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Free boundary and American options in a jump-diffusion model

YANG CHENGRONG[1,2], JIANG LISHANG[3] and BIAN BAOJUN[3]

[1] *Institute of Mathematics, Jilin University, Changchun, 130012, China*
email: wytr@email.jlu.edu.cn
[2] *Business School of Jilin University, Changchun, 130012, China*
[3] *Department of Applied Mathematics, Tongji University, Shanghai, 200092, China*
email: jianglsk@online.sh.cn, bianbj@yahoo.com.cn

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The purpose of this paper is to give a pricing analysis for the American option in a jump-diffusion model by PDE arguments. Existence and uniqueness of the solution to the obstacle problem for the associated model is shown in suitable spaces. We also prove the unique existence of the solution of the corresponding free boundary problem. Furthermore, smoothness and monotonicity of the free boundary which is the optimal exercise boundary of the option are deduced.

1 Introduction

Black & Scholes [1] tackled the problem of pricing a European option on a non-dividend paying stock. In the Black-Scholes model, the underlying stock price is a continuous function of time, and is controlled by the following stochastic differential equation:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t,$$

where S_t is the underlying stock price at time t, μ, σ are respectively drift and volatility terms, and $\{W_t\}_{t\geqslant 0}$ is a standard real-valued Brownian motion. However, many empirical studies exhibit some biases in this kind of model. Cox & Ross [5] introduced a pure jump model in which the underlying stock is modeled by a known size jump process without diffusion term. The market is then complete. Merton [12] developed a jump-diffusion model in which the 'normal' vibration in the price, namely, the diffusion part, is still modeled by a standard geometric Brownian motion and has a continuous path, and the 'abnormal' vibration in the price is modeled by a 'jump' process. In such a model the price is not a continuous function of time. Such a model allows us to take into account brusque variations in market prices due to some rare events, such as nature disaster in a major economy, or major political changes, and can be used to explain the bias exhibited in the Black–Scholes model. Merton [12] established a pricing formula for the European option assuminng that jump risk is unpriced. Generalizations of his result can be found in Aase [1] and Naik & Lee [13].

The American option pricing problem has been given much attention in recent economic and finance literature. Nevertheless, there is no explicit formula for the price of the

American option. The earliest works on this problem are due to Mckean [11], and further to Van Moerbeke [15], who transformed the American option pricing analysis into a free-boundary problem, within the framework of the diffusion model. In addition to the free-boundary method, the formulation of the optimal-stopping problem by variational inequalities, as developed by Bensoussan & Lions [2], and applied to the American option in diffusion models by Jaillet *et al.* [9], provided numerical computations for the pricing of American options. This approach was further applied by Zhang [17] to the American option in Merton's jump-diffusion model. However variational inequalities lead to a somewhat less-explicit characterization of the American option price. Using the free-boundary approach as well as results in probability theory, Pham [14] studied the behavior of the optimal-stopping boundary for this problem, and proved the continuity of the corresponding free boundary with some restriction on the size of jump risk.

This paper studies the problem of pricing the American option with dividend in a jump-diffusion model by a PDE argument. The existence and uniqueness of the solution to the obstacle problem for the associated model is shown in suitable spaces. We also prove the unique existence of the solution of the corresponding free boundary problem. Furthermore, strict monotonicity and regularity of the free boundary which is the optimal exercise boundary of the considered option are deduced without more restriction. Because of some technical difficulty, we are merely able to prove the smoothness of the optimal exercise boundary under some additional condition.

The paper is organized as follows: § 1 outlines the problem of pricing the American option in the jump-diffusion model, and relates this pricing problem to a parabolic variational inequality. § 2 studies a penalized problem corresponding to this variational inequality. § 3 proves the existence and uniqueness and other properties of the solution to this variational inequality. § 4 relates this variational inequality to a free boundary problem, in which the free boundary is the optimal exercise boundary of American options in the jump-diffusion model, and establishes some basic properties of the free boundary and the price function. § 5 derives further properties of the optimal exercise boundary, such as continuity, differentiability and strict monotonicity etc.

We consider a financial market where two assets (B, S) are traded continuously up to some fixed time horizon T, B is a riskless asset, such as a bond, whose price B_t at time t is governed by the differential equation

$$\frac{dB_t}{B_t} = r dt,$$

where r is the constant positive interest rate, and S is a risky asset, such as a stock, with price S_t at time t. Here, $(S_t)_{t\geqslant 0}$ is assumed to be a stochastic process and is governed by the following stochastic differential equation:

$$\frac{dS_t}{S_{t^-}} = (\mu - q)dt + \sigma dW_t + d\left(\sum_{j=1}^{N_t} U_j\right),$$

where coefficients μ, q, σ are positive constants, q is a dividend yield, $(W_t)_{t\geqslant 0}$ is a standard Brownian motion, $(N_t)_{t\geqslant 0}$ is a Poisson process with parameter λ and $(U_j)_{j\geqslant 1}$ is a sequence of square integrable independent, identically distributed random variables, with values in $(-1, +\infty)$. The parameter λ of the Poisson process $(N_t)_{t\geqslant 0}$ accounts for the frequency of jumps, and the random variable $(U_j)_{j\geqslant 1}$ accounts for the relative amplitude of jumps. The model described above can be interpreted as follows: the underlying stock price in this model allows a discontinuous path, with jump times controlled by a Poisson process, which is modeled by a geometric Brownian motion between two jump times, and can leap a random value at jump times. Here, We assume that processes $(W_t)_{t\geqslant 0}$, $(N_t)_{t\geqslant 0}$, $(U_j)_{j\geqslant 1}$ are independent.

Similar to the argument in Willmott [16], we can relate the American option pricing problem concerned here to some obstacle problem by the risk minimizing approach. That is, the American option price, within the jump-diffusion model, $V = V(S,\tau)$ solves the following parabolic variational inequality:

$$\begin{cases} \min\{-\mathscr{L}V, V-\phi\} = 0, \quad S > 0, 0 \leqslant \tau < T, \\ V(S,T) = \phi(S), \quad S > 0, \end{cases} \tag{P}$$

where $K > 0$ is the striking price in this option, S is the price of the underlying stock, T is a maturity date, $\phi(S)$ is the payoff of this American option and $\mathscr{L}$ is the parabolic integro-differential operator

$$\mathscr{L}V = \frac{\partial V}{\partial \tau} + \frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2} + (r-q-\lambda k)S\frac{\partial V}{\partial S} - (r+\lambda)V + \lambda\int_{-1}^{+\infty} V(S(1+y),\tau)dN(y),$$

in which $N(y)$ is the distribution function of the random variable U_1, namely, $N(y)$ is a nondecreasing function satisfying $N(-1) = 0$, $N(+\infty) = 1$, $k = \int_{-1}^{+\infty} y dN(y)$.

In this paper, we focus our discussions on put options, so the payoff function is

$$\phi(S) = (K-S)^+.$$

Let $t = T-\tau$, $x = \ln S$, $u(x,t) = V(S,\tau)$, $\psi(x) = \phi(S)$. Then (P) changes into

$$\begin{cases} \min\{\mathscr{L}_1 u, u-\psi\} = 0, -\infty < x < \infty, 0 < t \leqslant T, \\ u(x,0) = \psi(x), -\infty < x < \infty, \end{cases} \tag{1.1}$$

where

$$\mathscr{L}_1 u = \frac{\partial u}{\partial t} - \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} - \left(r - q - \frac{\sigma^2}{2} - \lambda k\right)\frac{\partial u}{\partial x} + ru - \lambda\int_{-1}^{+\infty}[u(x+\ln(1+y),t) - u(x,t)]dN(y),$$

or after changing the variable to $z = \ln(1+y)$ in the integral on the right,

$$\mathscr{L}_1 u = \frac{\partial u}{\partial t} - \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} - (r - q - \frac{\sigma^2}{2} - \lambda k)\frac{\partial u}{\partial x} + (r+\lambda)u - \lambda\int_{-\infty}^{+\infty} u(x+z,t)d\tilde{N}(z)$$

with $\tilde{N}(z) = N(e^z - 1)$; $\tilde{N}(z)$ is a nondecreasing function satisfying $\tilde{N}(-\infty) = 0$, $\tilde{N}(+\infty) = 1$, $\int_{-\infty}^{\infty} e^z d\tilde{N}(z) = k+1$, and

$$\psi(x) = (K - e^x)^+.$$

Throughout this paper, we always assume that the constants σ, r, q, λ and k are positive, and $\tilde{N}(z)$ satisfies

$$\int_{-\infty}^{\infty} e^{|z|} d\tilde{N}(z) < \infty. \tag{A}$$

In addition, we always denote $H_T = R \times (0,T]$, where $R = (-\infty,\infty)$.

2 Approximation

In this section, we study the penalty problem corresponding to (1.1):

$$\begin{cases} \mathscr{L}_1 u^\epsilon + \beta_\epsilon(u^\epsilon - \psi_\epsilon) = 0, -\infty < x < \infty, t > 0, \\ u^\epsilon(x,0) = \psi_\epsilon(x), -\infty < x < \infty \end{cases} \tag{2.1}$$

with $0 < \epsilon < 1$, $\psi_\epsilon(x) = \Pi_\epsilon(K - e^x)$, where $\beta_\epsilon(x)$, $\Pi_\epsilon(y)$ satisfy the following conditions:

$$\beta_\epsilon(x) \in C^\infty(R);\ \beta_\epsilon(x) \leqslant 0;\ \beta_\epsilon(x) = 0,\ \text{if } x \geqslant \epsilon;$$
$$\beta_\epsilon(0) = -C_\epsilon, (C_\epsilon \geqslant (r+\lambda)K + r\epsilon);\ \beta'_\epsilon(x) \geqslant 0;\ \beta''_\epsilon(x) \leqslant 0;$$
$$\lim_{\epsilon \to 0} \beta_\epsilon(x) = \begin{cases} 0, & x > 0, \\ -\infty, & x < 0; \end{cases} \tag{A1}$$
$$\Pi_\epsilon(y) \in C^\infty(R),\ 0 \leqslant \Pi'_\epsilon(y) \leqslant 1;\ \Pi''_\epsilon(y) \geqslant 0;\ \Pi_\epsilon(y) = \begin{cases} y, & y \geqslant \epsilon, \\ 0, & y \leqslant -\epsilon. \end{cases} \tag{A2}$$

First, we have a lemma on integro-differential equations, which is similar to the maximum principle of parabolic equations in the unbounded region. In financial terms it states that the value of a non-negative payoff is non-negative, which is clear if the time-t value is regarded as a risk-neutral expectation of the payoff.

Lemma 2.1 *Assume that $u(x,t) \in C(\overline{H}_T) \bigcap C^{2,1}(H_T)$ satisfies*

$$Lu = \frac{\partial u}{\partial t} - a^2(x,t)\frac{\partial^2 u}{\partial x} + b(x,t)\frac{\partial u}{\partial x} + (c(x,t) + \lambda)u - \lambda \int_{-\infty}^{+\infty} u(x+z,t)d\tilde{N}(z) \geqslant 0,$$

in which $0 < |a(x,t)| < M$, $|b(x,t)| \leqslant M$, $c(x,t) \geqslant \delta > 0$, and λ, σ, M are positive constants. If $u(x,t) \geqslant -m$, ($m > 0$ is a constant) in H_T, then $u(x,0) \geqslant 0$ implies $u(x,t) \geqslant 0$.

Proof For any $r_0 > 0$, let

$$w(x,t) = \frac{m}{r_0^2}(x^2 + \beta t)e^{\alpha t} + u(x,t),$$

where α, β will be determined later. We have

$$\begin{aligned} &Lw(x,t) \\ &\geqslant \frac{me^{\alpha t}}{r_0^2}[\beta + (x^2+\beta t)\alpha - 2a^2 + 2bx + (c+\lambda)(x^2+\beta t) - \lambda \int_{-\infty}^{+\infty} ((x+z)^2 + \beta t)d\tilde{N}(z)] \\ &= \frac{me^{\alpha t}}{r_0^2}[(\alpha + c + \lambda)x^2 + 2bx + (\beta - 2a^2) + \beta(\alpha + c)t - \lambda \int_{-\infty}^{+\infty} (x+z)^2 d\tilde{N}(z)] \\ &\geqslant \frac{me^{\alpha t}}{r_0^2}[(\alpha + \delta - \lambda - 1)x^2 + (\beta - 2a^2 - M^2) + (\alpha + \delta)\beta t - \lambda \int_{-\infty}^{\infty} z^2 d\tilde{N}(z)]. \end{aligned}$$

From (A), we can take α, $\beta > 0$ so large that $Lw \geqslant 0$ in H_T.

Since $w(x,0) = \frac{m}{r_0^2}x^2 + u(x,0) \geqslant 0$, $w(x,t)|_{|x|=r_0} \geqslant m + u(x,t)|_{|x|=r_0} \geqslant 0$ and $w(x,t) \geqslant m + u(x,t) \geqslant 0$ when $|x| > r_0$, we may assert that $w(x,t) \geqslant 0$ in $[-r_0, r_0] \times [0,T]$. In fact, if $w(x,t)$ has a negative minimum at the point $(x_0,t_0) \in (-r_0,r_0) \times (0,T]$, then, since

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$\int_{-\infty}^{\infty} d\tilde{N}(z) = 1$, we have

$$Lw(x_0,t_0) \leqslant (c(x_0,t_0)+\lambda)w(x_0,t_0) - \lambda\int_{-\infty}^{\infty} w(x_0+z,t)d\tilde{N}(z)$$
$$\leqslant c(x_0,t_0)w(x_0,t_0) < \delta w(x_0,t_0) < 0,$$

which contradicts $Lw \geqslant 0$. For any $(\xi,\tau) \in H_T$, we take $r_0 > 0$ so large that $(\xi,\tau) \in (-r_0,r_0)\times(0,T]$. From the above argument, we obtain $w(\xi,\tau) \geqslant 0$, i.e., $\frac{m}{r_0^2}(\xi^2+\beta\tau)e^{\alpha\tau} + u(\xi,\tau) \geqslant 0$. Letting $r_0 \to +\infty$ then gives $u(\xi,\tau) \geqslant 0$ and the desired conclusion follows from the arbitrariness of $(\xi,\tau) \in H_T$.

Theorem 2.1 *For any* $\epsilon \in (0,1)$, *the problem* (2.1) *has a unique solution* $u^\epsilon(x,t) \in C^\infty(\overline{H}_T) \cap L^\infty(\overline{H}_T)$.

Proof Uniqueness of the solution to (2.1) is a direct corollary of Lemma 2.1. Now we prove the existence.

Let $B = \{u \in C^{\alpha,\frac{\alpha}{2}}(\overline{H}_\tau),\ \|u\|_{C^{\alpha,\frac{\alpha}{2}}(\overline{H}_\tau)} \leqslant U_0\}$. Then B is a bounded closed convex set in the space $C^{\alpha,\frac{\alpha}{2}}(\overline{H}_\tau)$, where $H_\tau = R\times(0,\tau]$ and $\tau, U_0 > 0$ will be determined below.

For any $v \in B$, we consider the following Cauchy problem

$$\begin{cases} \tilde{L}u = \lambda\int_{-\infty}^{\infty} v(x+z,t)d\tilde{N}(z) - \beta_\epsilon(v-\psi_\epsilon), \\ u(x,0) = \psi_\epsilon(x), \end{cases} \tag{2.2}$$

where $\tilde{L}u = \frac{\partial u}{\partial t} - \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x} - (r-q-\frac{\sigma^2}{2}-\lambda k)\frac{\partial u}{\partial x} + (r+\lambda)u$. It is clear that (2.2) has a unique solution $u = Tv \in C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{H}_\tau)$. By properties of the fundemental solution of the operator $\tilde{L}$, it is easily proved that for small $\tau \in (0,T)$, we have

$$\|u\|_{C^{\alpha,\frac{\alpha}{2}}(\overline{H}_\tau)} \leqslant C + \theta\|v\|_{C^{\alpha,\frac{\alpha}{2}}(\overline{H}_\tau)},$$

where $C > 0$, $\theta \in (0,1)$ are some constants. Therefore, if we choose $U_0 \geqslant \frac{C}{1-\theta}$, then $u = Tv \in B$ for any $v \in B$.

Denote

$$g(x,t) = \lambda\int_{-\infty}^{+\infty} v(x+z,t)d\tilde{N}(z) - \beta_\epsilon(v-\psi_\epsilon).$$

Then

$$\|g\|_{C^{\alpha,\frac{\alpha}{2}}(\overline{H}_\tau)} \leqslant C(\|v\|_{C^{\alpha,\frac{\alpha}{2}}(\overline{H}_\tau)} + 1).$$

By the standard theory of parabolic equations, we have

$$\|u\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{H}_\tau)} = \|Tv\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{H}_\tau)} \leqslant C(\|v\|_{C^{\alpha,\frac{\alpha}{2}}(\overline{H}_\tau)} + 1),$$

which implies that the set B is precompact. Besides, for any $v_1, v_2 \in B$, we have

$$\|T(v_1-v_2)\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{H}_\tau)} \leqslant C\|v_1-v_2\|_{C^{\alpha,\frac{\alpha}{2}}(\overline{H}_\tau)},$$

which implies the continuity of the operator Tv. Therefore, T has a fixed-point $u^\epsilon \in C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{H}_\tau)$ by Schauder's Fixed-Point Theorem.

Since $\tau \in (0,T)$ can be choosen depending only on ϵ, λ and the bound of the initial value, we can obtain a solution u^ϵ of (2.1) in $\overline{H}_T$ and $u^\epsilon \in C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{H}_T)$. Using the theory of parabolic equations, we further conclude that $u^\epsilon \in C^\infty(\overline{H}_T)$.

