, \dots, x_k \in \mathbb{R}^d$, we introduce a random graph $\Gamma(x_1, \dots, x_k)$ with vertex set $\{x_1, \dots, x_k\}$ by taking independent random variables $U_{i,j}$, $i, j \in [k]$, with the uniform distribution on $[0, 1]$ and by taking $\{x_i, x_j\}$ as an edge if $x_i < x_j$ and $U_{i,j} \le \varphi(x_i - x_j)$. This is just the RCM with a finite deterministic vertex set. The next result is a version of Proposition 6.2 of [9]. For the convenience of the reader we give a short proof.

Proposition 4.1. *Let $n \in \mathbb{N}_0$ and set $x_0 := 0$. Then*

$$\begin{aligned}\mathbb{P}(|C(0, \chi_t^0)| = n + 1) &= \frac{t^n}{n!} \int \mathbb{P}(\Gamma(x_0, \dots, x_n) \textit{ is connected}) \\ &\qquad \times \exp\biggl[-t \int \biggl(1 - \prod_{i=0}^{n} (1 - \varphi(y - x_i))\biggr) \,\mathrm{d}y\biggr] \,\mathrm{d}(x_1, \dots, x_n). \end{aligned} \tag{4.5}$$

In the $n = 0$ case, the right-hand side has to be read as $\exp(-t \int \varphi(y) \,\mathrm{d}y)$.

Proof. We assume that $n \ge 1$. (The $n = 0$ case is trivial.) We have $|C(0, \chi_t^0)| = n + 1$ if and only if there are n distinct points $x_1, \dots, x_n \in \eta_t$ such that $\mathcal{G}(\chi_t^0)$ restricted to those points is connected and none of the x_i is connected to a point in $\eta_t \setminus \{x_1, \dots, x_n\}$. Given η_t, these two events are (conditionally) independent and have respective probabilities $\mathbb{P}(\Gamma(x_0, \dots, x_n)$ is connected) and

$$\prod_{y \in \eta_t \setminus \{x_1, \dots, x_n\}} \prod_{i=0}^{n} (1 - \varphi(y - x_i)).$$

After conditioning, we obtain, from the multivariate Mecke equation (A.3),

$$\begin{aligned}\mathbb{P}(|C(0, \chi_t^0)| = n + 1) &= \frac{t^n}{n!} \int \mathbb{P}(\Gamma(x_0, \dots, x_n) \text{ is connected}) \mathbb{E} \prod_{y \in \eta_t} \prod_{i=0}^{n} (1 - \varphi(y - x_i)) \,\mathrm{d}(x_1, \dots, x_n).\end{aligned}$$

Using the well-known formula for the characteristic functional of η_t (see, e.g. [8, Chapter 3]), we obtain the asserted formula (4.5). □

Next we need to discuss a Margulis–Russo-type formula for $\chi_t^{0,x}$. This formula provides a power series expansion of expectations of functions of $\chi_t^{0,x}$. Adding just two points $0, x$ is enough for our purposes. It would be no problem to extend the result to a random connection model with any fixed number of points added. Let $n \in \mathbb{N}$ and $\mathbb{N}_{-n-1} := \mathbb{N} \cup \{0, -1, \dots, -n-1\}$. Extend the (double) sequence $(U_{i,j})_{i,j=1}^{\infty}$ featuring in (3.2) to a sequence $(U_{i,j})_{i,j \in \mathbb{N}_{-n-1}}$ of independent random variables uniformly distributed on $[0, 1]$, independent of the Poisson process η_t. Let $x_0, x_{n+1} \in \mathbb{R}^d$ and $\boldsymbol{x} = (x_1, \dots, x_n) \in (\mathbb{R}^d)^n$. For $J \subset [n]$, we define $\boldsymbol{x}_J := (x_i)_{i \in J}$ and

$$\chi_t^{x_0, \boldsymbol{x}_J, x_{n+1}} := \{(\{X_i, X_j\}, U_{i,j}) \colon vX_i < X_j,\ i, j \in \mathbb{N}_J\}, \tag{4.6}$$

where $\mathbb{N}_J := \mathbb{N} \cup \{0, -n-1\} \cup \{-i \colon i \in J\}$ and $(X_0, \dots, X_{-n-1}) := (x_0, \dots, x_{n+1})$. In the $t = 0$ case, the indices i, j are restricted to $\{-i \colon i \in J\}$. Similarly, we define the

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point process $\chi_t^{x_0,\boldsymbol{x}_J}$. For $J = \varnothing$, we set $\chi_t^{x_0} := \chi_t^{x_0,\boldsymbol{x}_\varnothing}$. For $x_{n+1} := x$ and $J = \varnothing$, the point process $\chi_t^{0,\boldsymbol{x}_\varnothing,x}$ has the same distribution as $\chi_t^{0,x}$ given by (4.4). Let $f\colon \boldsymbol{N}(\mathbb{R}^{[2d]} \times [0,1]) \to \mathbb{R}$ be measurable and fix some $x \in \mathbb{R}^d$. Define $F_t := f(\chi_t^{0,x})$ and

$$\Delta_{\boldsymbol{x}}^n F_t := \sum_{J \subset [n]} (-1)^{n-|J|} f(\chi_t^{0,\boldsymbol{x}_J,x}), \qquad \boldsymbol{x} \in (\mathbb{R}^d)^n. \tag{4.7}$$

We say that $f\colon \boldsymbol{N}(\mathbb{R}^{[2d]} \times [0,1]) \to \mathbb{R}$ is *determined* by a compact subset $W \subset \mathbb{R}^d$ if $f(\mu) = f(\mu_W)$ for all $\mu \in \boldsymbol{N}(\mathbb{R}^{[2d]} \times [0,1])$, where

$$\mu_W := \{(e,u) \in \mu\colon e \subset W\}, \tag{4.8}$$

that is, if the value of f only depends on the edges with endpoints in W.

Theorem 4.1. *Let $f\colon \boldsymbol{N}(\mathbb{R}^{[2d]} \times [0,1]) \to \mathbb{R}$ be measurable and let $x \in \mathbb{R}^d$. Assume that f is determined by a compact set $W \subset \mathbb{R}^d$ with $\{0,x\} \subset W$. Let $s \geq 0$ and $t \geq -s$ be such that $\mathbb{E}|F_{s+|t|}| < \infty$, where $F_t := f(\chi_t^{0,x})$. Then*

$$\mathbb{E}F_{s+t} = \mathbb{E}F_s + \sum_{n=1}^{\infty} \frac{t^n}{n!} \int_{W^n} \mathbb{E}\Delta_{\boldsymbol{x}}^n F_s \,\mathrm{d}\boldsymbol{x}. \tag{4.9}$$

Proof. First we recall the Poisson process analogue of the Margulis–Russo formula to be found in [10] and for a general phase space and more general integrability assumptions in [7] and [8]. Let $f\colon \boldsymbol{N}(\mathbb{R}^d) \to \mathbb{R}$ be measurable, $n \in \mathbb{N}$, and $\boldsymbol{x} = (x_1,\dots,x_n) \in (\mathbb{R}^d)^n$. Then we define a measurable function $D_{\boldsymbol{x}}^n f\colon \boldsymbol{N}(\mathbb{R}^d) \to \mathbb{R}$ by

$$D_{\boldsymbol{x}}^n f(\mu) := \sum_{J \subset [n]} (-1)^{n-|J|} f(\mu \cup \{x_j\colon j \in J\}).$$

Assume now that there is a compact set $W \subset \mathbb{R}^d$ such that $f(\mu)$ depends for each $\mu \in \boldsymbol{N}(\mathbb{R}^d)$ only on the restriction of μ to W. Then we have, for all $s \geq 0$ and $t \geq -s$,

$$\mathbb{E}f(\eta_{s+t}) = \mathbb{E}f(\eta_s) + \sum_{n=1}^{\infty} \frac{t^n}{n!} \int_{W^n} (\mathbb{E}D_{\boldsymbol{x}}^n f(\eta_s)) \,\mathrm{d}\boldsymbol{x}, \tag{4.10}$$

provided that $\mathbb{E}|f(\eta_{s+|t|})| < \infty$.

