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The complexity of order-computable sets

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Abstract. This paper studies the conjecture of Hirschfeldt, Miller and Podzorov in [13] on the complexity of order-computable sets, where a set A is order-computable if there is a computable copy of the structure $(\mathbb{N}, <, A)$ in the language of linear orders together with a unary predicate. The class of order-computable sets forms a subclass of Δ_2^0 sets. Firstly, we study the complexity of computably enumerable (c.e.) order-computable sets and prove that the index set of c.e. order-computable sets is Σ_4^0-complete. Secondly, as a corollary of the main result on c.e. order-computable sets, we obtain that the index set of general order computable sets is Σ_4^0-complete within the index set of Δ_2^0 sets. Finally, we continue to study the complexity of more general Δ_2^0 sets and prove that the index set of Δ_2^0 sets is Π_3^0-complete.

Key words: Computability theory; order-computable sets; Δ_2^0 sets
MSC(2020) 03D80, 68Q15, 68Q17

1 Introduction

Computability theory is a useful tool to classify mathematical structures according to their intrinsic complexity. For the work on computable model theory, refer to books, e.g., Ash and Knight [1], Harizanov Section 15 of [11]. For the work on special algebraic structures such as abelian groups, rings and fields, see, for instance, [2–4, 8–10, 12, 14, 15]. In [14], Lempp showed that the complexity of the problem of being torsion-free for a finitely presented group is Π_2^0-complete. Downey and Melnikov proved in [9] that a computable completely decomposable group is Δ_5^0-categorical; that is, any two computable presentations of such a group is isomorphic through a Δ_5^0-isomorphism. Riggs investigated the decomposability problem for torsion-free abelian groups in [17]; he obtained that the decomposability problem for torsion-free abelian groups of finite rank is Σ_3^0-complete, however, the same problem for general torsion-free abelian groups is Σ_1^1-complete. Conidis investigated the complexity of radicals of computable rings and of computable modules in [5, 6]; for instance, he constructed a computable noncommutative ring whose prime radical is Π_1^1-complete in [5]. Recently, we studied the complexity of decomposability of computable rings in [20] and obtained that the index set of computable decomposable rings is Σ_2^0-complete within the index set of computable rings.

In [13], Hirschfeldt, Miller and Podzorov proposed a seemingly simple notion of order-computable sets from the computable-model-theoretic point of view. However, their results reveal that there are no easy characterizations of order-computable sets from the perspective of pure computability theory. Hirschfeldt, Miller and Podzorov also analyzed the order-computability problem and left the conjecture on the complexity of order-computable sets open. In this paper, we affirmatively answer the conjecture on the complexity of order-computable sets in [13].

Fix the standard linear order $(\mathbb{N}, <)$ on natural numbers of order type ω.

Definition 1. *[13] A set A is order-computable if there is a computable structure $(\mathbb{N}, \prec, R)$ in the language of linear orders together with a unary predicate such that $(\mathbb{N}, \prec, R) \cong (\mathbb{N}, <, A)$.*

As proved in [13], an order-computable set A can be computed via a special copy $(\mathbb{N}, \prec, R)$ of $(\mathbb{N}, <, A)$ with R the following standard sets:

(1) $R = F_k := \{0, 1, \ldots, k-1\}$ if A is a finite set with size $|A| = k$;
(2) $\overline{R} = F_k := \{0, 1, \ldots, k-1\}$ if A is a cofinite set such that the size of the complement $\overline{A}$ of A is $|\overline{A}| = k$;
(3) $R = E := \{2m : m \in \mathbb{N}\}$ if A is an infinite coinfinite set.

In the above standard cases, say that $\prec$ *order-computes* A. Let f be the predecessor function of the linear order $(\mathbb{N}, \prec)$ isomorphic to $(\mathbb{N}, <)$; that is, $f(x)$ is the number of predecessors of x under $\prec$ for all x. Then f is a permutation on the set of natural numbers and it can be computed via a $\emptyset'$ oracle, i.e., $f \leq_T \emptyset'$. Furthermore, for any n,

$$n \in A \Leftrightarrow f^{-1}(n) \in R.$$

Then $A \leq_T \emptyset'$. This implies that order-computable sets are Δ^0_2.

Computable sets are order-computable, but there are order-computable sets that are not computable. In [13], the authors proved various results on order-computable sets from the perspective of high/low hierarchy and Ershov hierarchy in pure computability theory. For instance, they proved that any low c.e. set is order-computable and that there is a low d.c.e. set that is not order-computable; they also showed that the Turing degree $\mathbf{0}'$ contains both order-computable sets and non order-computable sets.

Using the tool of computable predecessor approximation functions, Hirschfeldt, Miller and Podzorov pointed out that the complexity of the property of being order-computable for a Δ^0_2 set is Σ^0_4, leaving the following two conjectures open.

- **Conjecture I:** The index set $\{e \in \mathbb{N} : W_e$ is order-computable$\}$ of c.e. order-computable sets is Σ^0_4-complete, where $W_0, W_1, \ldots, W_e, \ldots$ is an effective listing of all c.e. sets.
- **Conjecture II:** The index set $\{e \in \mathbb{N} : A_e$ is order-computable$\}$ of order computable sets is Σ^0_4-complete, where A_e is a Δ^0_2 set with index e. That is, the e-th partial computable binary function φ^2_e is total and it satisfies the condition $(\forall x)[\lim_s \varphi^2_e(x, s) \downarrow= 0$ or $1]$; in this case, $A_e(x) = \lim_s \varphi^2_e(x, s)$ for all x.

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In Section 3, we affirmatively answer the Conjecture I and Conjecture II on the complexity of order-computable sets above.

The class of Δ^0_2 sets plays an important role in pure computability theory. It is also interesting to consider the complexity of Δ^0_2 sets themselves. We continue to prove that the index set of Δ^0_2 sets is Π^0_3-complete in Section 4.

The remaining sections are organized as follows. Section 2 provides basic notions of computability theory as well as basic properties of order-computable sets. Section 3 answers the conjectures of Hirschfeldt, Miller and Podzorov on the complexity of c.e. order-computable sets and of general order-computable sets. Finally, Section 4 proves the result on the complexity of Δ^0_2 sets.

2 Preliminary

For basic notions of computability theory, refer to books such as [7, 16, 18, 19].

A function $f : A \to \mathbb{N}$ with domain A is *partial computable* if there is a Turing program that computes the value $f(x)$ for any $x \in A$. Based on the effective coding of all Turing programs, partial computable functions can be effectively listed as $\varphi_0, \varphi_1, \ldots, \varphi_e, \ldots$, where φ_e is the function computed by the e-th Turing program. Similarly, all partial computable binary functions are listed as $\varphi^2_0, \varphi^2_1, \ldots, \varphi^2_e, \ldots$. A set is *computably enumerable* if it is the domain of a partial computable function. All computably enumerable (often abbreviated as c.e.) sets can be effectively enumerated as $W_0, W_1, \ldots, W_e, \ldots$ with $W_e = dom(\varphi_e) = \{x \in \mathbb{N} : \varphi_e(x) \downarrow\}$. Computable functions are those partial computable functions φ_e with domain the set of all natural numbers, i.e., φ_e is total. Computable sets are those with computable characteristic functions.

We now briefly introduce arithmetical hierarchy on sets. For $n \geq 2$, let $\langle \cdot, \cdots, \cdot \rangle : \mathbb{N}^n \to \mathbb{N}$ be an effective coding of n-tuples of natural numbers into natural numbers. For a set A,

(1) A is Σ^0_1 if there is a computable set B such that $A = \{x : \exists y[\langle x, y\rangle \in B]\}$.
(2) A is Π^0_1 if there is a computable set B such that $A = \{x : \forall y[\langle x, y\rangle \in B]\}$.
(3) For $n \geq 2$, A is Σ^0_n if there is a Π^0_{n-1} set B such that $A = \{x : \exists y[\langle x, y\rangle \in B]\}$.
   - Σ^0_2 sets are of the form $\{x : \exists y_1 \forall y_2 \langle x, y_1, y_2\rangle \in B\}$ with B a computable set.
   - Σ^0_4 sets are of the form $\{x : \exists y_1 \forall y_2 \exists y_3 \forall y_4 \langle x, y_1, y_2, y_3, y_4\rangle \in B\}$ with B a computable set.
(4) For $n \geq 2$, A is Π^0_n if there is a Σ^0_{n-1} set B such that $A = \{x : \forall y[\langle x, y\rangle \in B]\}$.
   - Π^0_2 sets are of the form $\{x : \forall y_1 \exists y_2 \langle x, y_1, y_2\rangle \in B\}$ with B a computable set.
   - Π^0_3 sets are of the form $\{x : \forall y_1 \exists y_2 \forall y_3 \langle x, y_1, y_2, y_3\rangle \in B\}$ with B a computable set.

For two sets $A, B \subseteq \mathbb{N}$, A is *many-one reducible* (i.e., *m-reducible*) to B, if there is a computable function $f : \mathbb{N} \to \mathbb{N}$ such that for all $x \in \mathbb{N}$, $x \in A \Leftrightarrow f(x) \in B$.

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Definition 2. *For a complexity class $\varPi$ (e.g., $\varPi^0_3$, $\varSigma^0_4$), a Γ set A is m-complete Γ (or simply Γ-complete) if any Γ set B is m-reducible to A.*

Let $W_0, W_1, \ldots, W_e, \ldots$ be the effective listing of all c.e. sets. The index sets of subclasses of c.e. sets provide typical examples of arithmetic complete sets.

(1) The index set $\mathrm{Tot} = \{e : W_e = dom(\varphi_e) = \mathbb{N}\}$ of (total) computable functions is $\varPi^0_2$-complete.
(2) The index set $\mathrm{Cof} = \{e : W_e \text{ is cofinite}\}$ of cofinite c.e. sets is $\varSigma^0_3$-complete.
(3) The index set of computable sets $\{e : W_e \text{ is computable}\}$ is $\varSigma^0_3$-complete.

By definition, $\varDelta^0_2$ sets are those that are both $\varSigma^0_2$ and $\varPi^0_2$. By Post's Theorem, a set A is $\varDelta^0_2$ if and only if $A \leq_T \emptyset'$; furthermore, by Shoenfield Limit Lemma, A is $\varDelta^0_2$ if and only if there is a computable binary function f such that the characteristic function of A is $A(x) = \lim_s f(x,s) \downarrow$ for all x. Based on the index of partial computable binary functions, we have the following index for a $\varDelta^0_2$ set.

Definition 3. *A number e is an index of a $\varDelta^0_2$ set A if the e-th partial computable binary function φ^2_e is total and the characteristic function of A is $A(x) = \lim_s \varphi^2_e(x,s) \downarrow$ for all x.*

For a number e, it becomes an index of some $\varDelta^0_2$ set if and only if the following conditions on e hold:

(1) φ^2_e is a computable function, that is, for all x, s, there is a stage t such that the computation $\varphi^2_{e,t}(x,s)$ converges at stage t;
(2) for all x, the limit $\lim_s \varphi^2_e(x,s)$ converges to 0 or 1.

First, (1) is $\varPi^0_2$; furthermore, similar to the fact that the index set of computable functions $\mathrm{Tot} = \{e \in \mathbb{N} : (\forall x)[\varphi_e(x)\downarrow]\}$ is $\varPi^0_2$-complete, the condition (1) is $\varPi^0_2$-complete. More specifically, the index set $\mathrm{Tot}^2 = \{e \in \mathbb{N} : (\forall x,y)[\varphi^2_e(x,y)\downarrow]\}$ of computable binary functions is $\varPi^0_2$-complete. Second, the condition (2) is $\varPi^0_4$ because it can be described by the following $\varPi^0_4$ formula:

$$(\forall x)(\exists s_x)(\forall s \geq s_x)(\exists t \geq s)[\varphi^2_{e,t}(x,s)\downarrow = \varphi^2_{e,t}(x,s_x)\downarrow = 0 \text{ or } 1].$$

The condition (2) is indeed $\varPi^0_4$-complete. We omit the detail proof here.

