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INVERSE SPECTRAL PROBLEMS FOR COMPACT HANKEL OPERATORS

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Abstract Given two arbitrary sequences $(\lambda_j)_{j\geqslant 1}$ and $(\mu_j)_{j\geqslant 1}$ of real numbers satisfying

$$|\lambda_1| > |\mu_1| > |\lambda_2| > |\mu_2| > \cdots > |\lambda_j| > |\mu_j| \to 0,$$

we prove that there exists a unique sequence $c = (c_n)_{n\in\mathbb{Z}_+}$, real valued, such that the Hankel operators Γ_c and $\Gamma_{\tilde{c}}$ of symbols $c = (c_n)_{n\geqslant 0}$ and $\tilde{c} = (c_{n+1})_{n\geqslant 0}$, respectively, are selfadjoint compact operators on $\ell^2(\mathbb{Z}_+)$ and have the sequences $(\lambda_j)_{j\geqslant 1}$ and $(\mu_j)_{j\geqslant 1}$, respectively, as non-zero eigenvalues. Moreover, we give an explicit formula for c and we describe the kernel of Γ_c and of $\Gamma_{\tilde{c}}$ in terms of the sequences $(\lambda_j)_{j\geqslant 1}$ and $(\mu_j)_{j\geqslant 1}$. More generally, given two arbitrary sequences $(\rho_j)_{j\geqslant 1}$ and $(\sigma_j)_{j\geqslant 1}$ of positive numbers satisfying

$$\rho_1 > \sigma_1 > \rho_2 > \sigma_2 > \cdots > \rho_j > \sigma_j \to 0,$$

we describe the set of sequences $c = (c_n)_{n\in\mathbb{Z}_+}$ of complex numbers such that the Hankel operators Γ_c and $\Gamma_{\tilde{c}}$ are compact on $\ell^2(\mathbb{Z}_+)$ and have sequences $(\rho_j)_{j\geqslant 1}$ and $(\sigma_j)_{j\geqslant 1}$, respectively, as non-zero singular values.

Keywords: Hankel operators; inverse spectral problems; compressed shift operator; action–angle variables

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1. Introduction

1.1. Eigenvalues of selfadjoint Hankel operators on $\ell^2(\mathbb{Z}_+)$

Let $c = (c_n)_{n\geqslant 0}$ be a sequence of complex numbers. The Hankel operator Γ_c of symbol c is formally defined on $\ell^2(\mathbb{Z}_+)$ by

$$\forall x = (x_n)_{n\geqslant 0} \in \ell^2(\mathbb{Z}_+), \quad \Gamma_c(x)_n = \sum_{p=0}^{\infty} c_{n+p} x_p.$$

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These operators frequently appear in operator theory and in harmonic analysis, and we refer to the books by Nikolskii [10] and Peller [13] for an introduction and their basic properties. By a well-known theorem of Nehari [9], Γ_c is well defined and bounded on $\ell^2(\mathbb{Z}_+)$ if and only if there exists a function $f \in L^\infty(\mathbb{T})$ such that, $\forall n \geqslant 0, \hat{f}(n) = c_n$, or equivalently if the Fourier series $u_c = \sum_{n\geqslant 0} c_n e^{inx}$ belongs to the space $BMO(\mathbb{T})$ of bounded mean oscillation functions. Moreover, by a well-known result of Hartman [4], Γ_c is compact if and only if there exists a continuous function f on $\mathbb{T}$ such that, $\forall n \geqslant 0, \hat{f}(n) = c_n$, or equivalently if u_c belongs to the space $VMO(\mathbb{T})$ of vanishing mean oscillation functions. Assume moreover that the sequence c is real valued. Then Γ_c is selfadjoint and compact, so it admits a sequence of non-zero real eigenvalues $(\lambda_j)_{j\geqslant 1}$, tending to zero. A natural inverse spectral problem is the following: *given any sequence of real numbers $(\lambda_j)_{j\geqslant 1}$, tending to zero, does there exist a compact selfadjoint Hankel operator Γ_c having this sequence as non-zero eigenvalues, repeated according to their multiplicity?*

A complete answer to this question can be found in the literature as a consequence of a more general theorem by Megretskii *et al.* [8] characterizing selfadjoint operators which are unitarily equivalent to bounded Hankel operators. Here, we state the part of their result which concerns the compact operators.

Theorem 1 (Megretskii *et al.* [8]). *Let Γ be a compact selfadjoint operator on a separable Hilbert space. Then Γ is unitarily equivalent to a Hankel operator if and only if the following conditions are satisfied.*

(1) *Either* $\ker(\Gamma) = \{0\}$ *or* $\dim\ker(\Gamma) = \infty$.
(2) *For any* $\lambda \in \mathbb{R}^*$, $|\dim\ker(\Gamma - \lambda I) - \dim\ker(\Gamma + \lambda I)| \leqslant 1$.

As a consequence of this theorem, any sequence of real numbers with distinct absolute values and converging to 0 is the sequence of non-zero eigenvalues of some compact selfadjoint Hankel operator.

In this paper, we are interested in finding additional constraints on the operator Γ_c which give rise to the uniqueness of c. With this aim in view, we introduce the shifted Hankel operator $\Gamma_{\tilde{c}}$, where $\tilde{c}_n := c_{n+1}$ for all $n \in \mathbb{Z}_+$. If we denote by $(\lambda_j)_{j\geqslant 1}$ the sequence of non-zero eigenvalues of Γ_c and by $(\mu_j)_{j\geqslant 1}$ the sequence of non-zero eigenvalues of $\Gamma_{\tilde{c}}$, one can check – see below – that

$$|\lambda_1| \geqslant |\mu_1| \geqslant |\lambda_2| \geqslant |\mu_2| \geqslant \cdots \to 0.$$

Our result reads as follows.

Theorem 2. *Let $(\lambda_j)_{j\geqslant 1}$, $(\mu_j)_{j\geqslant 1}$ be two sequences of real numbers tending to zero so that*

$$|\lambda_1| > |\mu_1| > |\lambda_2| > |\mu_2| > \cdots \to 0.$$

There exists a unique real valued sequence $c = (c_n)$ such that Γ_c and $\Gamma_{\tilde{c}}$ are compact selfadjoint operators, the sequence of non-zero eigenvalues of Γ_c is $(\lambda_j)_{j\geqslant 1}$, and the sequence of non-zero eigenvalues of $\Gamma_{\tilde{c}}$ is $(\mu_j)_{j\geqslant 1}$.

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Furthermore, the kernel of Γ_c is reduced to zero if and only if the following conditions hold:

$$\sum_{j=1}^{\infty}\left(1-\frac{\mu_j^2}{\lambda_j^2}\right)=\infty, \quad \sup_N \frac{1}{\lambda_{N+1}^2}\prod_{j=1}^{N}\frac{\mu_j^2}{\lambda_j^2}=\infty. \tag{1}$$

Moreover, in that case, the kernel of $\Gamma_{\tilde{c}}$ is also reduced to 0*.*

In complement to the above statement, let us mention that an explicit formula for c is available, as well as an explicit description of the kernel of Γ_c when it is non-trivial; see Theorems 3 and 4 below.

1.2. Singular values of Hankel operators on the Hardy space

Theorem 2 is in fact a consequence of a more general result concerning the singular values of non-necessarily selfadjoint compact Hankel operators. Recall that the singular values of a bounded operator T on a Hilbert space $\mathcal{H}$ are given by the following min–max formula. For every $m \geqslant 1$, denote by $\mathcal{F}_m$ the set of linear subspaces of $\mathcal{H}$ of dimension at most m. The mth singular value of T is given by

$$s_m(T)=\min_{F\in\mathcal{F}_{m-1}}\ \max_{f\in F^{\perp},\|f\|=1}\|T(f)\|. \tag{2}$$

In this paper, we construct a homeomorphism between some set of symbols c and the singular values of Γ_c and $\Gamma_{\tilde{c}}$ up to the choice of an element in an infinite-dimensional torus.

In order to state this general result we complexify and reformulate the problem in the Hardy space.

1.2.1. The setting. We identify $\ell^2(\mathbb{Z}_+)$ with

$$L^2_+(\mathbb{T})=\left\{u : u=\sum_{n=0}^{\infty}\hat{u}(n)\,\mathrm{e}^{inx}, \sum_{n=0}^{\infty}|\hat{u}(n)|^2<+\infty\right\},$$

and we denote by Π the orthogonal projector from $L^2(\mathbb{T})$ onto $L^2_+(\mathbb{T})$.

Here, and in the following, for any space of distributions E on $\mathbb{T}$, the notation E_+ stands for the subspace of E consisting of those elements u of E such that $\hat{u}(n)=0$ for every $n<0$, or equivalently which can be holomorphically extended to the unit disc. In that case, we will still denote by $u(z)$ the value of this holomorphic extension at the point z of the unit disc.

We endow $L^2_+(\mathbb{T})$ with the scalar product

$$(u|v) := \int_{\mathbb{T}} u\bar{v}\,\frac{dx}{2\pi}$$

and with the associated symplectic form

$$\omega(u,v)=\mathrm{Im}(u|v).$$

For u sufficiently smooth, we define a $\mathbb{C}$-antilinear operator on L^2_+ by

$$H_u(h)=\Pi(u\bar{h}), \quad h\in L^2_+.$$

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If $u = u_c$,

$$\widehat{H_u(h)}(n) = \Gamma_c(x)_n, \quad x_p := \overline{\hat{h}(p)}.$$

Because of this equality, H_u is called the Hankel operator of symbol u. Similarly, $\Gamma_{\tilde{c}}$ corresponds to the operator $K_u = H_u T_z$, where T_z denotes multiplication by z. Note that, by definition, $H_u = H_{\Pi(u)}$. In the following, we always consider holomorphic symbols $u = \Pi(u)$.

