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Positive steady states of the Holling–Tanner prey–predator model with diffusion

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This paper is concerned with the Holling–Tanner prey–predator model with diffusion subject to the homogeneous Neumann boundary condition. We obtain the existence and non-existence of positive non-constant steady states.

1. Introduction

From last century, all kinds of biological models have received extensive concerns, and, in particular, the prey–predator models have been of great interest to both applied mathematicians and ecologists. Since the classical Lotka–Volterra models have the unavoidable limitation to precisely describe many realistic phenomena in biology, in some cases, they should make way to more sophisticated models from both a mathematical and biological point of view. Robert May developed a model in which he incorporated Holling's rate [8, 9]. This model, also known as the Holling–Tanner prey–predator model [21], has been studied both for its mathematical properties and its efficacy for describing real ecological systems such as mite and spider mite, lynx and hare, sparrow and sparrow hawk, etc., by Tanner [22] and Wollkind *et al*. [27].

The May or Holling–Tanner prey–predator model is

$$\left.\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}t} &= r_1 u\left(1-\frac{u}{k}\right)-\frac{quv}{m+u},\\ \frac{\mathrm{d}v}{\mathrm{d}t} &= r_2 v\left(1-\frac{v}{\gamma u}\right). \end{aligned}\right\} \tag{1.1}$$

Here, $u(t)$ and $v(t)$, respectively, represent the concentrations of the prey and predator. The parameters r_1 and r_2 are the respective intrinsic growth rates. The constant k is the carrying capacity of the prey and γ takes on the role of the prey-dependent carrying capacity for the predator. The rate at which the predator consumes the prey, $quv/(m+u)$, is known as the Holling type-II functional response [9, 10, 17, 21]. The parameter q is the maximum number of the prey that can be eaten per predator per time, and m is the saturation value that corresponds to the number of the prey necessary to achieve one half of the maximum rate q.

The dynamics of the Holling–Tanner prey–predator model have proven quite interesting and received intensive study. So far, this ordinary differential equation (ODE) model has been investigated by many authors in either qualitative or numerical analysis, and lots of interesting phenomena, such as stable limit cycles, semi-stable limit cycles, bifurcation, global stability of the unique constant positive steady-state, periodic solutions, have been uncovered (we refer interested readers to [1, 4, 9, 10, 16, 17, 27]).

In the case that the concentrations of the prey and predator are spatially inhomogeneous, and taking into account the effect of diffusion, instead of ODE (1.1), we need to consider the following reaction-diffusion system,

$$\left.\begin{aligned} \frac{\partial u}{\partial t} - d_1\Delta u &= r_1 u\left(1-\frac{u}{k}\right) - \frac{quv}{m+u}, && x\in\Omega, \quad t>0,\\ \frac{\partial v}{\partial t} - d_2\Delta v &= r_2 v\left(1-\frac{v}{\gamma u}\right), && x\in\Omega, \quad t>0,\\ \partial_\eta u &= \partial_\eta v = 0, && x\in\partial\Omega, \quad t>0, \end{aligned}\right\} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$, η is the outward unit normal vector on $\partial\Omega$ and $\partial_\eta = \partial/\partial\eta$. d_i $(i = 1, 2)$ is the diffusion coefficient corresponding to u and v, and all the parameters appearing in model (1.2) are assumed to be positive constants. The homogeneous Neumann boundary condition means that model (1.2) is self-contained and has no population flux across the boundary $\partial\Omega$.

Clearly, choosing some kind of scaling, model (1.2) can take the form:

$$\left.\begin{aligned} \frac{\partial u}{\partial t} - d_1\Delta u &= au - u^2 - \frac{quv}{m+u}, && x\in\Omega, \quad t>0,\\ \frac{\partial v}{\partial t} - d_2\Delta v &= bv - \frac{v^2}{\gamma u}, && x\in\Omega, \quad t>0,\\ \partial_\eta u &= \partial_\eta v = 0, && x\in\partial\Omega, \quad t>0, \end{aligned}\right\} \tag{1.3}$$

where d_1, d_2, a, b, q, m and γ are positive constants.

It is obvious that non-negative solutions of model (1.3) are of real interest. Since the asymptotical behaviours of non-negative solutions of model (1.3) are closely related to the non-negative steady states of model (1.3), in the present paper, we shall study the positive steady states of model (1.3) that satisfy the following (for simplicity, we take $q = 1$):

$$\left.\begin{aligned} -d_1\Delta u &= au - u^2 - \frac{uv}{m+u} && \text{in } \Omega, \quad \partial_\eta u = 0 \quad \text{on } \partial\Omega,\\ -d_2\Delta v &= bv - \frac{v^2}{\gamma u} && \text{in } \Omega, \quad \partial_\eta v = 0 \quad \text{on } \partial\Omega. \end{aligned}\right\} \tag{1.4}$$

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The positive solution (u, v) satisfying (1.4) refers to a classical one with $u > 0$, $v > 0$ on $\bar{\Omega}$. Clearly, the system (1.4) has two trivial non-negative solutions, namely, the semi-trivial constant solution $(a, 0)$ and the unique positive constant solution $(u, v) = (u^*, v^*)$, where

$$u^* = \tfrac{1}{2}\{a - m - b\gamma + \sqrt{(a - m - b\gamma)^2 + 4am}\} \quad \text{and} \quad v^* = b\gamma u^*.$$

The organization of our paper is as follows. In § 2, we first give some preliminary results, including the *a priori* estimates of upper and lower bounds. In § 3, we obtain the non-existence of positive non-constant solutions, while § 4 is devoted to the existence of positive non-constant solutions.

The role of diffusion in the modelling of many physical, chemical and biological processes has been extensively studied. Starting with Turing's seminal 1952 paper [23], diffusion has been observed as causes of the spontaneous emergence of ordered structures, called patterns, in a variety of non-equilibrium situations. These include the Gierer–Meinhardt model [11, 26], the Sel'kov model [5, 25], the Brusselator model [2], the chemotactic diffusion model [24], the Lotka–Volterra competition model [14, 15], the Lotka–Volterra predator–prey model [6, 7, 12, 19, 20], as well as models of semiconductors, plasmas, chemical waves, combustion systems, embryogenesis, etc. (see, for example, [3] and the references therein).

2. Preliminary results

We first state a lemma that is due to Lou and Ni [14].