Proposition 1. *The index set $\{e \in \mathbb{N} : (\forall x)[\lim_s \varphi^2_e(x,s)\downarrow = 0 \text{ or } 1]\}$ is $\varPi^0_4$-complete.*

For e to be a $\varDelta^0_2$ index, both conditions (1) and (2) hold. Then under the condition (1) that φ^2_e is total, the condition (2) becomes $\varPi^0_3$ because $\varphi^2_e(x,s)$ is always defined for all x, s. From the analysis above, we obtain the upper bound on the complexity of the index set $\mathcal{I}$ of $\varDelta^0_2$ sets. We further prove that $\mathcal{I}$ is $\varPi^0_3$-complete in Section 4.

Proposition 2. *The index set $\mathcal{I} = \{e \in \mathbb{N} : e \text{ is an index of a } \varDelta^0_2 \text{ set}\}$ is $\varPi^0_3$.*

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We next review basic properties of order-computable sets, for more details, refer to [13]. Order-computable sets are always Δ^0_2 and they are closely connected with predecessor functions of linear orders that order-compute them. Such linear orders are of order type ω.

For a computable linear order of order type ω, the predecessor function of it can be approximated via partial computable binary functions in a natural way.

Definition 4. *[13] A partial computable binary function f is a computable predecessor approximation function if it satisfies the following properties:*

(1) for all n, s, $f(n,s)\downarrow$ if and only if $n < s$;
(2) for all s, $f(\cdot, s)$ is a permutation on $\{0, 1, \dots, s-1\}$;
(3) for all $i < s$, $f(i,s) \le f(i, s+1) \le f(i,s)+1$;
(4) for all n, $\lim_s f(n,s)$ exists.

For an order-computable set A order-computed via the linear order $\prec$, that is, $(\mathbb{N}, <, A)$ is isomorphic to the computable structure $(\mathbb{N}, \prec, R)$ with R the standard sets defined in introduction above, let

$$f_A(n,s) := \begin{cases} |\{x < s : x \prec n\}|, & \text{if } n < s \\ \uparrow, & \text{otherwise} \end{cases}$$

For $n < s$, $f_A(n,s)$ is the number of predecessors of n under the restricted order $(\mathbb{N}, \prec) \restriction_s$ on the finite set $\{0, 1, \dots, s-1\}$, and $f_A(\cdot,\cdot)$ is a computable predecessor approximation function such that

$$A := \begin{cases} \{\lim_s f_A(m,s) : 0 \le m \le k-1\}, & \text{if } (\exists k)[|A| = k] \\ \{\lim_s f_A(m,s) : m \ge k\}, & \text{if } (\exists k)[|\overline{A}| = k] \\ \{\lim_s f_A(2m,s) : m \ge 0\}, & \text{otherwise} \end{cases}$$

Conversely, consider the e-th partial computable binary function $\varphi^2_e(\cdot,\cdot)$, if it is a computable predecessor approximation function under Definition 4, then for each $s \ge 1$, $\varphi^2_e(\cdot, s)$ is a permutation on $\{0, 1, \dots, s-1\}$, so it determines a finite linear order $\mathcal{L}_s = (\{0, 1, \cdots, s-1\}, \prec_s)$ such that

$$|\{x < s : x \prec_s n\}| = \varphi^2_e(n,s),$$

and $\mathcal{L} = \bigcup_s \mathcal{L}_s = (\mathbb{N}, \prec)$ is a computable linear order on $\mathbb{N}$ of order type ω such that the number of predecessors of n is just $\lim_s \varphi^2_e(n,s)$. Then the following set

$$O_e := \{\lim_s \varphi^2_e(2m,s) : m \in \mathbb{N}\}$$

is an infinite coinfinite order-computable set, which is order-computed via the constructed order $\prec$. From this analysis, we see that any infinite coinfinite order-computable sets are of the form O_e for some e with φ^2_e a computable predecessor approximation function.

The class of order-computable sets forms a subclass of Δ^0_2 sets. To study the complexity of order-computable sets by using indices of Δ^0_2 sets, rather than using usual m-reducibility, it is more suitable to consider the notion of m-reducibility within a given set proposed by Calvert and Knight, see Definitions 3.1, 3.2 in [3], see also Definition 3.6 in [4].

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Definition 5. *[3, 4] Let $A \subseteq B$. A set S is m-reducible to A within B if there is a computable function $f : \mathbb{N} \to B$ such that for all x, $x \in S \Leftrightarrow f(x) \in A$.*

Similar to the usual m-reducibility, we have the following Γ-completeness within a given set.

Definition 6. *[3, 4] Let $A \subseteq B$ and Γ a complexity class (e.g., Π^0_3, Σ^0_4). A set A is m-complete Γ within a set B (or simply Γ-complete within B) if*

(1) A is Γ within B; that is, there is a Γ set C such that $A = C \cap B$;
(2) any Γ set S is m-reducible to A within B.

We now consider the complexity of the index set of order-computable sets within the index set $\mathcal{I}$ of Δ^0_2 sets.

Proposition 3. *[13] The index set $\{e \in \mathcal{I} : A_e$ is order-computable$\}$ of order computable sets is Σ^0_4 within $\mathcal{I}$.*

Proof. Let $e \in \mathcal{I}$. For all x, $A_e(x) = \lim_s \varphi^2_e(x, s) \downarrow$. A_e is order-computable if and only if one of the following three conditions hold:

(1) A_e is finite;
(2) A_e is cofinite;
(3) there is an index i such that
  (3.1) φ^2_i is a computable predecessor approximation function, and
  (3.2) for any n, $n \in A_e$ if and only if $n = \lim_s \varphi^2_i(2m, s)$ for some m.

First of all, φ^2_e is a computable approximation for the Δ^0_2 set A_e, that is, φ^2_e is a total function such that for all x, $\lim_s \varphi^2_e(x, s)$ converges to 0 or 1. This can be described by the following Π^0_3 formula:

$$(\forall x, y)(\exists t)[\varphi^2_{e,t}(x, y) \downarrow] \wedge (\forall z)(\exists s_z)(\forall s \geq s_z)[\varphi^2_e(z, s) = \varphi^2_e(z, s_z)].$$

For (1), A_e is finite if and only if there is a large number N such that for all $n > N$, $A_e(n) = \lim_s \varphi^2_e(n, s) = 0$. This can be described by the Σ^0_4 formula:

$$(\exists N)(\forall n > N)(\exists s_n)(\forall s \geq s_n)[\varphi^2_e(n, s) = \varphi^2_e(n, s_n) = 0].$$

Similarly, A_e is cofinite if and only if

$$(\exists N)(\forall n > N)(\exists s_n)(\forall s \geq s_n)[\varphi^2_e(n, s) = \varphi^2_e(n, s_n) = 1].$$

Therefore, both (1) and (2) are Σ^0_4 properties on e.

Now consider the condition (3): there exists an index i satisfying (3.1) and (3.2). First, (3.1) says that φ^2_i is a computable predecessor approximation function, which can be described by the following Π^0_3 formula:

$$\begin{aligned}&(\forall s)(\forall n < s)(\exists t)[\varphi^2_{i,t}(n, s) \downarrow] \wedge\\ &(\forall s)(\forall n \geq s)(\forall t)[\varphi^2_{i,t}(n, s) \uparrow] \wedge\\ &(\forall s)(\forall n, m < s)[\varphi^2_i(n, s) < s \wedge (n \neq m \to \varphi^2_i(n, s) \neq \varphi^2_i(m, s))] \wedge\\ &(\forall s)(\forall n < s)[\varphi^2_i(n, s) \leq \varphi^2_i(n, s + 1) \leq \varphi^2_i(n, s) + 1] \wedge\\ &(\forall n)(\exists s_n > n)(\forall s \geq s_n)[\varphi^2_i(n, s) = \varphi^2_i(n, s_n)].\end{aligned}$$

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Second, (3.2) says that for all n, $A_e(n) = 1 \Leftrightarrow n = \lim_s \varphi_i^2(2m, s)$ for some m. This condition can be expressed as

$$(\forall n)(\exists m)(\exists t > n + 2m)(\forall s \geq t)[\varphi_e^2(n, s) = \varphi_e^2(n, t) = 1 \to \varphi_i^2(2m, s) = \varphi_i^2(2m, t) = n] \wedge$$
$$(\forall n)(\forall m)(\exists t > n + 2m)(\forall s \geq t)[\varphi_e^2(n, s) = \varphi_e^2(n, t) = 0 \to \varphi_i^2(2m, s) = \varphi_i^2(2m, t) \neq n].$$

Therefore, the condition (3) is a Σ_4^0 property on e.

For $e \in \mathcal{I}$, A_e is order-computable if and only the Σ_4^0 properties (1)-(3) on e hold. This shows that the index set of order-computable sets is Σ_4^0 within $\mathcal{I}$. □

Similar to the case of general order-computable sets, we have the same upper bound for the complexity of c.e. order-computable sets.

Proposition 4. *[13] The index set* $\{e \in \mathbb{N} : W_e$ *is order-computable*$\}$ *of c.e. order-computable sets is* Σ_4^0.

3 Order-computable sets

Let $\langle \Phi_e : e \in \mathbb{N} \rangle$ be an effective listing of all oracle Turing functionals. Fix the Halting set K. Then $\Phi_0^K, \Phi_1^K, \ldots, \Phi_e^K, \ldots$ is an effective listing of all partial computable functions relative to K and $W_0^K, W_1^K, \ldots, W_e^K, \ldots$ is an effective listing of all c.e. sets relative to K with $W_e^K = \{x \in \mathbb{N} : \Phi_e^K(x) \downarrow\}$, the domain of Φ_e^K. At a stage s, a number x goes into $W_e^K[s]$ if $\Phi_e^K(x)[s]$ is defined with use $\phi_e^K(x)[s]$; we also say that $x \in W_e^K[s]$ with use $\phi_e^K(x)[s]$. For any number x, it belongs to W_e^K if and only if there is a stage s_x such that at any stage $t \geq s_x$, $x \in W_e^K[t]$ with use the same as $\phi_e^K(x)[s_x]$. In this case, the convergent computation $\Phi_e^K(x)[s_x] \downarrow$ does not change after stage s_x, and thus, x stays in W_e^K forever. Let $\mathrm{Cof}^K = \{e \in \mathbb{N} : W_e^K$ is cofinite$\}$. Then Cof^K is Σ_3^0-complete relative to the c.e. complete set K, and thus, it is Σ_4^0-complete.

Theorem 1. *The index set of c.e. order-computable sets is* Σ_4^0*-complete.*

Proof. An infinite coinfinite set A is order-computable if and only if there is a computable predecessor approximation function f such that for all n,

$$n \in A \Leftrightarrow (\exists m)[n = \lim_s f(2m, s)].$$

Consider the Σ_4^0-complete set $\mathrm{Cof}^K = \{e \in \mathbb{N} : W_e^K$ is cofinite$\}$. To prove the theorem, we enumerate a uniform sequence $\langle C_e : e \in \mathbb{N} \rangle$ of noncomputable c.e. sets such that for all e,

$$e \in \mathrm{Cof}^K \Leftrightarrow C_e \text{ is order-computable.}$$

For $e \in \mathrm{Cof}^K$, we build C_e to be low so that it is order-computable, because low c.e. sets are order-computable (see Theorem 2.6, [13]). On the other hand, for $e \notin \mathrm{Cof}^K$, to ensure the non order-computability of C_e, we construct C_e to be not order-computed by φ_i^2 for any i; that is, $C_e \neq O_i = \{\lim_s \varphi_i^2(2m, s) : m \in \mathbb{N}\}$ if φ_i^2 does become a computable predecessor approximation function.

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3.1 Requirements and strategies

We enumerate a uniform sequence of c.e. sets $\langle C_e : e \in \mathbb{N}\rangle$ such that the following requirements are satisfied.

(1) $\mathcal{P}_{\langle e,i\rangle}$: $C_e \neq \varphi_i$.
(2) $\mathcal{R}_{\langle e,i,i+j\rangle}$: If $[i, i+j+1] \subseteq W_e^K$ and $(\exists^\infty s)[\Phi_j^{C_e}(j)[s]\downarrow]$, then $\Phi_j^{C_e}(j)\downarrow$.
If $[i, i+j+1] \nsubseteq W_e^K$ and φ_i^2 is a computable predecessor approximation function, then $C_e \neq O_i = \{\lim_s \varphi_i^2(2m, s) : m \in \mathbb{N}\}$.