For $\mu \in \boldsymbol{N}(\mathbb{R}^d)$ and $x_1,\dots,x_k \in \mathbb{R}^d$, $k \in \mathbb{N}$, we define $\mu^{x_1,\dots,x_k} := \mu \cup \{x_1,\dots,x_k\}$. There is a probability kernel K from $\boldsymbol{N}(\mathbb{R}^d)$ to $\boldsymbol{N}(\mathbb{R}^{[2d]} \times [0,1])$ such that, for all $r \geq 0$,

$$\mathbb{P}((\eta_r, \chi_r) \in \cdot) = \mathbb{E} \int \mathbf{1}\{(\eta_r, \psi) \in \cdot\}\, K(\eta_r, \mathrm{d}\psi), \tag{4.11}$$

and, for any $x_0 \in \mathbb{R}^d$, $n \geq 0$, and $\boldsymbol{x} = (x_1,\dots,x_n) \in (\mathbb{R}^d)^n$,

$$\mathbb{P}((\eta_t^{x_0,\boldsymbol{x}}, \chi_t^{x_0,\boldsymbol{x}}) \in \cdot) = \mathbb{E} \int \mathbf{1}\{(\eta_t^{x_0,\boldsymbol{x}}, \psi) \in \cdot\}\, K(\eta_t^{x_0,\boldsymbol{x}}, \mathrm{d}\psi). \tag{4.12}$$

Define a measurable function $f^*\colon \boldsymbol{N}(\mathbb{R}^d) \to \mathbb{R}$ by

$$f^*(\mu) := \int f(\psi)\, K(\mu^{0,x}, \mathrm{d}\psi).$$

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By the triangle inequality and (4.12) for $n = 0$,

$$\mathbb{E}|f^*(\eta_{s+|t|})| \le \mathbb{E}\int |f(\psi)|\, K(\eta^{0,x}_{s+|t|}, \mathrm{d}\psi) = \mathbb{E}|F_{s+|t|}| < \infty.$$

The properties of the kernel K imply that

$$\begin{aligned} f^*(\mu_W) &= \int f(\psi_W) K(\mu_W^{0,x}, \mathrm{d}\psi) \\ &= \int f(\psi_W) K(\mu^{0,x}, \mathrm{d}\psi) \\ &= \int f(\psi) K(\mu^{0,x}, \mathrm{d}\psi) \\ &= f^*(\mu). \end{aligned}$$

We can now apply (4.10) with f^* to obtain

$$\mathbb{E} f^*(\eta_{s+t}) = \mathbb{E} f^*(\eta_s) + \sum_{n=1}^{\infty} \frac{t^n}{n!} \int_{W^n} \mathbb{E} D^n_{\boldsymbol{x}} f^*(\eta_s)\, \mathrm{d}\boldsymbol{x}.$$

By (4.12), we have $\mathbb{E} f^*(\eta_{s+t}) = \mathbb{E}F_{s+t}$ and $\mathbb{E} f^*(\eta_s) = \mathbb{E}F_s$. Furthermore,

$$\begin{aligned} \mathbb{E} D^n_{\boldsymbol{x}} f^*(\eta_s) &= \sum_{J \subset [n]} (-1)^{n-|J|} \mathbb{E} f^*(\eta_s \cup \{x_j : j \in J\}) \\ &= \sum_{J \subset [n]} (-1)^{n-|J|} \mathbb{E} \int f(\psi)\, K(\eta_s^{0,x} \cup \{x_j : j \in J\}, \mathrm{d}\psi) \\ &= \sum_{J \subset [n]} (-1)^{n-|J|} \mathbb{E} f(\chi_s^{0,\boldsymbol{x}_J,x}). \end{aligned}$$

In view of definition (4.7), we obtain the assertion. □

We also need the following version of Proposition 4.1. The proof is omitted.

Proposition 4.2. *Let $n \in \mathbb{N}_0$ and $x \in \mathbb{R}^d$. Then*

$$\begin{aligned} &\mathbb{P}(x \in C(0, \chi_t^{0,x}), |C(0, \chi_t^{0,x})| = n+2) \\ &\quad = \frac{t^n}{n!} \int \mathbb{P}(\Gamma(x_0, \dots, x_{n+1}) \text{ is connected}) \\ &\qquad\qquad \times \exp\bigg[-t \int \bigg(1 - \prod_{i=0}^{n+1} (1 - \varphi(y - x_i))\bigg)\, \mathrm{d}y\bigg]\, \mathrm{d}(x_1, \dots, x_n), \end{aligned}$$

where $x_0 := 0$ and $x_{n+1} := x$.

5. The OZE for the random connection model

In this section we consider Poisson processes η_t with intensity $t \ge 0$ and the associated RCM $\mathcal{G}(\chi_t)$ as introduced in the previous section. We assume that

$$0 < m_\varphi < \infty,$$

where $m_\varphi := \int \varphi(x)\, \mathrm{d}x$.

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The *critical intensity* is given by

$$t_c := \sup\{t \geq 0 \colon \mathbb{P}(|C(0, \chi_t^0)| < \infty) = 0\}.$$

For the Gilbert graph (in fact for general Boolean models) it was proved in [4] that $0 < t_c < \infty$. The same is true for the more general RCM; see [9, Theorem 6.1]. By [9, Theorem 6.2], we have

$$t_c = \sup\{t \geq 0 \colon \mathbb{E}|C(0, \chi_t^0)| < \infty\}. \tag{5.1}$$

We need to consider another critical intensity, namely

$$t_* := \sup\{t \geq 0 \colon \mathbb{E}\exp(z|C(0, \chi_t^0)|) < \infty \text{ for some } z > 0\}.$$

Clearly, we have $t_* \leq t_c$. For the Gilbert graph it is well known that $t_* = t_c$. (While an early proof for fixed convex and symmetric gauge bodies can be found in [14], [19] covers the case of a random but deterministically bounded gauge body.) We are not aware of a similar result for the RCM. However, one can show that

$$t_* \geq m_\varphi^{-1}.$$

This is due to the fact that, for $t < m_\varphi^{-1}$, the number of points in the cluster of the origin can be dominated by the total progeny of a subcritical Galton–Watson process with a Poisson offspring distribution with mean $tm_\varphi < 1$; see the proof of Theorem 6.1 of [9]. It is well known that this progeny has exponential moments; see [12].

By (A.3), the pair correlation function ρ_t of η_t satisfies $\rho_t \equiv 1$, so that the two-point Palm probability measures $\mathbb{P}_{\eta_t}^{x,y}$ of η_t are well defined. They are given by the following lemma. Recall the definition (4.3) of χ_t^0 and the definition (4.6) of $\chi_t^{x,y}$.

Lemma 5.1. *We have* $\mathbb{P}_{\eta_t}^0(\chi_t \in \cdot) = \mathbb{P}(\chi_t^0 \in \cdot)$. *Moreover, the Palm probability* $\mathbb{P}_{\eta_t}^{x,y}$ *can be chosen such that*

$$\mathbb{P}_{\eta_t}^{x,y}(\chi_t \in \cdot) = \mathbb{P}(\chi_t^{x,y} \in \cdot), \qquad x, y \in \mathbb{R}^d. \tag{5.2}$$

Proof. We prove the second formula. Let $f \colon \mathbf{N}(\mathbb{R}^{[2d]} \times [0, 1]) \times \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty)$ be measurable. Then, from (4.11) and the Mecke equation (A.3), we obtain

$$\begin{aligned}
\mathbb{E} \sum_{x,y\in\eta_t}^{\neq} f(\chi_t, x, y) &= \mathbb{E} \sum_{x,y\in\eta_t}^{\neq} \int f(\psi, x, y)\, K(\eta_t, \mathrm{d}\psi) \\
&= t^2 \mathbb{E} \iint f(\psi, x, y) K(\eta_t^{x,y}, \mathrm{d}\psi)\, \mathrm{d}(x, y) \\
&= t^2 \mathbb{E} \int f(\chi_t^{x,y}, x, y)\, \mathrm{d}(x, y),
\end{aligned}$$

where we have used (4.12) to obtain the final identity. Comparing this with (2.3) (and using the fact that the pair correlation function ρ_t of η_t satisfies $\rho_t \equiv 1$), shows that (5.2) holds for almost every (x, y) (with respect to Lebesgue measure on $\mathbb{R}^d \times \mathbb{R}^d$). □

By Lemma 5.1, the pair-connectedness function P_t of the RCM $\mathcal{G}(\chi_t)$ is given by $P_t(x, y) = P_t(y - x)$, where

$$P_t(x) := \mathbb{P}(x \in C(0, \chi_t^{0,x})).$$

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Theorem 5.1. *Let $t < t_c$. Then there is a unique $Q_t \in L^1 \cap L^\infty$ such that*

$$P_t = Q_t + tQ_t * P_t.$$

Proof. We wish to apply Theorem 3.1. For any $x \in \mathbb{R}^d$, we define $\tau(x) := x$ if x is not a member of a finite cluster in $\mathcal{G}(\chi_t)$. Otherwise, we define $\tau(x)$ as the lexicographic minimum of the cluster $C(x, \chi_t)$. Then we have, almost surely, $C(x) = C(x, \chi_t)$ for all $x \in \eta_t$, where $C(x)$ is given by (3.2). Since $t < t_c$, the integrability assumption (3.4) follows from (5.1). Since the factorial moment measures of η_t coincide with Lebesgue measure (see (A.3)), assumption (3.8) follows from Remark 3.1. □