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For the need of the following sections, we proceed to establish a series of estimates for $u^\epsilon(x,t)$ which is always assumed to be the bounded solution of (2.1) in H_T.

As an immediate corollary of Lemma 2.1, we have

Lemma 2.2 *$u^\epsilon(x,t)$ is bounded uniformly in $\epsilon \in (0,1)$, more precisely, $0 \leqslant u^\epsilon(x,t) \leqslant K+\epsilon$.*

Proof From (A2) and the definition of $\psi_\epsilon(x)$, we have $u^\epsilon(x,0) = \psi_\epsilon(x) \in [0,K]$. From (A1), there holds

$$\mathscr{L}_1 u^\epsilon = -\beta_\epsilon(u^\epsilon - \psi_\epsilon) \geqslant 0.$$

Thus we may apply Lemma 2.1 to the operator $\mathscr{L}_1$ to conclude that $u^\epsilon(x,t) \geqslant 0$.

Let $w(x,t) = K+\epsilon - u^\epsilon(x,t)$. Then

$$\mathscr{L}_1 w = \mathscr{L}_1(K+\epsilon) - \mathscr{L}_1 u^\epsilon = r(K+\epsilon) + \beta_\epsilon(u^\epsilon - \psi_\epsilon) \geqslant \beta_\epsilon(u^\epsilon - \psi_\epsilon).$$

Since $\psi_\epsilon(x) \leqslant K$, the definition of $\beta_\epsilon(x)$ implies that $\beta_\epsilon(K+\epsilon-\psi^\epsilon(x) = 0$. Hence

$$\mathscr{L}_1 w + \beta_\epsilon(K+\epsilon-\psi_\epsilon) - \beta_\epsilon(u^\epsilon - \psi_\epsilon) = \mathscr{L}_1 w + \beta'_\epsilon(\cdot) w \geqslant 0,$$

where the variable of $\beta'_\epsilon(\cdot)$ is some value between $K+\epsilon-\psi_\epsilon$ and $u^\epsilon - \psi_\epsilon$. Denote $Lw = \mathscr{L}_1 w + \beta'_\epsilon(\cdot) w$. Then $Lw \geqslant 0$. Thus we can apply Lemma 2.1 to the operator L to conclude $w(x,t) \geqslant 0$, i.e., $u^\epsilon(x,t) \leqslant K+\epsilon$ from $w(x,0) = K+\epsilon-\psi_\epsilon(x) \geqslant 0$. It is to be noted that, since $\beta'_\epsilon(\cdot) \geqslant 0$ and hence the coefficients of w in Lw is bounded from below, Lemma 2.1 can be used to the operator L. □

Lemma 2.3 *$u^\epsilon(x,t) \geqslant \psi_\epsilon(x)$ and $\beta_\epsilon(u^\epsilon - \psi_\epsilon)$ is bounded uniformly in ϵ.*

Proof It is obvious from (A2) that $|\Pi_\epsilon(y) - y\Pi'_\epsilon(y)| < \epsilon$ and

$$\begin{aligned}
&\mathscr{L}_1 \psi_\epsilon(x) \\
&\quad = -\frac{\sigma^2}{2}\Pi''_\epsilon(K-e^x)e^{2x} + \frac{\sigma^2}{2}\Pi'_\epsilon(K-e^x)e^x + (r-q-\lambda k - \frac{\sigma^2}{2})\Pi'_\epsilon(K-e^x)e^x \\
&\qquad + (r+\lambda)\Pi_\epsilon(K-e^x) - \lambda\int_{-\infty}^{\infty} \Pi_\epsilon(K-e^{x+z})d\tilde{N}(z) \\
&\quad \leqslant r\Pi'_\epsilon(K-e^x)e^x + (r+\lambda)\Pi_\epsilon(K-e^x) \\
&\quad = r[\Pi_\epsilon(K-e^x) - (K-e^x)\Pi'_\epsilon(K-e^x)] + rK\Pi'_\epsilon(K-e^x) + \lambda\Pi_\epsilon(K-e^x) \\
&\quad \leqslant r\epsilon + (r+\lambda)K.
\end{aligned}$$

So, if we denote $w = u^\epsilon - \psi_\epsilon$, then

$$\begin{aligned}
\mathscr{L}_1 w = \mathscr{L}_1 u^\epsilon - \mathscr{L}_1 \psi_\epsilon &\geqslant -\beta_\epsilon(u^\epsilon - \psi_\epsilon) - (r\epsilon + rK + \lambda K) \\
&= \beta_\epsilon(0) - \beta_\epsilon(u^\epsilon - \psi_\epsilon) - (r\epsilon + rK + \lambda K) - \beta_\epsilon(0).
\end{aligned}$$

By the definition of C_ϵ,

$$\mathscr{L}_1 w + \beta_\epsilon(u^\epsilon - \psi_\epsilon) - \beta_\epsilon(0) = \mathscr{L}_1 w + \beta'_\epsilon(\cdot) w \geqslant -\beta_\epsilon(0) - (r\epsilon + rK + \lambda K) \geqslant 0.$$

Thus using Lemma 2.1, we obtain $w(x,t) \geqslant 0$, i.e. $u^\epsilon(x,t) \geqslant \psi_\epsilon(x)$. Besides, by virtue of the monotonicity of β_ϵ and Lemma 2.3, we see that $\beta_\epsilon(u^\epsilon - \psi_\epsilon)$ is bounded uniformly in ϵ. □

Lemma 2.4 *u_x^ϵ is bounded uniformly in $\epsilon \in (0,1)$, more precisely, $-(K+1) \leqslant u_x^\epsilon \leqslant 0$, and $u_t^\epsilon \geqslant 0$.*

Proof Let $w = u_x^\epsilon$. Then w satisfies

$$\begin{cases} \mathscr{L}_1 w + \beta_\epsilon'(u^\epsilon - \psi_\epsilon)(w - \psi_\epsilon') = 0, \\ w(x,0) = \psi_\epsilon'(x). \end{cases}$$

From (A2), $w(x,0) = -e^x \Pi_\epsilon'(K - e^x) \leqslant 0$. Since $\Pi_\epsilon'(K-e^x) = 0$ when $e^x \geqslant K + \epsilon$, we have $w(x,0) = -e^x \Pi_\epsilon'(K-e^x) \geqslant -(K+\epsilon) \geqslant -(K+1)$. Using Lemma 2.1 to the operator $\mathscr{L}_1 + \beta_\epsilon'(u^\epsilon - \psi_\epsilon)$, we obtain $-(K+1) \leqslant w(x,t) = u_x^\epsilon(x,t) \leqslant 0$.

Let $v = u_t^\epsilon$. Then v satisfies

$$\begin{cases} \mathscr{L}_1 v + \beta_\epsilon'(u^\epsilon - \psi_\epsilon) v = 0, \\ v(x,0) = f(x), \end{cases}$$

where

$$f(x) = \left[\frac{\sigma^2}{2}\psi_\epsilon'' + \left(r - q - \lambda k - \frac{\sigma^2}{2} \right) \psi_\epsilon' - (r+\lambda)\psi_\epsilon + \lambda \int_{-\infty}^{\infty} \psi_\epsilon(x+z,t) d\tilde{N}(z) - \beta_\epsilon(0) \right].$$

From the proof of Lemma 2.4, $v(x,0) = f(x) = \mathscr{L}_1 \psi_\epsilon - \beta_\epsilon(0) \geqslant C_\epsilon - (r\epsilon + rK + \lambda K) \geqslant 0$. Thus, by Lemma 2.1, we obtain $v = u_t^\epsilon \geqslant 0$. □

Lemma 2.5 $$\frac{\partial^2 u^\epsilon}{\partial x^2} - \frac{\partial u^\epsilon}{\partial x} \geqslant 0.$$

Proof Let $w = \frac{\partial^2 u^\epsilon}{\partial x^2} - \frac{\partial u^\epsilon}{\partial x}$. Then

$$\begin{cases} \mathscr{L}_1 w + \beta_\epsilon'(u^\epsilon - \psi_\epsilon) \cdot w = f(x,t), \\ w(x,0) = \phi(x) \end{cases}$$

where

$$\begin{aligned} f(x,t) &= -\beta_\epsilon''(u^\epsilon - \psi_\epsilon)(\tfrac{\partial u^\epsilon}{\partial x} + \Pi_\epsilon'(K-e^x)e^x)^2 + \beta_\epsilon'(u^\epsilon - \psi_\epsilon)\Pi_\epsilon''(K-e^x)e^{2x} \geqslant 0, \\ \phi(x) &= e^{2x}\Pi_\epsilon''(K-e^x). \end{aligned}$$

By Lemma 2.1, we have $w(x,t) \geqslant 0$, that is , $\frac{\partial^2 u^\epsilon}{\partial x^2} \geqslant \frac{\partial u^\epsilon}{\partial x}$. □

Lemma 2.6 *For any $a < b$, small positive constant $\delta \in (0,T)$ and $p \in (1,\infty)$, $u^\epsilon(x,t) \in W_p^{2,1}(Q_T)$ and*

$$\|u^\epsilon\|_{W_p^{2,1}(Q_T)} \leqslant C$$

with constant C independent of ϵ, where $Q_T = (a,b) \times [\delta, T]$.

Proof Choose cut-off functions $\xi(t) \in C_0^\infty(0,T]$, $\eta(x) \in C_0^\infty(a-1, b+1)$ such that $0 \leqslant \xi, \eta \leqslant 1$, $\xi(t) = 1$ if $t \in [\delta, T]$, $\eta(x) = 1$ if $x \in [a,b]$.

Let $w(x,t) = \xi(t)\eta(x)u^\epsilon(x,t)$. Then

$$\begin{cases} \frac{\partial w}{\partial t} - \frac{\sigma^2}{2}\frac{\partial^2 w}{\partial x^2} = f(x,t), & (x,t) \in \tilde{Q}_T \\ w(x,0) = 0, & x \in (a-1, b+1) \\ w(x,t) = 0, & x = a-1, b+1, t \in (0,T), \end{cases}$$

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where $\tilde{Q}_T = (a-1, b+1) \times (0, T)$ and

$$f(x,t) = (-\sigma^2 \xi \eta' + \xi \eta (r - q - \lambda k - \tfrac{\sigma^2}{2}))\tfrac{\partial u^\epsilon}{\partial x} + (-\tfrac{\sigma^2}{2}\xi\eta'' + \xi'\eta - (r+\lambda)\xi\eta)u^\epsilon \\ + \lambda\xi\eta \int_{-\infty}^{\infty} u^\epsilon(x+z,t)d\tilde{N}(z) - \xi\eta\beta_\epsilon(u^\epsilon - \psi_\epsilon).$$

Using $W_p^{2,1}$ estimates for parabolic equations gives

$$\|w\|_{W_p^{2,1}(\tilde{Q}_T)} \leqslant C(\|u^\epsilon\|_{L^p(\tilde{Q}_T)} + \|f\|_{L^p(\tilde{Q}_T)}).$$

Since by Lemmas 2.3–2.5, u^ϵ, u_x^ϵ and $\beta_\epsilon(u^\epsilon - \psi_\epsilon)$ are bounded uniformly in ϵ, we have

$$\|u^\epsilon\|_{W_p^{2,1}(Q_T)} \leqslant \|w\|_{W_p^{2,1}(\tilde{Q}_T)} \leqslant C,$$

where C is independent of ϵ. Thus Lemma 2.7 is proved. □

Lemma 2.7 *For any small $\delta \in (0, T)$ and $p \in (1, \infty)$, $u^\epsilon(x,t) \in W_p^{2,1}(Q, e^{-|x|}dx)$ and*

$$\|u^\epsilon\|_{W_p^{2,1}(Q, e^{-|x|}dx)} \leqslant C$$

with constant C independent of ϵ, where $Q = Q_1 \cup Q_2$, $Q_1 = (\ln K + \delta, +\infty) \times (0, T]$, $Q_2 = (-\infty, \ln K - \delta) \times (0, T]$ and $W_p^{2,1}(Q, e^{-|x|}dx)$ is the weighted space with weight $e^{-|x|}$.

Proof Choose a cut-off function $\eta(x) \in C_0^\infty(\ln K + \frac{\delta}{2}, N+1)(\ln K + \delta < N)$ such that $\eta(x) = 1$ if $x \in [\ln K + \delta, N]$, $\eta(x) = 0$ if $x \geqslant N+1$ and let $w = \eta u^\epsilon e^{-\frac{|x|}{p}}$. Then

$$\begin{cases} \frac{\partial w}{\partial t} - \frac{\sigma^2}{2}\frac{\partial^2 w}{\partial x^2} = \tilde{f}(x,t), & (x,t) \in \tilde{Q}_1 = (\ln K + \frac{\delta}{2}, N+1) \times (0, T) \\ w(x,0) = \psi_\epsilon(x) = 0, & x \in (\ln K + \frac{\delta}{2}, N+1) \\ w(x,t) = 0, & x = \ln K + \frac{\delta}{2}, N+1, t \in (0,T), \end{cases}$$

where

$$\tilde{f}(x,t) = (-\sigma^2(\eta' - \tfrac{\eta}{p}) + (r - q - \lambda k - \tfrac{\sigma^2}{2})\eta)\tfrac{\partial u^\epsilon}{\partial x} e^{-\frac{|x|}{p}} \\ + (-\tfrac{\sigma^2}{2}(\eta'' - (1 + \tfrac{1}{p})\eta' + \tfrac{\eta}{p^2}) - (r+\lambda)\eta)u^\epsilon e^{-\frac{|x|}{p}} \\ - \eta\beta_\epsilon(u^\epsilon - \psi_\epsilon)e^{-\frac{|x|}{p}} + \lambda\eta e^{-\frac{|x|}{p}} \int_{-\infty}^{\infty} u^\epsilon(x+z,t)d\tilde{N}(z)$$

The same reasoning which leads to Lemma 2.7 then gives

$$\|w\|_{W_p^{2,1}(\tilde{Q}_1)} \leqslant C.$$

Clearly, we can choose $\eta(x)$ such that η, η', η'' have a bound independent of N and hence the constant C does not depend upon ϵ and N. From this estimate, it follows that

$$\|u^\epsilon\|_{W_p^{2,1}(\tilde{\tilde{Q}}_1, e^{-|x|}dx)} \leqslant C,$$

where $\tilde{\tilde{Q}}_1 = (\ln K + \delta, N) \times (0, T]$. Since C is independent of N, we finally obtain

$$\|u^\epsilon\|_{W_p^{2,1}(Q_1, e^{-|x|}dx)} \leqslant C.$$

Similarly, we can prove

$$\|u^\epsilon\|_{W_p^{2,1}(Q_2, e^{-|x|}dx)} \leqslant C.$$ □

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Let $P = (\ln K, 0)$, and $B_\delta(P)$ be the disc in (x, t) plane with radius δ, centered at P.

Remark 2.1 Combining Lemma 2.7 with Lemma 2.8, we can affirm that for any small $\delta > 0$ and $p \in (1, \infty)$, $u^\epsilon \in W_p^{2,1}(H_1, e^{-|x|}dx)$ and

$$\|u^\epsilon\|_{W_p^{2,1}(H_1, e^{-|x|}dx)} \leqslant C$$

with constant C independent of ϵ, where $H_1 = R \times [0, T]/B_\delta(P)$.

Lemma 2.8 *For any $a < b$, $p \in [2, \infty)$ and almost all $t \in (0, T]$,*

$$\int_b^a \left(\left| \frac{\partial^2 u^\epsilon}{\partial x^2} \right|^p + \left| \frac{\partial u^\epsilon}{\partial t} \right|^p + \left| \frac{\partial u^\epsilon}{\partial x} \right|^p \right) dx \leqslant C$$

with constant C independent of ϵ.

Proof Differentiating the equation in (2.1) once with respect to t, multiplying by $e^{-|x|}\xi^2(t)\eta^2(x)(\frac{\partial u^\epsilon}{\partial t})^{2n-1}$ (n is a positive integer) and integrating over $\tilde{Q}_t = (a-1, b+1) \times (0, t)$, $t \in (0, T]$, we obtain

$$\begin{aligned}
&\iint_{\tilde{Q}_t} e^{-|x|}\xi^2(\tau)\eta^2(x)(\tfrac{\partial u^\epsilon}{\partial t})^{2n-1}\tfrac{\partial^2 u^\epsilon}{\partial t^2}dxd\tau - \tfrac{\sigma^2}{2}\iint_{\tilde{Q}_t} e^{-|x|}\xi^2(\tau)\eta^2(x)(\tfrac{\partial u^\epsilon}{\partial t})^{2n-1}\tfrac{\partial^3 u^\epsilon}{\partial x^2 \partial t}dxd\tau \\
&\quad -(r - q - \lambda k - \tfrac{\sigma^2}{2})\iint_{\tilde{Q}_t} e^{-|x|}\xi^2(\tau)\eta^2(x)(\tfrac{\partial u^\epsilon}{\partial t})^{2n-1}\tfrac{\partial^2 u^\epsilon}{\partial x \partial t}dxd\tau \\
&\quad +(r + \lambda)\iint_{\tilde{Q}_t} e^{-|x|}\xi^2(\tau)\eta^2(x)(\tfrac{\partial u^\epsilon}{\partial t})^{2n}dxd\tau \\
&\quad -\lambda\iint_{\tilde{Q}_t} e^{-|x|}\xi^2(\tau)\eta^2(x)(\tfrac{\partial u^\epsilon}{\partial t}(x, \tau))^{2n-1}(\textstyle\int_{-\infty}^{\infty}\tfrac{\partial u^\epsilon}{\partial t}(x + z, \tau)d\tilde{N}(z))dxd\tau \\
&\quad +\iint_{\tilde{Q}_t} e^{-|x|}\xi^2(\tau)\eta^2(x)(\tfrac{\partial u^\epsilon}{\partial t})^{2n}\beta'_\epsilon(u^\epsilon - \psi_\epsilon)\,dxd\tau \\
&= I_1 + I_2 + I_3 + I_4 + I_5 + I_6 = 0,
\end{aligned} \tag{2.3}$$

where $\xi(t) \in C_0^\infty(0, T]$, $\eta(x) \in C_0^\infty(a-1, b+1)$ are cut-off functions such that $0 \leqslant \xi(t)$, $\eta(x) \leqslant 1$, $\xi(t) = 1$ if $t \in [\delta, T]$ $(\delta > 0)$, $\eta(x) = 1$ if $x \in [a, b]$.