Here, $\langle \varphi_i : i \in \mathbb{N}\rangle$, $\langle \varphi_i^2 : i \in \mathbb{N}\rangle$ and $\langle \Phi_j : j \in \mathbb{N}\rangle$ are effective listing of partial computable functions, partial computable binary functions and oracle Turing functionals, respectively.

For each e, all $\mathcal{P}_{\langle e,i\rangle}$-requirements with $i \in \mathbb{N}$ together ensure the noncomputability of C_e. For $e \in \mathrm{Cof}^K$, W_e^K is cofinite; that is, there is a number i such that $[i, \infty) \subseteq W_e^K$. In this case, for all j, $[i, i+j+1] \subseteq W_e^K$, and the requirement $\mathcal{R}_{\langle e,i,j\rangle}$ ensures that the computation $\Phi_j^{C_e}(j)$ converges if and only if there are infinitely many stages s such that $\Phi_j^{C_e}(j)$ converges at stage s. This implies that $C_e' = \{j \in \mathbb{N} : \Phi_j^{C_e}(j)\downarrow\}$ is Δ_2^0 and thus $C_e' \leq_T \emptyset'$, i.e., C_e is low. As low c.e. sets are order-computable, C_e is order-computable. On the other hand, for $e \notin \mathrm{Cof}^K$, W_e^K is not cofinite. Then for all $i \in \mathbb{N}$, there exists a number j_i such that $[i, i+j_i+1] \nsubseteq W_e^K$, and the requirement $\mathcal{R}_{\langle e,i,j_i\rangle}$ ensures that C_e is not order-computed by φ_i^2. So C_e is not order-computable.

The construction of c.e. sets $\langle C_e : e \in \mathbb{N}\rangle$ proceeds on a priority tree T whose nodes are assigned $\mathcal{P}$- and $\mathcal{R}$-requirements. We view each node α as a strategy for satisfying the assigned requirement whose possible outcomes are immediate successors of α on T. A $\mathcal{P}$-strategy has two possible outcomes d, f with priority $d <_L f$ on the priority tree T. While an $\mathcal{R}$-strategy has three possible outcomes ∞, d, f with priority $\infty <_L d <_L f$ on T. We assign $\mathcal{P}$-requirements to nodes of length even and assign $\mathcal{R}$-requirements to nodes of length odd. So the priority tree T is simply a subtree of the full tree $S = \{\infty, d, f\}^{<\omega}$ of finite strings on $\{\infty, d, f\}$, where S has the usual lexicographic order such that $\alpha^\frown\infty <_L \alpha^\frown d <_L \alpha^\frown f$ for any $\alpha \in S$.

Definition 7. *The priority tree T is defined as follows.*

(1) Let the empty node $\lambda \in T$.
(2) Let $\alpha \in T$ with length $|\alpha| \geq 0$.
- *If $|\alpha|$ is even, add $\alpha^\frown d$ and $\alpha^\frown f$ into T.*
- *If $|\alpha|$ is odd, add $\alpha^\frown\infty$, $\alpha^\frown d$ and $\alpha^\frown f$ into T.*

At each stage s of the construction, we will define a current approximation $\delta_s \in T$ of the true path $\mathrm{TP} = \liminf_s \delta_s$ of the construction. TP will be an infinite path on the priority tree T. For any node $\alpha \in T$, when $\alpha \subseteq \delta_s$, say that α is *visited* at stage s, and call s an *α-stage*.

For ease of description, we often write the $\mathcal{P}_{\langle e,i\rangle}$-requirement assigned to a node $\alpha \in T$ as $\mathcal{P}_{\langle e(\alpha),i(\alpha)\rangle}$ (or simply $\mathcal{P}_\alpha$) with $e(\alpha) = e$ and $i(\alpha) = i$. Similarly,

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we write the $\mathcal{R}_{\langle e,i,j\rangle}$-requirement assigned to a node $\beta \in T$ as $\mathcal{R}_{\langle e(\beta),i(\beta),j(\beta)\rangle}$ (or simply $\mathcal{R}_\beta$) with $e(\beta) = e$, $i(\beta) = i$ and $j(\beta) = j$. In the following, we first give a basic $\mathcal{P}$- and $\mathcal{R}$-strategy for satisfying a single $\mathcal{P}$- and $\mathcal{R}$-requirement, respectively, and then define the assignment of the requirements to the priority tree T.

A basic $\mathcal{P}_{\langle e,i\rangle}$-strategy. Let α be a node of length even on the priority tree T assigned the $\mathcal{P}_{\langle e,i\rangle}$-requirement. A basic α-strategy works to diagonalize against $C_e = \varphi_i$ at α-stages.

(1) At the first α-stage s, choose a new witness $w_{\alpha,s}$ and a set of fresh *trigger* elements $\{x^k_{\alpha,s} : 1 \le k \le q_\alpha = 2^{3^{|\alpha|+1}}\}$ for higher priority $\mathcal{R}$-strategies β with $C_{e(\beta)} = C_{e(\alpha)}$ on the priority tree T to enumerate into $C_{e(\beta)}$ later such that
$$N_{\alpha,s} < x^1_{\alpha,s} < x^2_{\alpha,s} < \cdots < x^{q_\alpha}_{\alpha,s} < w_{\alpha,s},$$
where $N_{\alpha,s}$ is a new number larger than all used numbers before; particularly, $N_{\alpha,s}$ is bigger than all restraints $r(\beta, s)$ appointed on $C_{e(\beta)}$ by $\mathcal{R}$-strategies $\beta < \alpha$ (i.e., $\beta <_L \alpha$ or $\beta \subset \alpha$).
The witness $w_{\alpha,s}$ and the triggers $x^k_{\alpha,s}(1 \le k \le q_\alpha)$ keep unchanged at later stages unless the node α is initialized by higher priority strategies.
(2) Wait for an α-stage $t > s$ such that $\varphi_{i,t}(w_{\alpha,s}) \downarrow= 0$.
(3) Put $w_{\alpha,s}$ into $C_{e,t}$.

The possible outcomes of a basic α-strategy are the following:

f : wait at Step (2) forever; in this case, either φ_i is not total or $\varphi_i(w_{\alpha,s}) \downarrow= 1 \ne C_e(w_{\alpha,s}) = 0$;
d : arrive at Step (3); in this case, $\varphi_i(w_{\alpha,s}) \downarrow= 0 \ne C_e(w_{\alpha,s}) = 1$.

In both cases, the strategy ensures that $C_e \ne \varphi_i$. The priority of the two outcomes on the priority tree is $d <_L f$.

A basic $\mathcal{R}_{\langle e,i,j\rangle}$-strategy. Let α be a node of length odd on the priority tree T assigned the $\mathcal{R}_{\langle e,i,j\rangle}$-requirement. First, define the following notion of expansionary stages to approximate the condition $[i, i+j+1] \subseteq W^K_e$. Since the condition $[i, i+j+1] \subseteq W^K_e$ is Σ^0_2, as in usual infinite injury constructions, we ensure that the Σ^0_2 condition $[i, i+j+1] \subseteq W^K_e$ holds if and only if there are finitely many expansionary stages during the construction.

Definition 8. *Let s be an α-stage. If there was a last α-stage s^- such that $[i, i+j+1] \subseteq W^K_e[s^-]$ and for all $x \in [i, i+j+1]$, $x \in W^K_e[s]$ with use the same as $\phi^K_e(x)[s^-]$, then s is called a non α-expansionary stage; otherwise, s is called an α-expansionary stage.*

Supposed that α is visited infinitely often during the construction. The Σ^0_2 condition $[i, i+j+1] \subseteq W^K_e$ is true if and only if there is an α-stage s such that every α-stage $t \ge s$ is a non α-expansionary stage. An α-strategy works to satisfy its requirement $\mathcal{R}_{\langle e,i,j\rangle}$. At a non α-expansionary stage s, it prevents numbers from enumerating into C_e to preserve convergent computations of the form $\Phi_j^{C_e}(j)[s]\downarrow$. While at an α-expansionary stage s, it devotes to diagonalize against φ_i^2 to be a computable predecessor approximation function for C_e by enumerating numbers into C_e.

Second, to measure the approximation of the order-computable set O_i computed by φ_i^2, define the following length function $l(i,s)$ of O_i.

(1) If there is a number $x \in [1,s]$ such that the following conditions hold:
   - $\varphi_{i,s}^2(n,x)\downarrow< x$ for all $n < x$, and $\varphi_{i,s}^2(m,x)\uparrow$ for all $x \leq m \leq s$;
   - $\varphi_{i,s}^2(\cdot,x)$ is a permutation on $\{0,1,\dots,x-1\}$;
   - $\varphi_{i,s}^2(n,t) \leq \varphi_{i,s}^2(n,t+1) \leq \varphi_{i,s}^2(n,t)+1$ for all $n < t < x$;

   let $l(i,s)$ be the largest such a number x, and define

$$O_{i,s} = \{\varphi_{i,s}^2(2m, l(i,s)) : 2m < l(i,s)\}.$$

(2) Otherwise, such a number x does not exist, let $l(i,s) = 0$ and $O_{i,s} = \emptyset$.

If φ_i^2 does become a computable predecessor approximation function, then we have $\lim_s l(i,s) = \infty$, and the corresponding order-computable set is

$$O_i = \lim_s O_{i,s} = \{\lim_s \varphi_i^2(2m,s) : m \in \mathbb{N}\}.$$

In order to diagonalize against φ_i^2 to be a computable predecessor approximation function for C_e, an α-strategy aims to make sure that either $\lim_s l(i,s) < \infty$, or $\lim_s \varphi_i^2(y,s) = \infty$ for some even number y, or φ_i^2 really becomes a computable predecessor approximation function with $\lim_s l(i,s) = \infty$ but $C_e(x) \neq O_i(x)$ for some x; that is, there is a large stage s_x such that for all $t \geq s_x$, $C_{e,t}(x) \neq O_{i,t}(x)$. This analysis motivates us to formally define the following approximation of the length of agreement between C_e and O_i. The idea behind this definition was already used in Theorem 3.11, [13] to build a Turing degree which contains no order-computable sets.

Definition 9. *Say that φ_i^2 appears to be a computable predecessor approximation function for C_e at an α-stage s if $l(i,s) > 0$, $l(i,s) > l(i,t)$ for all α-stages $t < s$, and $C_{e,s} = \{\varphi_{i,s}^2(2m, l(i,s)) : 2m < l(i,s)\} = O_{i,s}$.*

A basic α-strategy acts to satisfy its requirement $\mathcal{R}_{\langle e(\alpha), i(\alpha), j(\alpha)\rangle}$ with $e(\alpha) = e$, $i(\alpha) = i$ and $j(\alpha) = j$ at α-stages as follows. There are two cases:

Case 1. At a non α-expansionary stage s, an α-strategy proceeds to preserve the computation $\Phi_j^{C_e}(j)[s]$. Let $\phi_j^{C_e}(j)[s]$ denote the use of the computation $\Phi_j^{C_e}(j)[s]$ if it converges.

(1) If $\Phi_j^{C_e}(j)[s]$ converges, appoint a restraint $r(\alpha,s) = \phi_j^{C_e}(j)[s]$ on C_e to preserve the computation $\Phi_j^{C_e}(j)[s]$.
(2) If $\Phi_j^{C_e}(j)[s]$ diverges, set $r(\alpha,s) = 0$.

Any lower priority β-strategy on T can only enumerate numbers $> r(\alpha,s)$ into $C_{e(\beta)}$ later.

Case 2. At α-expansionary stages, a basic α-strategy acts to diagonalize against φ_i^2 to be a computable predecessor approximation function for C_e.