As stated before, by the Nehari theorem [9], H_u is well defined and bounded on $L^2_+(\mathbb{T})$ if and only if u belongs to $\Pi(L^\infty(\mathbb{T}))$ or to $BMO_+(\mathbb{T})$. Moreover, by the Hartman theorem [4], it is a compact operator if and only if u is the projection of a continuous function on the torus, or equivalently if and only if it belongs to $VMO_+(\mathbb{T})$ with equivalent norms. Furthermore, note that this operator H_u is selfadjoint as an antilinear operator in the sense that, for any $h_1, h_2 \in L^2_+$,

$$(h_1|H_u(h_2)) = (h_2|H_u(h_1)). \tag{3}$$

A crucial property of Hankel operators is that

$$H_u T_z = T_z^* H_u. \tag{4}$$

As a consequence,

$$K_u^2 = H_u T_z T_z^* H_u = H_u(I - (\cdot|1))H_u,$$

hence

$$K_u^2 = H_u^2 - (\cdot|u)u. \tag{5}$$

Assume that $u \in VMO_+(\mathbb{T})$ and denote by $(\rho_j)_{j\geqslant 1}$ the sequence of singular values of H_u labelled according to the min–max formula (2). Notice that, by identity (3), the sequence $(\rho_j^2)_{j\geqslant 1}$ is the sequence of eigenvalues of H_u^2. Since, via the Fourier transform, H_u^2 identifies to $\Gamma_c\Gamma_c^*$ with $c = \hat{u}$, $(\rho_j)_{j\geqslant 1}$ is also the sequence of singular values of $\Gamma_{\hat{u}}$. Similarly, K_u is compact, so it has a sequence $(\sigma_j)_{j\geqslant 1}$ of singular values tending to 0, which are the singular values of $\Gamma_{\tilde{c}}$, since K_u^2 identifies to $\Gamma_{\tilde{c}}\Gamma_{\tilde{c}}^*$. From equality (5) and the min–max formula (2), one obtains

$$\rho_1 \geqslant \sigma_1 \geqslant \rho_2 \geqslant \sigma_2 \geqslant \cdots \to 0.$$

1.2.2. Generic symbols. We denote by $VMO_{+,\mathrm{gen}}$ the set of $u \in VMO_+(\mathbb{T})$ such that H_u and K_u admit only simple singular values with strict inequalities, or equivalently such that H_u^2 and $K_u^2 := H_u^2 - (\cdot|u)u$ admit only simple positive eigenvalues $\rho_1^2 > \rho_2^2 > \cdots > \cdots \to 0$ and $\sigma_1^2 > \sigma_2^2 > \cdots > \cdots \to 0$ so that

$$\rho_1^2 > \sigma_1^2 > \rho_2^2 > \sigma_2^2 > \cdots \to 0.$$

For any integer N, we denote by $\mathcal{V}(2N)$ the set of symbols u such that the rank of H_u and the rank of K_u are both equal to N. By a theorem of Kronecker (see [5]), $\mathcal{V}(2N)$ is a complex manifold of dimension $2N$ consisting of rational functions. One can consider as well the set $\mathcal{V}(2N-1)$ of symbols u such that H_u is of rank N and K_u is of rank $N-1$. It defines a complex manifold of rational functions of complex dimension $2N-1$.

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By the arguments developed in [2], it is straightforward to verify that $VMO_{+,\text{gen}}$ is a dense G_δ subset of $VMO_+(\mathbb{T})$. Indeed, let us consider the set $\mathcal{U}_N$ which consists of functions $u \in VMO_+(\mathbb{T})$ such that the N first eigenvalues of H_u^2 and of K_u^2 are simple. This set is obviously open in $VMO_+(\mathbb{T})$. Moreover, in Lemma 4 of [2], it is proved that $\mathcal{U}_N \cap \mathcal{V}(2N) := \mathcal{V}(2N)_{\text{gen}}$ is a dense open subset of $\mathcal{V}(2N)$. Now any element u in VMO_+ may be approximated by an element in $\mathcal{V}(2N')$, $N' > N$, which can be itself approximated by an element in $\mathcal{V}(2N')_{\text{gen}} \subset \mathcal{U}_N$, since $N' \geqslant N$. Eventually, $VMO_{+,\text{gen}}$ is the intersection of the $\mathcal{U}_N$ which are open and dense; hence $VMO_{+,\text{gen}}$ is a dense G_δ set.

1.2.3. Spectral data for generic symbols. Let $u \in VMO_{+,\text{gen}}$. Denote by $(\rho_j)_{j\geqslant 1}$ the singular values of H_u and by $(\sigma_j)_{j\geqslant 1}$ the singular values of K_u. Denote by $(\tilde{e}_j)$ an orthonormal family of corresponding eigenvectors,

$$H_u^2 \tilde{e}_j = \rho_j^2 \tilde{e}_j.$$

It is clear that $H_u \tilde{e}_j$ is also an eigenvector of H_u^2 for the same eigenvalue ρ_j^2; hence, since the eigenspace is assumed to be one dimensional due to the genericity assumption, we infer that

$$H_u \tilde{e}_j = \xi_j \tilde{e}_j,$$

with, by applying H_u to both sides and using the antilinearity,

$$|\xi_j|^2 = \rho_j^2.$$

Consequently, one can write, for some $\psi_j \in \mathbb{T}$,

$$\xi_j = \rho_j \,\mathrm{e}^{i\psi_j}.$$

We now replace our orthonormal family $(\tilde{e}_j)$ by (e_j) defined by

$$e_j = \mathrm{e}^{i\psi_j/2} \tilde{e}_j,$$

so that

$$H_u(e_j) = \rho_j e_j, \quad j \geqslant 1. \tag{6}$$

Notice that this orthonormal family is determined by u up to a change of sign on some of the e_j. We claim that $(1|e_j) \neq 0$. Indeed, if $(1|e_j) = 0$, then $(u|e_j) = \rho_j(e_j|1) = 0$ and, in view of (5), ρ_j^2 would be an eigenvalue of K_u^2, which contradicts the assumption. Therefore we can define the angles

$$\varphi_j(u) := 2\arg(1|e_j), \quad j \geqslant 1. \tag{7}$$

We do the same analysis with the operator $K_u = H_u T_z$. As before, by the antilinearity of K_u there exists an orthonormal family $(f_j)_{j\geqslant 1}$ of the range of K_u such that

$$K_u(f_j) = \sigma_j f_j, \quad j \geqslant 1, \tag{8}$$

and the family is determined by u up to a change of sign on some of the f_j. One has also $(u|f_j) \neq 0$ because of the assumption on the ρ_j and the σ_j. We set

$$\theta_j(u) := 2\arg(u|f_j), \quad j \geqslant 1. \tag{9}$$

1.2.4. The main result. We define

$$\Xi := \{(\zeta_j)_{j\geqslant 1} \in \mathbb{C}^{\mathbb{Z}_+}, |\zeta_1| > |\zeta_2| > |\zeta_3| > |\zeta_4| > \cdots \to 0\},$$

endowed with the topology induced by the Banach space $c_0(\mathbb{Z}_+)$ of sequences of complex numbers tending to 0.

Theorem 3. *The mapping*

$$\chi : u \in VMO_{+,\mathrm{gen}} \longmapsto \zeta = ((\zeta_{2j-1} = \rho_j \mathrm{e}^{-i\varphi_j})_{j\geqslant 1}, (\zeta_{2j} = \sigma_j \mathrm{e}^{-i\theta_j})_{j\geqslant 1})$$

is a homeomorphism onto Ξ. Moreover, one has an explicit formula for the inverse mapping. Namely, if ζ is given in Ξ, then the Fourier coefficients of u are given by

$$\hat{u}(n) = X \cdot A^n Y, \tag{10}$$

where $A = (A_{jk})_{j,k\geqslant 1}$ is the bounded operator on ℓ^2 defined by

$$A_{jk} = \sum_{m=1}^{\infty} \frac{\nu_j \nu_k \zeta_{2k-1} \kappa_m^2 \zeta_{2m}}{(|\zeta_{2j-1}|^2 - |\zeta_{2m}|^2)(|\zeta_{2k-1}|^2 - |\zeta_{2m}|^2)}, \quad j, k \geqslant 1, \tag{11}$$

with

$$\nu_j^2 := \left(1 - \frac{\sigma_j^2}{\rho_j^2}\right) \prod_{k\neq j} \left(\frac{\rho_j^2 - \sigma_k^2}{\rho_j^2 - \rho_k^2}\right), \tag{12}$$

$$\kappa_m^2 := \left(\rho_m^2 - \sigma_m^2\right) \prod_{\ell\neq m} \left(\frac{\sigma_m^2 - \rho_\ell^2}{\sigma_m^2 - \sigma_\ell^2}\right), \tag{13}$$

$$X = (\nu_j \zeta_{2j-1})_{j\geqslant 1}, \quad Y = (\nu_j)_{j\geqslant 1}, \tag{14}$$

and

$$V \cdot W := \sum_{j=1}^{\infty} v_j w_j \quad \text{if } V = (v_j)_{j\geqslant 1}, W = (w_j)_{j\geqslant 1}.$$

Theorem 3 calls for several comments. First, it is not difficult to see that the first part of Theorem 2 is a direct consequence of Theorem 3 (see the end of § 3 below). More generally, as an immediate corollary of Theorem 3, one shows that, for any given sequences $(\rho_j)_{j\geqslant 1}$ and $(\sigma_j)_{j\geqslant 1}$ satisfying

$$\rho_1 > \sigma_1 > \rho_2 > \sigma_2 > \cdots \to 0,$$

there exists an infinite-dimensional torus of symbols c such that the $(\rho_j)_{j\geqslant 1}$ are the non-zero singular values of Γ_c, and the $(\sigma_j)_{j\geqslant 1}$ are the non-zero singular values of $\Gamma_{\tilde{c}}$.

Next, we make the connection with previous results. In a previous article [3], we have obtained an analogue of Theorem 3 in the more restricted context of Hilbert–Schmidt Hankel operators. This result arises in [3] as a by-product of the study of the dynamics of some completely integrable Hamiltonian system called the cubic Szegő equation (see [2, 3]). In this setting, the phase space of this Hamiltonian system is the Sobolev space

$$H_+^{1/2} = \left\{ u \in L_+^2(\mathbb{T}) : \sum_{k=0}^{\infty} (1+k)|\hat{u}(k)|^2 < \infty \right\}, \tag{15}$$

which is the space of symbols of Hilbert–Schmidt Hankel operators, and the restriction of the mapping χ to the phase space can be interpreted as an action–angle map. In the present paper, we extend this result to compact Hankel operators, which is the natural setting for an inverse spectral problem.

Finally, we would like to comment on the explicit formula above giving $\hat{u}(n)$. The boundedness of the operator A defined by (11) is not trivial. In fact, it is a consequence of the proof of the theorem. However, it is possible to give a direct proof of this boundedness; see appendix B. Furthermore, from the complicated structure of formula (10), it seems difficult to check directly that the corresponding Hankel operators have the right sequences of singular values, namely that the map χ is onto. Our proof is in fact completely different and is based on some compactness argument, while, as in [3], the explicit formula is only used to establish the injectivity of χ.

1.2.5. Description of the kernel. We now state our last result, which describes the kernel of H_u in terms of the $\zeta = \chi(u)$.

Since $\ker H_u$ is invariant by the shift, the Beurling theorem – see e.g. [14] – provides the existence of an inner function φ so that $\ker H_u = \varphi L_+^2$. We use the notation of Theorem 3 to describe φ. Denote by R the range of H_u.