Since $\beta'_\epsilon(\cdot) \leqslant 0$ and $\frac{\partial u^\epsilon}{\partial t} \geqslant 0$, we have $I_6 \geqslant 0$. Let $\Phi(t) = \int_{a-1}^{b+1} e^{-|x|}\xi^2(t)\eta^2(x)(\frac{\partial u^\epsilon}{\partial t})^{2n}dx$. Then $I_4 = (r + \lambda)\int_0^t \Phi(\tau)d\tau$.

Integrating by parts and using Lemma 2.7 and Hölder's inequality gives

$$\begin{aligned}
I_1 &= \iint_{\tilde{Q}_t} e^{-|x|}\xi^2(\tau)\eta^2(x)(\tfrac{\partial u^\epsilon}{\partial t})^{2n-1}\tfrac{\partial^2 u^\epsilon}{\partial t^2}dxd\tau \\
&= \tfrac{1}{2n}\int_{a-1}^{b+1} e^{-|x|}\xi^2(t)\eta^2(x)(\tfrac{\partial u^\epsilon}{\partial t})^{2n}dx - \tfrac{1}{n}\iint_{\tilde{Q}_t} e^{-|x|}\xi(\tau)\xi'(\tau)\eta^2(x)(\tfrac{\partial u^\epsilon}{\partial t})^{2n}dxd\tau \\
&\geqslant \tfrac{1}{2n}\Phi(t) - C_1\int_0^t \Phi(\tau)d\tau - C_2.
\end{aligned}$$

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Again integrating by parts gives

$$
\begin{aligned}
I_2 &= -\frac{\sigma^2}{2}\iint\limits_{\tilde{Q}_t} e^{-|x|}\xi^2(\tau)\eta^2(x)\Big(\frac{\partial u^\epsilon}{\partial t}\Big)^{2n-1}\frac{\partial^3 u^\epsilon}{\partial x^2\partial t}dxd\tau \\
&= \frac{(2n-1)\sigma^2}{2}\iint\limits_{\tilde{Q}_t} \xi^2(\tau)\eta^2(x)e^{-|x|}\Big(\frac{\partial u^\epsilon}{\partial t}\Big)^{2n-2}\Big(\frac{\partial^2 u^\epsilon}{\partial x\partial t}\Big)^2 dxd\tau \\
&\quad + \frac{\sigma^2}{2}\iint\limits_{\tilde{Q}_t} \xi^2(\tau)\Big(\frac{\partial u^\epsilon}{\partial t}\Big)^{2n-1}\frac{\partial^2 u^\epsilon}{\partial x\partial t}\eta(x)e^{-|x|}(2\eta'(x)-\eta(x)sgn(x))dxd\tau \\
&= I_{21}+I_{22}.
\end{aligned}
$$

Note that $I_{21}\geqslant 0$. Using Lemma 2.7 and Hölder's inequality, we have

$$|I_{22}|\leqslant \frac{1}{3}I_{21}+C_2$$

which implies

$$I_2\geqslant \frac{2}{3}I_{21}-C_2.$$

Similar reasoning leads to

$$|I_3|\leqslant \frac{1}{3}I_{21}+C_1\int_0^t \Phi(\tau)d\tau.$$

Using Hölder's inequality and Young's inequality, Remark 2.1 and condition (A), we can easily derive

$$
\begin{aligned}
|I_5| &= \lambda\iint\limits_{\tilde{Q}_t} e^{-|x|}\xi^2(\tau)\eta^2(x)\Big(\frac{\partial u^\epsilon}{\partial t}(x,\tau)\Big)^{2n-1}\Big(\int_{-\infty}^{\infty}\frac{\partial u^\epsilon}{\partial t}(x+z,\tau)d\tilde{N}(z)\Big)dxd\tau \\
&= \lambda\int_{-\infty}^{\infty}\Big(\iint\limits_{\tilde{Q}_t} e^{-|x|}\xi^2(\tau)\eta^2(x)\Big(\frac{\partial u^\epsilon}{\partial t}(x,\tau)\Big)^{2n-1}\frac{\partial u^\epsilon}{\partial t}(x+z,\tau)dxd\tau\Big)d\tilde{N}(z) \\
&\leqslant \lambda\int_{-\infty}^{\infty}\Big[\int_0^t \xi^2(\tau)\Big(\int_{a-1}^{b+1} e^{-|x|}\eta^2(x)\Big(\frac{\partial u^\epsilon}{\partial t}(x,\tau)\Big)^{2n}dx\Big)^{\frac{2n-1}{2n}} \\
&\qquad\qquad \Big(\int_{a-1}^{b+1} e^{-|x|}\eta^2(x)\Big(\frac{\partial u^\epsilon}{\partial t}(x+z,\tau)\Big)^{2n}dx\Big)^{\frac{1}{2n}}d\tau\Big]d\tilde{N}(z) \\
&\leqslant \lambda\int_{-\infty}^{\infty}\Big[\int_0^t \xi^2(\tau)\Big(\frac{2n-1}{2n}\Phi(\tau)+\frac{1}{2n}\int_{a-1}^{b+1} e^{-|x|}\eta^2(x)\Big(\frac{\partial u^\epsilon}{\partial t}(x+z,\tau)\Big)^{2n}dx\Big)d\tau\Big]d\tilde{N}(z) \\
&\leqslant \frac{2n-1}{2n}\lambda\int_0^t \Phi(\tau)d\tau+\frac{1}{2n}\lambda\int_{-\infty}^{\infty}\int_0^t \xi^2(\tau)\int_{a-1}^{b+1} e^{-|x|}\eta^2(x)\Big(\frac{\partial u^\epsilon}{\partial t}(x+z,\tau)\Big)^{2n}dxd\tau d\tilde{N}(z) \\
&\leqslant \frac{2n-1}{2n}\lambda\int_0^t \Phi(\tau)d\tau+\frac{\lambda}{2n}\int_{-\infty}^{\infty}e^{|z|}\Big(\int_0^t \xi^2(\tau)\int_{a-1+z}^{b+1+z} e^{-|y|}\Big(\frac{\partial u^\epsilon}{\partial t}(y,\tau)\Big)^{2n}dyd\tau\Big)d\tilde{N}(z) \\
&\leqslant \frac{2n-1}{2n}\lambda\int_0^t \Phi(\tau)d\tau+C_2.
\end{aligned}
$$

Then combining these estimates for I_i $(i=1,\cdots,6)$ with (2.3), we are led to

$$\Phi(t)\leqslant C_1\int_0^t \Phi(\tau)d\tau+C_2,$$

which implies

$$\Phi(t)\leqslant C_2e^{C_1t}$$

by Gronwall's inequality. So, for any $\delta\leqslant t\leqslant T$,

$$\int_a^b\Big(\frac{\partial u^\epsilon}{\partial t}(x,\tau)\Big)^{2n}dx\leqslant C$$

with constant C independent of ϵ. From this, it follows from Lemma 2.4, by using (2.1) and Lemma 2.5 that

$$\int_a^b \left|\frac{\partial^2 u^\epsilon}{\partial x^2}(x,\tau)\right|^p dx \leqslant C, \qquad 2 \leqslant p < \infty.$$

Combining these estimates with Lemma 2.5 leads to Lemma 2.9. □

Lemma 2.9 *$\frac{\partial^2 u^\epsilon}{\partial x^2}$, $\frac{\partial u^\epsilon}{\partial t}$ are bounded uniformly in $\epsilon \in (0,1)$ in Q_T^δ , where $Q_T^\delta = (a,b) \times (0,T)/\overline{B_\delta(P)}$.*

Proof By Lemmas 2.5 and 2.6, we have $\frac{\partial^2 u^\epsilon}{\partial x^2} \geqslant \frac{\partial u^\epsilon}{\partial x} \geqslant -(K+1)$, so it suffices to prove that $\frac{\partial^2 u^\epsilon}{\partial x^2}$ has a uniform upper bound in Q_T^δ. Let $w = \frac{\partial u^\epsilon}{\partial t}$. Then w satisfies

$$\mathscr{L}_1 w + w \cdot \beta_\epsilon'(u^\epsilon - \psi_\epsilon) = 0.$$

Denote $Lw = \frac{\partial w}{\partial t} - \frac{\sigma^2}{2}\frac{\partial^2 w}{\partial x^2} - (r - q - \lambda k - \frac{\sigma^2}{2})\frac{\partial w}{\partial x} + (r+\lambda)w$. Use the same notations and choose a cut-off function $\eta(x,t) \in C_0^\infty(R \times (0,T])$ such that $0 \leqslant \eta(x,t) \leqslant 1$, $\eta(x,t) = 1$ if $(x,t) \in Q_T^\delta$, $\eta(x,t) = 0$ if $(x,t) \in B_{\frac{\delta}{2}}(P)$, $x < \tilde{a}$ or $x > \tilde{b}$ for some constants $\tilde{a}$, $\tilde{b}$. Denote $\tilde{Q}_T = (\tilde{a},\tilde{b}) \times (0,T)$. Then

$$\begin{aligned}
&L(\eta w) + \eta\beta_\epsilon'(u^\epsilon - \psi_\epsilon)\cdot w\\
&\quad = \eta(Lw + \beta_\epsilon'(u^\epsilon - \psi_\epsilon)\cdot w) - w\cdot(\tfrac{\sigma^2}{2}\eta_{xx} - (r-q-\lambda k - \tfrac{\sigma^2}{2})\eta_x - \eta_t) - \sigma^2\eta_x w_x\\
&\quad = \lambda\eta\textstyle\int_{-\infty}^{\infty} w(x+z,t)d\tilde{N}(z) + w\cdot(\tfrac{3\sigma^2}{2}\eta_{xx} + (r-q-\lambda k - \tfrac{\sigma^2}{2})\eta_x + \eta_t) - \sigma^2\tfrac{\partial}{\partial x}(\eta_x w)\\
&\quad = g_0 + (g_1)_x,
\end{aligned}$$

where $g_0 = \lambda\eta\int_{-\infty}^{\infty} w(x+z,t)d\tilde{N}(z) + w\cdot(\frac{3\sigma^2}{2}\eta_{xx} + (r-q-\lambda k-\frac{\sigma^2}{2})\eta_x + \eta_t)$, $g_1 = -\sigma^2\eta_x w$.

Since $\beta_\epsilon' \geqslant 0$, we have

$$L(\eta w) \leqslant g_0 + (g_1)_x.$$

This means that ηw is a subsolution of the problem

$$\begin{cases} Lv = g_0 + (g_1)_x, & (x,t) \in \tilde{Q}_T,\\ v(x,t) = 0, & x = \tilde{a},\tilde{b},\\ v(x,0) = 0, & x \in [\tilde{a},\tilde{b}]. \end{cases}$$

Lemma 2.9 implies that $|g_0|_{L^{\frac{p}{2},\infty}(\tilde{Q}_T)} + |g_1|_{L^{p,\infty}(\tilde{Q}_T)} \leqslant C$ with constant C independent of ϵ. Therefore, by the theory of parabolic equations, we obtain

$$\sup_{\tilde{Q}_T} \eta w \leqslant C(|g_0|_{L^{\frac{p}{2},\infty}(\tilde{Q}_T)} + |g_1|_{L^{p,\infty}(\tilde{Q}_T)}),$$

where $C > 0$ is independent of ϵ. This, together with Lemma 2.5, proves the uniform boundedness of $\frac{\partial u^\epsilon}{\partial t}$ on Q_T^δ.

The uniform boundedness of $\frac{\partial^2 u^\epsilon}{\partial x^2}$ on Q_T^δ then follows from the same property of u^ϵ, $\frac{\partial u^\epsilon}{\partial x}$, $\frac{\partial u^\epsilon}{\partial t}$ and $\beta_\epsilon(u^\epsilon - \psi_\epsilon)$ and equation (2.1). Lemma 2.10 is proved. □

Lemma 2.10 *For any $a < b < \ln K$, $0 < t_1 < t_2 < T$,*

$$\int_{t_1}^{t_2}\int_a^b \left(\frac{\partial^2 u^\epsilon}{\partial x \partial t}\right)^2 dxdt \leqslant C \tag{2.4}$$

with constant C independent of ϵ.

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Proof Let $v = \frac{\partial u^\epsilon}{\partial x}$. Then v satisfies

$$\mathscr{L}_1 v + \beta'_\epsilon(u^\epsilon - \psi)(v - \psi'_\epsilon) = 0.$$

Choose a cut-off function $\eta(x,t) \in C_0^\infty((-\infty, \ln K) \times (0,T))$ such that $\eta(x,t) = 1$ for $x \in [a,b], t \in [t_1, t_2]$, multiply the above equation by $\eta^2 v_t$ and integrate over H_T. Then

$$\begin{aligned}
&\iint\limits_{H_T} \eta^2 (\tfrac{\partial v}{\partial t})^2 dxdt - \tfrac{\sigma^2}{2} \iint\limits_{H_T} \tfrac{\partial^2 v}{\partial x^2} \eta^2 \tfrac{\partial v}{\partial t} dxdt - (r - q - \lambda k - \tfrac{\sigma^2}{2}) \iint\limits_{H_T} \tfrac{\partial v}{\partial x} \eta^2 \tfrac{\partial v}{\partial t} dxdt \\
&\quad + (r+\lambda) \iint\limits_{H_T} \eta^2 v \tfrac{\partial v}{\partial t} dxdt + \iint\limits_{H_T} \eta^2 \beta'_\epsilon(u^\epsilon - \psi_\epsilon)(v - \psi'_\epsilon) \tfrac{\partial v}{\partial t} dxdt \\
&\quad - \lambda \iint\limits_{H_T} \eta^2 \tfrac{\partial v}{\partial t} (\textstyle\int_{-\infty}^{+\infty} v(x+z,t) d\tilde{N}(z)) dxdt \\
&= I_1 + I_2 + I_3 + I_4 + I_5 + I_6 = 0.
\end{aligned} \tag{2.5}$$

Integrating by parts gives

$$\begin{aligned}
I_2 &= -\tfrac{\sigma^2}{2} \iint\limits_{H_T} \tfrac{\partial^2 v}{\partial x^2} \eta^2 \tfrac{\partial v}{\partial t} dxdt = \tfrac{\sigma^2}{4} \iint\limits_{H_T} \eta^2 \tfrac{\partial}{\partial t} [(\tfrac{\partial v}{\partial x})^2] dxdt + \tfrac{\sigma^2}{2} \iint\limits_{H_T} 2\eta \tfrac{\partial \eta}{\partial x} \tfrac{\partial v}{\partial x} \tfrac{\partial v}{\partial t} dxdt \\
&= \tfrac{\sigma^2}{4} \iint\limits_{H_T} \tfrac{\partial}{\partial t} [(\tfrac{\partial v}{\partial x})^2 \eta^2] dxdt + \sigma^2 \iint\limits_{H_T} \eta \tfrac{\partial \eta}{\partial x} \tfrac{\partial v}{\partial x} \tfrac{\partial v}{\partial t} dxdt - \tfrac{\sigma^2}{2} \iint\limits_{H_T} (\tfrac{\partial v}{\partial x})^2 \eta \tfrac{\partial \eta}{\partial t} dxdt \\
&= -\tfrac{\sigma^2}{2} \iint\limits_{H_T} (\tfrac{\partial v}{\partial x})^2 \eta \tfrac{\partial \eta}{\partial t} dxdt + \sigma^2 \iint\limits_{H_T} \eta \tfrac{\partial \eta}{\partial x} \tfrac{\partial v}{\partial x} \tfrac{\partial v}{\partial t} dxdt
\end{aligned}$$

Lemma 2.10 implies the uniform boundedness in ϵ of $\eta \frac{\partial v}{\partial x} = \eta \frac{\partial^2 u^\epsilon}{\partial x^2}$. Hence

$$|I_2| \leqslant C + \frac{1}{4} \iint\limits_{H_T} \eta^2 \left(\frac{\partial v}{\partial t} \right)^2 dxdt.$$

Similarly, we have

$$|I_3| + |I_4| \leqslant C(R) + \frac{1}{4} \iint\limits_{H_T} \eta^2 \left(\frac{\partial v}{\partial t} \right)^2 dxdt.$$

Again integrating by parts gives

$$\begin{aligned}
I_5 &= \iint\limits_{H_T} \eta^2 \beta'_\epsilon(u^\epsilon - \psi_\epsilon)(v - \psi'_\epsilon) \tfrac{\partial v}{\partial t} dxdt = \tfrac{1}{2} \iint\limits_{H_T} \eta^2 \beta'_\epsilon(u^\epsilon - \psi_\epsilon) \tfrac{\partial}{\partial t} [(v - \psi'_\epsilon)^2] dxdt \\
&= -\tfrac{1}{2} \iint\limits_{H_T} \eta^2 \beta''_\epsilon(u^\epsilon - \psi_\epsilon) \tfrac{\partial u^\epsilon}{\partial t} (v - \psi'_\epsilon)^2 dxdt - \iint\limits_{H_T} \eta \tfrac{\partial \eta}{\partial t} \beta'_\epsilon(u^\epsilon - \psi_\epsilon)(v - \psi'_\epsilon)^2 dxdt.
\end{aligned}$$

Since $\beta''_\epsilon(\cdot) \leqslant 0$ and by Lemma 2.5, $\frac{\partial u^\epsilon}{\partial t} \geqslant 0$, the first term of the right side is nonnegative. Thus, after integrating by parts, we obtain

$$\begin{aligned}
I_5 &\geqslant -\iint\limits_{H_T} \eta \tfrac{\partial \eta}{\partial t} \beta'_\epsilon(u^\epsilon - \psi_\epsilon)(v - \psi'_\epsilon)^2 dxdt \\
&= \iint\limits_{H_T} \beta_\epsilon(u^\epsilon - \psi_\epsilon) \tfrac{\partial}{\partial x} [(v - \psi'_\epsilon) \eta \tfrac{\partial \eta}{\partial t}] dxdt \geqslant -C.
\end{aligned}$$

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Here we have used the uniform boundedness in in ϵ of $\eta\frac{\partial v}{\partial x} = \eta\frac{\partial^2 u^\epsilon}{\partial x^2}$ and $\eta\psi_\epsilon''$, which follow from Lemma 2.10 and the definition of Π_ϵ.