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(1) At the first α-expansionary stage s_0, choose a large new witness w_{α,s_0} and a set of fresh *trigger* elements $\{x^k_{\alpha,s_0} : 1 \leq k \leq q_\alpha = 2^{3^{|\alpha|+1}}\}$ for higher priority $\mathcal{R}$-strategies β with $e(\beta) = e(\alpha)$ and $\beta \leq \alpha$ (i.e., $\beta <_L \alpha$ or $\beta \subseteq \alpha$) to enumerate into $C_{e(\beta)}$ later such that

$$N_{\alpha,s_0} < x^1_{\alpha,s_0} < x^2_{\alpha,s_0} < \cdots < x^{q_\alpha}_{\alpha,s_0} < w_{\alpha,s_0},$$

where N_{α,s_0} is a new number larger than all used numbers before; particularly, N_{α,s_0} is bigger than all restraints $r(\beta, s_0)$ appointed on $C_{e(\beta)}$ by $\mathcal{R}$-strategies β with $\beta < \alpha$.

- Enumerate w_{α,s_0} into $C_{e(\alpha),s_0}$.

The witness w_{α,s_0} and the triggers $x^k_{\alpha,s_0}(1 \leq k \leq q_\alpha)$ keep unchanged at later stages unless the node α is initialized by higher priority strategies.

(2) Wait for an α-expansionary stage $t_0 > s_0$ such that the length approximation of the order-computable set $l(i(\alpha), t_0) > w_{\alpha,s_0}$, and $\varphi^2_{i(\alpha)}$ appears to be a computable predecessor approximation function for $C_{e(\alpha)}$ at stage t_0.

(3) Let y_{α,t_0} be the unique even number such that $\varphi^2_{i(\alpha)}(y_{\alpha,t_0}, l(i(\alpha), t_0)) = w_{\alpha,s_0} \in C_{e(\alpha),s_0}$.

- Pick the largest trigger element $x^k_{\alpha,s_0} < w_{\alpha,s_0}$ that has not been enumerated into $C_{e(\beta)}$ by β-strategies with $\beta < \alpha$.
- Enumerate x^k_{α,s_0} into $C_{e(\alpha),t_0}$.
- Go to Step $(4.n)$ with $n = 1$.

The even number y_{α,t_0} remains the same at later stages as long as α is not initialized.

(4.n) Wait for an α-expansionary stage $t_n > t_{n-1}$ such that $\varphi^2_{i(\alpha)}$ appears to be a computable predecessor approximation function for $C_{e(\alpha)}$ at stage t_n.

(5.n) Let $u_{\alpha,t_0} = w_{\alpha,s_0}$ and $u_{\alpha,t_n} = \varphi^2_{i(\alpha)}(y_{\alpha,t_0}, l(i(\alpha), t_n))$. Then we have

$$\begin{aligned} w_{\alpha,s_0} \leq u_{\alpha,t_{n-1}} &= \varphi^2_{i(\alpha)}(y_{\alpha,t_0}, l(i(\alpha), t_{n-1})) \\ &< \varphi^2_{i(\alpha)}(y_{\alpha,t_0}, l(i(\alpha), t_n)) = u_{\alpha,t_n} \in O_{i(\alpha),t_n} = C_{e(\alpha),t_n} \end{aligned}$$

because the enumeration of the trigger element $x^k_{\alpha,s_0} < w_{\alpha,s_0}$ in Step (3) for $n = 1$ or $x^k_{\beta,t} \in (w_{\alpha,s_0}, u_{\alpha,t_{n-1}})$ in Step (5. $(n-1)$) for $n \geq 2$ forces the number of predecessors of y_{α,t_0} computed by $\varphi^2_{i(\alpha)}$ to move up.

As $u_{\alpha,t_n} \in C_{e(\alpha),t_n}$ and $u_{\alpha,t_n} > w_{\alpha,s_0}$, there is a unique node $\beta > \alpha$ on the priority tree T such that

$$u_{\alpha,t_n} = w_{\beta,t} \text{ or } u_{\alpha,t_n} = x^k_{\beta,t} \text{ for some } k,$$

where $w_{\beta,t}$ and $x^k_{\beta,t}(1 \leq k \leq q_\beta = 2^{3^{|\beta|+1}})$ were first appointed on β at some β-stage $t \in (t_0, t_n)$.

- Pick the largest trigger number $x^k_{\beta,t} < u_{\alpha,t_n} \leq w_{\beta,t}$ that has not been enumerated into $C_{e(\beta')}$ for any $\beta' \leq \beta$.
- Enumerate $x^k_{\beta,t}$ into $C_{e(\alpha),t_n}$.
- Go to Step $(4.(n+1))$.

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For each node $\beta \in T$, since the number of nodes $\beta' \in T$ with $\beta' <_L \beta$ or $\beta' \subseteq \beta$ is strictly less than $3^{|\beta|+1}$, we appoint enough (i.e., $2^{3^{|\beta|+1}}$ many) trigger elements $x^k_{\beta,t}$ on node β for higher priority β'-strategies working for $\mathcal{R}$-requirements to enumerate into $C_{e(\beta')}$ during the construction. That is, the trigger elements in Step (3) or Step (5.n) always exist if needed (see Lemma 1 below).

The possible outcomes of a basic α-strategy is listed as follows:

- f: finitely many α-expansionary stages;
- d: wait at Step (2) or Step (4.n) for some $n \geq 1$ forever at α-expansionary stages;
- ∞: go to Step (4.$(n+1)$) from Step (5.n) infinitely often at α-expansionary stages.

The priority of the outcomes on the priority tree T is $\infty <_L d <_L f$.

If there are finitely many α-expansionary stages, α has the finite outcome f, and the computation $\Phi^{C_{e(\alpha)}}_{j(\alpha)}(j(\alpha))$ will be preserved after certain stage if it converges at infinitely many α-stages. If there are infinitely many α-expansionary stages, α has two possible outcomes $\infty <_L d$.

(i) If α has the outcome d, then either $\lim_s l(i(\alpha), s) < \infty$ or $C_{e(\alpha)} \neq O_{i(\alpha)}$ in which case α only enumerates finitely many triggers into $C_{e(\alpha)}$.
(ii) If α has the outcome ∞, then $\lim_s \varphi^2_{i(\alpha)}(y_{\alpha,t_0}, s) = \lim_n u_{\alpha,t_n} = \infty$ in which case α enumerates infinitely many triggers into $C_{e(\alpha)}$.

In both cases, $\varphi^2_{i(\alpha)}$ cannot be a computable predecessor approximation function for $C_{e(\alpha)}$. In other words, $C_{e(\alpha)}$ is not order-computed via $\varphi^2_{i(\alpha)}$.

This ends the description of basic strategies.

Fix a bijection $\langle \cdot, \cdot \rangle : \mathbb{N}^2 \to \mathbb{N}$ such that $\langle e, i \rangle < \langle e, i' \rangle$ for all e, i, i' with $i < i'$. For $e, i, j \in \mathbb{N}$, let $\langle e, i, j \rangle = \langle \langle e, i \rangle, j \rangle$. Then we can fix a linear order $\prec$ on all $\mathcal{P}_{\langle e,i \rangle}$- and $\mathcal{R}_{\langle e,i,j \rangle}$-requirements ($e, i, j \in \mathbb{N}$) of order type ω as follows.

(1) For all e, i, e', i', let $\mathcal{P}_{\langle e,i \rangle} \prec \mathcal{P}_{\langle e',i' \rangle}$ if and only if $\langle e, i \rangle < \langle e', i' \rangle$, and let the priority of all $\mathcal{P}$-requirements be listed as

$$\mathcal{P}_{\langle e_0,i_0 \rangle} \prec \mathcal{P}_{\langle e_1,i_1 \rangle} \prec \cdots \prec \mathcal{P}_{\langle e_n,i_n \rangle} \prec \cdots .$$

Particularly, for any number e, we have

$$\mathcal{P}_{\langle e,0 \rangle} \prec \mathcal{P}_{\langle e,1 \rangle} \prec \cdots \prec \mathcal{P}_{\langle e,i \rangle} \prec \cdots .$$

(2) For all e, i, j, e', i', j', let $\mathcal{R}_{\langle e,i,j \rangle} \prec \mathcal{R}_{\langle e',i',j' \rangle}$ if and only if $\langle e, i, j \rangle < \langle e', i', j' \rangle$, and let the priority of all $\mathcal{R}$-requirements be listed as

$$\mathcal{R}_{\langle e'_0,i'_0,j'_0 \rangle} \prec \mathcal{R}_{\langle e'_1,i'_1,j'_1 \rangle} \prec \cdots \prec \mathcal{R}_{\langle e'_n,i'_n,j'_n \rangle} \prec \cdots .$$

Particularly, for fixed numbers e, i, we have

$$\mathcal{R}_{\langle e,i,0 \rangle} \prec \mathcal{R}_{\langle e,i,1 \rangle} \prec \cdots \prec \mathcal{R}_{\langle e,i,j \rangle} \prec \cdots .$$

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(3) Define the priority of all $\mathcal{P}$- and $\mathcal{R}$- requirements as

$$\mathcal{P}_{\langle e_0,i_0\rangle} \prec \mathcal{R}_{\langle e'_0,i'_0,j'_0\rangle} \prec \mathcal{P}_{\langle e_1,i_1\rangle} \prec \mathcal{R}_{\langle e'_1,i'_1,j'_1\rangle} \prec \cdots \prec \mathcal{P}_{\langle e_n,i_n\rangle} \prec \mathcal{R}_{\langle e'_n,i'_n,j'_n\rangle} \prec \cdots .$$

The tree of strategies. The priority tree T of the construction is a subtree of $S = \{\infty, d, f\}^{<\omega}$ such that any node $\alpha \in T$ of length even has two immediate extensions $\alpha^\frown d$ and $\alpha^\frown f$ on T and any node $\alpha \in T$ of length odd has three immediate extensions $\alpha^\frown\infty$, $\alpha^\frown d$ and $\alpha^\frown f$ on T. Say that a node $\alpha \in T$ is a $\mathcal{P}_{\langle e,i\rangle}$- or an $\mathcal{R}_{\langle e,i,j\rangle}$-strategy if it is assigned the corresponding requirement. As before, we write the $\mathcal{P}_{\langle e,i\rangle}$-requirement assigned to a node α of T with length even as $\mathcal{P}_{\langle e(\alpha),i(\alpha)\rangle}$ (or simply $\mathcal{P}_\alpha$). Similarly, we write the $\mathcal{R}_{\langle e,i,j\rangle}$-requirement assigned to a node β of T with length odd as $\mathcal{R}_{\langle e(\beta),i(\beta),j(\beta)\rangle}$ (or simply $\mathcal{R}_\beta$).

According to the priority of $\mathcal{P}$-, $\mathcal{R}$-requirements above, assign the requirements to nodes of the priority tree T by induction on their length as follows.

(1) For the empty node $\lambda \in T$, assign the highest priority $\mathcal{P}$-requirement $\mathcal{P}_{\langle e_0,i_0\rangle}$ to node λ.
(2) For nodes $d, f \in T$, assign the highest priority $\mathcal{R}$-requirement $\mathcal{R}_{\langle e'_0,i'_0,j'_0\rangle}$ to d, f.
(3) Assume that $\alpha = \alpha^{-\frown} o \in T$ with length $|\alpha| \geq 1$. There are two cases depending on the parity of $|\alpha|$.
   - If $|\alpha|$ is odd, then $\alpha^\frown\infty$, $\alpha^\frown d$ and $\alpha^\frown f \in T$. Suppose that the immediate predecessor α^- of α is assigned the $\mathcal{P}_{\langle e_n,i_n\rangle}$-requirement with $n \geq 0$.
     * For $o \in \{\infty, d, f\}$, assign $\mathcal{P}_{\langle e_{n+1},i_{n+1}\rangle}$ to node $\alpha^\frown o$.
   - If $|\alpha|$ is even, then $\alpha^\frown d$ and $\alpha^\frown f \in T$. Suppose that the immediate predecessor α^- of α is assigned the $\mathcal{R}_{\langle e'_n,i'_n,j'_n\rangle}$-requirement with $n \geq 0$.
     * For $o \in \{d, f\}$, find the least $m > n$ such that $\langle e'_m, i'_m\rangle \neq \langle e(\beta), i(\beta)\rangle$ for all nodes β of length odd assigned the $\mathcal{R}_{\langle e(\beta),i(\beta),j(\beta)\rangle}$-requirement with $\beta^\frown d \subseteq \alpha$ or $\beta^\frown\infty \subseteq \alpha$, and assign $\mathcal{R}_{\langle e'_m,i'_m,j'_m\rangle}$ to node $\alpha^\frown o$.