Proposition 5.1. *We have*

$$0 \le \int Q_t(x)\,\mathrm{d}x < t^{-1}, \qquad 0 < t < t_c,$$

and

$$\mathbb{E}|C(0, \chi_t^0)| = \left(1 - t\int Q_t(x)\,\mathrm{d}x\right)^{-1}. \tag{5.3}$$

Proof. The two assertions follow from Proposition 3.3 and Lemma 5.1. □

Remark 5.1. It is a fair conjecture that $\lim_{t\uparrow t_c} \mathbb{E}|C(0, \chi_t^0)| = \infty$, but we have not found this in the literature. Under this hypothesis, (5.3) would show that

$$t_c \lim_{t\to t_c-} \int Q_t(x)\,\mathrm{d}x = 1.$$

In what follows, we consider a measurable function $g\colon \mathbf{N}(\mathbb{R}^d) \to \mathbb{R}$ and fix some $x \in \mathbb{R}^d$. We study the function $t \mapsto \mathbb{E}g(C(0, \chi_t^{0,x}))$. The results will imply that $t \mapsto P_t(x)$ and $t \mapsto Q_t(x)$ are analytic functions on $[0, t_*)$. We assume that, for all $\varepsilon > 0$, there is an $n_1 = n_1(\varepsilon) \in \mathbb{N}$ such that

$$|g(\mu)| \le \exp(\varepsilon \max\{\mu(\mathbb{R}^d), n_1\}). \tag{5.4}$$

Theorem 5.2. *Suppose that $g\colon \mathbf{N}(\mathbb{R}^d) \to \mathbb{R}$ satisfies (5.4) and let $x \in \mathbb{R}^d$. Then the function $t \mapsto \mathbb{E}g(C(0, \chi_t^{0,x}))$ is analytic on $[0, t_*)$. The expansion at $s \in [0, t_*)$ is given by (4.9) with $F_t := g(C(0, \chi_t^{0,x}))$.*

For the proof of Theorem 5.2, we derive some preliminary results that might be of independent interest. Let $t \in [0, t^*)$ and define

$$G(x, t) := \mathbb{E}g(C(0, \chi_t^{0,x})).$$

We take a compact set $W \subset \mathbb{R}^d$ with $\{0, x\} \subset W$ and approximate the function G with

$$G_W(x, t) := \mathbb{E}g(C(0, \chi_{t,W}^{0,x})),$$

where $\chi_{t,W}^{0,x} := (\chi_t^{0,x})_W$; see (4.8). Note that $\mu \mapsto g(C(0, \mu_W))$ is determined by W. Let $z > 0$ be such that $\mathbb{E}\exp(2z|C(0, \chi_t^0)|) < \infty$. Since

$$|C(0, \chi_t^{0,x})| \le |C(0, \chi_t^0)| + |C(x, \chi_t^x)| \tag{5.5}$$

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and $C(x, \chi_t^x) \overset{\mathrm{D}}{=} C(0, \chi_t^0) + x$, we have

$$\mathbb{E} \exp(z|C(0, \chi_t^{0,x})|) < \infty.$$

Choosing $\varepsilon = z$ in assumption (5.4), we obtain

$$\mathbb{E}|g(C(0, \chi_{t,W}^{0,x}))| \le \mathbb{E} \exp(z \max\{|C(0, \chi_{t,W}^{0,x})|, n_1\}) \le \mathbb{E} \exp(z \max\{|C(0, \chi_t^{0,x})|, n_1\}) < \infty.$$

Therefore, we can apply Theorem 4.1 to obtain

$$G_W(x, t) = \mathbb{E} g(C(0, \chi_{s,W}^{0,x})) + \sum_{n=1}^{\infty} (t - s)^n g_{W,n}(x, s), \qquad s, t < t_*, \tag{5.6}$$

where

$$g_{W,n}(x, s) := \frac{1}{n!} \int_{W^n} \mathbb{E}\biggl[\sum_{J \subset [n]} (-1)^{n-|J|} g(C(0, \chi_{s,W}^{0,\boldsymbol{x}_J,x})) \biggr] \,\mathrm{d}\boldsymbol{x},$$

and $\chi_{s,W}^{0,\boldsymbol{x}_J,x} := (\chi_s^{0,\boldsymbol{x}_J,x})_W$. We use this definition for all Borel sets $W \subset \mathbb{R}^d$.

To bound the coefficients $g_{W,n}(x, t)$, we use the following integral inequality. Recall that $\Gamma(0, x_1, \dots, x_n)$ denotes an RCM with vertex set $\{0, x_1, \dots, x_n\}$.

Lemma 5.2. *Let $n \in \mathbb{N}$. Then*

$$\int \mathbb{P}(\Gamma(0, x_1, \dots, x_n) \textit{ is connected}) \,\mathrm{d}(x_1, \dots, x_n) \le n!\, m_\varphi^n \mathrm{e}^{n+1}.$$

Furthermore, we have, for any $x \in \mathbb{R}^d$,

$$\int \mathbb{P}(\Gamma(0, x_1, \dots, x_n, x) \textit{ is connected}) \,\mathrm{d}(x_1, \dots, x_n) \le n!\, m_\varphi^n \mathrm{e}^{n+2}.$$

Proof. We prove the second inequality only. For all $a_0, \dots, a_{n+1} \in [0, 1]$, we have the inequality

$$1 - \prod_{i=0}^{n+1} (1 - a_i) \le \sum_{i=0}^{n+1} a_i.$$

Taking $t > 0$ and defining $x_0 := 0$, we therefore obtain, from Proposition 4.2,

$$\begin{aligned} 1 &\ge \frac{t^n}{n!} \int \mathbb{P}(\Gamma(x_0, \dots, x_n, x) \text{ is connected}) \exp\biggl(-t \sum_{i=0}^{n+1} \int \varphi(y - x_i) \,\mathrm{d}y \biggr) \,\mathrm{d}(x_1, \dots, x_n) \\ &\ge \frac{t^n}{n!} \int \mathbb{P}(\Gamma(x_0, \dots, x_n, x) \text{ is connected}) \exp(-t(n + 2) m_\varphi) \,\mathrm{d}(x_1, \dots, x_n). \end{aligned}$$

Choosing $t = m_\varphi^{-1}$ yields the asserted inequality. □

There is a qualitative difference between the study of analyticity of G at $s = 0$ and $s > 0$. In fact, condition (5.4) can be relaxed slightly for $s = 0$.

Lemma 5.3. *Let $n \in \mathbb{N}$ and assume that there is a constant $c \geq 1$ such that*

$$|g(\mu)| \leq c^{\mu(\mathbb{R}^d)}, \qquad \mu \in \boldsymbol{N}(\mathbb{R}^d), \qquad \mu(\mathbb{R}^d) \leq n+1. \tag{5.7}$$

Let $x \in \mathbb{R}^d$ and $W \subset \mathbb{R}^d$ be a Borel set such that $\{0, x\} \subset W$. Then

$$|g_{W,n}(0,x)| \leq c(c+1)(1+e)((1+c)m_\varphi e)^n.$$

Proof. Let $x_0 := 0$ and $x_{n+1} := x$. Take $\boldsymbol{x} = (x_1, \dots, x_n) \in W^n$. We recall that, for $s = 0$, the point process η_s is the zero measure and $\Gamma(x_0, \dots, x_{n+1}) = \mathcal{G}(\chi_0^{x_0,\boldsymbol{x},x_{n+1}})$ is the RCM with vertex set $\{x_0, \dots, x_{n+1}\}$. Let $i \in [n]$ be such that x_0 and x_i are not connected by a path in $\Gamma(x_0, \dots, x_{n+1})$. Then we have, for any $J \subset [n] \setminus \{i\}$,

$$(-1)^{n-|J|} g(C(0, \chi_0^{x_0,\boldsymbol{x}_J,x_{n+1}})) + (-1)^{n-|J\cup\{i\}|} g(C(0, \chi_0^{x_0,\boldsymbol{x}_{J\cup\{i\}},x_{n+1}})) = 0,$$

since the cluster of 0 is the same in both summands. If, on the other hand, x_0 and x_i are connected in $\Gamma(x_0, \dots, x_{n+1})$ for each $i \in [n]$, then either $\Gamma(x_0, \dots, x_n)$ is connected (and x is not connected to any of the points $x_0, \dots, x_n$) or $\Gamma(x_0, \dots, x_{n+1})$ is connected. Hence, we have

$$|g_{W,n}(0,x)| \leq \frac{1}{n!} \mathbb{E} \int h(x_1, \dots, x_n, x) \sum_{J \subset [n]} |g(C(0, \chi_0^{x_0,\boldsymbol{x}_J,x_n}))| \, \mathrm{d}\boldsymbol{x},$$

where

$$h(x_1, \dots, x_n, x) := \mathbf{1}\{\Gamma(x_0, \dots, x_n) \text{ is connected}\} + \mathbf{1}\{\Gamma(x_0, \dots, x_{n+1}) \text{ is connected}\}.$$