Finally, using Lemma 2.3 derives

$$\begin{aligned} I_6 &= -\lambda \iint_{H_T} \eta^2 \tfrac{\partial v}{\partial t}\left(\int_{-\infty}^{+\infty} v(x+z,t)d\tilde{N}(z)\right)dxdt \geqslant -\lambda(K+1)\iint_{H_T} \eta^2 |\tfrac{\partial v}{\partial t}|dxdt \\ &\geqslant -C - \tfrac{1}{4}\int_{H_T} \eta^2 (\tfrac{\partial v}{\partial t})^2 dxdt. \end{aligned}$$

Combining these estimates for I_i $(i = 2, \cdots, 6)$ with (2.5), we deduce

$$\iint_{H_T} \eta^2 \left(\frac{\partial v}{\partial t}\right)^2 dxdt \leqslant C.$$

Hence

$$\int_a^b \int_{t_1}^{t_2} \left(\frac{\partial^2 u^\epsilon}{\partial x \partial t}\right)^2 dxdt \leqslant C,$$

where the constant C is independent of ϵ. □

Lemma 2.11 *For any $a < b < \ln K$, $0 < t_1 < t_2 < T$,*

$$\int_a^b \int_t^{t_2} \left(\frac{\partial^2 u^\epsilon}{\partial t^2}\right)^2 dxd\tau + \int_a^b \left(\frac{\partial^2 u^\epsilon}{\partial x \partial t}\right)^2 (x,t)dx \leqslant C, \qquad t \in (t_1, t_2) \tag{2.6}$$

with constant C independent of ϵ.

Proof Let $w = \frac{\partial u^\epsilon}{\partial t}$. Then w satisfies

$$\mathscr{L}_1 w + \beta_\epsilon'(u^\epsilon - \psi_\epsilon)w = 0.$$

Multiply by $\eta^2 w_t$ and integrate over $\Omega_t = (-\infty,\infty)\times(0,t)$ $(0 < t \leqslant t_2)$. Here $\eta(x,t) \in C_0^\infty((-\infty, \ln K)\times(0,T))$ is the same cut-off function as in Lemma 2.10. Then

$$\begin{aligned} &\iint_{\Omega_t} \eta^2 (\tfrac{\partial w}{\partial t})^2 dxd\tau - \tfrac{\sigma^2}{2}\iint_{\Omega_t} \eta^2 \tfrac{\partial^2 w}{\partial x^2}\tfrac{\partial w}{\partial t}dxd\tau - (r - q - \lambda k - \tfrac{\sigma^2}{2})\iint_{\Omega_t} \eta^2 \tfrac{\partial w}{\partial x}\tfrac{\partial w}{\partial t}dxd\tau \\ &\quad + (r+\lambda)\iint_{\Omega_t} \eta^2 w \tfrac{\partial w}{\partial t}dxd\tau + \iint_{\Omega_t} \eta^2 \beta_\epsilon'(u^\epsilon - \psi_\epsilon)w\tfrac{\partial w}{\partial t}dxd\tau \\ &\quad - \lambda \iint_{\Omega_t} \eta^2 \tfrac{\partial w}{\partial t}\left(\int_{-\infty}^{+\infty} w(x+z,t)d\tilde{N}(z)\right)dxd\tau \\ &= I_1 + I_2 + I_3 + I_4 + I_5 + I_6 = 0. \end{aligned} \tag{2.7}$$

Integrating by parts gives

$$\begin{aligned} I_2 &= -\tfrac{\sigma^2}{2}\iint_{\Omega_t} \eta^2 \tfrac{\partial^2 w}{\partial x^2}\tfrac{\partial w}{\partial t}dxd\tau = \tfrac{\sigma^2}{2}\iint_{\Omega_t} \eta^2 \tfrac{\partial w}{\partial x}\tfrac{\partial^2 w}{\partial x \partial t}dxd\tau + \sigma^2 \iint_{\Omega_t} \eta \tfrac{\partial \eta}{\partial x}\tfrac{\partial w}{\partial x}\tfrac{\partial w}{\partial t}dxd\tau \\ &= \tfrac{\sigma^2}{4}\int_{-\infty}^{\infty} \eta^2 (\tfrac{\partial w}{\partial x})^2(x,t)dx - \tfrac{\sigma^2}{2}\iint_{\Omega_t} \eta \tfrac{\partial \eta}{\partial t}(\tfrac{\partial w}{\partial x})^2 dxd\tau + \sigma^2 \iint_{\Omega_t} \eta \tfrac{\partial \eta}{\partial x}\tfrac{\partial w}{\partial x}\tfrac{\partial w}{\partial t}dxd\tau. \end{aligned}$$

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Using Lemma 2.11, we obtain

$$I_2 \geqslant \frac{\sigma^2}{4}\int_{-\infty}^{\infty}\eta^2\left(\frac{\partial w}{\partial x}\right)^2(x,t)dx - C - \frac{1}{6}\iint\limits_{\Omega_t}\eta^2\left(\frac{\partial w}{\partial t}\right)^2 dxd\tau.$$

Similarly, we have

$$I_3 \geqslant -C - \tfrac{1}{6}\iint\limits_{\Omega_t}\eta^2(\tfrac{\partial w}{\partial t})^2 dxd\tau,$$
$$I_4 \geqslant -C - \tfrac{1}{6}\iint\limits_{\Omega_t}\eta^2(\tfrac{\partial w}{\partial t})^2 dxd\tau.$$

To estimate I_5, we integrate by parts again and note that $\beta'_\epsilon(\cdot) \geqslant 0$, $\beta''_\epsilon(\cdot) \leqslant 0$ and $\frac{\partial u^\epsilon}{\partial t} \geqslant 0$,

$$\begin{aligned}
I_5 &= \iint\limits_{\Omega_t}\eta^2\beta'_\epsilon(u^\epsilon - \psi_\epsilon)w\tfrac{\partial w}{\partial t}dxd\tau\\
&\geqslant \tfrac{1}{2}\int_{-\infty}^{\infty}\eta^2\beta'_\epsilon(u^\epsilon-\psi_\epsilon)w^2(x,t)dx - \iint\limits_{\Omega_t}\eta\tfrac{\partial \eta}{\partial t}\beta'_\epsilon(u^\epsilon-\psi_\epsilon)w^2 dxd\tau\\
&\geqslant -\iint\limits_{\Omega_t}\eta\tfrac{\partial \eta}{\partial t}\beta'_\epsilon(u^\epsilon-\psi_\epsilon)w^2 dxd\tau\\
&= -\int_{-\infty}^{\infty}\eta\tfrac{\partial \eta}{\partial t}\beta_\epsilon(u^\epsilon-\psi_\epsilon)w(x,t)dx + \iint\limits_{\Omega_t}(\tfrac{\partial}{\partial t}(\eta\tfrac{\partial \eta}{\partial t})w + \eta\tfrac{\partial \eta}{\partial t}\tfrac{\partial w}{\partial t})\beta_\epsilon(u^\epsilon-\psi_\epsilon)dxd\tau\\
&\geqslant -C - \tfrac{1}{6}\iint\limits_{\Omega_t}\eta^2(\tfrac{\partial w}{\partial t})^2 dxd\tau.
\end{aligned}$$

Finally

$$I_6 = -\lambda\iint\limits_{\Omega_t}\eta^2\frac{\partial w}{\partial t}\left(\int_{-\infty}^{+\infty}w(x+z,t)d\tilde{N}(z)\right)dxd\tau \geqslant -C - \frac{1}{6}\iint\limits_{\Omega_t}\eta^2\left(\frac{\partial w}{\partial t}\right)^2 dxd\tau.$$

Combining these estimates with (2.7), we deduce

$$\frac{1}{6}\iint\limits_{\Omega_t}\eta^2\left(\frac{\partial w}{\partial t}\right)^2 dxd\tau + \frac{\sigma^2}{4}\int_{-\infty}^{\infty}\eta^2\left(\frac{\partial w}{\partial x}\right)^2(x,t)dx \leqslant C, \qquad t \in (0, t_2]$$

and complete the proof of (2.6). □

3 Solutions to the problem (P)

In this section, we return to the problem (P). We first prove the uniqueness of solutions to (P) or equivalently, (1.1), which is valid for slightly generalized solutions. To introduce such solutions, as we did in Lemma 2.8, we consider a certain kind of weighted function spaces with weight $e^{-|x|}$, denoted by

$$H = L^2(R, e^{-|x|}dx), \quad V = \{u \in H | u_x \in H\}.$$

The inner product and the norm in H are denoted by $(\cdot,\cdot)_H$ and $\|\cdot\|_H$.

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A function $u \in L^2((0,T];V)$ with $u_t \in L^2([\delta,T];H)$ for any $\delta \in (0,T)$ is said to be a generalized solution to (1.1), if

$$(u_t, v-u)_H + a(u,v-u) + b(u,v-u) \geqslant 0, \; a.e. \text{ in } (0,T], \text{ for any } v \in V, \; v \geqslant \psi(x), \quad (3.1)$$

$$\lim_{t\to 0^+} \int_R |u(x,t)-\psi(x)|^2 e^{-|x|} dx = 0, \quad (3.2)$$

$$u(x,t) \geqslant \psi(x), \quad a.e. \text{ in } R \times (0,T], \quad (3.3)$$

where

$$\begin{aligned} a(u,v) =& \tfrac{\sigma^2}{2} \textstyle\int_{-\infty}^{\infty} u_x v_x e^{-|x|} dx + (r+\lambda) \int_{-\infty}^{\infty} uv e^{-|x|} dx \\ &- \textstyle\int_{-\infty}^{\infty} (\tfrac{\sigma^2}{2} sgn(x) + r - q - \lambda k - \tfrac{\sigma^2}{2}) u_x v e^{-|x|} dx \\ b(u,v) =& -\lambda \textstyle\int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} u(x+z,t) d\tilde{N}(z) \right] v e^{-|x|} dx \end{aligned}$$

Theorem 3.1 (1.1) *admits at most one generalized solution* $u \in L^\infty((0,T];V)$.

Proof Suppose that u_1, u_2 are generalized solutions of (1.1). Let $v = u_2 + (u_1 - u_2)^+$. Then $v \in V, v \geqslant \psi$, and from (3.1),

$$\begin{aligned} &(\tfrac{\partial u_2}{\partial t}, v - u_2)_H + a(u_2, v-u_2) + b(u_2, v-u_2) \\ &= (\tfrac{\partial u_2}{\partial t}, (u_1-u_2)^+)_H + a(u_2,(u_1-u_2)^+) + b(u_2,(u_1-u_2)^+) \\ &\geqslant 0, \quad a.e. \quad \text{in } (0,T] \end{aligned} \quad (3.4)$$

Denote $v_- = u_1 - \epsilon(u_1-u_2)^+, 0 < \epsilon < 1$. It is obvious that $v_- \in V$ and $v_- \geqslant \psi$. Again from (3.1) by taking $v = v_-$, we obtain

$$\left(\frac{\partial u_1}{\partial t}, -\epsilon(u_1-u_2)^+)_H + a(u_1, -\epsilon(u_1-u_2)^+) + b(u_1, -\epsilon(u_1-u_2)^+ \right) \geqslant 0, \; a.e. \text{ in } (0,T],$$

and hence

$$\left(\frac{\partial u_1}{\partial t}, (u_1-u_2)^+)_H + a(u_1,(u_1-u_2)^+) + b(u_1,(u_1-u_2)^+ \right) \leqslant 0, \quad a.e.\text{in}(0,T]. \quad (3.5)$$

Combining (3.4) with (3.5) gives

$$\left(\frac{\partial(u_2-u_1)}{\partial t}, (u_1-u_2)^+ \right)_H + a(u_2-u_1,(u_1-u_2)^+) + b(u_2-u_1,(u_1-u_2)^+ \Big) \geqslant 0, \; a.e. \text{ in } (0,T].$$

Let $w = u_1 - u_2$. Then

$$\left(\frac{\partial w}{\partial t}, w^+ \right)_H + a(w,w^+) + b(w,w^+) \leqslant 0 \quad a.e. \text{ in } (0,T]. \quad (3.6)$$

Now we observe that

$$\left(\frac{\partial w}{\partial t}, w^+\right)_H = \frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty}(w^+)^2 e^{-|x|}dx = \frac{1}{2}\frac{d}{dt}\left(\|w^+\|_H^2\right) \tag{3.7}$$

and

$$a(w, w^+) = a(w^+, w^+) \geqslant \frac{\sigma^2}{4}\|w_x^+\|_H^2 - C\|w^+\|_H^2.$$

In addition,

$$\begin{aligned} b(w,w^+) &= -\lambda \int_R \left(\int_R w(x+z,t)d\tilde{N}(z)\right) w^+(x,t)\cdot e^{-|x|}dx \\ &\geqslant -\lambda \int_R \int_R w^+(x+z,t)w^+(x,t)e^{-|x|}d\tilde{N}(z)dx \\ &\geqslant -\lambda\|w^+\|_H \left(\int_R(\int_R w^{+2}(x+z,t)e^{-|x|}dx)d\tilde{N}(z)\right)^{\frac{1}{2}} \\ &\geqslant -\lambda\|w^+\|_H \left(\int_R(\int_R w^{+2}(x+z,t)e^{-|x+z|}dx)\cdot e^{|z|}d\tilde{N}(z)\right)^{\frac{1}{2}} \\ &= -\lambda\|w^+\|_H^2 \int_R e^{|z|}d\tilde{N}(z). \end{aligned}$$

Hence, using condition (A) gives

$$b(w, w^+) \geqslant -C\|w^+\|_H^2. \tag{3.9}$$

Combining (3.7), (3.8), (3.9) with (3.6), we derive

$$\frac{1}{2}\frac{d}{dt}\left(|w^+|_H^2\right) + \frac{\sigma^2}{4}\|w_x^+\|_H^2 - C\|w^+\|_H^2 \leqslant 0, \quad a.e. \text{ in } (0,T],$$

in particular,

$$\frac{d}{dt}\left(|w^+|_H^2\right) \leqslant C\|w^+\|_H^2, \quad a.e. \text{ in } (0,T].$$

Using (3.2), we see that $|w^+|_H^2 \to 0$ as $t \to 0$, so, by Gronwall's inequality, we obtain $|w^+|_H^2 = 0$ for $t \in (0,T]$ and hence $u_1 \leqslant u_2$ a.e. in $R \times (0,T]$. Similarly, we can prove that $u_2 \leqslant u_1$ a.e. in $R \times (0,T]$. Thus we obtain the uniqueness of solutions to (1.1). □

Corollary 3.1 *Suppose that u is a solution to (1.1), u, $\frac{\partial u}{\partial x}$ are bounded and continuous and $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial t} \in L^2([\delta,T],H)$ for any $\delta \in (0,T)$. Then u is a generalized solution to* (1.1).

Proof It suffices to verify that u satisfies (3.1) for any $v \in V$ with $v \geqslant \psi$. Since $\mathscr{L}_1 u \geqslant 0$ and $\mathscr{L}_1 u = 0$ whenever $u > \psi$, $v - u \geqslant 0$ whenever $u = \psi$, we have

$$(\mathscr{L}_1 u, v-u)_H \geqslant 0. \tag{3.10}$$

Integrating by parts gives

$$\begin{aligned} &\int_R \tfrac{\partial^2 u}{\partial x^2}(v-u)e^{-|x|}dx \\ &\quad = -\int_R \tfrac{\partial u}{\partial x}\left(\tfrac{\partial v}{\partial x} - \tfrac{\partial u}{\partial x}\right)e^{-|x|}dx + \int_R sgn(x)\tfrac{\partial u}{\partial x}(v-u)e^{-|x|}dx \end{aligned}$$

Thus from (3.10) it is easy to derive (3.1). □

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As before, let $P = (\ln K, 0)$, and $B_\delta(P)$ be the disc in (x,t) plane with radius δ , centered at P.