   Note that a node $\beta^\frown o \in T$ with $|\beta|$ odd and $o \in \{d, \infty\}$ represents that the β-strategy has an infinite outcome o. In this case, $[i(\beta), i(\beta)+j(\beta)+1] \nsubseteq W^K_{e(\beta)}$ and the β-strategy works to diagonalize against $\varphi^2_{i(\beta)}$ to be a computable predecessor approximation function for $C_{e(\beta)}$. So we do not need to assign $\mathcal{R}_{\langle e(\beta),i(\beta),\cdot\rangle}$-requirements to nodes of T below $\beta^\frown o$.

We now describe possible conflictions between multiple $\mathcal{R}$-strategies on the priority tree T. Let α be an $\mathcal{R}_{\langle e,i,j\rangle}$-strategy on T. Suppose that $[i, i+j+1] \nsubseteq W^K_e$. For all $j' > j$, as $[i, i+j+1] \subseteq [i, i+j'+1]$, we have $[i, i+j'+1] \nsubseteq W^K_e$. So we do not need to assign $\mathcal{R}_{\langle e,i,j'\rangle}$ requirements with $j' > j$ to nodes below the infinite outcomes of α (i.e., below $\alpha^\frown d$ or $\alpha^\frown\infty$) on the priority tree. However, for $i' \neq i$, $\mathcal{R}_{\langle e,i',\cdot\rangle}$-requirements are still assigned to nodes below the infinite outcomes of α. Let $\alpha' \supset \alpha$ be an $\mathcal{R}_{\langle e,i',j'\rangle}$-strategy on T.

(i) If $\alpha^\frown d \subset \alpha'$, then there are no conflictions because α only enumerates finitely many triggers of nodes with lower priority into C_e.

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(ii) If $\alpha^\frown\infty \subset \alpha'$, then α acts infinitely often to enumerate triggers of nodes $\beta \in T$ with $\beta > \alpha$ into C_e to force $\lim_s \varphi_i^2(y,s) = \infty$ for some even number y; this may injure a convergent computation $\Phi_{j'}^{C_e}(j')[s]\downarrow$, which α' needs to preserve, infinitely often. Since the triggers enumerated by α during the construction tend to infinity, to deal with this problem, we only preserve believable computations $\Phi_{j'}^{C_e}(j')[s]\downarrow$ where α will not enumerate triggers $< \phi_{j'}^{C_e}(j')[s]$, the use of the computation, after stage s.

Definition 10. *Let $\alpha' \in T$ be an $\mathcal{R}_{\langle e,i',j'\rangle}$-strategy. A convergent computation $\Phi_{j'}^{C_e}(j')[s]\downarrow$ is believable at a non α'-expansionary stage s if for any $\mathcal{R}$-strategy $\alpha \in T$ with $e(\alpha) = e$ and $\alpha^\frown\infty \subset \alpha'$, the α-strategy acts as in Step (5.n) for some $n \geq 1$ to enumerate a trigger $x_{\beta,t}^k$ of some node $\beta > \alpha$ into C_e at stage s, then the use of the convergent computation $\phi_{j'}^{C_e}(j')[s]$ is strictly less than $x_{\beta,t}^1$, where $x_{\beta,t}^1 < \cdots < x_{\beta,t}^{q_\beta}$ with $q_\beta = 2^{3^{|\beta|+1}}$ are all triggers of the node β first defined at some β-stage $t < s$.*

3.2 Construction and verification

We are ready to provide a formal construction of the c.e. sets $C_e (e \in \mathbb{N})$ by stages according to the tree of strategies.

Construction.

Stage 0. Let $\delta_0 = \lambda$, the empty node on T. Initialize all nodes on T. For all e, let $C_{e,0} = \emptyset$. For all nodes $\alpha \in T$ with length $|\alpha|$ odd, set the $C_{e(\alpha)}$-restraint $r(\alpha, 0) = 0$.

Stage $s \geq 1$. Define the current true path $\delta_s \in T$ with length $|\delta_s| \leq s$ at substages m with $0 \leq m \leq s-1$, starting at substage 0. Let $\alpha_0 = \lambda$, the empty node on T, and assume that we have defined $\alpha_m = \delta_s(0)\cdots\delta_s(m-1)$ by the end of substage $m-1$ for $1 \leq m \leq s$.

Substage $m (0 \leq m \leq s-1)$. Define the current true outcome $\delta_s(m)$ of α_m and enumerate numbers into $C_{e(\alpha_m),s}$ according to the basic α_m-strategy with additional initializations to strategies with lower priority on the priority tree T.

- If a node $\alpha \in T$ is initialized at stage s, then the witness $w_{\alpha,s}$ and triggers $x_{\alpha,s}^k (1 \leq k \leq q_\alpha = 2^{3^{|\alpha|+1}})$ are undefined.
- If $m = s-1$, let $\delta_s = \alpha_m{}^\frown\delta_s(m) = \alpha_{s-1}{}^\frown\delta_s(s-1) = \delta_s(0)\cdots\delta_s(s-1)$. Initialize all nodes $\alpha \geq \delta_s$, also initialize all nodes of length $\geq s+1$. End current stage s and go to next stage.

Substage m proceeds as follows. There are two cases depending on the parity of m.

Case I. If m is even, then α_m is a $\mathcal{P}$-strategy. Check whether there was a last α_m-stage s^- such that α_m has not been initialized since then.

(1) If such a last α_m-stage s^- did not exist, then s is the first α_m-stage after last initialization. Act as in the basic α_m-strategy.

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- Choose a large new witness $w_{\alpha_m,s}$ and a set of fresh *trigger* elements $\{x^k_{\alpha_m,s} : 1 \leq k \leq q_{\alpha_m} = 2^{3^{|\alpha_m|+1}}\}$ for β-strategies with $|\beta|$ odd and $\beta \leq \alpha_m$ (i.e., $\beta <_L \alpha_m$ or $\beta \subseteq \alpha_m$) on the priority tree T to enumerate into $C_{e(\beta)}$ later such that

$$N_{\alpha_m,s} < x^1_{\alpha_m,s} < x^2_{\alpha_m,s} < \cdots < x^{q_{\alpha_m}}_{\alpha_m,s} < w_{\alpha_m,s},$$

  where $N_{\alpha_m,s}$ is a new number larger than all used numbers before; particularly, $N_{\alpha_m,s}$ is bigger than all restraints $r(\beta, s)$ appointed on $C_{e(\beta)}$ by higher priority β-strategies.
- Go to Step (2); that is, wait for $\varphi_{i(\alpha_m)}(w_{\alpha_m,s}) \downarrow= 0$.
- Let $\delta_s(m) = f$. Define $\delta_s = \alpha_m{}^\frown\delta_s(m) = \alpha_m{}^\frown f$. Initialize all nodes $\alpha \geq \alpha_m{}^\frown f$, also initialize all nodes of length $\geq s+1$.
- End Stage s and go to next stage.

The witness $w_{\alpha_m,s}$ and the triggers $x^k_{\alpha_m,s}(1 \leq k \leq q_{\alpha_m})$ keep unchanged at later stages unless the node α_m is initialized by higher priority strategies.

(2) If such a last α_m-stage s^- did exist, then there was a largest α_m-stage $s_0 \leq s^- < s$ such that the witness w_{α_m,s_0} and trigger elements $x^k_{\alpha_m,s_0}$ with $1 \leq k \leq q_{\alpha_m} = 2^{3^{|\alpha_m|+1}}$ were defined at stage s_0 and α_m has not been initialized since then. Define the current outcome $\delta_s(m)$ of the α_m-strategy started at stage s_0 according to the current state of the strategy.
- If the α_m-strategy waits at Step (2), let $\delta_s(m) = f$.
  - Initialize all nodes $\alpha >_L \alpha_m{}^\frown f$. Go to substage $m+1$ of current stage s if $m+1 < s$.
- If the α_m-strategy currently arrives at Step (2), act as in Step (3).
  - Enumerate w_{α_m,s_0} into $C_{e(\alpha_m),s}$.
  - Let $\delta_s(m) = d$. Define $\delta_s = \alpha_m{}^\frown\delta_s(m) = \alpha_m{}^\frown d$. Initialize all nodes $\alpha \geq \alpha_m{}^\frown d$ and all nodes of length $\geq s+1$.
  - End Stage s and go to next stage.
- Otherwise, the α_m-strategy was already arrived at Step (2) at last α_m-stage s^-, let $\delta_s(m) = d$.
  - Initialize all nodes $\alpha >_L \alpha_m{}^\frown d$. Go to substage $m+1$ of current stage s if $m+1 < s$.

Case II. If m is odd, then α_m is an $\mathcal{R}$-strategy. Act as follows.

Case 1. If s is a non α_m-expansionary stage, let $\delta_s(m) = f$. Act as in the basic α_m-strategy.

(1) If the computation $\Phi^{C_{e(\alpha_m)}}_{j(\alpha_m)}(j(\alpha_m))[s]$ newly converges and it is a believable computation at stage s, let $\phi^{C_{e(\alpha_m)}}_{j(\alpha_m)}(j(\alpha_m))[s]$ be the use of the computation.
- Appoint a new $C_{e(\alpha_m)}$-restraint $r(\alpha_m, s) = \phi^{C_{e(\alpha_m)}}_{j(\alpha_m)}(j(\alpha_m))[s]$.
- Define $\delta_s = \alpha_m{}^\frown\delta_s(m) = \alpha_m{}^\frown f$. Initialize all nodes $\alpha \geq \delta_s$ (i.e., $\alpha >_L \delta_s$ or $\alpha \supseteq \delta_s$). Also initialize all nodes of length $\geq s+1$.
- End Stage s and go to next stage.

(2) Otherwise, there are two possible subcases:

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(i) if $\Phi_{j(\alpha_m)}^{C_{e(\alpha_m)}}(j(\alpha_m))[s]$ diverges or it converges but not believable at current stage, let the $C_{e(\alpha_m)}$-restraint $r(\alpha_m, s) = 0$.

(ii) otherwise, it is a believable computation already convergent at last non α_m-expansionary stage s^-, let the $C_{e(\alpha_m)}$-restraint $r(\alpha_m, s) = r(\alpha_m, s^-)$.

In both subcases, initialize all nodes $\alpha >_L \alpha_m{}^\frown f$. Go to substage $m+1$ of current stage s if $m+1 < s$.

Case 2. If s is an α_m-expansionary stage, check whether there was a last α_m-expansionary stage s^- such that α_m has not been initialized since then.

(1) If such a last α_m-expansionary stage s^- did not exist, then s is the first α_m-expansionary stage after last initialization. Act as in the basic α_m-strategy.

- Choose a large new witness $w_{\alpha_m,s}$ and a set of fresh *trigger* elements $\{x^k_{\alpha_m,s} : 1 \le k \le q_{\alpha_m} = 2^{3^{|\alpha_m|+1}}\}$ for higher priority β-strategies of length odd with $\beta \le \alpha_m$ (i.e., $\beta <_L \alpha_m$ or $\beta \subseteq \alpha_m$) on the priority tree T to enumerate into $C_{e(\beta)}$ later such that

$$N_{\alpha_m,s} < x^1_{\alpha_m,s} < x^2_{\alpha_m,s} < \cdots < x^{q_{\alpha_m}}_{\alpha_m,s} < w_{\alpha_m,s},$$

  where $N_{\alpha_m,s}$ is a new number larger than all used numbers before; particularly, $N_{\alpha_m,s}$ is bigger than all restraints $r(\beta, s)$ appointed on $C_{e(\beta)}$ by $\mathcal{R}$-strategies β with priority strictly higher than α_m.
- Enumerate the witness $w_{\alpha_m,s}$ into $C_{e(\alpha_m),s}$.
- Define $\delta_s = \alpha_m{}^\frown\delta_s(m) = \alpha_m{}^\frown d$. Initialize all nodes $\alpha \ge \alpha_m{}^\frown d$ and all nodes of length $\ge s+1$.
- End Stage s and go to next stage.