Our assumption (5.7) and the binomial formula imply that

$$|g_{W,n}(0,x)| \leq \frac{c(c+1)^n}{n!} \int h(x_1, \dots, x_n, x) \, \mathrm{d}(x_1, \dots, x_n).$$

An application of Lemma 5.2 concludes the proof. □

In the following it is convenient to introduce a function $c_V \colon [0,\infty) \to [0,\infty)$ with $c_V(t) > 0$ for $t < t_*$ satisfying

$$\mathbb{P}(|C(0, \chi_t^{0,x})| = n) \leq \mathrm{e}^{-c_V(t)n}, \qquad n \in \mathbb{N},\ x \in \mathbb{R}^d,\ t < t_*. \tag{5.8}$$

By the definition of t_* and (5.5), such a function exists. Next, we bound $g_n(t, W)$ for $t > 0$.

Lemma 5.4. *Let $t \in (0, t_*)$ and $x \in \mathbb{R}^d$. Assume that (5.4) holds. Then there is an $n_1(t) \in \mathbb{N}$ such that, for all $n > n_1(t)$ and all Borel sets $W \subset \mathbb{R}^d$ with $\{0, x\} \subset W$,*

$$|g_{W,n}(x,t)| \leq \frac{\mathrm{e}^{-c_V(t)/2}}{1 - \mathrm{e}^{-c_V(t)/2}} \left(\frac{2\mathrm{e}^{-c_V(t)/2}}{t(1 - \mathrm{e}^{-c_V(t)/2})} \right)^n.$$

Proof. As before, we set $x_0 := 0$ and $x_{n+1} := x$. Let $n \in \mathbb{N}$. With the same argument as in the proof of Lemma 5.3, we conclude that

$$|g_{W,n}(x,t)| \leq \frac{1}{n!} \int_{W^n} \mathbb{E}\Bigg[\mathbf{1}\{x_i \in C(0, \chi_{t,W}^{x_0,\boldsymbol{x},x_{n+1}}) \text{ for all } i \in [n]\} \times \sum_{J \subset [n]} |g(C(0, \chi_{t,W}^{x_0,\boldsymbol{x}_J,x_{n+1}}))| \Bigg] \mathrm{d}\boldsymbol{x}. \tag{5.9}$$

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Setting $\varepsilon := c_V(t)/2$ and using (5.4), we find an $n_1 \in \mathbb{N}$ such that

$$\sum_{J\subset[n]} |g(C(0, \chi_{t,W}^{x_0,\boldsymbol{x}_J,x_{n+1}}))| \le \sum_{J\subset[n]} \exp\biggl(\frac{c_V(t)}{2}\max\{n_1, |C(0, \chi_{t,W}^{x_0,\boldsymbol{x}_J,x_{n+1}})|\}\biggr)$$
$$\le 2^n \exp\biggl(\frac{c_V(t)}{2}\max\{n_1, |C(0, \chi_{t,W}^{x_0,\boldsymbol{x},x_{n+1}})|\}\biggr).$$

Inserting this in (5.9) and using (4.12) yields

$$|g_{W,n}(x,t)| \le \frac{2^n}{n!}\mathbb{E}\iint \mathbf{1}\{x_i \in C(0,\psi_W) \text{ for all } i \in [n]\}$$
$$\times \exp\biggl(\frac{c_V(t)}{2}\max\{n_1, |C(0,\psi_W)|\}\biggr)K(\eta_t^{x_0,\boldsymbol{x},x_{n+1}}, \mathrm{d}\psi)\,\mathrm{d}\boldsymbol{x}.$$

The Mecke equation (A.3) yields

$$|g_{W,n}(x,t)| \le \frac{2^n}{t^n n!}\mathbb{E}\sum_{x_1,\dots,x_n\in\eta_t}^{\ne}\int \mathbf{1}\{x_i \in C(0,\psi_W) \text{ for all } i \in [n]\}$$
$$\times \exp\biggl(\frac{c_V(t)}{2}\max\{n_1, |C(0,\psi_W)|\}\biggr)K(\eta_t^{0,x}, \mathrm{d}\psi)$$
$$= \frac{2^n}{t^n n!}\mathbb{E}\sum_{x_1,\dots,x_n\in\eta_t}^{\ne}\mathbf{1}\{x_i \in C(0,\chi_{t,W}^{0,x}) \text{ for all } i \in [n]\}$$
$$\times \exp\biggl(\frac{c_V(t)}{2}\max\{n_1, |C(0,\chi_{t,W}^{0,x})|\}\biggr),$$

where we have used (4.12) to achieve the final identity. Therefore,

$$|g_{W,n}(x,t)| \le \frac{2^n}{t^n n!}\sum_{k=1}^{\infty}\mathbb{E}\mathbf{1}\{|C(0,\chi_t^{0,x})| = k\}\exp\biggl(\frac{c_V(t)}{2}\max\{n_1, |C(0,\chi_{t,W}^{0,x})|\}\biggr)$$
$$\times \sum_{x_1,\dots,x_n\in\eta_t}^{\ne}\mathbf{1}\{x_i \in C(0,\chi_t^{0,x}) \text{ for all } i \in [n]\}.$$

On the event $\{|C(0,\chi_t^{0,x})| = k\}$, the above integral simplifies to $(k-1)\cdots(k-n)$. Hence,

$$|g_{W,n}(x,t)| \le \frac{2^n}{t^n n!}\sum_{k=n+1}^{\infty}\exp\biggl(\frac{c_V(t)}{2}\max\{n_1,k\}\biggr)(k-1)\cdots(k-n)\mathbb{P}(|C(0,\chi_t^{0,x})| = k).$$

Finally, we apply (5.8) and use the well-known formula for the factorial moment of the geometric distribution to obtain, for $n > n_1$,

$$|g_{W,n}(x,t)| \le \frac{2^n}{t^n n!}\sum_{k=n+1}^{\infty}\exp\biggl(-\frac{c_V(t)}{2}k\biggr)(k-1)\cdots(k-n)$$
$$= \frac{\mathrm{e}^{-c_V(t)/2}}{1-\mathrm{e}^{-c_V(t)/2}}\biggl(\frac{2\mathrm{e}^{-c_V(t)/2}}{t(1-\mathrm{e}^{-c_V(t)/2})}\biggr)^n,$$

as asserted. □

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Proof of Theorem 5.2. Let W_k, $k \in \mathbb{N}$, be a sequence of compact sets with union $\mathbb{R}^d$. Since $\mathbb{P}(|C(0, \chi_t^{0,x})| < \infty) = 1$, we have

$$\lim_{k\to\infty} g(C(0, \chi_{t,W_k}^{0,x})) = g(C(0, \chi_t^{0,x})), \qquad \mathbb{P}\text{-almost surely.}$$

By (5.4),

$$|g(C(0, \chi_{t,W_k}^{0,x}))| \le \exp(c_V(t) \max\{|C(0, \chi_{t,W_k}^{0,x})|, n_1\}) \exp(c_V(t) \max\{|C(0, \chi_t^{0,x},)|, n_1\}).$$

From (5.8), it follows that

$$\mathbb{E} \exp(c_V(t)|C(0, \chi_t^{0,x})|) < \infty.$$

Dominated convergence implies that

$$\lim_{k\to\infty} G_{W_k}(x, t) = \lim_{k\to\infty} \mathbb{E} g(C(0, \chi_{t,W_k}^{0,x})) = \mathbb{E} g(C(0, \chi_t^{0,x})) = G(x, t).$$

Similarly, dominated convergence implies that, for any $n \in \mathbb{N}$,

$$\lim_{k\to\infty} g_{W_k,n}(x, s) = g_{\mathbb{R}^d,n}(x, s).$$

Now we use the series representation (5.6) for $W = W_k$. Using Lemmas 5.3 and 5.4, we can apply dominated convergence to show that

$$\lim_{k\to\infty} G_{W_k}(t) = G(s) + \sum_{n=1}^{\infty} (t-s)^n \lim_{k\to\infty} g_{W_k,n}(x, s) = G(s) + \sum_{n=1}^{\infty} (t-s)^n g_{\mathbb{R}^d,n}(x, s)$$

holds for all t in some open neighborhood of $s \in [0, t_c)$. This completes the proof. □

Note that, due to the relaxed growth bound of Lemma 5.3 in comparison to (5.4), any functional that grows exponentially in the size of the cluster of the origin is analytic at least in $s = 0$. Lemmas 5.3 and 5.4 also yield a lower bound on the radius of convergence of the series representation of $G(t)$, which is rather small though.