Theorem 3.2 *For any $a < b$ and small positive constant δ, there exists a subsequence $\{\epsilon_k\}$, as $\epsilon_k \to 0$, such that the solution $u^{\epsilon_k}(x,t)$ of* (2.1) *converges uniformly in $Q_T = (a,b) \times (0,T)/\overline{B}_\delta(P)$. The limit function $u(x,t)$ is a solution of* (1.1)*, $u(x,t) \in W_p^{2,1}(Q_T)$, for any $p \in (1,\infty)$, and $\mathscr{L}_1 u = 0$, in $\{u > \psi\}$.*

Proof By Lemma 2.7, there exists a subsequence $\{\epsilon_k\}$ with $\epsilon_k \to 0$ and a function $u \in W_p^{2,1}(Q_T)$ such that

$$u^{\epsilon_k} \overset{\text{weakly}}{\longrightarrow} u, \quad \text{in} \quad W_p^{2,1}(Q_T).$$

Since $p \in (1,+\infty)$ is arbitrary, again by Lemma 2.7, there exists $\alpha \in (0,1)$ such that $u \in C^{1+\alpha,\frac{1+\alpha}{2}}(Q_T)$ by the imbedding theorem. From Lemmas 2.5 and 2.3, for any $x, y \in R, t \in (0,T]$, there holds

$$|u^\epsilon(x,t) - u^\epsilon(y,t)| \leqslant C|x-y|,$$

where C is independent of ϵ. Using an argument in treating parabolic equations without nonlocal term, we can obtain

$$|u^\epsilon(x,s) - u^\epsilon(x,t)| \leqslant C|s-t|^{\frac{1}{2}}$$

for any $x \in R$, s, $t \in (0,T]$, where C is independent of ϵ. The uniform boundedness and equicontinuity of $\{u^\epsilon\}$ imply that there exists a uniformly convergent subsequence $\{u^{\epsilon_k}\}(\epsilon_k \to 0)$ of $\{u^\epsilon\}$, supposed to be the same subsequence as above,

$$u^{\epsilon_k} \overset{\text{uniformly}}{\longrightarrow} u.$$

Now we prove that $u(x,t)$ is a solution of (1.1). Since $\beta_\epsilon \leqslant 0$, we have $\mathscr{L}_1 u^\epsilon \geqslant 0$. Letting $\epsilon = \epsilon_k \to 0$ gives $\mathscr{L}_1 u \geqslant 0$ in Q_T in the sense of distributions. Noting that $u \in W_p^{2,1}(Q_T)$, we can assert that $\mathscr{L}_1 u \geqslant 0$, *a.e.* in Q_T. Since a, b, δ are arbitrary, $\mathscr{L}_1 u \geqslant 0$, *a.e.* in H_T. By Lemma 2.4, $u^\epsilon(x,t) \geqslant \psi_\epsilon(x)$. Letting $\epsilon \to 0$, we find that $u(x,t) \geqslant \psi(x)$. It remains to prove that $\mathscr{L}_1 u = 0$ in $\{u > \psi\}$. In fact, for any $(x_0,t_0) \in \{u > \psi\}$, from the continuity of u, ψ, there exists a neighborhood $\mathscr{N}$ of (x_0,t_0) such that $u(x,t) > \psi(x)$ in $\mathscr{N}$. From the uniform convergence of $u^{\epsilon_k}(x,t)$ and $\psi_{\epsilon_k}(x)$, there exists $\delta > 0$ such that $u^{\epsilon_k}(x,t) \geqslant \psi_{\epsilon_k}(x) + \delta$ for ϵ_k small enough. So $\beta_{\epsilon_k}(u^{\epsilon_k} - \psi_{\epsilon_k}) = 0$ and $\mathscr{L}_1 u^{\epsilon_k} = 0$ in $\mathscr{N}$. Letting $\epsilon_k \to 0$ and noting that $u \in W_p^{2,1}(Q_T)$, we obtain $\mathscr{L}_1 u = 0$ *a.e.* in $\mathscr{N}$.

Thus we have proved that as the limit of a uniformly convergent subsequence of $\{u^\epsilon\}$, u is a solution of (1.1).

Similar to parabolic equations, we can prove by Bernstein's Method that

$$\|u^\epsilon\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\mathscr{N})} \leqslant C$$

for any $\mathcal{N} \subset \{u > \psi\}$, where C is independent of ϵ. Therefore, $\frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}$ exist, are continuous and $\mathcal{L}_1 u = 0$ in $\{u > \psi\}$. □

Remark 3.1 The solution u obtained in Theorem 3.3 as the limit of a subsequence of u^ϵ possesses all properties assumed in Corollary 3.2, some properties have been presented above and others will be given later. So we can further affirm that, for any $a < b$ and small positive constant δ, the whole family $\{u^\epsilon\}$ is uniformly convergent to u in $Q_T = (a,b) \times (0,T)/\overline{B}_\delta(P)$, as $\epsilon \to 0$.

In what follows, we discuss properties of the solution $u(x,t)$ to (1.1) thus obtained. Let $x = \ln S, t = T - \tau, u(x,t) = V(S,\tau)$. Then $V(S,\tau)$ is a solution to (P). We will describe properties of solutions sometimes in terms of $u(x,t)$, sometimes in terms of $V(S,\tau)$.

Theorem 3.3 *For any $a < b$ and small positive constant δ, $\frac{\partial u}{\partial t}$, $\frac{\partial^2 u}{\partial x^2} \in L^\infty(Q_T)$ and $\frac{\partial u}{\partial x}$ is continuous in H_T, where $Q_T = (a,b) \times (0,T]/\overline{B}_\delta(P)$,*

Proof By Lemma 2.10 , Theorem 3.3 , we can infer $\frac{\partial u}{\partial t}$, $\frac{\partial^2 u}{\partial x^2} \in L^\infty(Q_T)$. Since $\frac{\partial^2 u^\epsilon}{\partial x^2}$ is uniformly bounded in Q_T, we can prove that $\frac{\partial u}{\partial x}^\epsilon$ is uniformly $\frac{1}{2}$-*Hölder* continuous in t and hence infer the continuity of $\frac{\partial u}{\partial x}$ in H_T. □

Theorem 3.4 *The solution $V(S,\tau)$ to (P) possesses the following properties:*
(1) $V(S,\tau)$ is nonincreasing both in S and in τ;
(2) For each $\tau \in [0,T)$, $V(S,\cdot)$ is a convex function of S;
(3) $V(S,\tau) \in [0,K]$, for any $(S,\tau) \in [0,\infty) \times [0,T)$.

Proof Let $x = \ln S$, $t = T - \tau$. Then $u(x,t) = V(S,\tau)$ is a solution to (1.1).

(1) By Lemma 2.5, we have $\frac{\partial u^\epsilon}{\partial x} \leqslant 0$, $\frac{\partial u^\epsilon}{\partial t} \geqslant 0$. So $u^\epsilon(x,t)$ is nonincreasing in x and nondecreasing in t. Since $u(x,t)$ is the limit of $u^\epsilon(x,t)$ as $\epsilon \to 0$, $u(x,t)$ possesses the same property.

(2) Denote $v^\epsilon(S,t) = u^\epsilon(\ln S,t) = u^\epsilon(x,t)$. Then by Lemma 2.6, $\frac{\partial^2 v^\epsilon}{\partial S^2} = \frac{1}{S^2}(\frac{\partial^2 u^\epsilon}{\partial x^2} - \frac{\partial u^\epsilon}{\partial x}) \geqslant 0$, which implies that $v^\epsilon(S,t)$ is a convex function of S. Using Theorem 3.3, we conclude that so is the limit function $V(S,\tau)$.

(3) Since $u(x,t) \geqslant \psi(x)$, we have $u(x,t) \geqslant 0$, so does $V(S,\tau)$. By Lemma 2.3, $u^\epsilon(x,t) \leqslant K+\epsilon$, for any $(x,t) \in R \times (0,T]$. Letting $\epsilon \to 0$ yields $u(x,t) \leqslant K$, so does $V(S,\tau)$ for any $(S,\tau) \in [0,+\infty) \times [0,T)$. □

Lemma 3.1 *Assume that $\mathcal{L}w \leqslant 0$ in $R^+ \times [0,T)$ with $\mathcal{L}$ being the operator in §1 and E is a subregion in $R^+ \times [0,T)$ in which $w(S,\tau) \geqslant 0$. If for some point $P \in \partial E$, $w(P) = 0$, the inward normal vector γ at P to ∂E is not parallel to the τ axis and there exists an inscribed disc $B \subset E$ such that $\overline{B} \cap \partial E = \{P\}$ and $w > 0$ on $\overline{B}/\{P\}$, then $\frac{\partial w}{\partial \gamma}(P) > 0$, where $\frac{\partial}{\partial \gamma}$ denotes the derivative along γ.*

Proof Let (S_1,τ_1) and R be the centre and radius of B. Consider a disc B_1 centered at $P = (S_0,\tau_0)$ with radius R_0 less than $|S_1 - S_0|$ (since γ is not parallel to the $\tau-$ axis,

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$|S_1 - S_0| > 0$). Denote $C' = \partial B_1 \cap \overline{B}$, $C'' = \partial B \cap B_1$, and D the region with boundary $C' \cup C''$. Since $w > 0$ on $\overline{B}/\{P\}$, there exists $\eta > 0$ such that

$$w(S,\tau) \geqslant \eta, \quad (S,\tau) \in \overline{B}/B_1. \tag{3.11}$$

Consider the auxilliary function

$$v(S,\tau) = e^{-\alpha R^2} - e^{-\alpha[(S-S_1)^2+(\tau-\tau_1)^2]}.$$

Then a simple calculation shows that

$$\begin{aligned}\mathscr{L}v =& e^{-\alpha[(S-S_1)^2+(\tau-\tau_1)^2]}\{-2\sigma^2S^2(S-S_1)^2\alpha^2 \\ &+\alpha[2(\tau-\tau_1)+\sigma^2S^2+2(r-q-\lambda k)S(S-S_1)]+(r+\lambda)\}-re^{-\alpha R^2} \\ &-\lambda\int_{-1}^{+\infty} e^{-\alpha[(S(1+y)-S_1)^2+(\tau-\tau_1)^2]}dN(y).\end{aligned}$$

Since $R_0 < |S_1 - S_0|$, we have $\mathscr{L}v < 0$ in D, if $\alpha > 0$ is chosen appropriately large.

Let $\overline{w} = w + \epsilon v(\epsilon > 0)$. Then for large $\alpha > 0$ and $\epsilon > 0$,

$$\mathscr{L}\overline{w} = \mathscr{L}w + \epsilon\mathscr{L}v < 0 \qquad \text{in} \quad D$$

from which it follows that $\overline{w}$ can not attain negative global minimum in D. Note that $v = 0$ on ∂B, so $\overline{w} \geqslant 0$ on C''. From (3.11), we can take $\epsilon > 0$ so small that $\overline{w} > 0$ on C'. If $\overline{w} < 0$ at some point in D, then $\overline{w}$ will attain negative minimum in D, which is impossible. Hence $\overline{w} \geqslant 0$ on $\overline{D}$ and $\overline{w}(P) = 0$. Thus

$$\frac{\partial \overline{w}}{\partial \gamma}(P) = \frac{\partial w}{\partial \gamma}(P) + \epsilon\frac{\partial v}{\partial \gamma}(P) \geqslant 0.$$

Since $\frac{\partial v(P)}{\partial \gamma} = -2\alpha R e^{-\alpha R^2} < 0$, we finally obtain $\frac{\partial w(P)}{\partial \gamma} > 0$. So the lemma is proved. □

Theorem 3.5 $V(S,\tau) > 0, \forall (S,\tau) \in R^+ \times [0,T)$.

Proof Since $V(S,\tau) \geqslant \phi(S) = (K-S)^+$(see the proof of Theorem 3.3), we have $V(S,\tau) > 0$ when $S < K$. So it remains to prove that $V(S,\tau) > 0$ when $S \geqslant K$. Since $V(S,\tau)$ is decreasing in S, it suffices to prove that $V(S,\tau) > 0$ when $S > K$, $\tau \in (0,T)$.

Let $\Omega_1 = \{(S,\tau)|S > K, \tau \in (0,T), V(S,\tau) = 0\}$ and $\Omega_2 = \{(S,\tau)|S > K, \tau \in (0,T), V(S,\tau) > 0\}$. Suppose that Ω_1 is nonempty, namely, there exists $(S_0,\tau_0) \in \Omega_1$. Denote $\Gamma = \partial\Omega_1/\{\tau = 0, T\}$. By the monotonicity of $V(S,\tau)$ and $V(S,\tau) \geqslant 0$, there exists $(S_1,\tau_0) \in \Gamma$ with $S_1 \leqslant S_0$ and $(S,\tau_0) \in \Omega_1$ when $S \geqslant S_1$.

First, we prove that there exist a point $P^*(S^*,\tau^*) \in \Omega_2$ and a constant $R > 0$ such that $B_R(P^*) \subset \Omega_2$ and $\partial B_R(P^*) \cap \Gamma$ is a set of singer point $\tilde{P}(\tilde{S},\tilde{\tau})$ with $\tilde{S} > S^*$, $\tilde{\tau} \geqslant \tau^*$.

If there exists $(S_1,\tau_1) \in \Gamma$ $(\tau_1 > \tau_0)$, then from the monotonicity of $V(S,\tau)$ and $V(S,\tau) \geqslant 0$, we have $(S_0,\tau) \in \Gamma$ when $\tau \in (\tau_0,\tau_1)$. In this case, the conclusion we want to prove is trivial. Suppose not. Then for any $\tau_1 \in (\tau_0,T)$, there exists $S_2 < S_1$ such that $(S_2,\tau_1) \in \Gamma$. Denote by l the segment with endpoints (S_2,τ_1) and (S_1,τ_0). Then we treat the following two cases separately.

(1) There are points of Γ (besides (S_2,τ_1) and (S_1,τ_0)) lying on l or on the left side of l. In this case, there must be a line l' parallel to l such that no point of Γ lies on the left side of l', but there exists a point of Γ lying on l', denoted by (S_3,τ_2) with $\tau_2 \in (\tau_0,\tau_1)$. Hence the existence of the point $P^*(S^*,\tau^*)$ and the constant $R > 0$ required follows immediately.

(2) All points of Γ except (S_1,τ_0) and (S_2,τ_1) lie on the right side of l.
For $P(S_P,\tau_P) \in l$, denote $d_P = dist(P,\Gamma)$. Suppose that $Q(S_Q,\tau_Q) \in \Gamma$ is a point such that $d_P = dist(P,Q)$. If there exists a point $P \in l, P \neq (S_1,\tau_0)$ and (S_2,τ_1) such that the corresponding $Q(S_Q,\tau_Q)$ satisfies $S_Q > S_P$, then we may take (S_Q,τ_Q) as $\tilde{P}(\tilde{S},\tilde{\tau})$, and obtain the desired conclusion. Now we prove by contradiction that the contrary case is impossible. Note that, in the contrary case, for any $P \in l, S_Q = S_P$. Since d_P is a continuous function of P, there exists $P_0(S_{P_0},\tau_{P_0}) \in l$ such that $d_{P_0} = \max_{P\in l} d_P$. Let $Q_0(S_{Q_0},\tau_{Q_0}) \in \Gamma$ be a point such that $d_{P_0} = dist(P_0,Q_0)$. By the definition of P_0, we have $B_{d_{P_0}}(P_0) \subset \Omega_2$, $Q_0 \in \partial B_{d_{P_0}}(P_0) \cap \Gamma$. Let $\tilde{P} \in l$ be a point with $\tau_{P_0} - \tau_{\tilde{P}} > 0$ small enough and $\tilde{Q} \in \Gamma$ be a point such that $d_{\tilde{P}} = dist(\tilde{P},\tilde{Q})$. If $\tilde{\tilde{Q}}$ is the intersecting point of $\partial B_{d_{P_0}}(P_0)$ and the segment $\overline{\tilde{P}\tilde{Q}}$ (since $d_{P_0} \leqslant dist(P_0,\tilde{Q})$, $\tilde{\tilde{Q}}$ exists!), then

$$\tau_{Q_0} - \tau_{\tilde{Q}} \leqslant \tau_{Q_0} - \tau_{\tilde{\tilde{Q}}} \leqslant \tau_{P_0} - \tau_{\tilde{P}}.$$

Therefore, since $S_{\tilde{Q}} = S_{\tilde{P}}$, $S_{Q_0} = S_{P_0}$, we obtain

$$\begin{aligned} d_{\tilde{P}} &= dist(\tilde{P},\tilde{Q}) = \tau_{\tilde{Q}} - \tau_{\tilde{P}} = (\tau_{Q_0} - \tau_{P_0}) - (\tau_{Q_0} - \tau_{\tilde{Q}}) + (\tau_{P_0} - \tau_{\tilde{P}}) \\ &> \tau_{Q_0} - \tau_{P_0} = d_{P_0}, \end{aligned}$$

which contradicts the definition of d_{P_0}.

Summing up, we have proved the conclusion in any cases.

Now we are ready to use Lemma 3.6. Since $\tilde{S} > S^*$, the inward normal vector γ at $\tilde{P}(\tilde{S},\tilde{\tau})$ to Γ is not parallel to the $\tau-$ axis. Thus, by Lemma 3.6, we have $\frac{\partial V}{\partial \gamma}(\tilde{S},\tilde{\tau}) > 0$ and hence $\frac{\partial V}{\partial S}(\tilde{S},\tilde{\tau}) > 0$, which contradicts $\frac{\partial V}{\partial S}(\tilde{S},\tilde{\tau}) = 0$ following from the fact that $V(S,\tau) = 0$ in Ω_1. Therefore, Theorem 3.7 is proved. □

Remark 3.2 It is clear that the value of a non-negative payoff is non-negative if the time-t value is regarded as a risk-neutral expectation of the payoff. The conclusion presented in Theorem 3.7 is that the option owns the positive value before the expiry date.

Theorem 3.6 $\lim_{S\to+\infty} V(S,\tau) = 0$, *for any* $\tau \in [0,T]$.

Proof Since the solution $u(x,t)$ is nonnegative and nonincreasing in x , the limit $u(+\infty,t) = \lim_{x\to+\infty} u(x,t)$ exists and $u(+\infty,t) \geqslant 0$.

We prove $u(+\infty,t) = 0$ by contradiction. Suppose that there exists some $\tilde{t} \in (0,T]$ such that $u(+\infty,\tilde{t}) > 0$. Since $u(x,t)$ is nonincreasing in x and nondecreasing in t, we have $u(x,\tilde{t}) > 0$ for any $x \in (-\infty,+\infty)$ and $u(x,t) > 0$ for $x \in (-\infty,+\infty)$, $t \in [\tilde{t},T]$. Denote $t_0 = \inf\{\tilde{t} \geqslant 0; u(x,t) > 0,\ x \in (-\infty,+\infty),\ t \in (\tilde{t},T]\}$. Clearly, $u(x,t) > 0$ when $x \in (-\infty,+\infty)$, $t \in (\tilde{t},T]$, but $u(+\infty,t_0) = 0$.