LEMMA 2.1 (maximum principle). *Suppose that* $g \in C(\bar{\Omega} \times \mathbb{R}^1)$.

(i) *Assume that* $w \in C^2(\Omega) \cap C^1(\bar{\Omega})$ *and satisfies*

$$\Delta w(x) + g(x, w(x)) \geqslant 0 \quad \text{in } \Omega, \qquad \partial_\eta w \leqslant 0 \quad \text{on } \partial\Omega.$$

If $w(x_0) = \max_{\bar{\Omega}} w$, *then* $g(x_0, w(x_0)) \geqslant 0$.

(ii) *Assume that* $w \in C^2(\Omega) \cap C^1(\bar{\Omega})$ *and satisfies*

$$\Delta w(x) + g(x, w(x)) \leqslant 0 \quad \text{in } \Omega, \qquad \partial_\eta w \geqslant 0 \quad \text{on } \partial\Omega.$$

If $w(x_0) = \min_{\bar{\Omega}} w$, *then* $g(x_0, w(x_0)) \leqslant 0$.

Assume that (u, v) is a positive solution of (1.4) and set

$$u(x_1) = \max_{\bar{\Omega}} u, v(x_2) = \max_{\bar{\Omega}} v, \quad u(y_1) = \min_{\bar{\Omega}} u \quad \text{and} \quad v(y_2) = \min_{\bar{\Omega}} v.$$

Applying lemma 2.1 to (1.4), we obtain that

$$a - u(x_1) - \frac{v(x_1)}{m + u(x_1)} \geqslant 0, \qquad a - u(y_1) - \frac{v(y_1)}{m + u(y_1)} \leqslant 0, \tag{2.1}$$

$$b - \frac{v(x_2)}{\gamma u(x_2)} \geqslant 0, \qquad b - \frac{v(y_2)}{\gamma u(y_2)} \leqslant 0. \tag{2.2}$$

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By virtue of the definitions of x_i and y_i ($i = 1, 2$), it follows from (2.1) and (2.2) that $u(x_1) < a$ and

$$u^2(x_1) - (a - m)u(x_1) + b\gamma u(y_1) - am \leqslant 0, \tag{2.3}$$

$$u^2(y_1) - (a - m)u(y_1) + b\gamma u(x_1) - am \geqslant 0, \tag{2.4}$$

$$v(x_2) \leqslant b\gamma u(x_1) < ab\gamma, \qquad v(y_2) \geqslant b\gamma u(y_1). \tag{2.5}$$

Furthermore, inequality (2.4) implies that

$$-u^2(y_1) + (a - m)u(y_1) + am \leqslant b\gamma u(x_1) < ab\gamma,$$

i.e.

$$u^2(y_1) - (a - m)u(y_1) + ab\gamma - am > 0. \tag{2.6}$$

If $m > b\gamma$ or $m = b\gamma$ and $a > b\gamma$, then, from (2.6), we get that

$$u(y_1) > \tfrac{1}{2}\{a - m + \sqrt{(a + m)^2 - 4ab\gamma}\} \equiv B. \tag{2.7}$$

Combing (2.3) with (2.7), we deduce that

$$u(x_1) \leqslant \tfrac{1}{2}\{a - m + \sqrt{(a + m)^2 - 4b\gamma u(y_1)}\} < A, \tag{2.8}$$

where

$$A = \tfrac{1}{2}\Big\{a - m + \sqrt{(a + m)^2 - 2b\gamma[a - m + \sqrt{(a + m)^2 - 4ab\gamma}]}\Big\}.$$

From (2.7), (2.8) and (2.5), we can claim the following a priori estimates.

THEOREM 2.2. *Assume that $m > b\gamma$ or $m = b\gamma$ and $a > b\gamma$. Then every positive solution (u, v) of (1.4) satisfies*

$$B < u(x) < A \quad \textit{and} \quad b\gamma B < v(x) < b\gamma A \quad \forall x \in \bar{\Omega}.$$

REMARK 2.3. From the definitions of A and B, we note that $A < a$ and, moreover, $A \to a$ and $B \to a$ as $m \to \infty$. Naturally, this fact motivates us to obtain the non-existence of positive non-constant solutions of (1.4) when m is large enough. In fact, theorem 3.5 (i) in § 3 will tell us that (1.4) has no positive non-constant solution when m is large sufficiently. Consequently, this information indicates that, to some extent, the above *a priori* estimates is optimal provided that either $m > b\gamma$ or $m = b\gamma$ and $a > b\gamma$ hold.

REMARK 2.4. From (2.4) and $u(x_1) \geqslant u(y_1)$, we easily see that, for every positive solution (u, v) of (1.4), there is a positive constant $C_1 = C_1(a, b, m, \gamma)$ such that $\max_{\bar{\Omega}} u \geqslant C_1$.

For the general case, we can state the following result of *a priori* estimates. First, we need a lemma, which comes from [13].

LEMMA 2.5 (Harnack inequality). *Let $w \in C^2(\Omega) \cap C^1(\bar{\Omega})$ be a positive solution to $\Delta w(x) + c(x)w(x) = 0$, where $c \in C(\bar{\Omega})$, satisfying the homogeneous Neumann boundary condition. Then there exists a positive constant C^*, depending only on $\max_{\bar{\Omega}} |c(x)|$ and Ω, such that*

$$\max_{\bar{\Omega}} w \leqslant C^* \min_{\bar{\Omega}} w.$$

Theorem 2.6. *Let d be an arbitrary fixed positive number. Then there exists a positive constant $C(a,b,m,\gamma,d,\Omega)$ such that if $d_1 \geqslant d$, $d_2 > 0$, every positive solution (u,v) of (1.4) satisfies*

$$C(a,b,m,\gamma,d,\Omega) < u(x) < a \quad \text{and} \quad C(a,b,m,\gamma,d,\Omega) < v(x) < ab\gamma \quad \forall x \in \bar{\Omega}.$$

Proof. We first note that $u(x) < a$ and $v(x) < ab\gamma$ from the proof of theorem 2.1. It suffices to verify the lower bounds of (u,v). We shall prove by contradiction.