Theorem 4. *We keep the notation of Theorem 3. Let $u \in VMO_{+,\mathrm{gen}}$. The kernel of H_u and the kernel of K_u are reduced to zero if and only if $1 \in \overline{R} \setminus R$ or if and only if the following conditions hold.*

$$\sum_{j=1}^{\infty} \left(1 - \frac{\sigma_j^2}{\rho_j^2}\right) = \infty, \quad \sup_N \frac{1}{\rho_{N+1}^2} \prod_{j=1}^{N} \frac{\sigma_j^2}{\rho_j^2} = \infty. \tag{16}$$

When these conditions are not satisfied, then $\ker H_u = \varphi L_+^2$ with φ inner satisfying the following.

(1) *If 1 does not belong to the closure of the range of H_u i.e. $1 \notin \overline{R}$, then*

$$\varphi(z) = \left(1 - \sum \nu_j^2\right)^{-1/2} \left(1 - \sum_{n \geqslant 0} \alpha_n z^n\right),$$

where

$$\alpha_n = Y \cdot A^n Y. \tag{17}$$

Furthermore, $\ker K_u = \ker H_u = \varphi L_+^2$.

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(2) *If* 1 *belongs to the range of* H_u, *i.e.* $1 \in R$, *then* $\varphi(z) = z\psi(z)$ *with*

$$\psi(z) = \left(\sum_{j=1}^{\infty} \frac{\nu_j^2}{\rho_j^2}\right)^{-1/2} \sum_{n\geqslant 0} \beta_n z^n,$$

where

$$\beta_n = W \cdot A^n Y, \quad W = (\nu_j \zeta_{2j-1} \rho_j^{-2})_{j\geqslant 1}. \tag{18}$$

Furthermore, $\ker K_u = \ker H_u \oplus \mathbb{C} H_u^{-1}(1) = \varphi L_+^2 \oplus \mathbb{C}\psi$.

1.3. Organization of the paper

We end this introduction by describing the organization of this paper. In § 2, we start the proof of Theorem 3. We first recall from [3] a finite-dimensional analogue to Theorem 3. Then we generalize from [3] an important trace formula to arbitrary compact Hankel operators. We then use this formula and the Adamyan–Arov–Krein theorem to derive a crucial compactness lemma about Hankel operators. Using this compactness lemma, we prove Theorem 3 in § 3, and we infer the first part of Theorem 2. § 4 is devoted to the proof of Theorem 4, from which the second part of Theorem 2 easily follows. Finally, for the convenience of the reader, we have gathered in appendix A the main steps of the proof of the finite-dimensional analogue of Theorem 3, while appendix B is devoted to a direct proof of the boundedness of operator A involved in Theorem 3.

2. Preliminary results

The proof of Theorem 3 is based on a finite rank approximation of H_u. We first recall the notation and a similar result obtained on finite rank operators in [3].

2.1. The finite rank result

By a theorem due to Kronecker [5], the Hankel operator H_u is of finite rank if and only if u is a rational function, holomorphic in the unit disc. As in subsection 1.2.2, we consider $\mathcal{V}(2N)$ the set of rational functions u, holomorphic in the unit disc, so that H_u and K_u are of finite rank N. It is elementary to check that $\mathcal{V}(2N)$ is a $2N$-dimensional complex submanifold of L_+^2 (we refer to [2] for a complete description of this set and for an elementary proof of Kronecker's Theorem). We denote by $\mathcal{V}(2N)_{\text{gen}}$ the set of functions $u \in \mathcal{V}(2N)$ such that H_u^2 and K_u^2 have simple distinct eigenvalues $(\rho_j^2)_{1\leqslant j\leqslant N}$ and $(\sigma_m^2)_{1\leqslant m\leqslant N}$, respectively, with

$$\rho_1^2 > \sigma_1^2 > \rho_2^2 > \cdots \rho_N^2 > \sigma_N^2 > 0.$$

As in the introduction, we can define new variables on $\mathcal{V}(2N)_{\text{gen}}$ and a corresponding mapping χ_N. The following result has been proven in [3].

Theorem 5. *The mapping*

$$\chi_N : u \in \mathcal{V}(2N)_{\text{gen}} \longmapsto \zeta = (\zeta_{2j-1} = \rho_j \mathrm{e}^{-i\varphi_j}, \zeta_{2j} = \sigma_j \mathrm{e}^{-i\theta_j})_{1\leqslant j\leqslant N}$$

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is a symplectic diffeomorphism onto

$$\Xi_N := \{\zeta \in \mathbb{C}^{2N}, |\zeta_1| > |\zeta_2| > |\zeta_3| > |\zeta_4| > \cdots > |\zeta_{2N-1}| > |\zeta_{2N}| > 0\}$$

in the sense that the image of the symplectic form ω *by* χ_N *satisfies*

$$(\chi_N)_*\omega = \frac{1}{2i}\sum_{1\leqslant j\leqslant 2N} d\zeta_j \wedge d\overline{\zeta_j}. \tag{19}$$

There is also an explicit formula for the inverse χ_N analogous to the one given in Theorem 3, except that the sums in formulae (10) run over the integers $1, \ldots, N$.

In order to prove the extension of Theorem 5 to $VMO_{+,\mathrm{gen}}$, we have to extend some tools introduced in [3].

2.2. The functional $J(x)$

Let H be a compact selfadjoint antilinear operator on a Hilbert space $\mathcal{H}$. Let $A = H^2$ and $e \in \mathcal{H}$ so that $\|e\| = 1$. Notice that A is selfadjoint, positive, and compact. We define the generating function of H for $|x|$ small by

$$J(x)(A) = 1 + \sum_{n=1}^{\infty} x^n J_n,$$

where $J_n = J_n(A) = (A^n(e)|e)$. Consider the operator

$$B := A - (\,\cdot\,|H(e))H(e),$$

which is also selfadjoint, positive, and compact. Denote by $(a_j)_{j\geqslant 1}$ (respectively, $(b_j)_{j\geqslant 1}$) the non-zero eigenvalues of A (respectively, of B) labelled according to the min–max principle:

$$a_1 \geqslant b_1 \geqslant a_2 \geqslant \cdots .$$

Notice that

$$J(x)(A) = ((I - xA)^{-1}(e)|e),$$

which shows that J extends as an entire meromorphic function, with poles at $x = \frac{1}{a_j}, j \geqslant 1$.

Proposition 1.

$$J(x)(A) = \prod_{j=1}^{\infty} \frac{1 - b_j x}{1 - a_j x}, \quad x \notin \left\{\frac{1}{a_j}, j \geqslant 1\right\}. \tag{20}$$

Proof. Assume first that A and B are in the trace class. In that case, we can compute the trace of the rank 1 operator $(I - xA)^{-1} - (I - xB)^{-1}$. We first write

$$[(I - xA)^{-1} - (I - xB)^{-1}](f) = \frac{x}{J(x)}(f|(I - xA)^{-1}H(e)) \cdot (I - xA)^{-1}H(e).$$

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Consequently, taking the trace, we get

$$\mathrm{Tr}[(I-xA)^{-1}-(I-xB)^{-1}]=\frac{x}{J(x)}\|(I-xA)^{-1}H(e)\|^2.$$

Since, on the one hand,

$$\|(I-xA)^{-1}H(e)\|^2=((I-xA)^{-2}A(e)|e)=J'(x),$$

and on the other hand

$$\begin{aligned}\mathrm{Tr}[(I-xA)^{-1}-(I-xB)^{-1}]&=x\mathrm{Tr}[A(I-xA)^{-1}-B(I-xB)^{-1}]\\&=x\sum_{j=1}^{\infty}\left(\frac{a_j}{1-a_jx}-\frac{b_j}{1-b_jx}\right),\end{aligned}$$

we get

$$\sum_{j=1}^{\infty}\left(\frac{a_j}{1-a_jx}-\frac{b_j}{1-b_jx}\right)=\frac{J'(x)}{J(x)},\quad x\notin\left\{\frac{1}{a_j},\frac{1}{b_j},j\geqslant 1\right\}. \tag{21}$$

From this equation, one gets easily formula (20) for A and B in the trace class. To extend it to compact operators, we first recall that

$$a_j\geqslant b_j\geqslant a_{j+1}.$$

Hence, $\sum_j(a_j-b_j)$ converges when A is compact since $0\leqslant a_j-b_j\leqslant a_j-a_{j+1}$ and a_j tends to zero by compactness of A. Hence, the infinite product in formula (20) converges, and the above computation makes sense for compact operators. □

Lemma 1. *Let $e\in\mathcal{H}$ with $\|e\|=1$. Let (H_p) be a sequence of compact selfadjoint antilinear operators on a Hilbert space $\mathcal{H}$ which converges strongly to H, namely*

$$\forall h\in\mathcal{H},\quad H_ph\xrightarrow[p\to\infty]{}Hh. \tag{22}$$

We assume that H is compact. Let $A_p=H_p^2$, $B_p=A_p-(\cdot|H_p(e))H_p(e)$, and $A=H^2$, $B=A-(\cdot|H(e))H(e)$ be their strong limits. For every $j\geqslant 1$, denote by $\mathcal{F}_j$ the set of linear subspaces of $\mathcal{H}$ of dimension at most j, and set

$$a_j^{(p)}=\min_{F\in\mathcal{F}_{j-1}}\max_{h\in F^\perp,\|h\|=1}(A_p(h)|h),$$
$$b_j^{(p)}=\min_{F\in\mathcal{F}_{j-1}}\max_{h\in F^\perp,\|h\|=1}(B_p(h)|h).$$

Assume that there exist $(\overline{a}_j)$ and $(\overline{b}_j)$ such that

$$\sup_{j\geqslant 1}|a_j^{(p)}-\overline{a}_j|\xrightarrow[p\to\infty]{}0,\quad \sup_{j\geqslant 1}|b_j^{(p)}-\overline{b}_j|\xrightarrow[p\to\infty]{}0,$$

and that the non-zero $\overline{a}_j$, $\overline{b}_m$ are pairwise distinct. Then the positive eigenvalues of A are simple and are exactly the $\overline{a}_j$; similarly, the positive eigenvalues of $B=A-(\cdot|H(e))H(e)$ are simple and are exactly the $\overline{b}_m$.