The witness $w_{\alpha_m,s}$ and the triggers $x^k_{\alpha_m,s}(1 \le k \le q_{\alpha_m})$ keep unchanged at later stages unless the node α_m is initialized by higher priority strategies.

(2) If such a last α_m-expansionary stage s^- did exist, then there was a largest α_m-expansionary stage $s_0 \le s^- < s$ such that the witness w_{α_m,s_0} and trigger elements $x^k_{\alpha_m,s_0}$ with $1 \le k \le q_{\alpha_m} = 2^{3^{|\alpha_m|+1}}$ were defined at stage s_0 and α_m has not been initialized since then. Define the current outcome $\delta_s(m)$ of the α_m-strategy started at stage s_0 according to the current state of the strategy.

- If the α_m-strategy waits at Step (2) or at Step ($4.n$) for some $n \ge 1$, let $\delta_s(m) = d$.
  * Initialize all nodes $\alpha >_L \alpha_m{}^\frown\delta_s(m) = \alpha_m{}^\frown d$. Go to substage $m+1$ of current stage s if $m+1 < s$.
- If the α_m-strategy currently arrives at Step (2), let $\delta_s(m) = d$. Act as in Step (3).
  * Let $t_0 = s$ and $u_{\alpha_m,t_0} = w_{\alpha_m,s} = w_{\alpha_m,s_0}$. Let $y_{\alpha_m,s}$ be the unique even number such that

$$\varphi^2_{i(\alpha_m)}(y_{\alpha_m,s}, l(i(\alpha_m), s)) = w_{\alpha_m,s} = w_{\alpha_m,s_0} \in O_{i(\alpha_m),s} = C_{e(\alpha_m),s}.$$

    The even number $y_{\alpha_m,s}$ remains the same at later stages as long as α_m is not initialized.

- $*$ Pick the largest trigger element $x^k_{\alpha_m,s} = x^k_{\alpha_m,s_0} < w_{\alpha_m,s_0}$ that has not been enumerated into $C_{e(\beta)}$ by β-strategies with $\beta < \alpha_m$ (i.e., $\beta <_L \alpha_m$ or $\beta \subset \alpha_m$).
- $*$ Enumerate $x^k_{\alpha_m,s}$ into $C_{e(\alpha_m),s}$. Go to Step (4.1).
  By Lemma 1 below, such a trigger $x^k_{\alpha_m,s_0}$ always exists.
- $*$ Define $\delta_s = \alpha_m{}^\frown\delta_s(m) = \alpha_m{}^\frown d$. Initialize all nodes $\alpha \geq \alpha_m{}^\frown d$ and all nodes of length $\geq s+1$.
- $*$ End Stage s and go to next stage.

- If the α_m-strategy arrives at Step $(4.n)$ for some $n \geq 1$, let $\delta_s(m) = \infty$. Act as in Step $(5.n)$.
  - $*$ Let $t_n = s$ and $u_{\alpha_m,t_n} = \varphi^2_{i(\alpha_m)}(y_{\alpha_m,t_0}, l(i(\alpha_m), s))$. Then we have

$$w_{\alpha_m,s_0} \leq u_{\alpha_m,t_{n-1}} = \varphi^2_{i(\alpha_m)}(y_{\alpha_m,t_0}, l(i(\alpha_m), t_{n-1})) < \varphi^2_{i(\alpha_m)}(y_{\alpha_m,t_0}, l(i(\alpha_m), t_n)) = u_{\alpha_m,t_n} \in C_{e(\alpha_m),t_n}$$

    because the enumeration of the trigger element $x^k_{\alpha_m,t_0} < w_{\alpha_m,s_0}$ in Step (3) for $n = 1$ or $x^k_{\beta,t} \in (w_{\alpha_m,s_0}, u_{\alpha_m,t_{n-1}})$ in Step (5. $(n-1)$) for $n \geq 2$ forces the number of predecessors of y_{α_m,t_0} computed by $\varphi^2_{i(\alpha_m)}$ to move up.
    As $u_{\alpha_m,t_n} \in C_{e(\alpha_m),t_n}$ and $u_{\alpha_m,t_n} > w_{\alpha_m,s_0}$, there is a unique node $\beta > \alpha_m$ on the priority tree T with $e(\beta) = e(\alpha_m)$ such that

$$u_{\alpha_m,t_n} = w_{\beta,t} \text{ or } u_{\alpha_m,t_n} = x^k_{\beta,t} \text{ for some } k,$$

    where $w_{\beta,t}$ and $x^k_{\beta,t}(1 \leq k \leq q_\beta = 2^{3^{|\beta|+1}})$ were first appointed on β at some β-stage $t \in (s_0, t_n)$.
  - $*$ Pick the largest trigger number $x^k_{\beta,t} < u_{\alpha_m,t_n} \leq w_{\beta,t}$ that has not been enumerated into $C_{e(\beta')}$ for any $\beta' \leq \beta$.
  - $*$ Enumerate $x^k_{\beta,t}$ into $C_{e(\alpha_m),t_n}$. Go to Step $(4.(n+1))$.
    By Lemma 1 below, such a trigger $x^k_{\beta,t}$ always exists.
  - $*$ Initialize all nodes $\alpha >_L \alpha_m{}^\frown\infty$. Go to substage $m+1$ of current stage s if $m+1 < s$.

This ends the construction.

Lemma 1. *For each node $\alpha \in T$, let $q_\alpha = 2^{3^{|\alpha|+1}}$. Suppose that s_0 is a first α-stage after last initialization during the construction such that the witness w_{α,s_0} and the set of trigger elements $X_{\alpha,s_0} = \{x^k_{\alpha,s_0} : 1 \leq k \leq q_\alpha\}$ were first chosen at stage s_0. Then there are enough trigger elements from X_{α,s_0} for higher priority β-strategies to enumerate into $C_{e(\beta)}$ after stage s_0. That is,*

$$\{x^k_{\alpha,s_0} \in C_{e(\beta)} : \beta \in T, \beta \leq \alpha, 1 \leq k \leq q_\alpha\} \subsetneq X_{\alpha,s_0}.$$

Proof. During the construction, only higher priority $\mathcal{R}$-strategies $\beta \leq \alpha$ with $e(\beta) = e(\alpha)$ can enumerate triggers from X_{α,s_0} into $C_{e(\beta)}$ after stage s_0. Furthermore, if $\beta \leq \alpha$ is initialized at some stage $s > s_0$, then it will initiate a new strategy later; thus β will not use triggers from X_{α,s_0} after stage s.

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Let β_1 be the first $\mathcal{R}$-strategy with $\beta_1 \leq \alpha$ that acts as in Step (3) or Step (5.n) for some n to enumerate a trigger x^k_{α,s_0} into $C_{e(\beta_1),s_1}$ at some stage $s_1 > s_0$. Then there is a unique even number y_{β_1,s_1} such that

$$\varphi^2_{i(\beta_1)}(y_{\beta_1,s_1}, l(i(\beta_1), s_1)) = w_{\alpha,s_0} \in O_{i(\beta_1),s_1} = C_{e(\beta_1),s_1},$$

so we have $e(\beta_1) = e(\alpha)$, and the β_1-strategy enumerates the largest $x^{q_\alpha}_{\alpha,s_0}$ into $C_{e(\beta_1),s_1} = C_{e(\alpha),s_1}$. This enumeration forces the number of predecessors of y_{β_1,s_1} approximated by $\varphi^2_{i(\beta_1)}$ to move up if the β_1-strategy next arrives at Step (4.1) from Step (3) or at Step (4.($n+1$)) from Step (5.n) at some stage $s > s_1$. That is, at stage s, we have

$$w_{\alpha,s_0} = \varphi^2_{i(\beta_1)}(y_{\beta_1,s_1}, l(i(\beta_1), s_1)) < \varphi^2_{i(\beta_1)}(y_{\beta_1,s_1}, l(i(\beta_1), s)) \in C_{e(\alpha),s}.$$

Since we only enumerate witnesses and triggers of nodes of T into $C_{e(\alpha)}$ during the construction, $\varphi^2_{i(\beta_1)}(y_{\beta_1,s_1}, l(i(\beta_1), s))$ is a trigger or witness of some node $\beta \in T$ with priority strictly lower than α. The β_1-strategy enumerates some trigger of β into $C_{e(\beta_1)} = C_{e(\alpha)}$ at stage s. From this analysis, we see that β_1 never enumerates triggers from the set X_{α,s_0} into $C_{e(\beta_1)}$ after stage s_1. So β_1 uses at most one trigger from X_{α,s_0}.

Let β_2 be the first $\mathcal{R}$-strategy in $\{\beta \in T : \beta \leq \alpha\} \setminus \{\beta_1\}$ that acts as in Step (3) or Step (5.n) for some n to enumerate a trigger x^k_{α,s_0} into $C_{e(\beta_2),s_2}$ at some stage $s_2 > s_1$. Then there is a unique even number y_{β_2,s_2} such that

$$\varphi^2_{i(\beta_2)}(y_{\beta_2,s_2}, l(i(\beta_2), s_2)) = w_{\alpha,s_0} \text{ or } x^{q_\alpha}_{\alpha,s_0} \in O_{i(\beta_2),s_2} = C_{e(\beta_2),s_2},$$

then $e(\beta_2) = e(\alpha)$ and $k = q_\alpha - 1$.

(i) If $\varphi^2_{i(\beta_2)}(y_{\beta_2,s_2}, l(i(\beta_2), s_2)) = w_{\alpha,s_0}$, as in the case of β_1-strategy above, β_2 never enumerates triggers from X_{α,s_0} into $C_{e(\beta_2)}$ after stage s_2.

(ii) If $\varphi^2_{i(\beta_2)}(y_{\beta_2,s_2}, l(i(\beta_2), s_2)) = x^{q_\alpha}_{\alpha,s_0} < w_{\alpha,s_0}$, the enumeration of $x^{q_\alpha-1}_{\alpha,s_0}$ forces $\varphi^2_{i(\beta_2)}(y_{\beta_2,s_2}, l(i(\beta_2), s)) > x^{q_\alpha}_{\alpha,s_0}$ if the β_2-strategy acts as in Step (5.m) for some m to enumerate triggers at some least stage $s > s_2$.
   - If $\varphi^2_{i(\beta_2)}(y_{\beta_2,s_2}, l(i(\beta_2), s)) > w_{\alpha,s_0}$, then β_2 acts to enumerate a trigger of node β with $\beta > \alpha$ at stage s.
   - If $\varphi^2_{i(\beta_2)}(y_{\beta_2,s_2}, l(i(\beta_2), s)) = w_{\alpha,s_0}$, then β_2 acts to enumerate the largest unused trigger x^k_{α,s_0} at stage s.

   Therefore, the β_2-strategy requires at most two triggers from X_{α,s_0} into $C_{e(\beta_2)}$ after stage s_0.

Let β_3 be the first $\mathcal{R}$-strategy in $\{\beta \in T : \beta \leq \alpha\} \setminus \{\beta_1, \beta_2\}$ that acts as in Step (3) or Step (5.n) for some n to enumerate a trigger x^k_{α,s_0} into $C_{e(\beta_3),s_3}$ at some stage $s_3 > s_2$. Then there is a unique even number y_{β_3,s_3} such that

$$\varphi^2_{i(\beta_3)}(y_{\beta_3,s_3}, l(i(\beta_3), s_3)) = w_{\alpha,s_0} \text{ or } x^k_{\alpha,s_0} \in O_{i(\beta_3),s_3} = C_{e(\beta_3),s_3} = C_{e(\alpha),s_3},$$

and there are at most 1+2=3 triggers $x^k_{\alpha,s_0} \in C_{e(\alpha),s_3}$. Now it requires at most 4 enumerations of new triggers from X_{α,s_0} into $C_{e(\beta_3)} = C_{e(\alpha)}$ to force the

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number of predecessors of y_{β_3,s_3} approximated by $\varphi^2_{i(\beta_3)}$, namely, $\varphi^2_{i(\beta_3)}(y_{\beta_3,s_3},\cdot)$, to exceed w_{α,s_0}. The total number of triggers from X_{α,s_0} used by β_1,β_2 and β_3 is at most $1+2+2^2=2^3-1$.