Theorem 5.2 shows that the pair-connectedness function and the expected cluster size are analytic functions on the whole interval $[0, t_*)$. In particular, given $x \in \mathbb{R}^d$, every $s \in [0, t_*)$ has a neighborhood $U(s)$ such that

$$P_t(x) = \sum_{n=0}^{\infty} (t-s)^n p_n(x, s), \qquad t \in U(s),$$

where $p_0(x, s) := P_s(x)$ and, for $n \in \mathbb{N}$,

$$p_n(x, s) := \frac{1}{n!} \int \mathbb{E} \sum_{J \subset [n]} (-1)^{n-|J|} \mathbf{1}\{x \in C(0, \chi_s^{0,\boldsymbol{x}_J,x})\} \, \mathrm{d}\boldsymbol{x}. \tag{5.10}$$

We summarize the integrability properties of the coefficients p_n in the following corollary.

Corollary 5.1. *For any $n \in \mathbb{N}_0$ and $t \in [0, t_*)$, there are constants $c_1(t)$, $c_2(t)$ such that*

$$\|p_n(\cdot, t)\|_\infty \le c_1(t)c_2(t)^n, \qquad \|p_n(\cdot, t)\|_1 \le c_1(t)c_2(t)^n.$$

Moreover, for any $s \in [0, t_*)$, there is a neighborhood $U(s)$ such that

$$P_t(\cdot) = \sum_{n=0}^{\infty} (t-s)^n p_n(\cdot, t), \qquad t \in U(s), \tag{5.11}$$

where the convergence holds in L^1 and L^∞.

Proof. In the proofs of Lemma 5.3 and 5.4, we observed that the bounds on g_n only depend on the growth bound (5.4), which immediately yields the bound of $\|p_n(\cdot, t)\|_\infty$.

From the arguments in the proof of Lemma 5.3, we have

$$\int_{\mathbb{R}^d} |p_n(x, 0)| \, \mathrm{d}x \le \frac{2^n}{n!} \int_{(\mathbb{R}^d)^{n+1}} \mathbf{1}\{\Gamma(0, x_1, \ldots, x_n, x) \text{ is connected}\} \, \mathrm{d}(x_1, \ldots, x_n, x),$$

which can be bounded using Lemma 5.2. The bound on $\|p_n(\cdot, t)\|_1$ with $t > 0$ can be derived in a similar way. It is clear that these bounds imply the L^1 and L^∞ convergence of the sum in (5.11) for t in a neighborhood $U(s)$ of s. □

With a good understanding of the analyticity of P_t, we are now able to show similar results for the solution Q_t of the OZE. We will write f^{*n} for an n-fold convolution of the function f with itself, that is, $f^{*(n+1)} := f^{*n} * f$ for $n \in \mathbb{N}$ and $f^{*1} := f$. In the same spirit, we define $*_{k=a}^{b} f_k := f_b * (*_{k=a}^{b-1} f_k)$ and $*_{k=a}^{a} f_k := f_a$ for $a < b \in \mathbb{Z}$ and functions $f_a, \ldots, f_b$.

Proposition 5.2. *If $t \ge 0$ is such that $\mathbb{E}|C(0, \chi_t^0)| < 2$ then*

$$Q_t = \sum_{n=0}^{\infty} (-t)^n P_t^{*(n+1)} \tag{5.12}$$

in L^1 and L^∞. Moreover, for any $s \in [0, t_*)$, there is a neighborhood $U(s)$ and functions $q_n(\cdot, t_*)$ such that

$$Q_t(\cdot) = \sum_{n=0}^{\infty} (t-s)^n q_n(\cdot, s), \qquad t \in U(s) \tag{5.13}$$

in L^1 and L^∞. The coefficients can be recursively determined by the solvable equations

$$q_0(\cdot, s) + s p_0(\cdot, s) * q_0(\cdot, s) = p_0(\cdot, s), \tag{5.14}$$

$$q_n(\cdot, s) + s q_n(\cdot, s) * p_0(\cdot, s) = p_n(\cdot, s) - \sum_{k=1}^{n} q_{n-k}(\cdot, s) * (p_{k-1}(\cdot, s) + s p_k(\cdot, s)). \tag{5.15}$$

Proof. From $\mathbb{E}|C(0, \chi_t^0)| < 2$ and (3.5), we obtain $t\|P_t\|_1 < 1$. By (3.6) and (3.7), we have

$$\|P_t^{*k+1}\|_\infty \le \|P_t\|_1^k \|P_t\|_\infty, \quad k \in \mathbb{N}, \qquad \|P_t^{*k+1}\|_1 \le \|P_t\|_1^{k+1}, \quad k \in \mathbb{N},$$

and, hence, the convergence of the right-hand side of (5.12) in L^1 and L^∞. A simple calculation shows that (5.12) solves the OZE.

To prove the second part of the claim, we start by solving (5.14) and (5.15) for $q_0(\cdot, s)$ and $q_n(\cdot, s)$, respectively. From the proof of Theorem 3.1, we know that there is a function $f \in L^1$

such that $(\delta_0 + sp_0(\cdot, s)) * (\delta_0 + sf) = \delta_0$. Hence, (5.14) and (5.15) are equivalent to

$$\begin{aligned} q_0(\cdot, s) &= (\delta_0 + sf) * p_0(\cdot, s), \\ q_n(\cdot, s) &= (\delta_0 + sf) * \left(p_n(\cdot, s) - \sum_{k=1}^{n} q_{n-k}(\cdot, s) * (p_{k-1}(\cdot, s) + sp_k(\cdot, s))\right). \end{aligned} \tag{5.16}$$

This implies that the q_n can be recursively determined.

In the next step, we show that the series in (5.13) converges. We fix s and write p_n for $p_n(\cdot, s)$ and q_n for $q_n(\cdot, s)$. We choose $p, c \in \mathbb{R}$ such that $\max\{\|p_n\|_1, \|p_n\|_\infty\} \le p^n$ for all $n \in \mathbb{N}$ as well as $\max\{\|p_0\|_1, \|q_0\|_1, \|\delta_0 + sf\|_1\} \le c$. This is possible due to Corollary 5.1 and Theorem 3.1. Moreover, we choose q such that

$$q > p, \qquad q > c(p + c^2 + scp), \qquad c\left(\frac{p}{q} + \frac{2c}{q} + \frac{p}{q(q-p)} + \frac{scp}{q} + \frac{sp}{q-p}\right) \le 1.$$

By (3.7) and (5.16), we have

$$\|q_1\|_1 = \|(\delta_0 + sf) * (p_1 - q_0 * p_0 - sq_0 * p_1)\|_1 \le c(p + c^2 + scp) < q.$$

By induction over n, we obtain

$$\begin{aligned} \|q_{n+1}\|_1 &= \left\|(\delta_0 + sf) * \left(p_{n+1} - \sum_{k=0}^{n} p_k * q_{n-k} - s\sum_{k=1}^{n+1} p_k * q_{n+1-k}\right)\right\|_1 \\ &\le c\left(p^{n+1} + p^n c + cq^n + \frac{p^n q - pq^n}{p-q} + sp^{n+1}c + s\frac{p^{n+1}q - pq^{n+1}}{p-q}\right) \\ &= q^{n+1}c\Bigg(\left(\frac{p}{q}\right)^{n+1} + \left(\frac{p}{q}\right)^n\frac{c}{q} + \frac{c}{q} + \frac{1}{q-p}\left(\left(\frac{p}{q}\right) - \left(\frac{p}{q}\right)^n\right) \\ &\qquad + sc\left(\frac{p}{q}\right)^{n+1} + s\frac{p}{q-p}\left(1 - \left(\frac{p}{q}\right)^n\right)\Bigg) \\ &\le q^{n+1}c\left(\frac{p}{q} + \frac{2c}{q} + \frac{p}{q(q-p)} + \frac{scp}{q} + \frac{sp}{q-p}\right) \\ &\le q^{n+1}. \end{aligned}$$

If we use (3.6) instead of (3.7), we obtain the same bound on $\|q_n\|_\infty$, which implies the convergence of the sum in (5.13).