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Proof. Notice that, by the uniform boundedness principle, the norm of H_p is uniformly bounded, and (22) holds uniformly for h in every compact subset of $\mathcal{H}$. Consequently, for every $h \in \mathcal{H}$, we have

$$\forall n \geqslant 1, \quad A_p^n(h) \xrightarrow[p\to\infty]{} A^n(h).$$

In particular, for every $n \geqslant 1$,

$$J_n(A_p) := (A_p^n(e)|e) \xrightarrow[p\to\infty]{} (A^n(e)|e) := J_n(A),$$

and there exists $C > 0$ such that

$$\forall n \geqslant 1, \quad \sup_p J_n(A_p) \leqslant C^n.$$

Choose $\delta > 0$ such that $\delta C < 1$. Then, for every real number x such that $|x| < \delta$, we have, by dominated convergence,

$$J(x)(A_p) := 1 + \sum_{n=1}^{\infty} x^n J_n(A_p) \xrightarrow[p\to\infty]{} 1 + \sum_{n=1}^{\infty} x^n J_n(A) := J(x)(A).$$

On the other hand, in view of the assumption about the convergence of $(a_j^{(p)})_{j\geqslant 1}$ and $(b_j^{(p)})_{j\geqslant 1}$ and the convergence of the product in formula (20), we also have, for $|x| < \delta$,

$$J(x)(A_p) = \prod_{j=1}^{\infty} \left(\frac{1 - b_j^{(p)} x}{1 - a_j^{(p)} x} \right) \xrightarrow[p\to\infty]{} \prod_{j=1}^{\infty} \left(\frac{1 - \overline{b}_j x}{1 - \overline{a}_j x} \right). \tag{23}$$

Hence, we obtain

$$J(x)(A) = \prod_{j=1}^{\infty} \left(\frac{1 - \overline{b}_j x}{1 - \overline{a}_j x} \right). \tag{24}$$

By assumption, the non-zero $\overline{a}_j, \overline{b}_m$ are pairwise distinct, so no cancellation can occur in the right-hand side of (20), and the poles are all distinct.

On the other hand, denote by (a_j) the family of eigenvalues of A and by (b_j) that of B. By a classical result (see e.g. Lemma 1, § 2.2 of [3]),

$$\{a_j, j \geqslant 1\} \subset \{\overline{a}_j, j \geqslant 1\}, \quad \{b_j, j \geqslant 1\} \subset \{\overline{b}_j, j \geqslant 1\},$$

and the multiplicity of positive eigenvalues is 1. Consequently, there is no cancellation in the expression of $J(x)(A)$, and all the poles are simple. We conclude that $a_j = \overline{a}_j$, $b_j = \overline{b}_j$ for every $j \geqslant 1$. □

2.3. A compactness result

From now on, we choose $\mathcal{H} = L^2_+$ and $e = 1$. As a first application of Proposition 1, we obtain the following.

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Lemma 2. *For any $u \in VMO_+(\mathbb{T})$, we have, for $x \notin \left\{\frac{1}{\rho_j^2(u)}\right\}_{j\geqslant 1}$,*

$$J(x) := J(x)(H_u^2) = \prod_{j=1}^{\infty} \frac{1 - x\sigma_j^2(u)}{1 - x\rho_j^2(u)} = 1 + x \sum_{j=1}^{\infty} \frac{\rho_j^2(u)\,\nu_j^2}{1 - x\rho_j^2(u)}. \tag{25}$$

Here, $\nu_j := |(1|e_j)|$. In particular,

$$\nu_j^2 = \left(1 - \frac{\sigma_j^2}{\rho_j^2}\right) \prod_{k \neq j} \left(\frac{\rho_j^2 - \sigma_k^2}{\rho_j^2 - \rho_k^2}\right).$$

The first equality in (25) is just a consequence of (20). For the second equality in (25), we use the formula

$$J(x) = ((I - xH_u^2)^{-1}(1)|1) = 1 + x((I - xH_u^2)^{-1}(u)|u), \tag{26}$$

and we expand u according to the decomposition

$$(\ker H_u)^{\perp} = \oplus_{j\geqslant 1}\mathbb{C}e_j.$$

Finally, the expression of ν_j^2 is obtained by multiplying both expressions of $J(x)$ in (25) by $(1 - x\rho_j^2(u))$, and by letting x go to $1/\rho_j^2(u)$.

From Lemma 1, we infer the following compactness result, which can be interpreted as a compensated compactness result.

Proposition 2. *Let (u_p) be a sequence of $VMO_+(\mathbb{T})$ weakly convergent to u in $VMO_+(\mathbb{T})$. We assume that, for some sequences $(\overline{\rho}_j)$ and $(\overline{\sigma}_j)$,*

$$\sup_{j\geqslant 1} |\rho_j(u_p) - \overline{\rho}_j| \xrightarrow[p\to\infty]{} 0, \quad \sup_{j\geqslant 1} |\sigma_j(u_p) - \overline{\sigma}_j| \xrightarrow[p\to\infty]{} 0,$$

and the following simplicity assumption: all the non-zero $\overline{\rho}_j$, $\overline{\sigma}_m$ are pairwise distinct. Then, for every $j \geqslant 1$, $\rho_j(u) = \overline{\rho}_j$, $\sigma_j(u) = \overline{\sigma}_j$, and u_p converges to u holds for the norm convergence in $VMO_+(\mathbb{T})$.

Remark 1. Let us emphasize that this result specifically uses the structure of Hankel operators. It is false in general for compact operators assumed to converge only strongly. One also has to remark that the simplicity of the eigenvalues is a crucial hypothesis, as the following example shows. Denote by (u_p), $|p| < 1$, p real, the sequence of functions defined by

$$u_p(z) = \frac{z - p}{1 - pz}.$$

Then, the selfadjoint Hankel operators H_{u_p} and K_{u_p} have eigenvalues $\lambda_1 = \mu_1 = 1$ and $\lambda_2 = -1$, and $\mu_m = \lambda_{m+1} = 0$ for $m \geqslant 2$, independently of p. As p goes to 1, $p < 1$, u_p tends weakly to the constant function -1; hence the convergence is not in the VMO norm, or equivalently not in the operator norm for the corresponding Hankel operator. Indeed, H_{-1} is the rank 1 operator given by $H_{-1}(h) = -(1|h)$; hence H_{-1}^2 is a rank 1 projector while $H_{u_p}^2$ is a rank 2 projector. Therefore,

$$\|H_{u_p}^2 - H_{-1}^2\| \geqslant 1 \quad \text{since } \mathrm{Ran} H_{u_p}^2 \cap \ker H_{-1}^2 \neq \{0\}.$$

Proof. We now start the proof of Proposition 2. Let us first recall the Adamyan–Arov–Krein (AAK) theorem on approximation of Hankel operators by finite rank Hankel operators.

Theorem 6 (Adamyan–Arov–Krein [1]). *Let Γ be a bounded Hankel operator on $L_+^2(\mathbb{T})$. Let $(s_m(\Gamma))_{m\geqslant 1}$ be the family of singular values of Γ labelled according to the min–max principle. Then, for any $m \geqslant 1$, there exists a Hankel operator Γ_m of rank $m-1$ such that*

$$s_m(\Gamma) = \|\Gamma - \Gamma_m\|.$$

In other words, the AAK theorem states that the mth singular value of a Hankel operator, as the distance of this operator to operators of rank $m-1$, is exactly achieved by some Hankel operator of rank $m-1$, and hence, by some Hankel operator with a rational symbol.

This result is crucial in order to obtain our compactness result. We want to apply Lemma 1 with $A = H_u^2$, $B = K_u^2$, and $e = 1$. One has to prove that, for any $h \in L_+^2$, $H_{u_p}^2(h) \to H_u^2(h)$. By the AAK theorem, for any p and any $j \geqslant 1$, there exists a function $u_{p,j} \in \mathcal{V}(2j) \cup \mathcal{V}(2j-1)$ so that

$$\|H_{u_p} - H_{u_{p,j}}\| = \rho_{j+1}(u_p).$$

In particular, we get

$$\|u_p - u_{p,j}\|_{L^2} \leqslant \rho_{j+1}(u_p).$$

On the other hand, one has

$$\|H_{u_{p,j}}\| \geqslant \frac{1}{\sqrt{j}} (\mathrm{Tr}(H_{u_{p,j}}^2))^{1/2} \geqslant \frac{1}{\sqrt{j}} \|u_{p,j}\|_{H_+^{1/2}},$$

where $H_+^{\frac{1}{2}}$ has been defined in (15). Hence, for fixed j, the sequence $(u_{p,j})_p$ is bounded in $H_+^{1/2}$. We are going to prove that the sequence $\{u_p\}_p$ is precompact in L_+^2. We show that, for any $\varepsilon > 0$, there exists a finite sequence $v_k \in L_+^2$, $1 \leqslant k \leqslant M$, so that $\{u_p\}_p \subset \cup_{k=1}^M B_{L_+^2}(v_k, \varepsilon)$. Let j be fixed so that $\sup_p \rho_{j+1}(u_p) \leqslant \varepsilon/2$. Since the sequence $(u_{p,j})_p$ is uniformly bounded in $H_+^{1/2}$, there is a subsequence which converges weakly in $H_+^{1/2}$. In particular, it is precompact in L_+^2; hence, there exists $v_k \in L_+^2$, $1 \leqslant k \leqslant M$ so that $\{u_{p,j}\}_p \subset \cup B_{L_+^2}(v_k, \varepsilon/2)$. Then, for every p, there exists a k such that

$$\|u_p - v_k\|_{L^2} \leqslant \rho_{j+1}(u_p) + \|u_{p,j} - v_k\|_{L^2} \leqslant \varepsilon.$$

Therefore $\{u_p\}$ is precompact in L_+^2 and, since L^2 is complete, some subsequence of (u_p) has a strong limit in L_+^2. Since u_p converges weakly to u, this limit has to be u, and we conclude that the whole sequence (u_p) is strongly convergent to u in L_+^2. Since $\|H_{u_p}\| \simeq \|u_p\|_{BMO}$ is bounded, we infer the strong convergence of operators:

$$\forall h \in L_+^2, \quad H_{u_p}(h) \xrightarrow[p\to\infty]{} H_u(h).$$

By Lemma 1, for every k we have $\rho_k(u) = \overline{\rho}_k$ and $\sigma_k(u) = \overline{\sigma}_k$. We now want to prove that

$$\|H_{u_p} - H_u\| \to 0.$$

Let us distinguish two cases.
First case: for every $j \geqslant 1$, $\overline{\rho}_j > 0$. We come back to the AAK situation above. For every j, we select $u_{p,j} \in \mathcal{V}(2j) \cup \mathcal{V}(2j-1)$ so that

$$\|H_{u_p} - H_{u_{p,j}}\| = \rho_{j+1}(u_p).$$

Since the operator norm is lower semicontinuous for the strong convergence of operators, we infer that any limit point $\tilde{u}_j$ of $u_{p,j}$ in L^2_+ as $p \to \infty$ satisfies

$$\|H_u - H_{\tilde{u}_j}\| \leqslant \overline{\rho}_{j+1}.$$

In particular, $|\overline{\sigma}_j - \sigma_j(\tilde{u}_j)| \leqslant \overline{\rho}_{j+1}$; hence $\sigma_j(\tilde{u}_j) > 0$, and thus $\tilde{u}_j \in \mathcal{V}(2j)$. Using the following elementary lemma, the proof is then completed by the triangle inequality.