Generally, for $m\geq 2$, suppose that β_m is the first $\mathcal{R}$-strategy in $\{\beta\in T:\beta\leq\alpha\}\setminus\{\beta_1,\dots,\beta_{m-1}\}$ that acts as in Step (3) or Step ($5.n$) for some n to enumerate a trigger x^k_{α,s_0} into $C_{e(\beta_m),s_m}$ at some stage $s_m>s_{m-1}$. Then there is a unique even number y_{β_m,s_m} such that

$$\varphi^2_{i(\beta_m)}(y_{\beta_m,s_m},l(i(\beta_m),s_m))=w_{\alpha,s_0} \text{ or } x^k_{\alpha,s_0}\in O_{i(\beta_m),s_m}=C_{e(\beta_m),s_m}=C_{e(\alpha),s_m},$$

and there are at most $1+2+\cdots+2^{m-2}=2^{m-1}-1$ triggers $x^k_{\alpha,s_0}\in C_{e(\alpha),s_m}$. Since the value of $\varphi^2_{i(\beta_m)}(y_{\beta_m,s_m},l(i(\beta_m),s_m))$ has at most 2^{m-1} many possible choices, it requires at most 2^{m-1} many enumerations of new triggers from X_{α,s_0} into $C_{e(\beta_m)}=C_{e(\alpha)}$ to force the number of predecessors of y_{β_m,s_m} to exceed w_{α,s_0}. So the total number of triggers from X_{α,s_0} used by $\beta_1,\dots,\beta_m$ is at most $1+2+\cdots+2^{m-1}=2^m-1$.

Since the number of nodes of length $\leq|\alpha|$ in the priority tree T is strictly less than

$$1+3+\cdots+3^{|\alpha|}=\frac{3^{|\alpha|+1}-1}{2}<3^{|\alpha|+1},$$

the number of higher priority nodes $\beta\in T$ with $\beta\leq\alpha$ is strictly less than $3^{|\alpha|+1}$, and we have $m<3^{|\alpha|+1}$. This implies that the total number of triggers from X_{α,s_0} used by all higher priority $\mathcal{R}$-strategies β with $\beta\leq\alpha$ is strictly less than

$$1+2+\cdots+2^{3^{|\alpha|+1}-1}=2^{3^{|\alpha|+1}}-1<2^{3^{|\alpha|+1}}=q_\alpha.$$

Therefore, we appoint enough trigger elements x^k_{α,s_0} on node α for higher priority β-strategies to enumerate into $C_{e(\beta)}=C_{e(\alpha)}$ during the construction. That is, the trigger elements in Step (3) or Step ($5.n$) of a basic α-strategy initiated at stage s_0 always exist if needed. This completes the proof of the lemma. □

Let $\mathrm{TP}=\liminf_s\delta_s$ be the leftmost path on T visited infinitely often during the construction, called the *true path* of the construction. That is, for any $n\geq 1$, the node $\mathrm{TP}\restriction_n=\mathrm{TP}(0)\cdots\mathrm{TP}(n-1)\in T$ is visited infinitely often, but any node $\alpha<_L\mathrm{TP}\restriction_n$ is visited finitely often during the construction.

Lemma 2. TP *is an infinite path on the priority tree T.*

Proof. The empty node $\lambda\subset\mathrm{TP}$ because every stage is a λ-stage; furthermore, λ has the highest priority on the priority tree T, it is not initialized at stages $s>0$. Let $\alpha\subset\mathrm{TP}$ with $|\alpha|\geq 0$. Assume that there is a least stage s_α such that α is not initialized at any stage $s>s_\alpha$. There are two cases.

Case 1. If $|\alpha|$ is even, α is a $\mathcal{P}$-strategy. Let t_α be the first α-stage $>s_\alpha$. The α-strategy picks a new witness w_{α,t_α} and a finite set $\{x^k_{\alpha,t_\alpha}:1\leq k\leq q_\alpha=2^{3^{|\alpha|+1}}\}$ of new triggers at stage t_α. Since α has the highest priority after stage s_α, the witness w_{α,t_α} and the triggers x^k_{α,t_α} are never redefined later.

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(1) If the α-strategy waits at Step (2) forever after stage t_α, then any α-stage $s \geq t_\alpha$ is an $\alpha^\frown f$-stage. In this case, $\alpha^\frown f$ is not initialized after stage t_α, α has true outcome f, and we have $\alpha^\frown f \subset \mathrm{TP}$.

(2) Otherwise, the α-strategy arrives at Step (2) and acts as in Step (3) at some stage $s > t_\alpha$; the strategy enumerates the witness w_{α,t_α} into $C_{e(\alpha),s}$, making $C_{e(\alpha)} \neq \varphi_{i(\alpha)}$, and the construction is terminated at stage s with the current true path $\delta_s = \alpha^\frown d$. Then any α-stage $s' \geq s$ is an $\alpha^\frown d$-stage, and $\alpha^\frown d$ is not initialized after stage s. So we have $\alpha^\frown d \subset \mathrm{TP}$.

Case 2. If $|\alpha|$ is odd, then α is an $\mathcal{R}$-strategy. There are two subcases.

(1) If there are finitely many α-expansionary stages, let t_α be the least stage $> s_\alpha$ such that any $s \geq t_\alpha$ is not an α-expansionary stage. The basic α-strategy acts at most once to appoint a new restraint on $C_{e(\alpha)}$ at some stage $s \geq t_\alpha$ if the believable computation $\Phi_{j(\alpha)}^{C_{e(\alpha)}}(j(\alpha))[s]$ newly converges in which case the α-strategy terminates current construction by setting the current true path $\delta_s = \alpha^\frown f$ and initializing all nodes $\geq \alpha^\frown f$. This action preserves the believable computation $\Phi_{j(\alpha)}^{C_{e(\alpha)}}(j(\alpha))[s]$ forever, and the α-strategy never sets new $C_{e(\alpha)}$-restraints later. Then $\alpha^\frown f$ is not initialized after stage s and any α-stage $s' \geq s$ is an $\alpha^\frown f$-stage. So we have $\alpha^\frown f \subset \mathrm{TP}$.

(2) If there are infinitely many α-expansionary stages, let t_α be the first α-expansionary stage $> s_\alpha$. The α-strategy picks a new witness w_{α,t_α} and a finite set $\{x^k_{\alpha,t_\alpha} : 1 \leq k \leq q_\alpha = 2^{3^{|\alpha|+1}}\}$ of new triggers, it first enumerates w_{α,t_α} into $C_{e(\alpha),t_\alpha}$ at the same stage t_α. Since α has the highest priority after stage s_α, the witness w_{α,t_α} and the triggers x^k_{α,t_α} are never redefined later.
   - If the α-strategy waits at Step (2) forever after stage t_α, then any α-expansionary stage $s \geq t_\alpha$ is an $\alpha^\frown d$-stage. In this case, $\alpha^\frown d$ is not initialized after stage t_α, and we have $\alpha^\frown d \subset \mathrm{TP}$.
   - Otherwise, the α-strategy arrives at Step (2) and acts as in Step (3) at some stage $t_0 > t_\alpha$. At stage t_0, the strategy enumerates the largest unused trigger x^k_{α,t_α} into $C_{e(\alpha),t_0}$; the construction is terminated by setting the current true path $\delta_{t_0} = \alpha^\frown d$, and initializing all nodes $\geq \alpha^\frown d$. There are two possible situations after stage t_0.
     * The α-strategy waits at Step $(4.n)$ forever after certain stage $s > t_0$. Then any α-expansionary stage $s' \geq s$ is an $\alpha^\frown d$-stage. In this case, $\alpha^\frown d$ is not initialized after stage s and $\alpha^\frown d \subset \mathrm{TP}$.
     * The α-strategy arrives at Step $(4.(n+1))$ from Step $(5.n)$ infinitely often after stage t_0. Then there are infinitely many α-expansionary stages $s > t_0$ such that the α-strategy acts as in Step $(5.n)$ for some n at stage s by enumerating a largest unused trigger of some node with lower priority into $C_{e(\alpha),s}$. Such a stage s is an $\alpha^\frown\infty$-stage. In this case, $\alpha^\frown\infty$ is not initialized after stage t_0 and $\alpha^\frown\infty \subset \mathrm{TP}$.

We have seen that for any $\alpha \subset \mathrm{TP}$ with $|\alpha| \geq 0$, there is a least stage s_α such that α is not initialized after stage s_α. The α-strategy acts after stage s_α

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with an outcome o such that $\alpha^\frown o \subset \mathrm{TP}$. This shows that TP is an infinite path on the priority tree T. □

Lemma 3. *For all e, C_e is noncomputable.*

Proof. For fixed numbers e, i, let α be the unique $\mathcal{P}_{\langle e,i\rangle}$-strategy along the true path TP of the construction, then α is initialized finitely often, and there are infinitely many α-stages during the construction. Suppose that α has the highest priority after some least stage s_α. The α-strategy acts to diagonalize against φ_i computing C_e after stage s_α.

(i) If the strategy waits at Step (2) forever after certain stage, then there is a witness w such that either $\varphi_i(w)$ diverges or it converges with $\varphi_i(w)\downarrow = 1 \neq C_e(w) = 0$; in this case, α has true outcome f.
(ii) If the strategy arrives at Step (2) and acts as in Step (3) at some stage $s > s_\alpha$, then there is a witness w such that $\varphi_i(w)\downarrow = 0 \neq C_e(w) = 1$; in this case, α has true outcome d.

The α-strategy succeeds to ensure that $C_e \neq \varphi_i$ after stage s_α. Then $C_e \neq \varphi_i$ for all i, and thus it is not computable. □

Lemma 4. *For all e, $e \in \mathrm{Cof}^K$ if and only if C_e is order-computable.*

Proof. Fix a number e. If $e \in \mathrm{Cof}^K$, W_e^K is cofinite. Then there exists a number i such that $[i, \infty) \subseteq W_e^K$, that is, $[i, i+j+1] \subseteq W_e^K$ for all j. For any node α assigned an $\mathcal{R}_{\langle e,i,\cdot\rangle}$-requirement on the priority tree, there are only finitely many α-expansionary stages. For each j, there is a unique node $\alpha_{\langle e,i,j\rangle}$ working for $\mathcal{R}_{\langle e,i,j\rangle}$ along the true path TP of the construction such that

$$\alpha_{\langle e,i,0\rangle}{}^\frown f \subset \alpha_{\langle e,i,1\rangle}{}^\frown f \subset \cdots \subset \alpha_{\langle e,i,j\rangle}{}^\frown f \subset \cdots \subset \mathrm{TP}.$$

For each j, the $\alpha_{\langle e,i,j\rangle}$-strategy is not initialized after certain stage s_j. Then $s_0 \leq s_1 \leq \cdots \leq s_j \leq \cdots$ because of the priority of the strategies. $\alpha_{\langle e,i,j\rangle}$ acts at most once to appoint a new C_e-restraint $\phi_j^{C_e}(j)[s]$ to preserve the believable computation $\Phi_j^{C_e}(j)[s]$ if it newly converges at some stage $s \in (s_j, s_{j+1}]$. Since the computation $\Phi_j^{C_e}(j)[s]\downarrow$ is believable, all higher priority $\mathcal{R}$-strategies β with $\beta^\frown\infty \subset \alpha_{\langle e,i,j\rangle}$ only enumerate triggers $x^k_{\beta',s'}$ of nodes $\beta' > \alpha_{\langle e,i,j\rangle}$ at any stage $s' \geq s$ such that $x^1_{\beta',s'} > \phi_j^{C_e}(j)[s]$, the use of the believable computation. Then the computation $\Phi_j^{C_e}(j)[s]\downarrow$ is not injured by β-strategy after stage s, and thus, it is preserved forever. Then C_e' is a Δ_2^0 set because for all j, $j \in C_e'$ if and only $\Phi_j^{C_e}(j)$ converges if and only if there are infinitely many stages s such that the computation $\Phi_j^{C_e}(j)[s]\downarrow$ is believable. Therefore, C_e is a low c.e. set, and thus, it is order-computable.