Suppose that theorem 2.2 does not hold. Then there exists a sequence

$$\{(d_{1,i}, d_{2,i})\}_{i=1}^{\infty},$$

with $d_{1,i} \geqslant d$, $d_{2,i} > 0$ and the positive solution (u_i, v_i) of (1.4) corresponding to $(d_1, d_2) = (d_{1,i}, d_{2,i})$, such that

$$\min_{\bar{\Omega}} u_i(x) \to 0 \quad \text{or} \quad \min_{\bar{\Omega}} v_i(x) \to 0 \quad \text{as } i \to \infty, \tag{2.9}$$

and (u_i, v_i) satisfies

$$\left.\begin{aligned} -d_{1,i}\Delta u_i &= au_i - u_i^2 - \frac{u_i v_i}{m+u_i} && \text{in } \Omega, \qquad \partial_\eta u_i = 0 \quad \text{on } \partial\Omega, \\ -d_{2,i}\Delta v_i &= bv_i - \frac{v_i^2}{\gamma u_i} && \text{in } \Omega, \qquad \partial_\eta v_i = 0 \quad \text{on } \partial\Omega. \end{aligned}\right\} \tag{2.10}$$

We observe that lemma 2.1 guarantees

$$b\gamma \min_{\bar{\Omega}} u_i \leqslant \min_{\bar{\Omega}} v_i \leqslant \max_{\bar{\Omega}} v_i(x) \leqslant b\gamma \max_{\bar{\Omega}} u_i(x) \tag{2.11}$$

for all $i \geqslant 1$ by use of the second equation in (2.10). On the other hand, applying lemma 2.5 and remark 2.4 to the first equation in (2.10), we see that there are positive constants $C_1(a,b,m,\gamma), C(a,b,m,\gamma,d,\Omega)$ such that

$$C_1(a,b,m,\gamma) \leqslant \max_{\bar{\Omega}} u_i(x) \leqslant C(a,b,m,\gamma,d,\Omega) \min_{\bar{\Omega}} u_i(x) \tag{2.12}$$

for all $i \geqslant 1$. Therefore, a contradiction occurs by (2.9), (2.11) and (2.12), and the proof is complete. □

3. Non-existence of positive non-constant solutions

In this section, we present the results of non-existence of non-trivial solutions. The techniques used here are the *implicit function theorem* and the energy method.

First, we can use the energy method to obtain the following results of non-existence of positive non-constant solutions of (1.4). Let $0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$ be all the eigenvalues of the operator $-\Delta$ subject to the homogeneous Neumann condition.

Theorem 3.1.

(i) *Let ϵ_1 be an arbitrary positive constant. Then there exists*

$$D_1^* = D_1^*(a,b,m,\gamma,\epsilon_1,\Omega) > 0$$

such that (1.4) has no positive non-constant solution provided that $d_1 > D_1^*$ and $\lambda_1 d_2 > b + \epsilon_1$.

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(ii) *Let ϵ_2 be an arbitrary positive constant. Then there exists*

$$D_2^* = D_2^*(a, b, m, \gamma, \epsilon_2, \Omega) > 0$$

such that (1.4) has no positive non-constant solution provided that $d_2 > D_2^*$ and $\lambda_1 d_1 > b + \epsilon_2$.

Proof. We only prove (i); the proof of (ii) can be accomplished similarly. Let (u, v) be a positive solution of (1.4) and write

$$\bar{u} = \frac{1}{|\Omega|}\int_\Omega u \,\mathrm{d}x.$$

We restrict $d_1 \geqslant 1$. Then, multiplying the first equation of (1.4) by $(u - \bar{u})$, integrating over Ω and using theorem 2.6, we have that

$$\begin{aligned} d_1 \int_\Omega |\nabla(u-\bar{u})|^2 \,\mathrm{d}x &= \int_\Omega \left\{ \left[a - (u+\bar{u}) - \frac{mv}{(m+u)(m+\bar{u})}\right](u-\bar{u})^2 - \frac{\bar{u}}{m+\bar{u}}(u-\bar{u})(v-\bar{v}) \right\} \mathrm{d}x \\ &\leqslant [a + c(\varepsilon)] \int_\Omega (u-\bar{u})^2 \,\mathrm{d}x + \varepsilon \int_\Omega (v-\bar{v})^2 \,\mathrm{d}x \end{aligned} \tag{3.1}$$

for any $\varepsilon > 0$ that depends only on a, b, m, γ and Ω. Similarly, from the second equation in (1.4), it follows that

$$d_2 \int_\Omega |\nabla(v-\bar{v})|^2 \,\mathrm{d}x \leqslant c(\varepsilon) \int_\Omega (u-\bar{u})^2 \,\mathrm{d}x + (b+\varepsilon) \int_\Omega (v-\bar{v})^2 \,\mathrm{d}x. \tag{3.2}$$

Hence, adding (3.1) and (3.2) and applying Poincaré's inequality, we have that

$$\begin{aligned} \lambda_1 d_1 \int_\Omega (u-\bar{u})^2 \,\mathrm{d}x + \lambda_1 d_2 \int_\Omega (v-\bar{v})^2 \,\mathrm{d}x \\ \leqslant [a + c(\varepsilon)] \int_\Omega (u-\bar{u})^2 \,\mathrm{d}x + (b+2\varepsilon) \int_\Omega (v-\bar{v})^2 \,\mathrm{d}x. \end{aligned} \tag{3.3}$$

If $\lambda_1 d_2 > b + \epsilon_1$, then it is easily seen from (3.3) that there exists

$$D_1^* = D_1^*(a, b, m, \gamma, \epsilon_1, \Omega) > 0$$

such that (1.4) has only the positive constant solution $(u, v) = (u^*, v^*)$ when $d_1 > D_1^*$. This concludes our proof. □

Now we can improve the results in theorem 3.1. For our purposes, we need to apply the *implicit function theorem*. we first establish some lemmas.

LEMMA 3.2. *Assume that $f(u)$ is a continuous real function in $[0, \infty)$ and, for some positive constant a, $f(u) > 0$ in $(0, a)$ and $f(u) < 0$ in (a, ∞). Then the following problem has a unique positive classical solution $u(x) = a$:*

$$-\Delta u = uf(u) \quad \textit{in } \Omega, \qquad \partial_\eta u = 0 \quad \textit{on } \partial\Omega.$$

Proof. The above result is easily obtained by the application of lemma 2.1. □

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LEMMA 3.3.