Lemma 3. *Let N be a positive integer and $w_p \in \mathcal{V}(2N) \cup \mathcal{V}(2N-1)$ such that $w_p \xrightarrow[p\to\infty]{} w$ in L^2_+. Assume that $w \in \mathcal{V}(2N) \cup \mathcal{V}(2N-1)$. Then $\|H_{w_p} - H_w\| \xrightarrow[p\to\infty]{} 0$.*

Let us postpone the proof of Lemma 3 to the end of the argument.
Second case: there exists $k \geqslant 1$ such that $\overline{\rho}_k$=0. We denote by j the greatest $k \geqslant 1$ such that $\overline{\rho}_k > 0$. Of course we may assume that there exists such a j, otherwise this would mean that $\|H_{u_p}\|$ tends to 0, a trivial case. For such a j, we again write

$$\|H_{u_p} - H_{u_{p,j}}\| = \rho_{j+1}(u_p),$$

and, passing to the limit, we conclude that $u_{p,j}$ is strongly convergent to u in L^2_+. Using again Lemma 3, we conclude that $\|H_{u_{p,j}} - H_u\|$ tends to 0, and the proof is again completed by the triangle inequality.

Finally, let us prove Lemma 3. Recall the explicit description of $\mathcal{V}(d)$; see e.g. [2]. Elements of $\mathcal{V}(2N)$ are rational functions of the following form,

$$w(z) = \frac{A(z)}{B(z)},$$

where A, B have no common factors, B has no zeros in the closed unit disc, $B(0) = 1$, and $\deg(A) \leqslant N-1$, $\deg(B) = N$. Elements of $\mathcal{V}(2N-1)$ have the same form, except that the last part is replaced by $\deg(A) = N-1$, $\deg(B) \leqslant N-1$.

Write similarly

$$w_p(z) = \frac{A_p(z)}{B_p(z)}.$$

By the Cauchy formula, we have, for every z in the unit disc,

$$w_p(z) \to w(z).$$

Since

$$B_p(z) = \prod_{k=1}^{N} (1 - b_{k,p} z)$$

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with $|b_{k,p}| < 1$, we may assume that, up to extracting a subsequence,

$$B_p(z) \to \tilde{B}(z) = \prod_{k=1}^{N} (1 - \tilde{b}_k z),$$

with $|\tilde{b}_k| \leqslant 1$. Multiplying by $B_p(z)$ and passing to the limit, we get, for every z in the unit disc,

$$A_p(z) \to \tilde{B}(z) \frac{A(z)}{B(z)} =: \tilde{A}(z).$$

Since $\tilde{B}A$ is divisible by B, $\tilde{B}$ is divisible by B. On the other hand, we claim that $\deg(\tilde{B}) \leqslant \deg(B)$. Indeed, either $w \in \mathcal{V}(2N)$, and $\deg(B) = N \geqslant \deg(\tilde{B})$, or $w \in \mathcal{V}(2N-1)$, and $\deg(A) = N - 1 \geqslant \deg(\tilde{A})$. In both cases, we conclude that $\tilde{B} = B$, which means that the numbers $b_{k,p}$ stay away from the unit circle. Consequently, the convergence of $w_p(z)$ to $w(z)$ holds uniformly on a disc $D(0, r)$ for some $r > 1$; thus, say, $w_p \to w$ in $H^s(\mathbb{T})$ for every $s > 0$. Choosing $s = \frac{1}{2}$, we conclude that H_{w_p} converges to H_w in the Hilbert–Schmidt norm, and hence in the operator norm. □

3. Proof of Theorem 3 and of the first part of Theorem 2

3.1. The surjectivity of χ

Let $(\zeta_p)_{p\geqslant 1}$ be an element in Ξ. We want to prove the existence of $u \in VMO_{+,\mathrm{gen}}$ so that $\chi(u) = (\zeta_p)_{p\geqslant 1}$. We are going to use the finite rank result. By Theorem 5, for every N we construct $u_N \in \mathcal{V}(2N)$ via the diffeomorphism χ_N by letting

$$\chi_N(u_N) = (\zeta_p)_{1\leqslant p\leqslant 2N}.$$

The sequence (u_N) satisfies $\|H_{u_N}\| = \rho_1(u_N) = |\zeta_1|$; hence it is bounded in VMO, and therefore has a subsequence, still denoted by (u_N), which is weakly convergent to u in VMO_+. We can then apply Proposition 2; hence u is the limit of (u_N) in the $VMO_+(\mathbb{T})$ norm, so that

$$\rho_j(u) = |\zeta_{2j-1}| := \rho_j, \quad \sigma_j(u) = |\zeta_{2j}| := \sigma_j.$$

In particular, $u \in VMO_{+,\mathrm{gen}}$. It remains to consider the convergence of the angles and hence of the eigenvectors. Let j be fixed. For $N > j$, denote by $e_{j,N}$ the normalized eigenvector of $H_{u_N}^2$ related to the simple eigenvalue ρ_j^2 so that $H_{u_N}(e_{j,N}) = \rho_j\, e_{j,N}$. Since $(e_{j,N})$ is a sequence of unitary vectors, it has a weakly convergent subsequence to some vector $\tilde{e}_j$. We now show that the convergence is in fact strong. Let us consider the operator

$$P_{j,N} = \int_{\mathcal{C}_j} (zI - H_{u_N}^2)^{-1} \frac{dz}{2i\pi},$$

where $\mathcal{C}_j$ is a small circle around ρ_j^2. If $\mathcal{C}_j$ is sufficiently small, then

$$P_{j,N}(h) = (h|e_{j,N})e_{j,N}.$$

By the convergence of H_{u_N} to H_u, we have, for any $h \in L^2$,

$$P_{j,N}(h) \to P_j(h),$$

where P_j is the projector onto the eigenspace of H_u^2 corresponding to ρ_j^2. Denoting by e_j a unitary vector of this eigenline, we get that, for any $h \in L^2$,

$$(h|e_{j,N})e_{j,N} \to (h|e_j)e_j.$$

Since $(h|e_{j,N})$ converges to $(h|\tilde{e}_j)$ by weak convergence, and on the other hand $|P_{j,N}(e_{j,N})| = |(h|e_{j,N})|$ tends to $\|P_j(h)\| = |(h|e_j)|$, we get that $|(h|\tilde{e}_j)| = |(h|e_j)|$ for any h in L^2; hence $\tilde{e}_j = \mathrm{e}^{i\Psi} e_j$ is unitary. We conclude that the convergence of $e_{j,N}$ to $\tilde{e}_j$ is strong, since the convergence is weak and the vectors are unitary. Hence $H_{u_N}(e_{j,N}) = \rho_{j,N} e_{j,N}$ converges to $H_u(\tilde{e}_j) = \rho_j \tilde{e}_j$, and the angles $\arg(1|e_{j,N})^2$ converge to $\arg(1|\tilde{e}_j)^2$. The same holds for the eigenvectors of K_{u_N}. We conclude that there exists $u \in VMO_{+,\mathrm{gen}}$ with $\chi(u) = (\zeta_p)_{p\geqslant 1}$. The mapping χ is onto.

The second step is to prove that χ is one-to-one. It comes from an explicit formula giving u in terms of $\chi(u)$.

3.2. An explicit formula via the compressed shift operator

We are going to use the well-known link between the shift operator and the Hankel operators. Namely, if T_z denotes the shift operator, recall formula (4):

$$H_u T_z = T_z^* H_u.$$

With the notation introduced in the introduction, it reads

$$K_u = T_z^* H_u,$$

which, as we already observed in the introduction, leads to formula (5). We introduce the compressed shift operator [10, 11, 13]

$$S := P_u T_z,$$

where P_u denotes the orthogonal projector onto the closure of the range of H_u. By property (4), $\ker H_u = \ker P_u$ is stable by T_z; hence

$$S = P_u T_z P_u,$$

so that S is an operator from the closure of the range of H_u into itself, and

$$S^n = P_u T_z^n P_u. \tag{27}$$

In what follows, we shall always denote by S the induced operator on the closure of the range of H_u, and by S^* the adjoint of this operator.

Now observe that the operator S arises in the Fourier series decomposition of u, namely

$$u(z) = \sum_{n=0}^{\infty} \hat{u}(n) z^n,$$

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where, using (27),

$$\hat{u}(n) = (u|z^n) = (u|T_z^n(1)) = (u|S^n P_u(1)). \tag{28}$$

As a consequence, we have, for $|z| < 1$,

$$u(z) = (u|(I - \bar{z}S)^{-1} P_u(1)), \tag{29}$$

which makes sense since $\|S\| \leqslant 1$. By studying the spectral properties of K_u^2, one obtains the following lemma.

Lemma 4. *The sequence* $(g_j)_{j\geqslant 1}$ *defined by* $g_j = (H_u^2 - \sigma_j I)^{-1}(u)$ *is an orthogonal basis of the closed range of* K_u*, on which the compressed shift operator acts as*

$$S(g_j) = \sigma_j \, \mathrm{e}^{i\theta_j} \, h_j, \quad h_j := (H_u^2 - \sigma_j^2 I)^{-1} P_u(1).$$

Proof. For $\sigma > 0$, we solve the eigenvalue equation

$$K_u^2 g = \sigma^2 g$$

by appealing to formula (5), which we reproduce here for the convenience of the reader:

$$K_u^2 = H_u^2 - (\cdot|u)u.$$

This yields

$$(H_u^2 - \sigma^2 I)g = (g|u)u,$$

and

$$g = (g|u)(H_u^2 - \sigma^2 I)^{-1} u.$$

The condition on σ is

$$((H_u^2 - \sigma^2 I)^{-1} u|u) = 1, \tag{30}$$

which, in view of formula (26), is equivalent to

$$J\left(\frac{1}{\sigma^2}\right) = 0,$$

where J was defined in (25). This means that σ has to be one of the σ_j. The eigenvector of K_u^2 is therefore given by

$$g_j = (H_u^2 - \sigma_j I)^{-1}(u).$$

As j varies, it gives rise to an orthogonal basis of the closed range of K_u^2, which is also the closed range of K_u. By the genericity assumption, g_j is proportional to the unit eigenvector f_j introduced in § 1.2.3; see formula (8). This reads

$$g_j = \|g_j\| \mathrm{e}^{i\psi_j} f_j.$$

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Notice that, from (30),

$$(u|g_j) = 1.$$

In view of (9), this implies that

$$\theta_j = 2\psi_j,$$

and

$$K_u(g_j) = \|g_j\| \mathrm{e}^{-i\psi_j} K_u(f_j) = \|g_j\| \mathrm{e}^{-i\psi_j} \sigma_j f_j = \sigma_j \mathrm{e}^{-i\theta_j} g_j.$$

Since

$$g_j = H_u(h_j), \quad h_j = (H_u^2 - \sigma_j^2 I)^{-1} P_u(1) \in (\ker H_u)^\perp,$$

and $K_u = H_u S$, and since H_u is one-to-one on $(\ker H_u)^\perp$, this completes the proof. □

To obtain an explicit formula from formula (29), it is sufficient to express the action of S on a basis of the closure $\overline{R}$ of the range of H_u.