It remains to show that the sum in (5.13) solves the OZE. This is achieved by rewriting the OZE in the form

$$P_t = Q_t + (t - s)P_t * Q_t + sP_t * Q_t.$$

Substituting for P_t and Q_t, the series expansion at s yields that the equation holds if, for all $n \in \mathbb{N}_0$,

$$p_n = q_n + \sum_{k=1}^{n} p_{k-1} * q_{n-k} + s\sum_{k=0}^{n} p_k * q_{n-k},$$

which is equivalent to (5.14) and (5.15). □

6. Combinatorics for small intensities

The coefficients p_n in (5.11) (given by (5.10)) are quite complex probabilistic objects. In the expansion of $P_t(x)$ around $s = 0$, however, η_s vanishes and the only random objects that remain are the random connections between the points $0, x_1, \ldots, x_n, x$. This leads to an almost combinatorial interpretation of the $p_n(x, 0)$. (In the Gilbert graph with a fixed gauge body all randomness disappears.) Moreover, this interpretation provides a simple combinatorial way to determine the coefficients $q_n(x, 0)$ in (5.13).

For $n \in \mathbb{N}_0$, let $\mathbb{G}_n$ be the set of connected graphs with $n+2$ vertices $\{0, \ldots, n+1\}$. For a graph $G = (V(G), E(G)) \in \mathbb{G}_n$ with vertex set $V(G)$ and edge set $E(G)$, we call 0 the start vertex and $n+1$ the end vertex. For $i, j \in V(G)$ and $I \subset [n]$, we write '$i \leftrightarrow j$ in $G \mid I$' if there is a path from i to j in G that uses only vertices in $I \cup \{i, j\}$. For $n \in \mathbb{N}_0$, we define the combinatorial functionals $\pi_n \colon \mathbb{G}_n \to \mathbb{Z}$ by

$$\pi_n(G) := \sum_{I \subset [n]} (-1)^{n-|I|} \mathbf{1}\{0 \leftrightarrow n+1 \text{ in } G|I\}.$$

By a slight abuse of notation, we write $G = \mathcal{G}(\chi_0^{x_0,\ldots,x_{n+1}})$ for $G \in \mathbb{G}_n$ if the two graphs are equal after changing the labels in G from i to x_i.

It was shown in Lemma 5.3 that the integrand in (5.10) vanishes if $\mathcal{G}(\chi_0^{x_0,\boldsymbol{x},x_{n+1}})$ is not connected. Hence, (5.10) is equivalent to

$$\begin{aligned} p_n(x,s) &= \frac{1}{n!} \int \mathbb{E} \sum_{G \in \mathbb{G}_n} \sum_{J \subset [n]} (-1)^{n-|J|} \mathbf{1}\{x \in C(0, \chi_0^{0,\boldsymbol{x},x})\} \mathbf{1}\{\mathcal{G}(\chi_0^{0,\boldsymbol{x},x}) = G\} \, \mathrm{d}\boldsymbol{x} \\ &= \frac{1}{n!} \int \sum_{G \in \mathbb{G}_n} \pi_n(G) \mathbb{E}\, \mathbf{1}\{\mathcal{G}(\chi_0^{0,\boldsymbol{x},x}) = G\} \, \mathrm{d}\boldsymbol{x} \\ &= \frac{1}{n!} \sum_{G \in \mathbb{G}_n} \pi_n(G) I_n(G, x), \end{aligned} \tag{6.1}$$

where $I_n \colon \mathbb{G}_n \times \mathbb{R}^d \to [0, \infty)$ is defined by

$$\begin{aligned} I_n(G, x) &:= \int \mathbb{P}(\mathcal{G}(\chi_0^{0,\boldsymbol{x},x}) = G) \, \mathrm{d}\boldsymbol{x} \\ &= \int \mathbb{E} \prod_{\{i,j\} \in E(G)} \mathbf{1}\{\{x_i, x_j\} \in E(\chi_0^{x_0,\boldsymbol{x},x_{n+1}})\} \\ &\qquad \times \prod_{\{i,j\} \notin E(G)} \mathbf{1}\{\{x_i, x_j\} \notin E(\chi_0^{x_0,\boldsymbol{x},x_{n+1}})\} \, \mathrm{d}\boldsymbol{x} \\ &= \int \prod_{\{i,j\} \in E(G)} \varphi(x_i - x_j) \prod_{\{i,j\} \notin E(G)} (1 - \varphi(x_i - x_j)) \, \mathrm{d}(x_1, \ldots, x_n), \end{aligned}$$

where again $x_0 := 0$ and $x_{n+1} := x$. By (6.1) we have found a representation of $p_n(x)$ as a sum over the graphs in $\mathbb{G}_n$ where each summand consists of a purely combinatorial factor and an integral-geometric factor. This representation looks rather natural, but is not well suited for the convolution. Therefore, we will derive a second representation that convolutes in a very simple way. This will also enable us to give a very simple representation of the $q_n(x)$.

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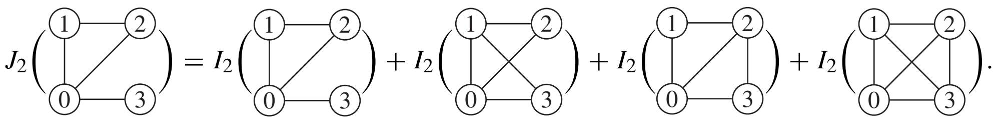
Let $J_n\colon \mathbb{G}_n \times \mathbb{R}^d \to [0,\infty)$ be defined by

$$J_n(G,x) := \int_{(\mathbb{R}^d)^n} \prod_{\{i,j\}\in E(G)} \varphi(x_i - x_j)\,\mathrm{d}(x_1,\dots,x_n).$$

By multiplying the integrand with a $1 = \varphi(x_i - x_j) + (1 - \varphi(x_i - x_j))$ for each edge $\{i,j\}$ which is not contained in $E(G)$, we obtain

$$J_n(G,x) = \sum_{H\in\mathbb{G}_n,\, E(H)\supset E(G)} I_n(H,x).$$

For example,



By a Möbius inversion (see, e.g. [13]), we have

$$I_n(G,x) = \sum_{H\in\mathbb{G}_n,\, E(H)\supset E(G)} (-1)^{|E(H)|-|E(G)|} J_n(H,x), \qquad G \in \mathbb{G}_n.$$

This leads to the announced second representation for $p_n(x)$, namely

$$\begin{aligned} p_n(x) &= \frac{1}{n!}\sum_{G\in\mathbb{G}_n} \pi_n(G) I_n(G,x) \\ &= \frac{1}{n!}\sum_{G\in\mathbb{G}_n}\sum_{H\in\mathbb{G}_n} \mathbf{1}\{E(H)\supset E(G)\}\pi_n(G)(-1)^{|E(H)|-|E(G)|} J_n(H,x) \\ &= \frac{1}{n!}\sum_{H\in\mathbb{G}_n} J_n(H,x) \sum_{G\in\mathbb{G}_n} \mathbf{1}\{E(G)\subset E(H)\}\pi_n(G)(-1)^{|E(H)|-|E(G)|}. \end{aligned}$$

In particular,

$$p_n(x) = \frac{1}{n!}\sum_{H\in\mathbb{G}_n} \kappa_n(H) J_n(H,x), \qquad n \in \mathbb{N}, \tag{6.2}$$

where

$$\kappa_n(H) := \sum_{G\in\mathbb{G}_n} \mathbf{1}\{E(G)\subset E(H)\}\pi_n(G)(-1)^{|E(H)|-|E(G)|}.$$

A vertex $i \in [n]$ of a graph $G \in \mathbb{G}_n$ is called *pivotal* if any path from 0 to $n+1$ contains i. The subset $\mathbb{G}_n^0 \subset \mathbb{G}_n$ of graphs which contain no pivotal vertex plays a significant role for determining the coefficients $q_n(x)$ from $p_n(x)$ as the next theorem shows.