(i) *Fix* d_1, d_2, a *and* γ. *Assume that* (u_i, v_i) *is the positive solution of (1.4) with* $m = m_i$ *and* $m_i \to \infty$ *as* $i \to \infty$. *Then* $(u_i, v_i) \to (a, ab\gamma)$ *in* $[C^2(\bar{\Omega})]^2$ *as* $i \to \infty$.

(ii) *Let* d *be an arbitrary positive constant. Fix* a, b, m, γ *and assume that* (u_i, v_i) *is the positive solution of (1.4) with* $d_1 = d_{1,i}$, $d_2 = d_{2,i}$ *and* $d_{1,i} \to \infty$, $d_{2,i} \to d_2^* \in [d, \infty]$ *as* $i \to \infty$. *Then* $(u_i, v_i) \to (u^*, v^*)$ *in* $[C^2(\bar{\Omega})]^2$ *as* $i \to \infty$.

(iii) *Let* d *be an arbitrary positive constant. Fix* a, b, m, γ *and* $m > b\gamma$ *or* $m = b\gamma$ *and* $a > b\gamma$. *Assume that* (u_i, v_i) *is the positive solution of (1.4) with* $d_1 = d_{1,i}$, $d_2 = d_{2,i}$ *and* $d_{1,i} \to d_1^* \in [d, \infty]$, $d_{2,i} \to \infty$ *as* $i \to \infty$. *Then* $(u_i, v_i) \to (u^*, v^*)$ *in* $[C^2(\bar{\Omega})]^2$ *as* $i \to \infty$.

Proof. From remark 2.3, we see that (i) holds. Now we prove (ii).

We first assume that $d_2^* \in [d, \infty)$. By theorem 2.6, the embedding theory and the standard regularity theory of elliptic equations, we have that there exists a subsequence of (u_i, v_i), also labelled by itself, such that

$$(u_i, v_i) \to (u, v) \quad \text{in } [C^2(\bar{\Omega})]^2 \quad \text{as } i \to \infty.$$

Furthermore, $u \equiv c$, which is a positive constant, $v > 0$ on $\bar{\Omega}$ and (c, v) solves

$$\left.\begin{aligned} \int_\Omega c\left(a - c - \frac{v}{m+c}\right) \mathrm{d}x = 0, \\ -d_2^* \Delta v = bv - \frac{v^2}{c\gamma} \quad \text{in } \Omega, \qquad \partial_\eta v = 0 \quad \text{on } \partial\Omega. \end{aligned}\right\} \tag{3.4}$$

Hence the second equation in (3.4) indicates that $v \equiv bc\gamma$, and so it follows that $(u, v) = (u^*, v^*)$ from the first equation in (3.4). When $d_2^* = \infty$, from the above proof, it is easy to see that our statement is also valid.

Next, we assert (iii). Similarly, it suffices to prove the case $d_1^* \in [d, \infty)$. Theorem 2.2, the embedding theory and the standard regularity theory of elliptic equations claim that there is a subsequence of (u_i, v_i), labelled by itself again, such that

$$(u_i, v_i) \to (u, v) \quad \text{in } [C^2(\bar{\Omega})]^2 \quad \text{as } i \to \infty.$$

In addition, $v \equiv c$, which is a positive constant, $u > 0$ on $\bar{\Omega}$ and (u, c) satisfies

$$\left.\begin{aligned} -d_1^* \Delta u = au - u^2 - \frac{cu}{m+u} \quad \text{in } \Omega, \qquad \partial_\eta u = 0 \quad \text{on } \partial\Omega, \\ \int_\Omega c\left(b - \frac{c}{\gamma u}\right) \mathrm{d}x = 0. \end{aligned}\right\} \tag{3.5}$$

Under the assumption of (iii), we can claim that (3.5) has only a positive solution, which shows that $(u, v) = (u^*, v^*)$. To this end, it is clear that it is sufficient to prove that the first equation in (3.5) has a unique positive constant solution for fixed $c > 0$. We observe that theorem 2.2 tells us that c satisfies $c \leqslant b\gamma A$.

We let c be fixed and take $f(u) = a - u - c/(m + u)$. As $A < a$ and $b\gamma \leqslant m$, f satisfies the condition imposed in lemma 3.2. Thus

$$u \equiv \tfrac{1}{2}\{a - m + \sqrt{(a-m)^2 + 4(am - c)}\},$$

hence our claim holds. Therefore, the proof is finished. □

LEMMA 3.4. *For any positive constants a, b, m and γ, the following holds:*

$$a - 2u^* - \frac{mv^*}{(m + u^*)^2} = \frac{u^*}{m + u^*}(a - m - 2u^*).$$

Moreover, if $m \geqslant b\gamma$, then $a - m - 2u^* < 0$.

Proof. From the definitions of u^* and v^*, we easily see that

$$(u^*)^2 + (m + b\gamma - a)u^* - am = 0, \quad b\gamma u^* = (a - u^*)(m + u^*) \quad \text{and} \quad v^* = b\gamma u^*.$$

Then direct computations deduce that

$$\begin{aligned} a - 2u^* - \frac{mv^*}{(m + u^*)^2} &= -u^* + \frac{b\gamma u^*}{m + u^*} - \frac{bm\gamma u^*}{(m + u^*)^2} \\ &= \frac{u^*}{(m + u^*)^2}[-(m + u^*)^2 + b\gamma(m + u^*) - bm\gamma] \\ &= \frac{u^*}{m + u^*}(a - m - 2u^*). \end{aligned}$$

In addition, easy computations also give that if $m \geqslant b\gamma$, then $a - m - 2u^* < 0$. This finishes the proof. □

Now, on the base of the above lemmas, we can apply the *implicit function theorem* to obtain the following results.

THEOREM 3.5.

(i) *Fix d_1, d_2, a, b and γ. Then there exists a positive constant*

$$M_0 = M_0(d_1, d_2, a, b, \gamma, \Omega)$$

such that (1.4) has no positive non-constant solution provided that $m > M_0$.

(ii) *Let ϵ_3 be an arbitrary positive constant. Fix a, b, m and γ. Then there exists a positive constant*

$$D_1 = D_1(a, b, m, \gamma, \epsilon_3, \Omega)$$

such that (1.4) has no positive non-constant solution provided that $d_1 > D_1$ and $d_2 > \epsilon_3$.