Hence, when the closure of the range of H_u and the closure of the range of K_u coincide, one can conclude from this lemma, Lemma 2, and equation (29) to obtain the explicit formula writing everything in the basis $(\tilde{e}_j)_{j\geqslant 1}$ of $\overline{R}$, where

$$\tilde{e}_j := \mathrm{e}^{i\varphi_j/2} e_j. \tag{31}$$

More precisely, in the basis $(\tilde{e}_j)$, one easily checks that the components of u are $(\nu_j \zeta_{2j-1})$, the components of $P_u(1)$ are (ν_j), and the matrix of S is $\overline{A}$.

If the range of K_u is strictly included in the range of H_u, there exists g in the range of H_u so that $K_u g = 0 = T_z^* H_u g$, and hence $H_u g$ is a non-zero constant; in particular, 1 belongs to the range of H_u. Let us write $1 = H_u g_0$. In this case, an orthogonal basis of the closure of the range of H_u is given by the sequence $(g_m)_{m\geqslant 0}$ and, since $K_u(g_0) = 0 = H_u S(g_0)$, $S(g_0) = 0$. So we obtain the same explicit formula for u in terms of $\chi(u)$. This proves that the mapping χ is one-to-one.

To prove that χ is a homeomorphism, it remains to prove that χ^{-1} is continuous on Ξ. One has to prove that, if $\chi(u_p)$ tends to $\chi(u)$, then (u_p) tends to u in VMO. It is straightforward from Proposition 2 that (u_p) has a subsequence which converges to v in the VMO norm. Since χ is continuous and one-to-one, we get $v = u$.

3.3. The case of real Fourier coefficients

Finally, let us infer the first part of Theorem 2 from Theorem 3. First, we claim that the elements of $VMO_{+,\mathrm{gen}}$ with real Fourier coefficients correspond via the map χ to elements $\zeta \in \Xi$ which are real valued. Indeed, if ζ is real valued, the explicit formula (10) clearly implies that $\hat{u}(n)$ is real for every n. Conversely, if $u \in VMO_{+,\mathrm{gen}}$ has real Fourier coefficients, then H_u and K_u are compact selfadjoint operators on the closed real subspace of L^2_+ consisting of functions with real Fourier coefficients. Consequently, they admit orthonormal bases of eigenvectors in this space. Therefore we can

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write

$$H_u(\tilde e_j)=\lambda_j\tilde e_j,\quad \lambda_j=\pm\rho_j,\quad K_u(\tilde f_m)=\mu_m\tilde f_m,\quad \mu_m=\pm\sigma_m,$$

where $\tilde e_j$ and $\tilde f_m$ are unitary vectors with real Fourier coefficients. Since ρ_j^2 and σ_m^2 are simple eigenvalues of H_u^2 and K_u^2, respectively, we conclude that $\tilde e_j$ is collinear to e_j, and similarly that $\tilde f_m$ is collinear to f_m. More precisely, since H_u and K_u are antilinear,

$$\tilde e_j=\begin{cases}\pm e_j & \text{if } \lambda_j=\rho_j\\ \pm ie_j & \text{if } \lambda_j=-\rho_j;\end{cases}\qquad \tilde f_m=\begin{cases}\pm f_m & \text{if } \mu_m=\sigma_m\\ \pm if_m & \text{if } \mu_m=-\sigma_m.\end{cases}$$

Since $(1|\tilde e_j)$ and $(u|\tilde f_m)$ are real, we conclude that

$$\varphi_j=\begin{cases}0 & \text{if } \lambda_j=\rho_j\\ \pi & \text{if } \lambda_j=-\rho_j;\end{cases}\qquad \theta_m=\begin{cases}0 & \text{if } \mu_m=\sigma_m\\ \pi & \text{if } \mu_m=-\sigma_m.\end{cases}$$

Therefore, $\zeta_{2j-1}=\lambda_j$ and $\zeta_{2j}=\mu_j$. This completes the proof.

4. Proof of Theorem 4

Proof. We already observed that $\ker H_u\subset\ker K_u$ and that the inclusion is strict if and only if $1\in R$, and, in that case, $\ker K_u=\ker H_u\oplus\mathbb{C}H_u^{-1}(1)$. Hence, in the following, we focus on the kernel of H_u.

We first prove that $\ker H_u=\{0\}$ if and only if $1\in\overline{R}\setminus R$.

Since $\ker H_u=\{0\}$ is equivalent to $\overline{R}=L^2_+$, $\ker H_u=\{0\}$ implies that $1\in\overline{R}$. If $1\in R$, then there exists $w\in L^2_+$ so that $1=H_u(w)$. If we introduce the function $\psi=zw$, then $H_u(\psi)=T_z^*H_u(w)=T_z^*(1)=0$. This implies that ψ belongs to $\ker H_u$ and that $\psi\neq 0$. Hence, $\ker H_u=\{0\}$ implies that $1\in\overline{R}\setminus R$.

Let us prove the converse. Assume that $\ker H_u\neq\{0\}$ and that $1\in\overline{R}$. Let us show that $1\in R$. By the Beurling theorem, we have $\ker H_u=\varphi L^2_+$ for some inner function φ. Since 1 belongs to $\overline{R}$, it is orthogonal to $\ker H_u$; hence $(1|\varphi)=0$. This implies that $\varphi=zw$ for some w and, since $H_u(\varphi)=0=T_z^*H_u(w)$, we get that $H_u(w)$ is a non-zero constant (if $H_u(w)=0$, w should be divisible by φ, which is impossible, since $\varphi=zw$). Eventually, we get that the constants are in R, and so is 1. Hence we proved that $\ker H_u\neq\{0\}$ if and only if either 1 belongs to R or 1 does not belong to $\overline{R}$.

It remains to prove that the property $1\in\overline{R}\setminus R$ is equivalent to formulae (16) that we recall here:

$$\sum_{j=1}^{\infty}\left(1-\frac{\sigma_j^2}{\rho_j^2}\right)=\infty,\qquad \sup_N\frac{1}{\rho_{N+1}^2}\prod_{j=1}^{N}\frac{\sigma_j^2}{\rho_j^2}=\infty.$$

First, $1\in\overline{R}$ if and only if $\sum_{j=1}^{\infty}\nu_j^2=1$, where $\nu_j^2=|(1|e_j)|^2$;, see Lemma 2. Letting x tend to $-\infty$ in formula (25), and using the monotone convergence theorem,

$$\prod_{j=1}^{\infty}\frac{\sigma_j^2}{\rho_j^2}=1-\sum_{j=1}^{\infty}\nu_j^2.$$

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This gives the first condition. We claim that 1 belongs to R if and only if

$$\sum_{j=1}^{\infty} \frac{\nu_j^2}{\rho_j^2} < \infty.$$

Indeed, it is a necessary and sufficient condition to be able to define

$$w = \sum_{j=1}^{\infty} \frac{\nu_j}{\rho_j} \mathrm{e}^{-i\varphi_j/2} e_j, \tag{32}$$

where the φ_j were defined in (7), so that $H_u(w) = 1$.

We now show that this condition is equivalent to

$$\sup_N \frac{1}{\rho_{N+1}^2} \prod_{j=1}^{N} \frac{\sigma_j^2}{\rho_j^2} < \infty.$$

Let us denote by p_N the quantity

$$p_N := \frac{1}{\rho_{N+1}^2} \prod_{j=1}^{N} \frac{\sigma_j^2}{\rho_j^2},$$

and let us show that $\sup_N p_N < \infty$. Indeed, the sequence (p_N) is increasing, and

$$\sum_{j=1}^{\infty} \frac{\nu_j^2}{\rho_j^2} = -\lim_{x\to\infty} xJ(x) = \lim_{y\to\infty} F(y), \quad F(y) := y \prod_{j=1}^{\infty} \frac{1 + y\sigma_j^2}{1 + y\rho_j^2}. \tag{33}$$

Here we used Lemma 2 and the equality $\sum_{j=1}^{\infty} \nu_j^2 = 1$ so that $J(x) = \sum_{j=1}^{\infty} \frac{\nu_j^2}{1 - x\rho_j^2(u)}$. Let us define

$$F_N(y) = \frac{y}{1 + y\rho_{N+1}^2} \prod_{j=1}^{N} \frac{1 + y\sigma_j^2}{1 + y\rho_j^2} = \frac{y}{1 + y\rho_1^2} \prod_{j=1}^{N} \frac{1 + y\sigma_j^2}{1 + y\rho_{j+1}^2}.$$

Then, this quantity is increasing with respect to N and to y; hence

$$\sup_N p_N = \sup_N \sup_y F_N(y) = \sup_y \sup_N F_N(y) = \sup_y F(y) < \infty.$$

Now, we prove the formulae (17) and (18), which give the generators of the kernels.

We first consider the case when $1 \notin \overline{R}$. Since $1 - P_u(1)$ belongs to $\ker H_u$, $1 - P_u(1) = \varphi f$ for some $f \in L_+^2$. Let us remark that, for any $h \in \ker H_u$, $\overline{(1 - P_u(1))h}$ is holomorphic. Indeed, for any $k \geqslant 1$, one has

$$(\overline{(1 - P_u(1))}h | \overline{z}^k) = (z^k h | 1 - P_u(1)) = 0 - (z^k h | P_u(1)) = 0,$$

the last equality coming from the fact that $z^k h \in \ker H_u$. In particular, picking $h = \varphi$ in $\ker H_u$, the modulus of which is 1 almost everywhere, this implies that $\overline{f}$ is holomorphic;

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hence it is a constant. We get that $\varphi = \frac{1-P_u(1)}{\|1-P_u(1)\|}$. One can write, as in formula (28),

$$1 - P_u(1) = 1 - \sum (P_u(1)|S^n P_u(1)) z^n.$$

In order to get the explicit formula for φ, we write this equality in the orthonormal basis $(\tilde{e}_j)$ defined by (31), keeping in mind that

$$\|1 - P_u(1)\|^2 = 1 - \|P_u(1)\|^2 = 1 - \sum_{j=1}^{\infty} \nu_j^2,$$

the components of $P_u(1)$ are (ν_j), and the matrix of S is $\overline{A}$.

It remains to consider the case $1 \in R$. Then, one can choose $w \in \overline{R}$ so that $H_u(w) = 1$. In particular, $H_u(zw) = T_z^* H_u(w) = 0$ so that $zw = \varphi f$ for some f in L^2_+. As before, one can prove that, for any $h \in \ker H_u$, $\overline{zw}h$ is holomorphic; hence $\overline{f}$ is holomorphic, and hence is constant. Eventually, in this case, we obtain $\varphi = z\frac{w}{\|w\|}$. The explicit formula follows from direct computation as before, since

$$\hat{w}(n) = (w|S^n P_u(1)),$$

and since, by equation (32), the components of w in the basis $(\tilde{e}_j)$ are $(\nu_j \zeta_{2j-1}/\rho_j^2)$. □

Appendix A. The finite rank case

In this appendix, we give a sketch of the proof of Theorem 5, referring to [3] for details. The mapping χ_N is of course well defined and smooth on $\mathcal{V}(2N)_{\text{gen}}$. The explicit formula of u in terms of $\chi_N(u)$ is obtained as before thanks to the compressed shift operator, and it proves that χ_N is one-to-one.