Theorem 6.1. *The coefficients $q_n(x) := q_n(x,0)$ of the series representation of Q_t at $t_0 = 0$ satisfy*

$$q_n(x) = \frac{1}{n!}\sum_{H\in\mathbb{G}_n^0} \kappa_n(H) J_n(H,x), \qquad x \in \mathbb{R}^d.$$

This means that $q_n(x)$ differs from $p_n(x)$ only by the sum over the graphs with pivotal vertices. The proof of Theorem 6.1 is based on the following three lemmas.

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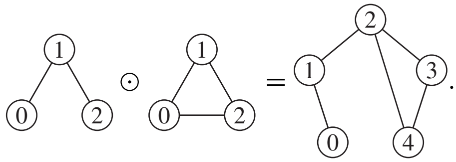
At first we define the concatenation of two graphs. For $n, m \in \mathbb{N}_0$, let $G_1 \in \mathbb{G}_n$ and $G_2 \in \mathbb{G}_m$. The concatenation $G_1 \odot G_2 \in \mathbb{G}_{n+m+1}$ of G_1 and G_2 is constructed in the following way.

- Relabel all nodes in G_2 with labels $\{n+1, \dots, n+m+2\}$ without changing the order.
- Define $V(G_1 \odot G_2) := V(G_1) \cup V(G_2)$.
- Define $E(G_1 \odot G_2) := E(G_1) \cup E(G_2)$.

For example,



In other words, we only combine the end vertex of G_1 and the start vertex of G_2 to a new vertex and adjust the labels.

Lemma 6.1. *For $n, m \in \mathbb{N}_0$ and $G_1 \in \mathbb{G}_n$, $G_2 \in \mathbb{G}_m$, we have*

$$\pi_{n+m+1}(G_1 \odot G_2) = \pi_n(G_1)\pi_m(G_2).$$

Proof. The vertex with label $n+1$ is by construction pivotal. If $0 \leftrightarrow n+m+2$ in $(G_1 \odot G_2)|I$ then $n+1 \in I$. Hence,

$$\begin{aligned}\pi_{n+m+1}(G_1 \odot G_2) &= \sum_{I \subset [n+m+1]} (-1)^{n+m+1-|I|}\,\mathbf{1}\{0 \leftrightarrow n+m+2 \text{ in } (G_1 \odot G_2) \mid I\} \\ &= \sum_{I_1 \subset [n]} \sum_{I_2 \subset [m]+n+1} (-1)^{n+m-|I_1|-|I_2|}\,\mathbf{1}\{0 \leftrightarrow n+1 \text{ in } (G_1 \odot G_2) \mid I_1\} \\ &\qquad\qquad \times \mathbf{1}\{n+1 \leftrightarrow n+m+2 \text{ in } (G_1 \odot G_2) \mid I_2\} \\ &= \sum_{I_1 \subset [n]} \sum_{I_2 \subset [m]} (-1)^{n+m-|I_1|-|I_2|}\,\mathbf{1}\{0 \leftrightarrow n+1 \text{ in } G_1 \mid I_1\} \\ &\qquad\qquad \times \mathbf{1}\{0 \leftrightarrow m+1 \text{ in } G_2 \mid I_2\} \\ &= \pi_n(G_1)\pi_m(G_2).\end{aligned}$$

□

Lemma 6.2. *For $n, m \in \mathbb{N}_0$ and $G_1 \in \mathbb{G}_n$, $G_2 \in \mathbb{G}_m$, we have*

$$\kappa_{n+m+1}(G_1 \odot G_2) = \kappa_n(G_1)\kappa_m(G_2).$$

Proof. In every graph $H \in \mathbb{G}_{n+m+1}$ with $E(H) \subset E(G_1 \odot G_2)$ the vertex $n+1$ is pivotal. Hence, there are uniquely determined graphs $H_1 \in \mathbb{G}_n$ and $H_2 \in \mathbb{G}_m$ such that $H = H_1 \odot H_2$. The graph H_1 consists of 0, $n+1$ and all vertices lying 'in front' of $n+1$, whereas H_2 is a relabeled version of the subgraph of H which consists of $n+1$, $n+m+2$, and all vertices lying 'behind' $n+1$. Hence, by Lemma 6.1,

$$\begin{aligned}\kappa_{n+m+1}(G_1 \odot G_2) &= \sum_{H \in \mathbb{G}_{n+m+1}} \mathbf{1}\{E(H) \subset E(G_1 \odot G_2)\}\pi_{n+m+1}(H)(-1)^{|E(H)|-|E(G_1 \odot G_2)|} \\ &= \sum_{H_1 \in \mathbb{G}_n} \sum_{H_2 \in \mathbb{G}_m} \mathbf{1}\{E(H_1 \odot H_2) \subset E(G_1 \odot G_2)\}\pi_{n+m+1}(H_1 \odot H_2) \\ &\qquad\qquad \times (-1)^{|E(H_1 \odot H_2)|-|E(G_1 \odot G_2)|}\end{aligned}$$

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$$= \sum_{H_1 \in \mathbb{G}_n} \sum_{H_2 \in \mathbb{G}_m} \mathbf{1}\{E(H_1) \subset E(G_1)\} \mathbf{1}\{E(H_2) \subset E(G_2)\} \times \pi_n(H_1)\pi_m(H_2)(-1)^{|E(H_1)|+|E(H_2)|-|E(G_1)|-|E(G_2)|}$$

$$= \kappa_n(G_1)\kappa_m(G_2).$$ □

Lemma 6.3. *For $n, m \in \mathbb{N}_0$ and $G_1 \in \mathbb{G}_n$, $G_2 \in \mathbb{G}_m$, we have*

$$J_n(G_1) * J_m(G_2) = J_{n+m+1}(G_1 \odot G_2).$$

Proof. For all $x \in \mathbb{R}^d$, we have

$$(J_n(G_1) * J_m(G_2))(x) = \int_{\mathbb{R}^d} \int_{(\mathbb{R}^d)^n} \prod_{\{i,j\} \in E(G_1)} \varphi(x_i - x_j)\, \mathrm{d}(x_1, \dots, x_n) \times \int_{(\mathbb{R}^d)^m} \prod_{\{i,j\} \in E(G_2)} \varphi(y_i - y_j)\, \mathrm{d}(y_1, \dots, y_m)\, \mathrm{d}x_{n+1},$$

where $x_0 := 0$ and $y_0 := 0$, $y_{m+1} := x - x_{n+1}$. By translation invariance, nothing changes if we redefine $y_0 := x_{n+1}$ and $y_{m+1} := x$. If we apply Fubinis theorem and rename the integration variables in the same way as we renamed the vertex labels in the definition of 'concatenation', we obtain

$$(J_n(G_1) * J_m(G_2))(x) = \int_{\mathbb{R}^d} \int_{(\mathbb{R}^d)^n} \int_{(\mathbb{R}^d)^m} \prod_{\{i,j\} \in E(G_1 \odot G_2)} \varphi(x_i - x_j)\, \mathrm{d}(x_1, \dots, x_n) \times \mathrm{d}(x_{n+2}, \dots, x_{n+m+1})\, \mathrm{d}x_{n+1}$$

$$= J_{n+m+1}(G_1 \odot G_2)(x).$$ □

We are now in a position to prove Theorem 6.1.

Proof of Theorem 6.1. For $t_0 = 0$, (5.14) and (5.15) simplify to $q_0 = p_0$ and

$$q_{n+1} = p_{n+1} - \sum_{k=0}^{n} q_{n-k} * p_k, \qquad n \in \mathbb{N}_0. \tag{6.3}$$

We will use this for an induction over n. First, we observe that trivially $\mathbb{G}_0 = G_0^0$ as the graph G_0 that connects 0 and 1 with a single bond is the only element of $\mathbb{G}_0$. Hence,

$$\frac{1}{0!} \sum_{G \in \mathbb{G}_0^0} \kappa_0(G) J_0(G, x) = J_0(G_0, x) = p_0 = q_0.$$

For the induction step, we define $\mathbb{G}_n^{>0} := \mathbb{G}_n \setminus \mathbb{G}_n^0$. From (6.2), it follows that

$$p_{n+1} = \frac{1}{(n+1)!} \sum_{G \in \mathbb{G}_{n+1}^0} \kappa_{n+1}(G) J_{n+1}(G) + \frac{1}{(n+1)!} \sum_{G \in \mathbb{G}_{n+1}^{>0}} \kappa_{n+1}(G) J_{n+1}(G).$$

Hence, by (6.3), it is enough to show

$$\frac{1}{(n+1)!} \sum_{G \in \mathbb{G}_{n+1}^{>0}} \kappa_{n+1}(G) J_{n+1}(G) = \sum_{k=0}^{n} p_k * q_{n-k}. \tag{6.4}$$