(iii) *Let ϵ_4 be an arbitrary positive constant. Fix a, b, m, γ and let $m > b\gamma$ or $m = b\gamma$ and $a > b\gamma$. Then there exists a positive constant*

$$D_2 = (a, b, m, \gamma, \epsilon_4, \Omega)$$

such that (1.4) has no positive non-constant solution provided that $d_2 > D_2$ and $d_1 > \epsilon_4$.

Proof. We first prove (iii). By theorem 3.1 (ii), for a fixed large constant d_1^* depending only on b, Ω, there exists $D_2^* = D_2^*(a, b, m, \gamma, \Omega)$ such that (1.4) has no positive non-constant solution when $d_1 \geqslant d_1^*$, $d_2 \geqslant D_2^*$. In the following, it suffices to consider the case $d_1 \in [\frac{1}{2}\epsilon_4, d_1^*]$.

Write $v = w + \xi$, with $\int_\Omega w = 0$ and $\xi \in \mathbb{R}_+^1$. We observe that finding the positive solution of (1.4) is equivalent to solving the following problem,

$$\left.\begin{aligned}
&d_1\Delta u + au - u^2 - \frac{u(w+\xi)}{m+u} = 0 \quad \text{in } \Omega, \qquad \partial_\eta u = 0 \qquad \text{on } \partial\Omega,\\
&\Delta w + \rho\left\{b(w+\xi) - \frac{(w+\xi)^2}{\gamma u}\right\} = 0 \quad \text{in } \Omega, \qquad \partial_\eta w = 0 \qquad \text{on } \partial\Omega,\\
&\int_\Omega \left\{b(w+\xi) - \frac{(w+\xi)^2}{\gamma u}\right\} \mathrm{d}x = 0, \qquad \xi > 0,\ u > 0 \quad \text{on } \bar{\Omega},
\end{aligned}\right\} \tag{3.6}$$

where $\rho = d_2^{-1}$. Clearly, $(u, w, \xi) = (u^*, 0, v^*)$ is a solution of (3.6) for $\rho > 0$ and $d_1 \in [\frac{1}{2}\epsilon_4, d_1^*]$.

From the above analysis, to verify our assertion, by the finite-covering argument, it is enough to prove that, for any fixed $\tilde{d}_1 \in [\frac{1}{2}\epsilon_4, d_1^*]$, there exists a small positive constant δ_0 such that, if $\rho \in (0, \delta_0)$, $d_1 \in (\tilde{d}_1 - \delta_0, \tilde{d}_1 + \delta_0)$, then $(u^*, 0, v^*)$ is the unique solution of (3.6). For this, we define

$$W_\nu^{2,2}(\Omega) = \{g \in W^{2,2}(\Omega) \mid \partial_\eta g = 0 \text{ on } \partial\Omega\},$$
$$L_0^2(\Omega) = \left\{g \in L^2(\Omega) \;\middle|\; \int_\Omega g \, \mathrm{d}x = 0\right\}$$

and

$$F(d_1, \rho, u, w, \xi) = (f_1, f_2, f_3)(d_1, \rho, u, w, \xi),$$

with

$$f_1(d_1, \rho, u, w, \xi) = d_1\Delta u + au - u^2 - \frac{u(w+\xi)}{m+u},$$
$$f_2(d_1, \rho, u, w, \xi) = \Delta w + \rho\left\{b(w+\xi) - \frac{(w+\xi)^2}{\gamma u}\right\},$$
$$f_3(d_1, \rho, u, w, \xi) = \int_\Omega \left\{b(w+\xi) - \frac{(w+\xi)^2}{\gamma u}\right\} \mathrm{d}x.$$

Then

$$F : \mathbb{R}_+^1 \times \mathbb{R}_+^1 \times W_\nu^{2,2} \times (L_0^2(\Omega) \cap W_\nu^{2,2}(\Omega)) \times \mathbb{R}_+^1 \to L^2(\Omega) \times L_0^2(\Omega) \times \mathbb{R}^1,$$

and (3.6) is equivalent to solving $F(d_1, \rho, u, w, \xi) = 0$. Moreover, problem (3.6) has a unique solution $(u, w, \xi) = (u^*, 0, v^*)$ when $\rho = 0$, $d_1 = \tilde{d}_1$. By simple computations, we have

$$\begin{aligned}\Phi \equiv D_{(u,w,\xi)}F(\tilde{d}_1, 0, u^*, 0, v^*) : W_\nu^{2,2} \times (L_0^2(\Omega) \cap W_\nu^{2,2}(\Omega)) \times \mathbb{R}_+^1 \\ \to L^2(\Omega) \times L_0^2(\Omega) \times \mathbb{R}^1,\end{aligned}$$

where

$$\Phi(y,z,\tau)=\begin{pmatrix}\tilde{d}_1\Delta y+\dfrac{u^*}{m+u^*}(a-m-2u^*)y-\dfrac{u^*}{m+u^*}z-\dfrac{u^*}{m+u^*}\tau\\ \Delta z\\ \displaystyle\int_\Omega\{b^2\gamma y-bz-b\tau\}\,\mathrm{d}x\end{pmatrix}.$$

Here we used the relations of u^* and v^* appearing in the proof of lemma 3.4.

In order to use the *implicit function theorem*, we have to verify that Φ is both invertible and surjective. In fact, assume that $\Phi(y,z,\tau)=(0,0,0)$. Then $z\equiv 0$. Our lemma 3.4 guarantees $a-m-2u^*<0$. Thus $\tau\in\mathbb{R}^1$ implies that y must be a constant and (y,z) satisfies

$$\tau=(a-m-2u^*)y \quad\text{and}\quad \tau=b\gamma y.$$

Hence $y=\tau=0$ and Φ is invertible. On the other hand, it is easily verified that Φ is also a surjection.

By the *implicit function theorem*, there exist positive constants δ_0 and ε_0 such that, for each $\rho\in[0,\delta_0]$, $d_1\in(\tilde{d}_1-\delta_0,\tilde{d}_1+\delta_0)$, $(u^*,0,v^*)$ is the unique solution of $F(d_1,\rho,u,w,\xi)=0$ in $B_{\varepsilon_0}(u^*,0,v^*)$, where $B_{\varepsilon_0}(u^*,0,v^*)$ is the ball in $W_\nu^{2,2}(\Omega)\times(L_0^2(\Omega)\cap W_\nu^{2,2}(\Omega))\times\mathbb{R}^1$ centred at $(u^*,0,v^*)$ with radius ε_0. Taking smaller δ_0 and ε_0 if necessary, we can conclude the proof by use of lemma 3.4 (iii).