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Take $X > 0$ such that $\psi(x) = (K - e^x)^+ = 0$ when $x \geqslant X$. Then

$$u(x,t) - \psi(x) = u(x,t) > 0, \quad \text{for} \quad x \in (X, +\infty), \ t \in (t_0, T].$$

Hence, by Theorem 3.3, $u(x,t)$ satisfies

$$\mathscr{L}_1 u = \frac{\partial u}{\partial t} - \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} - \left(r - q - \lambda k - \frac{\sigma^2}{2}\right)\frac{\partial u}{\partial x} + ru + \lambda\left(u - \int_{-\infty}^{+\infty} u(x+z,t)d\tilde{N}(z)\right) = 0$$

for $x \in (X, +\infty), t \in (t_0, T]$.

Choose $\eta(x) \in C^\infty(-\infty, +\infty)$ such that

$$\eta(x) = 0 \quad \text{when} \quad x \leqslant X; \eta = 1 \quad \text{when} \quad x \geqslant 2X. \tag{3.12}$$

Let $v = \eta u$. Then v satisfies

$$Lv = f(x,t) + g(x,t),$$

where

$$\begin{aligned} Lv &= \tfrac{\partial v}{\partial t} - \tfrac{\sigma^2}{2}\tfrac{\partial^2 v}{\partial x^2} - (r - q - \lambda k - \tfrac{\sigma^2}{2})\tfrac{\partial v}{\partial x} + rv, \\ f(x,t) &= -\tfrac{\sigma^2}{2}(2\eta' \tfrac{\partial u}{\partial x} + \eta'' u) - (r - q - \lambda k - \tfrac{\sigma^2}{2})\eta' u, \\ g(x,t) &= -\lambda\eta\big(u - \textstyle\int_{-\infty}^{+\infty} u(x+z,t)d\tilde{N}(z)\big). \end{aligned}$$

From (3.12),

$$f(x,t) = 0 \quad \text{when} \quad x \geqslant 2X. \tag{3.13}$$

It is clear that

$$g(+\infty, t) = \lim_{x \to +\infty} g(x,t) = 0, \qquad \text{when} \quad t \in (0, T). \tag{3.14}$$

Using the fundmental solution $\Gamma(x,t;\xi,\tau)$ of the parabolic operator L, we can express v as

$$v(x,t) = \int_{-\infty}^{+\infty} \Gamma(x, t-t_0; \xi, 0) v(\xi, t_0) d\xi - \int_{t_0}^{t}\int_{-\infty}^{+\infty} \Gamma(x, t-t_0; \xi, \tau - t_0)(f(\xi,\tau) + g(\xi,\tau)) d\xi d\tau$$

for $x \in (-\infty, +\infty), t \in (t_0, T]$.

For $\Gamma(x,t;\xi,\tau)$,the following estimates holds (see Friedman [6]):

$$|\Gamma(x,t;\xi,\tau)| \leqslant C(t-\tau)^{-\frac{1}{2}} \exp\left(-\frac{\lambda_0(x-\xi)^2}{4(t-\tau)}\right), \qquad x \in (-\infty, +\infty), \quad 0 \leqslant \tau < t \leqslant T,$$

where C, λ_0 are positive constants. So we have

$$\begin{aligned} &|v(x,t)| \\ &\quad \leqslant C \textstyle\int_{-\infty}^{+\infty} (t-t_0)^{-\frac{1}{2}} \exp\big(-\frac{\lambda_0(x-\xi)^2}{4(t-t_0)}\big)|v(\xi,t_0)|d\xi \\ &\qquad + C \textstyle\int_{t_0}^{t}\int_{-\infty}^{+\infty} (t-\tau)^{-\frac{1}{2}} \exp\big(-\frac{\lambda_0(x-\xi)^2}{4(t-\tau)}\big)(|f(\xi,\tau)| + |g(\xi,\tau)|)d\xi d\tau \\ &\quad = 2C \textstyle\int_{-\infty}^{+\infty} \exp(-\lambda_0\eta^2)|v(x + 2(t-t_0)^{-\frac{1}{2}}\eta, t_0)|d\eta \\ &\qquad + 2C \textstyle\int_{t_0}^{t}\int_{-\infty}^{+\infty} \exp(-\lambda_0\eta^2)(|f(x + 2(t-t_0)^{-\frac{1}{2}}\eta, \tau)| + |g(x + 2(t-t_0)^{-\frac{1}{2}}\eta, \tau)|)d\eta d\tau \end{aligned}$$

for $x \in (-\infty, +\infty)$, $t \in (t_0, T]$. Since $v(+\infty, t_0) = u(+\infty, t_0) = 0$, it is easy to see that

$$2C \int_{-\infty}^{+\infty} \exp(-\lambda_0 \eta^2) |v(x + 2(t - t_0)^{-\frac{1}{2}} \eta, t_0)| d\eta \to 0, \text{ as } x \to +\infty.$$

Similarly, from (3.13), (3.14), we have

$$2C \int_{t_0}^{t} \int_{-\infty}^{+\infty} \exp(-\lambda_0 \eta^2) \big(|f(x+2(t-t_0)^{-\frac{1}{2}} \eta, \tau)| + |g(x+2(t-t_0)^{-\frac{1}{2}} \eta, \tau)|\big) d\eta d\tau \to 0, \text{ as } x \to +\infty.$$

Hence,

$$u(+\infty, t) = v(+\infty, t) = 0, \quad \text{when} \quad t \in (t_0, T).$$

The contradiction shows that the conclusion of Theorem 3.8 is true. □

Remark 3.3 A put option allows its holder to sell the underlying asset on a specific date for a precribed amount and a specified price. Whereas the holder of a put option wants the underlying asset price to fall as low as possible. The conclusion of Theorem 3.8 describe the fact in finance that the put option is of no value when the underlying asset price is large enough.

The following conclusions about the relationship between the solution V of (P) and its related parameters, such as K, σ, T, can be similarly proved by the argument in Chaper 6 of Jiang [10]. Dependence upon its related parameters are explicitly indicated, if necessary.

Proposition 3.1 (1) $0 \leqslant V(S, \tau; K_1) - V(S, \tau; K_2) \leqslant K_1 - K_2$ when $K_1 \geqslant K_2$;
(2) $V(S, \tau; T_2) \leqslant V(S, \tau; T_1)$ when $T_1 \geqslant T_2$, where $\tau \in [0, T_2]$;
(3) $V(S, \tau; \sigma_2) \leqslant V(S, \tau; \sigma_1)$ when $\sigma_1 \geqslant \sigma_2$,
where $V(S, \tau; i)$ denotes the value of $V(S, \tau)$ with all parameters unvaried except the parameter i.

Proof Let $w = u_1^\epsilon(x, t; K_1) - u_2^\epsilon(x, t; K_2)$. Then w satisfies

$$\begin{cases} \mathscr{L}_1 w + \beta_\epsilon(u_1^\epsilon - \psi_{1\epsilon}) - \beta_\epsilon(u_2^\epsilon - \psi_{2\epsilon}) = 0, \\ w(x, 0) = \psi_{1\epsilon} - \psi_{2\epsilon} = \Pi_\epsilon(K_1 - e^x) - \Pi_\epsilon(K_2 - e^x). \end{cases}$$

Note that

$$\begin{aligned} &\beta_\epsilon(u_1^\epsilon - \psi_{1\epsilon}) - \beta_\epsilon(u_2^\epsilon - \psi_{2\epsilon}) \\ &\quad = \beta_\epsilon'(\xi)[u_1^\epsilon - u_2^\epsilon - \Pi_\epsilon(K_1 - e^x) - \Pi_\epsilon(K_2 - e^x)] \\ &\quad = \beta_\epsilon'(\xi)[w - \Pi_\epsilon'(\eta)(K_1 - K_2)], \end{aligned}$$

and

$$\Pi_\epsilon(K_1 - e^x) - \Pi_\epsilon(K_2 - e^x) = \Pi_\epsilon'(\eta)(K_1 - K_2),$$

where ξ is some value between $u_1^\epsilon - \psi_{1\epsilon}$ and $u_2^\epsilon - \psi_{2\epsilon}$, and η is between $K_1 - e^x$ and $K_2 - e^x$. Let

$$\begin{aligned} Lw = &\tfrac{\partial w}{\partial t} - \tfrac{\sigma^2}{2} \tfrac{\partial^2 w}{\partial x^2} - (r - q - \lambda k - \tfrac{\sigma^2}{2}) \tfrac{\partial w}{\partial x} \\ &+ (r + \lambda + \beta_\epsilon'(\xi)) w - \lambda \textstyle\int_{-\infty}^{+\infty} w(x + z, t) d\tilde{N}(z). \end{aligned}$$

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Then by conditions (A1) and (A2), we have

$$Lw = \beta_\epsilon'(\xi)\Pi_\epsilon'(\eta)(K_1 - K_2) \geqslant 0$$

whenever $K_1 \geqslant K_2$. Therefore using Lemma 2.1 leads to $w(x,t) \geqslant 0$, i.e., $u_1^\epsilon(x,t;K_1) \geqslant u_2^\epsilon(x,t;K_2)$ from $w(x,0) \geqslant 0$ and the conclusion (1) is proved.

Similary, we can prove conclusions (2) and (3). □

4 Free boundary problem

In this section, we are concerned with a free boundary problem corresponding to the problem (P). We first derive a solution of this problem from the solution $V(S,\tau)$ to (P), whose uniquely existence has been established in Theorems 3.3 and 3.1, and then prove the uniqueness. In addition, some properties of solutions of the free boundary problem are discussed.

Theorem 4.1 *For each $\tau \in [0,T)$, there exists $s(\tau) \in [0,K)$ such that $V(S,\tau) = (K-S)^+$ when $0 \leqslant S \leqslant s(\tau)$; $V(S,\tau) > (K-S)^+$ when $S > s(\tau)$. Moreover, $s(\tau)$ is a nondecreasing function of τ.*

Proof Noting that $V(S,T) = (K-S)^+$ and using Theorem 3.4, we have $(K-S)^+ \leqslant V(S,\tau) \leqslant V(0,\tau)$ for $\tau \in [0,T)$. In particular, $V(0,\tau) \geqslant K$. Since by Theorem 3.4, $V(S,\tau) \in [0,K]$, we obtain $V(0,\tau) = K$.

For any $\tau \in [0,T)$, define

$$\begin{aligned} s_1(\tau) &= \sup \Omega_1(\tau), \quad \Omega_1(\tau) = \{S_0 | V(S,\tau) = (K-S)^+, 0 \leqslant S \leqslant S_0\}, \\ s_2(\tau) &= \inf \Omega_2(\tau), \quad \Omega_2(\tau) = \{S_0 | V(S,\tau) > (K-S)^+, S_0 < S < +\infty\}. \end{aligned}$$

Since $V(0,\tau) = K$, we have $0 \in \Omega_1(\tau)$. On the other hand, by Theorem 3.7, $V(S,\tau) > 0$, for any $(S,\tau) \in R^+ \times [0,T)$, which implies $K \in \Omega_2(\tau)$. Hence $\Omega_1(\tau)$, $\Omega_2(\tau)$ are nonempty, and $s_1(\tau)$, $s_2(\tau)$ are well-defined. From the definitions of $s_1(\tau)$ and $s_2(\tau)$, it is clear that $s_1(\tau) \leqslant s_2(\tau)$. We further prove that $s_1(\tau) = s_2(\tau)$.

Suppose that there exists $\tau_0 \in [0,T)$ such that $s_1(\tau_0) < s_2(\tau_0)$. Since $V(S,\tau) > 0$, for any $(S,\tau) \in R^+ \times [0,T)$, it is easy to see that $s_1(\tau) \leqslant K$ for any $\tau \in [0,T)$. Also we have $s_2(\tau) \leqslant K$, for any $\tau \in [0,T)$ which follows from the fact $K \in \Omega_2(\tau)$. Therefore, at $S = s_1(\tau_0)$ and $S = s_2(\tau_0)$, we have $V(S,\tau_0) = K - S$. This and the convexity of $V(S,\tau)$ in S imply that $V(S,\tau_0) = K - S$ in $(s_1(\tau_0), s_2(\tau_0))$, which contradicts the definition of $s_1(\tau_0)$. So $s_1(\tau) = s_2(\tau)$ for any $\tau \in [0,T)$. In addition, by the continuity of $V(S,\tau)$, $V(s(\tau),\tau) = (K - s(\tau))^+$ for any $\tau \in [0,T)$, which implies $s(\tau) < K$ by Thoerem 3.6.

Denote

$$s(\tau) = s_1(\tau) = s_2(\tau).$$

Since $V(S,\tau)$ is nonincreasing in τ, we have $\Omega_2(\tau_1) \subset \Omega_2(\tau_2)$ and hence $s(\tau_1) \geqslant s(\tau_2)$ when $\tau_1 > \tau_2$. □

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Remark 4.1 From Theorem 4.1, we see that $R^+ \times [0, T)$ can be divided into two parts by $S = s(\tau)$:

- The continuation region:

$$\Sigma_1 = \{(S, \tau) | V(S, \tau) > (K - S)^+\} = \{(S, \tau) | S > s(\tau), 0 \leqslant \tau < T\},$$

- The stopping region:

$$\Sigma_2 = \{(S, \tau) | V(S, \tau) = (K - S)^+\} = \{(S, \tau) | 0 < S \leqslant s(\tau), 0 \leqslant \tau < T\}.$$

$S = s(\tau)$ is just the optimal exercise boundary of American options.

Remark 4.2 Theorem 3.3 and 3.4 show that $V(S, \tau)$ is a smooth function and $\mathscr{L}V = 0$ in Σ_1, $\frac{\partial V}{\partial S}$ is continuous in $R^+ \times [0, T)$ and $V(S, \tau) = K - S$ in Σ_2, $\frac{\partial V}{\partial S}(s(\tau), \tau) = -1$. Thus, from (3) of Theorem 3.4, $-1 \leqslant \frac{\partial V}{\partial S} \leqslant 0$, for any $(S, \tau) \in R^+ \times [0, T)$.

A similar argument of Lemma 3.6 leads to the following Lemma:

Lemma 4.1 *Assume that $\mathscr{L}w \leqslant 0$ in $R^+ \times [0, T)$ with $\mathscr{L}$ being the operator in § 1 and E is a subregion in $R^+ \times [0, T)$ in which $w(S, \tau) \geqslant 0$. If $w = 0$ at some interior point $(S_0, \tau_0) \in E$, then $w(S, \tau) = 0$ for any $(S, \tau) \in \{\tau \geqslant \tau_0\} \cap E$.*

Proposition 4.1 $V_\tau < 0, V_{SS} > 0$ in Σ_1.

Proof Note that $\mathscr{L}V_\tau = 0$ in Σ_1 and $V_\tau \leqslant 0$ in $R^+ \times (0, T]$. If there exists $(S_0, \tau_0) \in \Sigma_1$ such that $V_\tau(S_0, \tau_0) = 0$, then by Lemma 4.2, we have $V_\tau(S, \tau) = 0$, for any $S > s(\tau), \tau \geqslant \tau_0$. So $\int_\tau^T V_\tau(S, t)dt = V(S, T) - V(S, \tau) = 0$, i.e. $V(S, \tau) = V(S, T) = (K - S)^+$ in $\Sigma_1 \cap \{\tau \geqslant \tau_0\}$, which contradicts $V(S, \tau) > (K - S)^+$ in Σ_1. Thus $V_\tau < 0$ in Σ_1.

Using a similar argument, we have $V_{SS} > 0$ in Σ_1. □

Theorem 4.2 *The free boundary problem*

$$\begin{cases} \mathscr{L}w(S, \tau) = 0, & S > b(\tau), 0 \leqslant \tau < T, \\ w(b(\tau), \tau) = K - b(\tau), & 0 \leqslant \tau < T, \\ \frac{\partial w}{\partial S}(b(\tau), \tau) = -1, & 0 \leqslant \tau < T, \\ w(S, T) = (K - S)^+, & S > 0, \\ w(\infty, \tau) = 0, & 0 \leqslant \tau < T, \\ w(S, \tau) > (K - S)^+, S > b(\tau); & w(S, \tau) = (K - S)^+, S \leqslant b(\tau). \end{cases} \qquad (\overline{P})$$

has a unique solution $(w(S, \tau), b(\tau))$, where $w(S, \tau)$ is smooth when $S \geqslant b(\tau)$, $0 \leqslant \tau \leqslant T$, nonincreasing and convex in S and $b(\tau) \in [0, K)$.

Proof By Theorem 3.3, Problem (P) has a solution $V(S, \tau)$. Theorem 3.8 shows that $\lim_{S \to +\infty} V(S, \tau) = 0$. Let $s(\tau)$ be the function defined in Theorem 4.1. Then from Remark 4.2, we see that $(V(S, \tau), s(\tau))$ is a solution of $(\overline{P})$. Now we prove the uniqueness.

Suppose that $(V_1(S, \tau),\ s_1(\tau))$ and $(V_2(S, \tau),\ s_2(\tau))$ are two solutions of $(\overline{P})$. Let

$$l_1(\tau) = \min\{s_1(\tau), s_2(\tau)\}, \quad l_2(\tau) = \max\{s_1(\tau), s_2(\tau)\},$$
$$w(S, \tau) = V_1(S, \tau) - V_2(S, \tau).$$

Then $V_1(S,\tau) = V_2(S,\tau) = K - S$ when $S \leqslant l_1(\tau)$, $0 \leqslant \tau < T$, that is, $w(S,\tau) = 0$; $V_1(S,\tau) > (K-S)^+$, $V_2(S,\tau) > (K-S)^+$, $\mathscr{L}w = \mathscr{L}V_1 - \mathscr{L}V_2 = 0$ when $S \geqslant l_2(\tau)$, $0 \leqslant \tau < T$.