We use the induction hypothesis, (6.2), and Lemmas 6.2 and 6.3 to obtain

$$\sum_{k=0}^{n} p_k * q_{n-k} = \sum_{k=0}^{n} \Big(\frac{1}{k!} \sum_{G_1 \in \mathbb{G}_k} \kappa_k(G_1) J_k(G_1) \Big) * \Big(\frac{1}{(n-k)!} \sum_{G_2 \in \mathbb{G}^0_{n-k}} \kappa_{n-k}(G_2) J_{n-k}(G_2) \Big)$$
$$= \sum_{k=0}^{n} \frac{1}{k!\,(n-k)!} \sum_{G_1 \in \mathbb{G}_k} \sum_{G_2 \in \mathbb{G}^0_{n-k}} \kappa_{n+1}(G_1 \odot G_2) J_{n+1}(G_1 \odot G_2). \qquad (6.5)$$

Finally, we have a look at the left-hand side of (6.4). Let $G \in \mathbb{G}^0_{n+1}$. Each path from 0 to $n+2$ runs through the pivotal vertices of G in the same order. Let $v \in [n+1]$ be the last of these pivotal vertices. We define the set of graphs $\mathbb{H}_k \subset \mathbb{G}^0_{n+1}$, $k \in \{0, \dots, n\}$, with the following properties.

- Each $H \in \mathbb{H}_k$ contains at least one pivotal vertex.
- The vertex with label $k+1$ is the last pivotal vertex in each $H \in \mathbb{H}_k$.
- The k vertices $\{1, \dots, k\}$ lie in front of the vertex $k+1$.
- The $n-k$ vertices $\{k+2, \dots, n+1\}$ lie behind the vertex $k+1$.

We partition the set $[n+1]$ of vertices in each $G \in \mathbb{G}^0_{n+1}$ into three sets M_1, M_2, and M_3. The set M_1 contains all vertices that lie in front of the last pivotal vertex of G. The set M_2 contains only the last pivotal vertex and M_3 contains the remaining vertices. Now we relabel the vertices in G to obtain a graph $\widetilde{G}$ in the following way: the vertices in M_1 are labeled with the numbers $1, \dots, |M_1|$ without changing the order. The vertex in M_2 is labeled $|M_1|+1$ and the vertices in M_3 are labeled with the numbers $|M_1|+2, \dots, n+1$, again without changing the order. By construction, we have $\widetilde{G} \in \mathbb{H}_{|M|_1}$ but $\kappa(G) = \kappa(\widetilde{G})$ and $J_{n+1}(G) = J_{n+1}(\widetilde{G})$. There are exactly

$$\binom{n+1}{k,\, n-k,\, 1}$$

graphs in $\mathbb{G}^0_{n+1}$, which become the same $\widetilde{G}$ by this relabeling procedure. Hence, we have, for the left-hand side of (6.4),

$$\frac{1}{(n+1)!} \sum_{G \in \mathbb{G}^{>0}_{n+1}} \kappa_{n+1}(G) J_{n+1}(G) = \frac{1}{(n+1)!} \sum_{k=0}^{n} \sum_{H \in \mathbb{H}_k} \binom{n+1}{k,\, n-k,\, 1} \kappa_{n+1}(H) J_{n+1}(H)$$
$$= \sum_{k=0}^{n} \sum_{H \in \mathbb{H}_k} \frac{1}{k!\,(n-k)!} \kappa_{n+1}(H) J_{n+1}(H),$$

which is equal to (6.5) due to the definition of the concatenation. □

Appendix A. Palm distributions

In this appendix we work in the setting of Section 2. The following result (Neveu's exchange formula; see, e.g. [6]) is a versatile tool of Palm theory.

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Proposition A.1. *Let η, η' be two invariant point processes with finite intensities and let $f\colon \Omega\times \mathbb{R}^d \to [0,\infty)$ be measurable. Then*

$$\gamma_\eta \mathbb{E}^0_\eta \sum_{x\in\eta'} f(\theta_0, x) = \gamma_{\eta'}\mathbb{E}^0_{\eta'} \sum_{x\in\eta} f(\theta_x, -x). \tag{A.1}$$

This remains true for any measurable $f\colon \Omega\times\mathbb{R}^d \to \mathbb{R}$ with $\mathbb{E}^0_\eta \sum_{x\in\eta'} |f(\theta_0, x)| < \infty$.

Let η be an invariant point process with a positive and finite intensity and assume that its second factorial moment measure $\alpha^{(2)}$ is locally finite and absolutely continuous. We explain one possible construction of the two-point Palm probability measures of η. An easy calculation shows that, for any $B \in \mathcal{B}^d$,

$$\mathbb{E}^0_\eta \sum_{x\in\eta\setminus\{0\}} \mathbf{1}\{x\in B\} = \gamma_\eta^2 \int \mathbf{1}\{x\in B\}\rho(x)\,\mathrm{d}x.$$

We now assume that $(\Omega, \mathcal{A})$ is a *Borel space*. This very weak assumption can be made without restricting generality. By a standard disintegration technique, we can then find a (measurable) family $\{\mathbb{P}^{0,x}_\eta : x\in\mathbb{R}^d\}$ of probability measures on $(\Omega,\mathcal{A})$ such that

$$\mathbb{E}^0_\eta \int f(\theta_0, x)\eta(\mathrm{d}x) = \mathbb{E}^0_\eta f(\theta_0, 0) + \gamma_\eta \int \mathbb{E}^{0,x}_\eta f(\theta_0, x)\rho(x)\,\mathrm{d}x \tag{A.2}$$

for all measurable $f\colon \Omega\times\mathbb{R}^d \to [0,\infty)$. We can then define

$$\mathbb{P}^{x,y}_\eta(A) := \mathbb{P}^{0,y-x}(\theta_x A), \qquad x,y\in\mathbb{R}^d,\ A\in\mathcal{A},$$

so that (2.4) holds. Using the refined Campbell theorem (2.1) and (A.2), it is then not difficult to check that (2.3) holds. It is also easy to see that $\mathbb{P}^{x,y}_\eta(x,y\in\eta) = 1$ for $\alpha^{(2)}$-almost everywhere, $(x,y)\in\mathbb{R}^d\times\mathbb{R}^d$.

Let us now assume that $\eta \equiv \eta_t$ is a stationary Poisson process of intensity $t > 0$. The multivariate Mecke equation (see, e.g. [15, Corollary 3.2.3] or [8, Theorem 4.4]) states that, for any $n\in\mathbb{N}$ and any measurable function $f\colon \boldsymbol{N}(\mathbb{R}^d)\times(\mathbb{R}^d)^n \to [0,\infty)$,

$$\mathbb{E} \sum_{x_1,\dots,x_n\in\eta}^{\neq} f(\eta_t, x_1,\dots,x_n) = t^n \int \mathbb{E} f(\eta_t\cup\{x_1,\dots,x_n\}, x_1,\dots,x_n)\,\mathrm{d}(x_1,\dots,x_n). \tag{A.3}$$

The $n = 1$ case easily implies (together with stationarity of η_t) that the Palm distribution of η_t is given by

$$\mathbb{P}^0_{\eta_t}(\eta_t\in\cdot) = \mathbb{P}(\eta_t\cup\{0\}\in\cdot).$$

For $n = 2$, we obtain from (A.3) that the pair correlation function ρ_t of η_t satisfies $\rho_t \equiv 1$ and that, moreover,

$$\mathbb{P}^{x,y}_{\eta_t}(\eta_t\in\cdot) = \mathbb{P}(\eta_t\cup\{x,y\}\in\cdot)$$

for almost every (x,y) with respect to Lebesgue measure on $(\mathbb{R}^d)^2$.

In this paper we work also with point processes on a metric space $\mathbb{X}$ different from $\mathbb{R}^d$. These are random elements of the space $\boldsymbol{N}(\mathbb{X})$ of all integer-valued locally finite measures μ on $\mathbb{X}$ equipped with the smallest σ-field making the mappings $\mu\mapsto\mu(B)$ measurable for all B in the Borel σ-field on $\mathbb{X}$. For more details on point processes we refer the reader to [8] and [15]. A survey of Palm theory can be found in [6].

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We wish to thank Salvatore Torquato for drawing our attention to the Ornstein–Zernike equation and its relevance for percolation theory. This work was supported by the German Research Foundation (DFG) through the research unit 'Geometry and Physics of Spatial Random System' under the grant LA 965/7-2.

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