A.1. A local diffeomorphism

To prove that χ_N is a local diffeomorphism, we establish some identities on the Poisson brackets. This set of identities implies that the differential of χ_N is of maximal rank so that χ_N is a local diffeomorphism. As a consequence, it is an open mapping.

Let us first recall some basic definitions on Hamiltonian formalism. Given a smooth real-valued function F on a finite-dimensional symplectic manifold $(\mathcal{M}, \omega)$, the Hamiltonian vector field of F is the vector field X_F on $\mathcal{M}$ defined by

$$\forall m \in \mathcal{M}, \forall h \in T_m\mathcal{M}, \quad dF(m) \cdot h = \omega(h, X_F(m)).$$

Given two smooth real valued functions F, G, the Poisson bracket of F and G is

$$\{F, G\} = dG.X_F = \omega(X_F, X_G).$$

The above identity is generalized to complex valued functions F, G by $\mathbb{C}$-bilinearity.

To obtain that the image of the symplectic form ω by χ_N is given by formula (19), one has to prove equivalently that

$$(\chi_N)_* \omega = \sum_j \rho_j d\rho_j \wedge d\varphi_j + \sigma_j d\sigma_j \wedge d\theta_j,$$

which includes the following identities.

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Proposition 3. *For any $j,k\in\{1,\dots,N\}$, one has*

$$\{\rho_j,\rho_k\}=\{\rho_j,\sigma_k\}=\{\sigma_j,\sigma_k\}=0$$
$$\{\rho_j,\varphi_k\}=\rho_j^{-1}\delta_{jk},\quad \{\sigma_j,\varphi_k\}=0,$$
$$\{\rho_j,\theta_k\}=0,\quad \{\sigma_j,\theta_k\}=\sigma_j^{-1}\delta_{jk}.$$

In order to compute for instance $\{\sigma_j,\theta_k\}$, one has for instance to differentiate θ_k along the direction of X_{σ_j}. Since the expression of X_{σ_j} is fairly complicated, we use the 'Szegő hierarchy' studied in [2]. More precisely, we use the generating function $J(x)=((I-xH_u^2)^{-1}(1)|1)=1+\sum_{n=1}^{\infty}x^nJ_{2n}$. In what follows, we shall restrict ourselves to real values of x, so that $J(x)$ is a real-valued function.

We proved in [2] that the Hamiltonian flow associated to $J(x)$ as a function of u admits a Lax pair involving the Hankel operator H_u. From this Lax pair, one can deduce easily a second one involving the operator K_u.

Theorem 7 (The Szegő hierarchy, [2], Theorem 8.1 and Corollary 8). *Let $s>\frac{1}{2}$. The map $u\mapsto J(x)$ is smooth on H^s_+. Moreover, the equation $\partial_t u=X_{J(x)}(u)$ implies that $\partial_t H_u=[B_u^x,H_u]$, or $\partial_t K_u=[C_u^x,K_u]$, where B_u^x and C_u^x are skew-adjoint if x is real.*

Remark 2. As a direct consequence, the spectrum of H_u as well as the spectrum of K_u is conserved by the Hamiltonian flow of $J(x)$. We infer that the Poisson brackets of $J(x)$ with ρ_j or σ_j are zero, which implies, in view of Lemma 2, that the brackets of ρ_k or σ_ℓ with ρ_j or σ_m are zero; hence this gives the first set of commutation properties stated in Proposition 3.

Using the Szegő hierarchy, we can also compute the Poisson brackets of $J(x)$ with the angles.

Lemma 5.

$$\{J(x),\varphi_j\}=\frac{1}{2}\frac{xJ(x)}{1-\rho_j^2x}\quad \{J(x),\theta_j\}=-\frac{1}{2}\frac{xJ(x)}{1-\sigma_j^2x}.$$

Using again the expression of $J(x)$, these commutation properties allow to obtain by identification of the polar parts the last commutation properties of Proposition 3.

To conclude that the image of the symplectic form ω is given by formula (19), we need to establish the following remaining commutation properties:

$$\{\varphi_j,\varphi_k\}=\{\varphi_j,\theta_k\}=\{\theta_j,\theta_k\}=0.$$

In [3], these identities are obtained as consequences of further calculations. Here we give a simpler argument. By Lemma 3, one can write

$$(\chi_N)_*\omega=\sum_j \rho_j d\rho_j\wedge d\varphi_j+\sigma_j d\sigma_j\wedge d\theta_j+\tilde{\omega},$$

where $\tilde{\omega}$ is a closed form depending only on variables ρ_j, σ_m. Consider the following real submanifold of $\mathcal{V}(2N)_{\rm gen}$:

$$\Lambda_N = \{u \in \mathcal{V}(2N)_{\rm gen} : \varphi_1 = \cdots = \varphi_N = \theta_1 = \cdots = \theta_N = 0\}.$$

By formula (10), every element u of Λ_N has real Fourier coefficients. Consequently, $\omega = 0$ on Λ_N. On the other hand, $(\chi_N)_* \omega = \tilde{\omega}$ on $\chi_N(\Lambda_N)$, and the ρ_j, σ_m are coordinates on Λ_N. We conclude that $\tilde{\omega} = 0$.

A.2. Surjectivity: a compactness result

Since Ξ_N is connected, it suffices to prove that χ_N is proper. Let us take a sequence $(\zeta^{(p)})_p$ in Ξ_N which converges to $\zeta \in \Xi_N$, and such that, for every p, there exists $u_p \in \mathcal{V}(2N)_{\rm gen}$ with

$$\chi_N(u_p) = \zeta^{(p)}.$$

Since

$$\|u_p\|_{VMO} = \|H_{u_p}\| = \max_{1 \leqslant j \leqslant N} (\rho_j^{(p)}) = \max_{1 \leqslant j \leqslant N} (|\zeta_{2j-1}^{(p)}|),$$

(u_p) is a bounded sequence in $VMO_+(\mathbb{T})$. Up to extracting a subsequence, we may assume that $(u_p)_{p \in \mathbb{Z}_+}$ converges weakly to some u in $VMO_+(\mathbb{T})$. At this stage we can appeal to Proposition 2 and conclude that the convergence of u_p to u holds for the VMO norm, and that

$$\rho_j(u) = |\zeta_{2j-1}|, \quad \sigma_j(u) = |\zeta_{2j}|, \quad j = 1, \ldots, N$$

with $\rho_j(u) = 0$, $\sigma_j(u) = 0$ if $j > N$. Therefore $u \in \mathcal{V}(2N)_{\rm gen}$. This completes the proof of the surjectivity of χ_N.

Appendix B. The boundedness of the operator A

In this appendix, we prove the boundedness of the operator A defined by (11) in Theorem 3. Of course, this boundedness follows from the theorem itself, since it implies that A is conjugated to the compressed shift operator. However, we found interesting to give a self-contained proof of this fact. We need the following two lemmas.

Lemma 6. *Let $(\rho_j)_{j \geqslant 1}$ and $(\sigma_j)_{j \geqslant 1}$ be two sequences such that*

$$\rho_1^2 > \sigma_1^2 > \rho_2^2 > \cdots \to 0.$$

Then, the following quantities are well defined and coincide respectively outside $\{\frac{1}{\rho_j^2}\}_{j \geqslant 1}$ and $\{\frac{1}{\sigma_j^2}\}_{j \geqslant 1}$:

$$\prod_{j=1}^{\infty} \frac{1 - x\sigma_j^2}{1 - x\rho_j^2} = 1 + x \sum_{j=1}^{\infty} \frac{\nu_j^2 \rho_j^2}{1 - x\rho_j^2} \tag{34}$$

$$\prod_{j=1}^{\infty} \frac{1 - x\rho_j^2}{1 - x\sigma_j^2} = 1 - x \left(C + \sum_{j=1}^{\infty} \frac{\kappa_j^2}{1 - x\sigma_j^2} \right), \tag{35}$$

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where

$$C = \begin{cases} 0 \quad if \sum_{j=1}^{\infty} \nu_j^2 < 1 \quad or \sum_{j=1}^{\infty} \nu_j^2 = 1 \quad and \sum_{j=1}^{\infty} \nu_j^2 \rho_j^{-2} = \infty \\ \left(\sum_{j=1}^{\infty} \nu_j^2 \rho_j^{-2} \right)^{-1} \quad if \sum_{j=1}^{\infty} \nu_j^2 = 1 \quad and \sum_{j=1}^{\infty} \nu_j^2 \rho_j^{-2} < \infty. \end{cases}$$

Here, the ν_j^2 *are given by formula (12) and the* κ_j^2 *by formula (13).*

Remark 3. Notice that formulae (34) and (35) can be interpreted in light of Theorem 3, as we did in Lemma 2. More precisely, formula (34) gives the value of $J(x) = ((I - xH_u^2)^{-1}(1)|1) = 1 + x((I - xH_u^2)^{-1}u|u)$, while formula (35) gives the value of $1/J(x) = 1 - x((I - xK_u^2)^{-1}u|u)$. This provides an interpretation of the constant C, as the contribution of $\ker(K_u) \cap \overline{\mathrm{Ran} H_u}$ in the expansion.

Proof. We first consider finite sequences $(\rho_j)_{1 \leqslant j \leqslant N}$ and $(\sigma_j)_{1 \leqslant j \leqslant N}$ such that $\rho_1^2 > \sigma_1^2 > \rho_2^2 > \cdots > \sigma_N^2 > 0$. We claim that, for $x \notin \{\frac{1}{\rho_j^2}\}_{j \geqslant 1}$,

$$\prod_{j=1}^{N} \frac{1 - x\sigma_j^2}{1 - x\rho_j^2} = 1 + x \sum_{j=1}^{N} \frac{(\nu_j^{(N)})^2 \rho_j^2}{1 - x\rho_j^2}, \tag{36}$$

where

$$(\nu_j^{(N)})^2 = \left(1 - \frac{\sigma_j^2}{\rho_j^2}\right) \prod_{\substack{k \neq j \\ 1 \leqslant k \leqslant N}} \frac{\rho_j^2 - \sigma_k^2}{\rho_j^2 - \rho_k^2}.$$

Indeed, both functions have the same poles and the same residue; hence their difference is a polynomial. Moreover, this polynomial function tends to a constant at infinity, and hence is a constant. Since both terms coincide at $x = 0$, they coincide everywhere. It remains to let $N \to \infty$. The left-hand side in (36) tends to

$$\prod_{j=1}^{\infty} \frac{1 - x\sigma_j^2}{1 - x\rho_j^2},$$

since this product converges in view of the assumption on the sequences (ρ_j) and (σ_j).