To prove the uniqueness, it suffices to show that $w(S,\tau) = 0$ when $S > l_1(\tau)$, $0 \leqslant \tau < T$. Suppose there exists some point (S,τ) with $S > l_1(\tau)$, $0 \leqslant \tau < T$ such that $w(S,\tau) \neq 0$. For defnteness, suppose that at this point $w(S,\tau) > 0$. Then since $w(l_1(\tau),\tau) = 0$, $w(S,T) = 0$, and $w(+\infty,\tau) = 0$, $w(S,\tau)$ will achieve positive maximum at some point (S_0,τ_0) with $S_0 > l_1(\tau_0)$, $0 \leqslant \tau_0 < T$. We consider the following cases separately:

(1) $S_0 > l_2(\tau_0)$.
Note that $\mathscr{L}w = 0$ when $S > l_2(\tau)$, $0 \leqslant \tau < T$. In particular, $\mathscr{L}w(S_0,\tau_0) = 0$. However, since (S_0,τ_0) is the positive maximum point of $w(S,\tau)$, we have $\frac{\partial w}{\partial \tau} \leqslant 0$, $\frac{\partial w}{\partial S} = 0$, $\frac{\partial^2 w}{\partial S^2} \leqslant 0$, at (S_0,τ_0), from which $\mathscr{L}w(S_0,\tau_0) \leqslant -rw(S_0,\tau_0) < 0$ and a contradiction is derived.

(2) $l_1(\tau_0) = s_1(\tau_0) < S_0 \leqslant l_2(\tau_0) = s_2(\tau_0)$.
In this case, for $S \in (s_1(\tau_0), s_2(\tau_0)]$, $V_2(S,\tau_0) = K - S$ and hence $\frac{\partial V_2(S,\tau_0)}{\partial S} = -1$. Note that $\frac{\partial w(S_0,\tau_0)}{\partial S} = 0$. Then $\frac{\partial V_1(S_0,\tau_0)}{\partial S} = -1$. From this and $\frac{\partial V_1(s_1(\tau_0),\tau_0)}{\partial S} = -1$, we obtain $\frac{\partial V_1(S,\tau_0)}{\partial S} = -1$ for $S \in [s_1(\tau_0), S_0]$ by the convexity in S of $V(S,\tau)$. But $V_1(s_1(\tau_0),\tau_0) = K - s_1(\tau_0)$, so $V_1(S,\tau_0) = K - S$ for $S \in [s_1(\tau_0), S_0]$, which contradicts $V_1(S,\tau_0) > K - S_0$.

(3) $l_1(\tau_0) = s_2(\tau_0) < S_0 \leqslant l_2(\tau_0) = s_1(\tau_0)$.
Similar reasoning as in the case (2) also derives a contradiction. □

From (2) of Proposition 3.1, we see that as the exipiry T increases, the correspongding options price $V(S,\tau;T)$ is getting expensive. $V(S,\tau;\infty)$ is the most expensive price, that is, $V(S,\tau;T) \leqslant V(S,\infty)$, where $V(S,\infty)$ denotes the American options price without expiration date; such options are called the perpetual American options. $V(S,\infty)$ satisfies the following variational inequality:

$$\begin{cases} \min\{-\mathscr{L}_\infty v, v - (K-S)^+\} = 0, & 0 < S < \infty, \\ v(0) = K, v(\infty) = 0, \end{cases} \qquad (P)_\infty$$

where

$$\mathscr{L}_\infty v = \frac{\sigma^2}{2} S^2 \frac{d^2 v}{dS^2} + (r - q - \lambda k) S \frac{dv}{dS} - (r+\lambda) v + \lambda \int_{-1}^{+\infty} v(S(1+y)) dN(y),$$

The corresponding free boundary problem is to find $(V(S,\infty), s_\infty)$ such that

$$\begin{cases} \mathscr{L}_\infty v = 0, & S > s_\infty, \\ v(s_\infty) = K - s_\infty, \\ v(\infty) = 0, \\ v'(s_\infty) = -1, \\ v(S) = K - S, S \leqslant s_\infty; \quad v(S) > (K-S)^+, S > s_\infty. \end{cases} \qquad (\overline{P})_\infty$$

Note that the free boundary problem $(\overline{P})_\infty$ has no explicit solution. In Jaiellet *et al.* [9] and elsewhere, the authors obtain such a solution only in some special cases. In order to discuss the regularity of $s(\tau)$ in §5, we need to estimate the lower bound of $s(\tau)$. For this purpose, we consider the following problem for fixed $S_0 \in [0, K]$

$$\begin{cases} L_1 v = \frac{\sigma^2}{2} S^2 \frac{d^2 v}{dS^2} + (r - q - \lambda k) S \frac{dv}{dS} - (r + \lambda) v + \lambda K = 0, & S > S_0, \\ v(S_0) = K - S_0, \\ v(\infty) \in (0, K), \\ v(S) = K - S, & S \leqslant S_0. \end{cases} \tag{4.1}$$

It is easy to find that the solution of (4.1) is

$$V_1(S; S_0) = \left(\frac{\lambda K}{r + \lambda} - S_0 \right) \left(\frac{S}{S_0} \right)^{\alpha_1} + \frac{\lambda K}{r + \lambda},$$

where

$$\alpha_1 = \frac{-r + q + \lambda k + \frac{\sigma^2}{2} - \sqrt{(r - q - \lambda k - \frac{\sigma^2}{2})^2 + 2\sigma^2 (r + \lambda)}}{\sigma^2},$$

$$\alpha_2 = \frac{-r + q + \lambda k + \frac{\sigma^2}{2} + \sqrt{(r - q - \lambda k - \frac{\sigma^2}{2})^2 + 2\sigma^2 (r + \lambda)}}{\sigma^2}$$

Let S_0^* be the maximum point of $V_1(S; S_0)$ in $[0, K]$ as a function of S_0. Then $\frac{dV_1}{dS_0}|_{S_0 = S_0^*} = 0$. Hence

$$S_0^* = \frac{r K \alpha_1}{(r + \lambda)(\alpha_1 - 1)}, \tag{4.2}$$

$$V_1(S; S_0^*) = \frac{r K}{(r + \lambda)(1 - \alpha_1)} \left(\frac{S}{S_0^*} \right)^{\alpha_1} + \frac{\lambda K}{r + \lambda}. \tag{4.3}$$

Notice that

$$V_1(S_0^*; S_0^*) = K - S_0^*,$$
$$\frac{dV_1}{dS}(S; S_0^*)|_{S = S_0^*} = \left(\frac{rK}{(r+\lambda)(1-\alpha_1)} \left(\frac{S}{S_0^*}\right)^{\alpha_1} \frac{\alpha_1}{S} \right) |_{S = S_0^*} = -1.$$

Theorem 4.3 *Assume that* $(V(S, \infty), s_\infty)$ *is a solution to* $(\overline{P})_\infty$. *Then* $V(S, \infty) \leqslant V_1(S; S_0^*)$, $S_0^* \leqslant s_\infty$, *where* S_0^*, $V_1(S; S_0^*)$ *are given by (4.2), (4.3).*

Proof First we have $V_1(S; S_0) < K$ for $S > S_0$. In fact, since $V_1(S_0; S_0) = K - S_0 \leqslant K$, $V_1(\infty; S_0) < K$, if $V_1(S; S_0)$(as a function of S) takes maximum K at some point $S_1 \in (S_0, \infty)$, then $L_1 V_1(S_1; S_0) \leqslant -rK < 0$, which contradicts $L_1 V_1(S_1; S_0) = 0$.

Now, we prove that $V(S; \infty) \leqslant V_1(S; s_\infty)$. In fact, $w(S) = V_1(S; s_\infty) - V(S; \infty)$ satisfies

$$\mathscr{L}_\infty w = \lambda \int_{-1}^{+\infty} [V_1(S(1 + y); s_\infty) - K] dN(y),$$

and $w(\infty) = 0, w(s_\infty) = 0$. Since $V_1(S; S_0) < K$ for $S > S_0$, we have

$$\mathscr{L}_\infty w \leqslant 0 \qquad \text{for} \quad S > s_\infty.$$

If $w(S)$ attains a negative minimum at $S_2 > s_\infty$, then

$$\mathscr{L}_\infty w(S_2) \geqslant -r w(S_2) > 0.$$

The contradiction shows that $w(S) \leqslant 0$, namely, $V(S; \infty) \leqslant V_1(S; s_\infty)$.

By virtue of the definition of S_0^*, we have $V_1(S;s_\infty) \leqslant V_1(S;S_0^*)$ and hence $V(S;\infty) \leqslant V_1(S;S_0^*)$. If $s_\infty < S_0^*$, then since $V(S,\infty) > (K-S)^+$ for $S > s_\infty$ and $V_1(S;S_0^*) = K - S$ for $s_\infty < S \leqslant S_0^*$, we have $V_1(S;S_0^*) < V(S,\infty)$, which is a contradiction. Hence $S_0^* \leqslant s_\infty$. Theroem 4.5 is proved. □

5 Further properties of the free boundary

In this section, we prove further properties of the free boundary $S = s(\tau)$, such as continuity, strict monotonicity and more regularity, etc. Denote

$$\Gamma = \{(x,t)|x = x(t), 0 < t \leqslant T\},$$

where $t = T - \tau, x(t) = \ln s(\tau)$. As before, let $u(x,t)$ and $V(S,\tau)$ be solutions to (1.1) and (P), respectively, and $S = s(\tau)$ be the free boundary defined in Theorem 4.1.

Theorem 5.1 *For any $a < b < \ln K$, $0 < t_1 < t_2 < T$,*

$$\frac{\partial u}{\partial t} \in L^\infty(t_1,t_2;H^1(a,b)), \frac{\partial^2 u}{\partial t^2} \in L^2(t_1,t_2;L^2(a,b)) \tag{5.1}$$

$u_t(x,t)$ is continuous across $x = x(t)$. Hence, $u_t(x,t) \in C((-\infty,+\infty) \times (0,T))$.

Proof (5.1) follows from Lemma 2.12 and the continuity of $u_t(x,t)$ cross $x = x(t)$ follows from Theorem 4.1 and Simon's Compactness theorem. □

Remark 5.1 From Theorem 5.1, we see that $\frac{\partial^2 u}{\partial x^2}$ is unilateral continuous up to $x = x(t)$. Coming back to variables (S,τ), it follows that V_τ is continuous in $R^+ \times [0,T)$, and V_{SS} is unilateral continuous up to the free boundary $S = s(\tau)$.

Theorem 5.2 *$s(\tau)$ is continuous in $[0,T)$.*

Proof First we prove that for any $\tau \in [0,T)$, $s(\tau^+) = s(\tau)$. Let $\{\tau_n\}$ be a sequence such that $\tau_n \to \tau^+$. Then $(s(\tau_n),\tau_n) \in \Sigma_2$, where Σ_2 is the stopping region defined in Remark 4.1. Hence, by the continuity of $V(S,\tau)$, $(s(\tau^+),\tau) \in \Sigma_2$ which implies that $s(\tau^+) \leqslant s(\tau)$. However, since $s(\tau)$ is nondecreasing, $s(\tau_n) \geqslant s(\tau)$, and hence $s(\tau^+) \geqslant s(\tau)$. So $s(\tau^+) = s(\tau)$.

Suppose that $s(\tau)$ is discontinuous. Then there exists $\tau_0 \in [0,T)$ such that $s(\tau_0^+) = s(\tau_0) > s(\tau_0^-)$. Denote $Q = \{(S,\tau)|s(\tau_0^-) < S < s(\tau_0), 0 < \tau < \tau_0\}$. Then $Q \subset \Sigma_1$ where Σ_1 is the continuation region defined in Remark 4.1. Thus

$$\begin{cases} \mathscr{L}V = 0, & (S,\tau) \in Q, \\ V(S,\tau_0) = K - S, & s(\tau_0^-) < S < s(\tau_0). \end{cases}$$

From this, one can verify, by using Theorem 3.3 and Theorem 5.1 and noting $k = \int_{-1}^{+\infty} y dN(y)$, that for any $s(\tau_0^-) < S < s(\tau_0)$,

$$\begin{aligned} V_\tau(S,\tau_0) &= rK - qS - \lambda \int_{\frac{s(\tau_0)}{S}-1}^{+\infty} [V(S(1+y),\tau_0) - (K - S(1+y))] dN(y) \\ &= 0. \end{aligned} \tag{5.2}$$

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Denote $f(S) = qS - rK + \lambda \int_{\frac{s(\tau_0)}{S}-1}^{+\infty} [V(S(1+y), \tau_0) - (K - S(1+y))] dN(y)$. Clearly, $f(0) = -rK < 0$, $f(+\infty) = +\infty$. In addition, since $\frac{\partial V}{\partial S} \geqslant -1$(see Remark 4.2), we have $f'(S) \geqslant q > 0$. So there is a unique zero point of $f(S)$ in $(0, +\infty)$, which contradicts (5.2). Thus Theorem 5.2 is proved. □

Define $s(T) = \lim_{\tau \to T^-} s(\tau)$. Then $s(\tau)$ is continuous in $[0, T]$.

Theorem 5.3 *$s(T)$ at $\tau = T$ is given by*

$$s(T) = \min\{K, S_0\} = \begin{cases} K, & \text{if} \quad r \geqslant q + \lambda \int_0^{+\infty} y dN(y); \\ S_0, & \text{if} \quad r \leqslant q + \lambda \int_0^{+\infty} y dN(y), \end{cases}$$

where S_0 is the unique solution of the following equation:

$$f(S) = qS - rK + \lambda \int_{\frac{K}{S}-1}^{+\infty} [S(1+y) - K] dN(y) = 0.$$

Proof Since $s(\tau) < K$, for $\tau \in [0, T)$, we have $s(T) \leqslant K$.

We consider the following cases seperately:

(1) $S_0 \geqslant K$.

Suppose $s(T) < K$. Then for small $\delta > 0$, $D_1^{(\delta)} = \{(S, \tau) | s(T) < S < K, T - \delta < \tau < T\} \subset \Sigma_1$. Noting that $\mathscr{L} V = 0$ in $D_1^{(\delta)}$ and $V(S, \tau) = (K - S)^+ = 0$ for $S \geqslant K$, which implies that $V(S(1+y), T) = 0$ for $y \geqslant \frac{K}{S} - 1$, we have

$$\frac{\partial V}{\partial \tau}|_{\tau=T} = rK - qS - \lambda \int_{\frac{K}{S}-1}^{+\infty} (S(1+y) - K) dN(y) = -f(S).$$

Since $f(0) = -rK < 0, f(+\infty) = +\infty$, $f'(S) \geqslant q > 0$, $f(S)$ is strictly increasing in $[0, +\infty)$ and has a unique zero point S_0. So $S_0 \geqslant K$ implies $f(K) \leqslant f(S_0) = 0$. However, for $s(\tau) < S < K$, $f(S) < f(K) \leqslant 0$ and hence $\frac{\partial V}{\partial \tau}(S, T) > 0$. Thus $V(S, \tau) < V(S, T) = (K - S)^+$ in $D_1^{(\delta)}$, which contradicts $V(S, \tau) \geqslant K - S$. Hence $s(T) = K$.

(2) $S_0 < K$.

If $s(T) > S_0$, then for small $\delta > 0$, $D_2^{(\delta)} = \{(S, \tau) | S_0 < S < s(\tau),\ T - \delta < \tau < T\} \subset \Sigma_2$, in which $V(S, \tau) = K - S$, $\mathscr{L} V \leqslant 0$. On the other hand, in $D_2^{(\delta)}$, we have

$$\begin{aligned} \mathscr{L} V(S, \tau) &= qS - rK + \lambda \textstyle\int_{-1}^{+\infty} [V(S(1+y), \tau) - (K - S(1+y))] dN(y) \\ &\geqslant qS - rK + \lambda \textstyle\int_{-1}^{+\infty} [(K - S(1+y))^+ - (K - S(1+y))] dN(y) \\ &= qS - rK + \lambda \textstyle\int_{\frac{K}{S}-1}^{+\infty} [(K - S(1+y))^+ - (K - S(1+y))] dN(y) \\ &= qS - rK + \lambda \textstyle\int_{\frac{K}{S}-1}^{+\infty} (S(1+y) - K) dN(y) = f(S). \end{aligned}$$

Since $f(S)$ is strictly increasing, we have $f(S) > f(S_0) = 0$ and hence $\mathscr{L} V(S, \tau) \geqslant f(S) > 0$, which contradicts $\mathscr{L} V \leqslant 0$. This shows that $s(T) > S_0$ is impossible.

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If $s(T) < S_0$, then a similar argument gives the existence of a region $D_3^{(\delta)} = \{(S,\tau)|s(\tau) < S < S_0, T-\delta < \tau < T\} \subset \Sigma_1$, in which $\frac{\partial V}{\partial \tau}|_{\tau=T} = -f(S)$. Since for $S < S_0$, $f(S) < f(S_0) = 0$, we have $\frac{\partial V}{\partial \tau}|_{\tau=T} > 0$ and hence $V(S,\tau) < V(S,T) = K - S$ in $D_3^{(\delta)}$, which is a contradiction.

Summing up, we conclude that $s(T) = S_0$ when $S_0 < K$. □

Remark 5.2 From Theorem 4.5 and Theorem 5.2, we have $s(\tau) \in (S_0^*, s(T)]$, for any $\tau \in [0,T]$, where $S_0^* > 0$ is given in (4.2).