The proofs of (i) and (ii) are similar to that of (iii) by applying lemmas 3.3, 3.4, theorem 2.6 and the *implicit function theorem*. In fact, we can define the analogous operator F to prove (ii). To prove (i), we can construct the operator F as in the proof of (iii) as follows.

Let $\rho=m^{-1}$ and define

$$F(\rho,u,v)=(f_1,f_2)(\rho,u,v):\mathbb{R}_+^1\times W_\nu^{2,2}(\Omega))\times W_\nu^{2,2}(\Omega)\to L^2(\Omega)\times L^2(\Omega),$$

where

$$\begin{aligned}f_1(\rho,u,v)&=d_1\Delta u+au-u^2-\frac{\rho uv}{1+\rho u},\\ f_2(\rho,u,v)&=d_2\Delta v+bv-\frac{v^2}{\gamma u}.\end{aligned}$$

It is clear that $(u,v)=(a,ab\gamma)$ is the unique positive solution of $F(0,u,v)=0$, and, moreover, we easily verify that $D_{(u,v)}F(0,a,ab\gamma)$ is a bijection. Then one can combine the *implicit function theorem* and lemma 3.3 (i) to yield our assertion. □

4. Existence of positive non-constant solutions

In this section, we shall use the topological degree theory to obtain the existence of positive non-constant solutions of (1.4). Throughout this section, we let $0=\lambda_0<\lambda_1<\lambda_2<\cdots$ be all the eigenvalues of the operator $-\Delta$ on Ω with the homogeneous Neumann boundary condition, and denote by ℓ_i the multiplicity of λ_i. We always fix a, b, m and γ from now on.

4.1. Existence of positive non-constant solutions

In this subsection, we study the existence of positive non-constant solutions. It is easy to see that (1.4) is equivalent to the equation

$$(\boldsymbol{I} - G)(u, v) = 0, \tag{4.1}$$

where

$$G(u, v) = \begin{pmatrix} (\boldsymbol{I} - d_1\Delta)^{-1}\left((a+1)u - u^2 - \dfrac{uv}{m+u}\right) \\ (\boldsymbol{I} - d_2\Delta)^{-1}\left((b+1)v - \dfrac{v^2}{\gamma u}\right) \end{pmatrix},$$

and $(\boldsymbol{I} - d_i\Delta)^{-1}$ $(i = 1, 2)$ is the inverse of $\boldsymbol{I} - d_i\Delta$ with the homogeneous Neumann boundary condition. Since the operators $(\boldsymbol{I} - d_i\Delta)^{-1} : C(\bar{\Omega}) \to C(\bar{\Omega})$ $(i = 1, 2)$ exist and are compact, $G : [C(\bar{\Omega})]^2 \to [C(\bar{\Omega})]^2$ is also compact.

In order to apply the degree theory to obtain the existence of positive non-constant solutions, our first aim is to compute the index of $\boldsymbol{I} - G$ at (u^*, v^*). Consider the eigenvalue problem

$$-(\boldsymbol{I} - D_{(u,v)}G(u^*, v^*))(w, z) = \mu(w, z), \qquad (w, z) \neq (0, 0). \tag{4.2}$$

By the Leray–Schauder theorem (see [18, pp. 37, 38]), we have that if 0 is not the eigenvalue of (4.2), then

$$\operatorname{index}(I - G, (u^*, v^*)) = (-1)^r, \quad \text{with } r = \sum_{\mu>0} n_\mu, \tag{4.3}$$

where n_μ is the multiplicity of the positive eigenvalue μ of (4.2). After some computations, we also see that, in order to solve (4.2), it is equivalent to solve

$$\left.\begin{aligned} -d_1(\mu+1)\Delta w + \left[\frac{u^*}{m+u^*}(m + 2u^* - a) + \mu\right]w + \frac{u^*}{m+u^*}z = 0 \quad \text{in } \Omega, \\ -d_2(\mu+1)\Delta z + (b+\mu)z - b^2\gamma w = 0 \quad \text{in } \Omega, \\ \partial_\eta w = \partial_\eta z = 0 \quad \text{on } \partial\Omega, \qquad (w, z) \neq (0, 0). \end{aligned}\right\} \tag{4.4}$$

We further see that μ is an eigenvalue of (4.4) if and only if $P_k(\mu) = 0$ for some $k \geqslant 0$, where

$$P_k(\mu) = \det \begin{pmatrix} \mu + \dfrac{d_1\lambda_k + (u^*/(m+u^*))(m + 2u^* - a)}{1 + d_1\lambda_k} & \dfrac{u^*}{(1 + d_1\lambda_k)(m+u^*)} \\ -\dfrac{b^2\gamma}{1 + d_2\lambda_k} & \mu + \dfrac{d_2\lambda_k + b}{1 + d_2\lambda_k} \end{pmatrix}$$

(see [19, 20] for the details). For the positive solution μ of $P_k(\mu) = 0$, we denote the multiplicity of μ by m_μ. Applying (4.3), we have that if $P_k(0) \neq 0$ for all $k \geqslant 0$, then

$$\operatorname{index}(\boldsymbol{I} - G, (u^*, v^*)) = (-1)^r, \qquad r = \sum_{k=0}^{\infty} \sum_{\mu>0,\, P_k(\mu)=0} m_\mu \ell_k \tag{4.5}$$

(see the proof of lemma 5.1 in [20]).

Lemma 4.1. *There exists a positive constant $\tilde{D}_1 = \tilde{D}_1(d_2, a, b, m, \gamma)$ such that, for all $d_1 \geqslant \tilde{D}_1$,*

$$\operatorname{index}(\boldsymbol{I} - G, (u^*, v^*)) = (-1)^{r_0},$$

where $r_0 = \sum_{\mu>0,\, P_0(\mu)=0} m_\mu$.

Proof. When $k = 0$, then $\lambda_0 = 0$ and the constant term of $P_0(\mu)$ is given by

$$\frac{bu^*}{m + u^*}(2u^* + m + b\gamma - a) > 0$$

from the definition of u^*, and thus $\mu = 0$ is not the root of $P_0(\mu) = 0$.