Let us consider the limit of the right-hand side in equality (36). Let x tend to $-\infty$ in equality (36). We get

$$\prod_{j=1}^{N} \frac{\sigma_j^2}{\rho_j^2} = 1 - \sum_{j=1}^{N} (\nu_j^{(N)})^2.$$

In particular, $\sum_{j=1}^{N} (\nu_j^{(N)})^2$ is bounded by 1, so $\sum_{j=1}^{\infty} \nu_j^2$ converges by Fatou's lemma. For $x \leqslant 0$,

$$\sum_{N \geqslant j \geqslant M} \frac{\rho_j^2 (\nu_j^{(N)})^2}{1 - x\rho_j^2} \leqslant \rho_M^2,$$

and hence the series $\sum_{N\geqslant j\geqslant 1}\frac{\rho_j^2(\nu_j^{(N)})^2}{1-x\rho_j^2}$ is uniformly summable on $\mathbb{R}^-$, and we infer that

$$\sum_{j=1}^{N}\frac{\rho_j^2(\nu_j^{(N)})^2}{1-x\rho_j^2}\to\sum_{j=1}^{\infty}\frac{\rho_j^2\nu_j^2}{1-x\rho_j^2},\qquad x\leqslant 0.$$

This gives the first equality (34) for $x\leqslant 0$ and for $x\notin\{\frac{1}{\rho_j^2}\}_{j\geqslant 1}$ by analytic continuation.

For equality (35), we do almost the same analysis. As before, as N tends to ∞,

$$\prod_{j=1}^{N}\frac{1-x\rho_j^2}{1-x\sigma_j^2}\to\prod_{j=1}^{\infty}\frac{1-x\rho_j^2}{1-x\sigma_j^2},\qquad x\notin\left\{\frac{1}{\sigma_j^2}\right\}_{j\geqslant 1}.$$

On the other hand, for $x\notin\{\frac{1}{\sigma_j^2}\}_{j\geqslant 1}$,

$$\prod_{j=1}^{N}\frac{1-x\rho_j^2}{1-x\sigma_j^2}=1-x\sum_{j=1}^{N}\frac{(\kappa_j^{(N)})^2}{1-x\sigma_j^2},$$

where

$$(\kappa_j^{(N)})^2=(\rho_j^2-\sigma_j^2)\prod_{\substack{k\neq j\\ 1\leqslant k\leqslant N}}\frac{\sigma_j^2-\rho_k^2}{\sigma_j^2-\sigma_k^2}.$$

Let H_N be the function defined by

$$H_N(x):=\sum_{j=1}^{N}\frac{(\kappa_j^{(N)})^2}{1-x\sigma_j^2}\qquad x\neq\sigma_j^{-2},\ 1\leqslant j\leqslant N. \tag{37}$$

The preceding equality reads

$$H_N(x)=\frac{1}{x}\left(1-\prod_{j=1}^{N}\frac{1-x\rho_j^2}{1-x\sigma_j^2}\right). \tag{38}$$

Using formula (38), and the expansion of the infinite product at $x=0$, we get that $H_N(0)=\sum_{j=1}^{N}(\rho_j^2-\sigma_j^2)$. Since, by formula (37),

$$H_N(0)=\sum_{j=1}^{N}(\kappa_j^{(N)})^2,$$

we obtain

$$\sum_{j=1}^{N}(\kappa_j^{(N)})^2=\sum_{j=1}^{N}(\rho_j^2-\sigma_j^2),$$

and this last sum is bounded independently of N, namely

$$\sum_{j=1}^{N}(\rho_j^2-\sigma_j^2)\leqslant\sum_{j=1}^{N}(\rho_j^2-\rho_{j+1}^2)\leqslant\rho_1^2.$$

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Hence the sum $\sum \kappa_j^2$ converges, by Fatou's lemma. We use this property to justify the convergence of $H_N'(x)$. Indeed, for $x \neq \sigma_j^{-2}$, $1 \leqslant j \leqslant N$,

$$H_N'(x) = \sum_{j=1}^{N} \frac{(\kappa_j^{(N)})^2 \sigma_j^2}{(1 - x\sigma_j^2)^2}.$$

So, a proof analogous to the one used before allows us to show that

$$H_N'(x) \to \sum_{j=1}^{\infty} \frac{\kappa_j^2\, \sigma_j^2}{(1 - x\sigma_j^2)^2}.$$

Furthermore, the convergence holds uniformly for $x \leqslant 0$. Therefore, on the one hand, as N tends to ∞,

$$H_N(x) = \frac{1}{x}\left(1 - \prod_{j=1}^{N} \frac{1 - x\rho_j^2}{1 - x\sigma_j^2}\right) \to \frac{1}{x}\left(1 - \prod_{j=1}^{\infty} \frac{1 - x\rho_j^2}{1 - x\sigma_j^2}\right)$$

and on the other hand, since

$$H_N(x) = \int_y^x H_N'(t)dt + H_N(y),$$

we get that, at the limit as N goes to ∞, for $x \leqslant 0$, and hence everywhere by analytic continuation,

$$\frac{1}{x}\left(1 - \prod_{j=1}^{\infty} \frac{1 - x\rho_j^2}{1 - x\sigma_j^2}\right) = \sum_{j=1}^{\infty} \frac{\kappa_j^2}{1 - x\sigma_j^2} + C.$$

It remains to compute C by taking the limit as x goes to $-\infty$.

$$C = -\lim_{x \to -\infty} \frac{1}{x} \prod_{j=1}^{\infty} \frac{1 - x\rho_j^2}{1 - x\sigma_j^2} = -\lim_{x \to -\infty} \frac{1}{xJ(x)},$$

where

$$J(x) := \prod_{j=1}^{\infty} \frac{1 - x\sigma_j^2}{1 - x\rho_j^2}.$$

This limit has been computed in (33) whenever

$$\sum_{j=1}^{\infty} \nu_j^2 = 1 \quad \text{and} \quad \sum_{j=1}^{\infty} \frac{\nu_j^2}{\rho_j^2} < \infty,$$

and is equal to

$$\left(\sum_{j=1}^{\infty} \frac{\nu_j^2}{\rho_j^2}\right)^{-1}.$$

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This calculation easily extends to the other cases, writing, for $x < 0$,

$$J(x) = 1 + x \sum_{j=1}^{\infty} \frac{\nu_j^2 \rho_j^2}{1 - x\rho_j^2}$$
$$= 1 - \sum_{j=1}^{\infty} \nu_j^2 + \sum_{j=1}^{\infty} \frac{\nu_j^2}{1 - x\rho_j^2}. \qquad \square$$

Corollary 1. *For any $m \geqslant 1$, we have*

$$\sum_j \frac{\rho_j^2 \nu_j^2}{\rho_j^2 - \sigma_m^2} = 1 \tag{39}$$

$$\sum_j \frac{\kappa_j^2}{\rho_m^2 - \sigma_j^2} + \frac{C}{\rho_m^2} = 1 \tag{40}$$

$$\sum_j \frac{\rho_j^2 \nu_j^2}{(\rho_j^2 - \sigma_m^2)(\rho_j^2 - \sigma_p^2)} = \frac{1}{\kappa_m^2} \delta_{mp} \tag{41}$$

$$\sum_j \frac{\sigma_j^2 \kappa_j^2}{(\sigma_j^2 - \rho_m^2)(\sigma_j^2 - \rho_p^2)} = \frac{1}{\nu_m^2} \delta_{mp} - 1. \tag{42}$$

Proof. The first two equalities (39) and (40) are obtained by making $x = \frac{1}{\sigma_m^2}$ and $x = \frac{1}{\rho_m^2}$, respectively, in formula (34) and formula (35). For equality (41) in the case when $m = p$, we first make the change of variable $y = 1/x$ in formula (34), then differentiate both sides with respect to y, and make $y = \sigma_m^2$. Equality (42) in the case when $m = p$ follows by differentiating equation (35) and making $x = \frac{1}{\rho_m^2}$. Both equalities in the case when $m \neq p$ follow directly, respectively, from equality (39) and equality (40). $\square$

Lemma 7. *Let m be a fixed positive integer. Let (φ_j) and (θ_m) be two sequences of elements of $\mathbb{T}$. Denote by $A^{(m)}$ the rank 1 operator of matrix*

$$A^{(m)} = \left(\frac{\nu_j}{\rho_j^2 - \sigma_m^2} \frac{\nu_k \rho_k e^{-i\varphi_k}}{\rho_k^2 - \sigma_m^2} \sigma_m \kappa_m^2 e^{-i\theta_m} \right)_{jk}.$$

Then $A := \sum_{m \geqslant 1} A^{(m)}$ defines a bounded operator on ℓ^2 with $AA^* \leqslant I$.

Proof. First we notice that $A^{(m)}$ satisfies $\|A^{(m)}\| \leqslant 1$. This follows from the Cauchy–Schwarz inequality, formula (41), and from the estimate

$$\sum_j \frac{\nu_j^2}{(\rho_j^2 - \sigma_m^2)^2} = -\frac{1}{\sigma_m^2} \sum_j \frac{\nu_j^2}{\rho_j^2 - \sigma_m^2} + \frac{1}{\sigma_m^2} \sum_j \frac{\rho_j^2 \nu_j^2}{(\rho_j^2 - \sigma_m^2)^2}$$
$$= \frac{1}{\sigma_m^2} \left(\frac{\sum_j \nu_j^2 - 1}{\sigma_m^2} + \frac{1}{\kappa_m^2} \right)$$
$$\leqslant \frac{1}{\sigma_m^2 \kappa_m^2},$$

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so that, in view of formula (41),

$$\|A^{(m)}\|^2 \leqslant \sum_j \frac{\nu_j^2}{(\rho_j^2-\sigma_m^2)^2} \sum_k \frac{\nu_k^2\rho_k^2}{(\rho_k^2-\sigma_m^2)^2}\sigma_m^2\kappa_m^4$$
$$\leqslant 1.$$

Let us consider the well-defined operator $A^{(m)}(A^{(p)})^*$. An elementary calculation gives

$$(A^{(m)}(A^{(p)})^*)_{jk} = \sum_\ell A^{(m)}_{j\ell}\overline{A^{(p)}_{k\ell}}$$
$$= \frac{\nu_j\nu_k}{(\rho_j^2-\sigma_m^2)(\rho_k^2-\sigma_m^2)}\sigma_m^2\kappa_m^2\delta_{mp}.$$

Taking the sum of both sides over m and p, we get by (42) that the sum converges and equals $\delta_{jk}-\nu_j\nu_k$. Consequently, the sum of $A^{(m)}(A^{(p)})^*$ defines a bounded positive operator estimated by I and coincides with the operator AA^*. This gives the boundedness of A and completes the proof. □

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