If $e \notin \mathrm{Cof}^K$, W_e^K is not cofinite. Now for all i, there is a least number j_i such that $[i, i+j_i+1] \nsubseteq W_e^K$. For all i, there are only finitely many nodes for $\mathcal{R}_{\langle e,i,\cdot\rangle}$-requirements along the true path TP of the construction, one node $\alpha_{\langle e,i,j\rangle}$ for

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$\mathcal{R}_{\langle e,i,j\rangle}$ with $0 \le j \le j_i$ such that each $\alpha_{\langle e,i,j\rangle}$ with $j < j_i$ has the finite outcome f and the longest node $\alpha_{\langle e,i,j_i\rangle}$ has an infinite outcome $o \in \{d, \infty\}$. That is,

$$\alpha_{\langle e,i,0\rangle}{}^\frown f \subset \alpha_{\langle e,i,1\rangle}{}^\frown f \subset \cdots \subset \alpha_{\langle e,i,j_i-1\rangle}{}^\frown f \subset \alpha_{\langle e,i,j_i\rangle}{}^\frown o \subset \mathrm{TP}.$$

Let $\alpha = \alpha_{\langle e,i,j_i\rangle}$ be the $\mathcal{R}_{\langle e,i,j_i\rangle}$-strategy along the true path TP of the construction, then α is initialized finitely often, and there are infinitely many α-expansionary stages. α has the highest priority after certain stage s_α. After stage s_α, the α-strategy acts to diagonalize against φ_i^2 to be a computable predecessor approximation function for C_e.

(i) If the strategy waits at Step (2) or Step $(4.n)$ forever after certain stage, then either $\lim_s l(i,s) < \infty$ or $C_e \neq O_i$, the order-computable set computed by φ_i^2; in this case, α has true outcome d, and $\alpha^\frown d \subset \mathrm{TP}$.
(ii) If the strategy first arrives at Step (2) and then arrives at Step $(4.(n+1))$ from Step $(5.n)$ infinitely often, then there is an even number y such that $\lim_s \varphi_i^2(y,s) = \infty$; in this case, α has true outcome ∞, and $\alpha^\frown \infty \subset \mathrm{TP}$.

The α-strategy succeeds in ensuring that C_e is not order-computed by φ_i^2 after stage s_α. Then C_e is not order-computed by φ_i^2 for all i, and thus it is not order-computable. □

We have constructed a uniform sequence of c.e. sets $\langle C_e : e \in \mathbb{N}\rangle$ such that $e \in \mathrm{Cof}^K$ if and only if C_e is order-computable for all e. Since Cof^K is Σ_4^0-complete, so is the index set of c.e. order-computable sets. This completes the proof of Theorem 1. □

Recall that $\mathcal{I} = \{e \in \mathbb{N} : \varphi_e^2 \text{ is total, and } (\forall x)[\lim_s \varphi_e^2(x,s) \downarrow\in \{0,1\}]\}$ is the index set of Δ_2^0 sets. For $e \in \mathcal{I}$, $A_e = \{x \in \mathbb{N} : (\forall x)[\lim_s \varphi_e^2(x,s) \downarrow= 1]\}$ is the Δ_2^0 set with index e. Since both c.e. sets and order-computable sets form subclasses of Δ_2^0 sets, based on Theorem 1, we obtain the exact complexity for general order-computable sets.

Corollary 1. *The index set $\{e \in \mathcal{I} : A_e$ is order-computable$\}$ is Σ_4^0-complete within $\mathcal{I}$.*

Proof. For a number e, $e \in \mathcal{I}$ if and only if φ_e^2 is a computable binary function such that $\lim_s \varphi_e^2(x,s)$ converges to 0 or 1 for all x. To prove the corollary, it suffices to construct a uniform sequence $\langle F_e : e \in \mathbb{N}\rangle$ of computable binary functions such that the following two conditions hold:

(1) for all e, $\lim_s F_e(x,s)$ converges to 0 or 1 for all x;
(2) for all e, $e \in \mathrm{Cof}^K \Leftrightarrow B_e = \{x \in \mathbb{N} : \lim_s F_e(x,s) \downarrow= 1\}$ is order-computable.

Let $\langle C_e : e \in \mathbb{N}\rangle$ be the uniform sequence of c.e. sets constructed in Theorem 1 such that $e \in \mathrm{Cof}^K \Leftrightarrow C_e$ is order-computable for all e. For any e, x, s and $t > x, s$, define $F_{e,t}(x,s) = C_{e,s}(x)$. Then the sequence $\langle F_e : e \in \mathbb{N}\rangle$ meets the desired conditions above. First, $F_e(x,s)$ is defined for all x, s, so F_e is a computable binary function. Second, since $C_{e,s}(x)$ changes from 0 to 1 for at most once when x is first enumerated into C_e at stage s, the condition (1) holds. Third, $\lim_s F_e(x,s)$ converges to 1 if and only if $x \in C_e$, so $B_e = \{x \in \mathbb{N} : \lim_s F_e(x,s) \downarrow= 1\} = C_e$ for all e; the condition (2) holds. □

4 Index set of Δ_2^0 sets

We continue to study the complexity of the index set $\mathcal{I}$ of Δ_2^0 sets.

Theorem 2. *The index set $\mathcal{I}$ of Δ_2^0 sets is Π_3^0-complete.*

Proof. For any number e, e is an index of a Δ_2^0 set if and only if φ_e^2 is total and $\lim_s \varphi_e^2(x,s)$ converges to 0 or 1 for all x in which case e is the index of the Δ_2^0 set $A_e = \{x \in \mathbb{N} : \lim_s \varphi_e^2(x,s) \downarrow= 1\}$. As $\mathrm{Tot}^K = \{e \in \mathbb{N} : (\forall x)[\Phi_e^K(x) \text{ converges}]\}$ is Π_2^0-complete relative the c.e. complete set K, it is Π_3^0-complete. To prove the theorem, it suffices to construct a uniform sequence $\langle F_e : e \in \mathbb{N}\rangle$ of computable binary functions such that for all e,

$$e \in \mathrm{Tot}^K \Leftrightarrow (\forall x)[\lim_s F_e(x,s) \downarrow= 0 \text{ or } 1].$$

We will build a computable ternary function $F(\cdot,\cdot,\cdot)$ satisfying the following requirements:

$\mathcal{R}_{\langle e,x\rangle}$: If $\Phi_e^K(x)$ is defined, then $\lim_s F(e,x,s)$ converges to 0.
If $\Phi_e^K(x)$ is undefined, then $\lim_s F(e,x,s)$ diverges.

Then $\langle F_e = F(e,\cdot,\cdot) : e \in \mathbb{N}\rangle$ is the desired sequence of computable binary functions.

If $e \in \mathrm{Tot}^K$, Φ_e^K is total, i.e., $\Phi_e^K(x)$ is defined for all x, then $\mathcal{R}_{\langle e,x\rangle}$ ensures that $\lim_s F_e(x,s)$ converges to 0. In this case, F_e approximates the empty set $\emptyset = \{x : \lim_s F_e(x,s) \downarrow= 1\}$. If $e \notin \mathrm{Tot}^K$, Φ_e^K is not total, i.e., there is a number x such that $\Phi_e^K(x)$ is undefined, then $\mathcal{R}_{\langle e,x\rangle}$ ensures that $\lim_s F(e,x,s)$ diverges. In this case, F_e cannot approximate a Δ_2^0 set.

Fix an effective bijection $\langle\cdot,\cdot\rangle : \mathbb{N}^2 \to \mathbb{N}$, and for all i,j,k, let $\langle i,j,k\rangle = \langle\langle i,j\rangle,k\rangle$. We construct the computable ternary function $F(\cdot,\cdot,\cdot)$ by stages.

Construction.

Stage 0. Define $F(0,0,0) = F(0,0,1) = 0$.

Assume that we have defined $F(e,x,y)$ for all e,x with $e,x \le s-1$ and $y \le 2s-1$ by the end of stage $s-1$.

Stage $s \ge 1$. For all $e,x \le s$, define $F(e,x,\cdot)$ as follows:

Step 1. For all $e,x < s$, define $F(e,x,2s)$ and $F(e,x,2s+1)$ as follows. There are two cases:

(1) If there was a last stage $s^- < s$ such that $\Phi_e^K(x)[s^-]$ was defined with use $\phi_e^K(x)[s^-]$ and the computation has not been changed since then, that is, $\Phi_e^K(x)[s]$ is defined with use $\phi_e^K(x)[s] = \phi_e^K(x)[s^-]$, define

$$F(e,x,2s) = F(e,x,2s+1) = 0.$$

(2) Otherwise, either $\Phi_e^K(x)[t]$ was undefined for all $t < s$, or there was a convergent computation $\Phi_e^K(x)[s^-]$ at last stage $s^- < s$ but the computation is changed currently, that is, either $\Phi_e^K(x)[s]$ is undefined or it is defined with use $\phi_e^K(x)[s] > \phi_e^K(x)[s^-]$, define

$$F(e,x,2s) = 0 \text{ and } F(e,x,2s+1) = 1.$$

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Step 2. For all $e, x \leq s$ and $t \leq 2s+1$, define $F(e,s,t) = F(s,x,t) = 0$.

This ends the construction.

For all $e, x \in \mathbb{N}$, at any stage s bigger than e, x of the construction, we have defined $F(e,x,y)$ for all $y \leq 2s+1$ by the end of stage s. Furthermore, the value of $F(e,x,y)$ is never changed during the construction. So F is a computable ternary function.

Lemma 5. *For all $e, x \in \mathbb{N}$, $\mathcal{R}_{\langle e,x\rangle}$ is satisfied.*

Proof. Fix numbers e, x. If $\Phi_e^K(x)$ is defined, then there is a least stage s such that the $\Phi_e^K(x)[s]$ is defined with use $\phi_e^K(x)[s]$ and the computation is not changed later; that is, at any stage $t \geq s$, the computation $\Phi_e^K(x)[t]$ is defined with use $\phi_e^K(x)[t]$ the same as $\phi_e^K(x)[s]$. Let $s_{e,x}$ be such a least stage bigger than e, x. Then at any stage $s > s_{e,x}$, we always defined $F(e,x,2s) = F(e,x,2s+1) = 0$. So $F(e,x,y) = 0$ for all $y > 2s_{e,x}$ and we have $\lim_s F(e,x,s) = 0$.

If $\Phi_e^K(x)$ is undefined, then there are infinitely many stages $s > e, x$ such that either the computation $\Phi_e^K(x)[s]$ is undefined or it is defined with use strictly bigger than that of the last convergent computation. At such a stage s, we defined $F(e,x,2s) = 0$, but $F(e,x,2s+1) = 1$. So $\lim_s F(e,x,s)$ does not exist. □

For a number e, there are two situations:

- If $e \in \mathrm{Tot}^K$, then $\Phi_e^K(x)$ is defined for all x. By Lemma 5, $\lim_s F(e,x,s) = 0$ for all x. In this case, $F_e = F(e,\cdot,\cdot)$ approximates the Δ_2^0 set $\{x : \lim_s F(e,x,s) \downarrow= 1\}$, which is exactly the empty set $\emptyset$.
- If $e \notin \mathrm{Tot}^K$, then $\Phi_e^K(x)$ is undefined for some x. By Lemma 5, $\lim_s F(e,x,s)$ does not exist for this x. In this case, $F_e = F(e,\cdot,\cdot)$ cannot approximate any Δ_2^0 set.

We have constructed a uniform sequence of computable binary functions $\langle F_e = F(e,\cdot,\cdot) : e \in \mathbb{N}\rangle$ such that for all e, Φ_e^K is total if and only if $\lim_s F_e(x,s)$ exists with limit 0 for all x. So the index set $\mathcal{I}$ of Δ_2^0 sets is Π_3^0-complete.

This completes the proof of Theorem 2. □

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