For $k \geqslant 1$, we have $\lambda_k \geqslant \lambda_1 > 0$ and

$$\lim_{d_1 \to \infty} P_k(\mu) = (\mu + 1)\left(\mu + \frac{d_2\lambda_k + b}{1 + d_2\lambda_k}\right) \quad \text{for all } k \geqslant 1.$$

Therefore, there exists a constant $\tilde{D}_1 > 0$ such that $P_k(\mu) = 0$ has no positive root for all $k \geqslant 1$. Consequently, lemma 4.1 follows from (4.5). □

Lemma 4.2. *Assume that $2u^* + m - a < 0$ and*

$$\frac{u^*}{d_1(m + u^*)}(a - m - 2u^*) \in (\lambda_{k^*}, \lambda_{k^*+1})$$

for some $k^* \geqslant 1$. Then there exists a positive constant $\tilde{D}_2 = \tilde{D}_2(d_1, a, b, m, \gamma)$ such that, for all $d_2 \geqslant \tilde{D}_2$,

$$\operatorname{index}(\boldsymbol{I} - G, (u^*, v^*)) = (-1)^{r_0 + r_1},$$

where $r_0 = \sum_{\mu>0,\, P_0(\mu)=0} m_\mu$ and $r_1 = \sum_{i=1}^{k^*} \ell_i$.

Proof. For $k \geqslant 1$, we have $\lambda_k \geqslant \lambda_1 > 0$ and thus, for all $k \geqslant 1$,

$$\begin{aligned}\lim_{d_2 \to \infty} P_k(\mu) &= (\mu + 1)\left[\mu + \frac{d_1\lambda_k + (u^*/(m + u^*))(2u^* + m - a)}{1 + d_1\lambda_k}\right] \\ &= \frac{1}{1 + d_1\lambda_k}(a_{1,k}\mu^2 + a_{2,k}\mu + a_{3,k}),\end{aligned}$$

where

$$\begin{aligned} a_{1,k} &= 1 + d_1\lambda_k > 0, \\ a_{2,k} &= 2d_1\lambda_k + \frac{u^*}{m + u^*}(2u^* + m - a) + 1, \\ a_{3,k} &= d_1\lambda_k + \frac{u^*}{m + u^*}(2u^* + m - a). \end{aligned}$$

We observe that $a_{3,k} < a_{2,k}$ for all $k \geqslant 1$. Hence, if there exists some $k^* \geqslant 1$ such that $a_{3,k^*} < 0$ and $a_{3,k^*+1} > 0$, then $a_{2,k} > 0$ for all $k \geqslant k^* + 1$. Therefore, $a_{1,k}\mu^2 + a_{2,k}\mu + a_{3,k} = 0$ has exactly one positive root for $1 \leqslant k \leqslant k^*$, while it has no positive root for $k \geqslant k^* + 1$. As a result, our conclusion easily follows from (4.5). □

THEOREM 4.3. *Assume that* $2u^* + m - a < 0$ *and*

$$\frac{u^*}{d_1(m+u^*)}(a - m - 2u^*) \in (\lambda_{k^*}, \lambda_{k^*+1})$$

for some $k^* \geqslant 1$. *If* $\sum_{i=1}^{k^*} \ell_i$ *is odd, then there exists a positive constant* $\bar{D}_2 = \bar{D}_2(d_1, a, b, m, \gamma)$ *such that, for all* $d_2 \geqslant \bar{D}_2$, *problem (1.4) has at least one positive non-constant solution.*

Proof. Fix $\bar{d}_2 > 0$ satisfying $\lambda_1 \bar{d}_2 > b$. Let $\bar{d}_1$ be a large positive number, which depends on $\bar{d}_2$, such that theorem 3.1 (i) and lemma 4.1 hold for $d_1 = \bar{d}_1$ and $d_2 = \bar{d}_2$. For $0 \leqslant t \leqslant 1$, we define

$$G((u,v);t) = \begin{pmatrix} (\boldsymbol{I} - [td_1 + (1-t)\bar{d}_1]\Delta)^{-1}\left((a+1)u - u^2 - \dfrac{uv}{m+u}\right) \\ (\boldsymbol{I} - [td_2 + (1-t)\bar{d}_2]\Delta)^{-1}\left((b+1)v - \dfrac{v^2}{\gamma u}\right) \end{pmatrix}.$$

We restrict $d_2 > d_1$. By virtue of theorem 2.6, there exists a positive constant $M = M(d_1, \bar{d}_1, a, b, m, \gamma)$ such that (1.4) has no positive solution on $\partial\Theta$, where

$$\Theta = \left\{(u,v) \in [C(\bar{\Omega})]^2 \;\middle|\; \frac{1}{M} < u,\ v < M\right\}.$$

Since

$$G((u,v);t) : \Theta \times [0,1] \to [C(\bar{\Omega})]^2$$

is compact, the degree $\deg(\boldsymbol{I} - G((u,v);t), \Theta, 0)$ is well defined. By the homotopy invariance of degree,

$$\deg(\boldsymbol{I} - G((u,v);0), \Theta, 0) = \deg(\boldsymbol{I} - G((u,v);1), \Theta, 0). \tag{4.6}$$

Using theorem 3.1 (i) and lemma 4.1, we have that

$$\deg(\boldsymbol{I} - G((u,v);0), \Theta, 0) = \operatorname{index}(G((u,v);0), (u^*, v^*)) = (-1)^{r_0}, \tag{4.7}$$

where r_0 is defined in lemma 4.1.

On the contrary, we assume that, for some $d_2 \geqslant \bar{D}_2$, problem (1.4) has no positive non-constant solution. By lemma 4.2, we have

$$\deg(\boldsymbol{I} - G((u,v);1), \Theta, 0) = \operatorname{index}(G((u,v);1), (u^*, v^*)) = (-1)^{r_0+r_1} = -(-1)^{r_0}, \tag{4.8}$$

where r_1 given by lemma 4.2 is odd. From (4.6)–(4.8), we get a contradiction, and the proof is completed. □

REMARK 4.4. By simple computations, $2u^* + m - a < 0$ indicates that

$$a < 2b\gamma \quad \text{and} \quad 0 < m < -(a + b\gamma) + \sqrt{b^2\gamma^2 + 4ab\gamma}.$$

If $a > b\gamma$, then $u^* \to a - b\gamma$ as $m \to 0$, while $u^* \to 0$ as $m \to 0$ if $a \leqslant b\gamma$. Consequently, when $b\gamma < a < 2b\gamma$, m is small enough, $d_1^{-1}(2b\gamma - a) \in (\lambda_{k^*}, \lambda_{k^*+1})$ and, in addition, $\sum_{i=1}^{k^*} \ell_i$ is odd, then (1.4) possesses at least one positive non-constant solution for large d_2.