Theorem 5.4 *$s(\tau)$ is strictly increasing in* $[0,T)$.

Proof We prove the conclusion by contradiction. Suppose that there exists $0 \leqslant t_1 < t_2 < T$ such that $s(\tau) = s(t_1)$ for $\tau \in [t_1,t_2]$. Choose $\tau_0 \in (t_1,t_2)$ and make a small enough disc $D \subset (s(t_1),+\infty) \times (t_1,t_2) \subset \Sigma_1$ tangent with $S = s(\tau)$ at $(s(\tau_0),\tau_0)$.

Let $w = V_\tau$. Then $\mathscr{L}w = 0$ in D. From Proposition 4.3, we have $w(S,\tau) < 0$ in D. However, by Theorem 5.1, $w(s(\tau_0),\tau_0) = 0$. So w achieves its maximum on $\overline{D}$ at $(s(\tau_0),\tau_0)$ and hence by Lemma 3.6, we obtain $\frac{\partial w}{\partial S}(s(\tau_0),\tau_0) > 0$. On the other hand, $\frac{\partial V}{\partial S}(s(\tau),\tau) = -1$, $s(\tau) = s(t_1)$ for $\tau \in (t_1,t_2)$, so $\frac{\partial w}{\partial S}(s(\tau),\tau) = 0$ for $\tau \in (t_1,t_2)$. In particular, $\frac{\partial w}{\partial S}(s(\tau_0),\tau_0) = 0$. This contradiction shows that $s(\tau)$ is strictly increasing. □

Theorem 5.5 *Assume that*

$$r \geqslant q + \lambda \int_0^{+\infty} y dN(y). \qquad (A3)$$

Then $s(\tau) \in C^{\frac{3}{4}}([0,T))$.

Proof Theorem 5.4 with condition (A3) implies $s(T) = K$. Since $s(\tau)$ is strictly increasing, we have $s(\tau) < K$, for any $\tau < T$. For any $\delta \in (0,T)$, there exists $\gamma > 0$ such that $s(T) - s(\tau) > \gamma$ for $\tau \in [0, T-\delta]$. Since $V_\tau < 0$ and $V_S < 0$, it is easy to verify that

$$\begin{aligned}
\frac{\sigma^2}{2}s^2(\tau)V_{SS}(s(\tau),\tau) &\geqslant rK - qs(\tau) - \lambda \int_0^{+\infty} [V(s(\tau)(1+y),\tau) - (K - s(\tau)(1+y))]dN(y) \\
&\geqslant rK - qs(\tau) - \lambda s(\tau) \int_0^{+\infty} y dN(y) \\
&= rs(T) - qs(\tau) - \lambda s(\tau) \int_0^{+\infty} y dN(y) \\
&> s(\tau)(r - q - \lambda \int_0^{+\infty} y dN(y)) + \gamma r \\
&\geqslant \gamma r > 0.
\end{aligned}$$

Here we have used the condition (A3) again. This combined with Proposition 4.3 yields $V_{SS}(S,\tau) > 0$ for $S \geqslant s(\tau)$, $\tau \in [0,T)$.

Given $\delta \in (0,T)$. Since $V_S(s(\tau),\tau) = -1$, we have $V_S(s(\tau+\eta),\tau+\eta) = V_S(s(\tau),\tau)$ and for any $\zeta > 0$, by the continuity of V_S, there exists $\epsilon > 0$, such that

$$|V_S(s(\tau+\eta)+\epsilon,\tau+\eta) - V_S(s(\tau)+\epsilon,\tau)| < \zeta$$

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and hence

$$\int_{s(\tau)+\epsilon}^{s(\tau+\eta)+\epsilon} V_{SS}(S,\tau+\eta)dS \leqslant \zeta - \int_{\tau}^{\tau+\eta} V_{S\tau}(s(\tau)+\epsilon,\sigma)d\sigma. \tag{5.3}$$

Let $\eta \in (0, \frac{\delta}{2})$. Then, since for $S \geqslant s(\tau)$, $\tau \in [0,T)$, V_{SS} is continuous and $V_{SS} > 0$, there exists $\rho > 0$, such that

$$\int_{s(\tau)+\epsilon}^{s(\tau+\eta)+\epsilon} V_{SS}(S,\tau+\eta)dS \geqslant \rho(s(\tau+\eta)-s(\tau)), \qquad \text{for} \tau \in [0, T-\delta]. \tag{5.4}$$

Now we combine (5.4) with (5.3) and use

$$V_{S\tau}(s(\tau)+\epsilon,\sigma) = V_{S\tau}(s(\tau)+\epsilon+x,\sigma) - \int_{s(\tau)+\epsilon}^{s(\tau)+\epsilon+x} V_{SS\tau}(S,\sigma)dS$$

to estimate $\int_{\tau}^{\tau+\eta} V_{S\tau}(s(\tau)+\epsilon,\sigma)d\sigma$. Then we obtain

$$\rho(s(\tau+\eta)-s(\tau)) \leqslant \zeta + \int_{\tau}^{\tau+\eta} |V_{S\tau}(s(\tau)+\epsilon+x,\sigma)|d\sigma + \int_{\tau}^{\tau+\eta}\int_{s(\tau)+\epsilon}^{s(\tau)+\epsilon+x} |V_{SS\tau}(S,\sigma)|dSd\sigma.$$

Integrating the above inequality over $[0,m]$ with respect to x and gives

$$\begin{aligned} &m\rho(s(\tau+\eta)-s(\tau)) \\ &\quad\leqslant m\zeta + \textstyle\int_{\tau}^{\tau+\eta}\int_0^m |V_{S\tau}(s(\tau)+\epsilon+x,\sigma)|dxd\sigma + \int_{\tau}^{\tau+\eta}\int_0^m\int_{s(\tau)+\epsilon}^{s(\tau)+\epsilon+x} |V_{SS\tau}(S,\sigma)|dSdxd\sigma \\ &\quad\leqslant m\zeta + \eta^{\frac{1}{2}}m^{\frac{1}{2}}(\textstyle\int_{\tau}^{\tau+\eta}\int_0^m |V_{S\tau}(s(\tau)+\epsilon+x,\sigma)|^2dxd\sigma)^{\frac{1}{2}} \\ &\qquad + \eta^{\frac{1}{2}}m(\textstyle\int_{\tau}^{\tau+\eta}\int_0^m\int_{s(\tau)+\epsilon}^{s(\tau)+\epsilon+x} |V_{SS\tau}(S,\sigma)|^2dSdxd\sigma)^{\frac{1}{2}}. \end{aligned}$$

Using Theorem 5.1 and Remark 5.1 on the integral of the right side, we then derive

$$m\rho(s(\tau+\eta)-s(\tau)) \leqslant m\zeta + C\eta^{\frac{3}{2}}m^{\frac{1}{2}} + C\eta^{\frac{1}{2}}m^2, \qquad \tau \in (0, T-\delta),$$

where the constant C is independent of small ϵ. Because of the arbitrariness of ζ, this implies

$$m\rho(s(\tau+\eta)-s(\tau)) \leqslant C\eta^{\frac{3}{2}}m^{\frac{1}{2}} + C\eta^{\frac{1}{2}}m^2,$$

namely,

$$s(\tau+\eta)-s(\tau) \leqslant C\eta^{\frac{3}{2}}m^{-\frac{1}{2}} + C\eta^{\frac{1}{2}}m.$$

Choose $m = \eta^{\frac{3}{2}}$. Then we obtain

$$s(\tau+\eta)-s(\tau) \leqslant C\eta^{\frac{3}{4}}, \qquad \tau \in (0, T-\delta)$$

Thus $s(\tau) \in C^{\frac{3}{4}}([0,T))$. □

Remark 5.3 The condition (A3) means that the riskless interest rate corrected by dividends and the jump risk, $\tilde{r} = r - q - \lambda\int_0^{+\infty} ydN(y)$, is nonnegative.

On the basis of the fact $s(\tau) \in C^{\frac{3}{4}}([0,T))$, we may further improve the regularity of $s(\tau)$.

Theorem 5.6 *Under the condition (A3),* $s(\tau) \in C^1([0, T))$.

Proof The crucial step is to prove that $V_{S\tau}$ is continuous in Σ_1 up to $S = s(\tau)$. To this purpose, we need to apply a result in Cannon *et al.* [4] to V_τ. It is to be noted that this result proved in Cannon *et al.* [4] for the heat equation is valid for more general parabolic equations and hence it can be applied to the present case.

Given $\delta \in (0, T)$. Since for $S \geqslant s(\tau)$, $\tau \in [0, T-\delta]$, V_{SS} is continuous and $V_{SS} > 0$, there exists $\rho > 0$, such that, for $\eta \in (0, \frac{\delta}{2})$,

$$\rho(s(\tau+\eta) - s(\tau)) \leqslant \int_{s(\tau)}^{s(\tau+\eta)} V_{SS}(S, \tau+\eta)dS = -\int_{\tau}^{\tau+\eta} V_{S\tau}(s(\tau), \sigma)d\sigma, \tag{5.5}$$

where the second equality follows from $V_S(s(\tau), \tau) = V_S(s(\tau+\eta), \tau+\eta) = -1$ and

$$0 = V_S(s(\tau+\eta), \tau+\eta) - V_S(s(\tau), \tau) = \int_{s(\tau)}^{s(\tau+\eta)} V_{SS}(S, \tau+\eta)dS + \int_{\tau}^{\tau+\eta} V_{S\tau}(s(\tau), \sigma)d\sigma.$$

Write (5.5) as

$$\frac{s(\tau+\eta) - s(\tau)}{\eta} \leqslant -\frac{1}{\rho\eta}\int_{\tau}^{\tau+\eta} V_{S\tau}(s(\tau), \sigma)d\sigma$$

and let $\eta \to 0^+$. Then we see that $s'(\tau)$ exists.

Differentiating $V_S(s(\tau), \tau) = -1$ gives $\frac{\partial V_S(s(\tau),\tau)}{\partial \tau} = V_{SS}(s(\tau), \tau)s'(\tau) + V_{S\tau}(s(\tau), \tau) = 0$. Hence

$$s'(\tau) = -\frac{V_{S\tau}(s(\tau), \tau)}{V_{SS}(s(\tau), \tau)}, \tag{5.6}$$

where

$$V_{SS}(s(\tau), \tau) = \frac{2}{\sigma^2 s^2(\tau)}[rK - qs(\tau) - \lambda\int_0^{+\infty}[V(s(\tau)(1+y), \tau) - (K - s(\tau)(1+y))]dN(y)].$$

Continuity of $s'(\tau)$ then follows from (5.6). □

Theorem 5.7 *Under the condition (A3),* $s(\tau) \in C^\infty([0, T))$.

Proof Let $z = S - s(\tau)$. Then the free boundary $S = s(\tau)$ changes into the fixed boundary $z = 0$. A simple calculation shows that $w(z, \tau) = V_\tau(z + s(\tau), \tau)$ satisfies, for $z > 0, 0 \leqslant \tau < T - \delta$,

$$\begin{aligned} &\tfrac{\partial w}{\partial \tau} + \tfrac{\sigma^2}{2}(z + s(\tau))^2\tfrac{\partial^2 w}{\partial z^2} + ((r - q - \lambda k)(z + s(\tau)) + s'(\tau))\tfrac{\partial w}{\partial z} \\ &\quad - (r+\lambda)w + \lambda\textstyle\int_{-1}^{\infty} w((z + s(\tau))(1+y), \tau)dN(y) = 0 \end{aligned} \tag{5.7}$$

and

$$w(0, \tau) = 0, \tag{5.8}$$

$$w(z, T-\delta) = h(z), \tag{5.9}$$

where $\delta \in (0, T)$ and $h(z) = V_\tau(z + s(T-\delta), T-\delta) \in C^1([0, +\infty))$, namely, w is the solution of the problem (5.7), (5.8), (5.9) in $(0, +\infty) \times (0, T-\delta)$ for any $\delta \in (0, T)$. we

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may regard $h(z)$ and $f(z,\tau) = -\lambda \int_{-1}^{\infty} w((z+s(\tau))(1+y),\tau)dN(y)$ as known functions and use the theory of parabolic equations to this problem to improve the regularity of $V(S,\tau)$ and hence, of $s(\tau)$ by (5.6). Once the regularity of $s(\tau)$ is improved, we may use the theory of parabolic equations again to further improve the regularity of $V(S,\tau)$. Repeating this argument, we finally conclude that $s(\tau) \in C^{\infty}([0,T-\delta))$ and hence $s(\tau) \in C^{\infty}([0,T))$ by the arbitrariness of $\delta \in (0,T)$. □

6 Conclusions

The pricing and hedging of derivative securities is a subject of much practical importance. As one basic type of derivatives, options have been around for many years, but it was only on 26th Apirl 1973 that they were first traded on an exchange. It was then that The Chicago Board Options Exchange (CBOE) first created standardized, listed options. Initially there were just calls on sixteen stocks and even no puts. Now worldwide, many kinds of exotic options are traded in over fifty exchange except for the standard option.

To enter into an option contract, there is cost referred to as premium, corresponding to the right purchased. As is known to all, the theory of arbitrage-free pricing establishes the option price. This theory imposes that the prices of different instruments must be related to one another in such a way that they offer no arbitrage opportunities. In practice to price the option we make use of a model describing the evolution through time of the underlying asset price and then impose no arbitrage arguments.

The risk associated with an option contract derives from the unknown evolution of the underlying asset price on the market. This risk is not reducible and is an intrinsic feature of the contract itself. Apart from this risk, neither controllable nor reducible, there is another part of risk which derives from the fact that the option price is an estimated quantity, potentially affected by an error, such as an error stemed from the evolution model of the underlying asset price. If for instance the call option price is overestimated, the option holder faces the risk of losing more money than what he should (in case of loss). Clearly, the more accurate the price estimate, the less the risk associated with the option.

Generally, the option price is calculated via a mathematical model (describing the evolution of the underlying asset) that contains a number of input variables whose values are affected by uncertainty. In this paper, we assume that the interest rate, the dividend yield and the volatility of the underlying are constants. Then in the standard option pricing model, the evolution through time of the underlying asset is described by the Brownian motion, i.e., the underlying asset price follows the lognormal random walk, that is, the path of the asset price is continuous in time. But there is plenty of evidence that such as currencies and equities do not follow the Brownian motion. One of the striking features of real financial market is that there is a sudden unexpected fall or crash inevitably. In this case, a jump process with a reasonable volatility is always added in the evolution model of the asset price. On all but the shortest timescales the movement looks discontinuous, that is, the prices of the asset have jumped. This is important for the theory and practice of options because it is usually impossible to hedge through the crash. So there is seldom a closed-form solution for the European option with jumps. At the same time, results show that jumps drive most of the uncertainty in the estimated option price, thus confirming their key role in the pricing process. The important of jumps is more evident for higher strike price. In addition, the pricing problem of American options is more complex than one of European options in virtue of the early exercise. In this paper, we have strictly proved some properties of the pricing function and the optimal exercise boundary of American options with dividend in a jump-diffusion model by using PDE arguments in mathematically. The main difficulty, compared with diffusion models without jumps, comes from the nonlocal term due to the presence of jump uncertainty in the stock price dynamics.

Reference

[1] Aase, K. K. (1988), Contingent claims valuation when the security price is a combination of an Itô process and a random point process. *Sochastic Process. Appl.* **28**, 185–220.

[2] Bensoussan, A. & Lions, J. L. (1978) *Applications des Iné quations Variationnelles en Contrôle stochastique*. Dunod, Paris.

[3] Black, F. & Scholes, M. (1973), The pricing of options and corporate liabilities. *J. Political Econ.* **81**, 635–654.

[4] Cannon, J. R., Henry, D. B. & Kotlow, D. B. (1974) Continuous differentiability of the free boundary for weak solutions of the Stefan problem. *Bull. Amer. Math. Soc.* **80**, 45–48.

[5] Cox, J. C. & Ross, S. A. (1976) The valuation of options for alternative stochastic processes. *J. Finance*, **3**, 145=-166.

[6] Friedman, A. V. (1964) *Partial Differential Equations of Parabolic Type*. Prentice-Hall.

[7] Friedman, A. V. (1975) Parabolic variational inequalities in one space dimension and smoothness of the free boundary. *J. Funct. Anal.* **18**, 151–176.

[8] Friedman, A. V. (1982) *Variational Principles and Free-boundary Problems*. Wiley Interscience.

[9] Jaillet, P., Lamberton, D. & Lapeyre, B. (1990) Variational inequalities and the pricing of American options. *Acta. Appl. Math.* **21**, 263–289.

[10] Jiang, L. (2003) *Mathematical Modeling and Methods of Option Pricing*. Beijing.

[11] McKean, H. P. Jr. (1965) Appendix: a free boundary problem for the heat equation arising from a problem in mathematical economics. *Indust. Manage. Rev.* **6**, 32–39.

[12] Merton, R. C. (1976) Option pricing when underlying stock returns are discontinuous. *J. Financial Econ.* **3**, 125–144.

[13] Naik, V. & Lee, M. (1990), General equilibrium pricing of options on the market portfolio with discontinuous returns. *Rev. Finan. Stud.* **3**, 493–521.

[14] Pham, H. (1995) Optimal stopping, free boundary and American option in a jump-diffusion model. *Appl. Math. Opt.* **35**, 145–164.

[15] Van Moerbeke, P. (1976) On optimal stopping and free boundary problem. *Arch. Rat. Mech. Anal.* **60**, 101–148.

[16] Willmott, P. (1999) *Derivatives: The theory and practice of financial engineering*. Wiley.

[17] Zhang, X. (1994) Analyse Numérique des Options Américaines dans un Modéle de Diffusion avec des Sauts *Doctoral dissertation*, Ecole Nationale des Ponts et Chaussées.