Remark 4.5. We note that theorem 3.1 (ii) tells us that when $\lambda_1 d_1 > a$ and d_2 is large enough, then (1.4) has no positive non-constant solution. On the other hand, we also observe that the condition

$$\frac{u^*}{d_1(m+u^*)}(a-m-2u^*) \in (\lambda_{k^*}, \lambda_{k^*+1})$$

in theorem 4.3 for some $k^* \geqslant 1$ implies that $\lambda_1 d_1 < a$.

4.2. Bifurcation

In this subsection, we use the bifurcation theory to establish the existence of positive non-constant solutions of (1.4). Throughout this subsection, we set $U=(u,v)$, $\tilde{U}=(u^*,v^*)$ and assume that $2u^*+m-a<0$. We fix d_2 and consider d_1 as the bifurcation parameter. Since the constant term of $P_k(\mu)$ defined in § 4.1 is given by

$$\frac{1}{(1+d_1\lambda_k)(1+d_2\lambda_k)}\left\{d_1d_2\lambda_k^2 + \left[bd_1 + \frac{u^*}{m+u^*}(2u^*+m-a)d_2\right]\lambda_k + \frac{bu^*}{m+u^*}(2u^*+b\gamma+m-a)\right\},$$

we denote $H(d_1,\lambda)$ by

$$H(d_1,\lambda) = d_1d_2\lambda^2 + \left[bd_1 + \frac{u^*}{m+u^*}(2u^*+m-a)d_2\right]\lambda + \frac{bu^*}{m+u^*}(2u^*+b\gamma+m-a).$$

Then, for fixed $d_1>0$, $H(d_1,\lambda)=0$ has at most two roots. Moreover, note that

$$2u^*+b\gamma+m-a = \sqrt{(a-m-b\gamma)^2+4am} > 0,$$

and thus, if

$$Q(d_1) = b^2d_1^2 - \frac{2d_2bu^*}{m+u^*}(2u^*+2b\gamma+m-a)d_1 + \left(\frac{u^*}{m+u^*}\right)^2(2u^*+m-a)^2d_2^2 > 0$$

holds, $H(d_1,\lambda)=0$ has two different real roots that are the same sign.

Now we introduce some notations and definitions.

$$\begin{aligned}
E(\lambda) &= \{\phi \mid -\Delta\phi = \lambda\phi \text{ in } \Omega,\ \partial_\eta\phi = 0 \text{ on } \partial\Omega\} \quad \forall \lambda \in R,\\
X &= [C(\bar{\Omega})]^2,\\
B_\delta(\tilde{U}) &= \{U \in X \mid \|U-\tilde{U}\|_X < \delta\},\\
S_p &= \bigcup_{i=1}^{\infty}\{\lambda_i\},\\
\Gamma &= \{d_1 \in (0,\infty) \mid H(d_1,\lambda)=0 \text{ for some } \lambda \in S_p\},\\
\Lambda(d_1) &= \{\lambda \in S_p \mid H(d_1,\lambda)=0\}.
\end{aligned}$$

We say that $(\tilde{d}_1,\tilde{U}) \in (0,\infty)\times \boldsymbol{X}$ is a regular point of (1.4) if there exists a positive constant δ such that, for any $d_1 \in [\tilde{d}_1-\delta, \tilde{d}_1+\delta]$, problem (1.4) has a unique positive constant solution $U=\tilde{U}$ in the set $B_\delta(\tilde{U})$. Otherwise, we say that $(\tilde{d}_1,\tilde{U}) \in (0,\infty)\times \boldsymbol{X}$ is a bifurcation point of (1.4).

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Similar to the treatments of papers [19, 20, 25], together with theorem 3.5 (ii), we have the following local and global bifurcation results.

THEOREM 4.6 (local bifurcation on d_1). *Let* $\tilde{d}_1 > 0$ *and consider the equilibrium* $(\tilde{d}_1, U(\cdot)) = (\tilde{d}_1, \tilde{U})$ *of (1.4).*

(i) *If* $\tilde{d}_1 \notin \Gamma$*, then* $(\tilde{d}_1, \tilde{U})$ *is a regular point of (1.4).*

(ii) *Assume* $2u^* + m - a < 0$ *and* $Q(\tilde{d}_1) > 0$*. If* $\tilde{d}_1 \in \Gamma$ *and* $\sum_{\lambda_i \in \Lambda(\tilde{d}_1)} \dim\{E(\lambda_i)\}$ *is odd. Then* $(\tilde{d}_1, \tilde{U})$ *is a bifurcation point of (1.4).*

THEOREM 4.7 (global bifurcation on d_1). *Let* $2u^* + m - a < 0$ *and* $Q(\tilde{d}_1) > 0$*. Suppose that* $\tilde{d}_1 \in \Gamma$ *and* $\sum_{\lambda_i \in \Lambda(\tilde{d}_1)} \dim\{E(\lambda_i)\}$ *is odd. Then there exists an interval* $(\alpha, \beta) \subset (0, \infty)$ *such that, for every* $d_1 \in (\alpha, \beta)$*, problem (1.4) has a positive non-constant solution. In addition, one of the following holds:*

(i) $\tilde{d}_1 = \alpha < \beta$ *and* $\beta \in \Gamma$*;*

(ii) $\alpha < \beta = \tilde{d}_1$ *and* $\alpha \in \Gamma$*;*

(iii) $(\alpha, \beta) = (0, \tilde{d}_1)$.

REMARK 4.8. If $Q(\tilde{d}_1) < 0$ for some $\tilde{d}_1 > 0$, then $H(\tilde{d}_1, \lambda) = 0$ has no real root. If $Q(\tilde{d}_1) = 0$, we have the remark similar to remark 5.1 in [25].

REMARK 4.9. Similarly, we can obtain the local and global bifurcations with respect to the parameter d_2.

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