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A convergence to the Navier–Stokes–Maxwell system with solenoidal Ohm's law from a two-fluid model

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We show the incompressible Navier–Stokes–Maxwell system with solenoidal Ohm's law can be derived from the two-fluid incompressible Navier–Stokes–Maxwell system when the momentum transfer coefficient tends to zero. The strategy is based on the decay and dissipative properties of the electromagnetic field.

Keywords: Navier–Stokes equations; Maxwell equations; global well-posedness; asymptotic process

2010 *Mathematics subject classification:* 35Q30, 35Q61, 35A01

1. Introduction

In this paper, we consider the asymptotic behaviour as ε tends to zero of the solutions of the following two-fluid incompressible Navier–Stokes–Maxwell system:

$$\begin{cases} \partial_t u^+ + u^+ \cdot \nabla u^+ - \mu\Delta u^+ + \dfrac{1}{2\sigma\varepsilon^2}(u^+ - u^-) = -\nabla p^+ + \dfrac{1}{\varepsilon}(cE + u^+ \times B), \\ \partial_t u^- + u^- \cdot \nabla u^- - \mu\Delta u^- - \dfrac{1}{2\sigma\varepsilon^2}(u^+ - u^-) = -\nabla p^- - \dfrac{1}{\varepsilon}(cE + u^- \times B), \\ \dfrac{1}{c}\partial_t E - \nabla \times B = -\dfrac{1}{2\varepsilon}(u^+ - u^-), \\ \dfrac{1}{c}\partial_t B + \nabla \times E = 0, \\ \mathrm{div} u^+ = 0,\ \mathrm{div} u^- = 0,\ \mathrm{div} E = 0,\ \mathrm{div} B = 0 \end{cases} \tag{NSM}$$

with the initial data

$$\begin{aligned} &u^+(t,x)|_{t=0} = u^{+,\mathrm{in}}(x),\ u^-(t,x)|_{t=0} = u^{-,\mathrm{in}}(x), \\ &E(t,x)|_{t=0} = E^{\mathrm{in}}(x),\ B(t,x)|_{t=0} = B^{\mathrm{in}}(x). \end{aligned}$$

This system models the evolution of a charged plasma. The first two equations are momentum equations. Here, $u^+, u^- \colon \mathbb{R}_t^+ \times \mathbb{R}_x^3 \to \mathbb{R}^3$ are the velocities of the cations and anions, respectively. The scalar functions p^+ and p^- represent pressures. The constants μ, c, and $\varepsilon > 0$ represent the kinematic viscosity, speed of light, and momentum transfer coefficient, respectively. The third equation is the Ampère–Maxwell equation describing the evolution of the electric field E: $\mathbb{R}^+_t \times \mathbb{R}^3_x \to \mathbb{R}^3$, and the fourth equation is Faraday's law for the magnetic field B: $\mathbb{R}^+_t \times \mathbb{R}^3_x \to \mathbb{R}^3$. For more details about the model, we refer to [**1**, **6**] and the references therein.

Defining $u = (1/2)(u^+ + u^-)$ and $j = (1/2\varepsilon)(u^+ - u^-)$, the bulk velocity and electrical current of the fluid, respectively, we rewrite system (NSM) in the following equivalent form:

$$\begin{cases} \partial_t u + u \cdot \nabla u + \varepsilon^2 j \cdot \nabla j - \mu \Delta u = -\nabla p + j \times B, \\ \varepsilon^2 \partial_t j + \varepsilon^2 (u \cdot \nabla j + j \cdot \nabla u) - \varepsilon^2 \mu \Delta j + \dfrac{1}{\sigma} j = -\nabla \tilde{p} + cE + u \times B, \\ \dfrac{1}{c} \partial_t E - \nabla \times B = -j, \\ \dfrac{1}{c} \partial_t B + \nabla \times E = 0, \\ \mathrm{div} u = 0, \ \mathrm{div} j = 0, \ \mathrm{div} E = 0, \ \mathrm{div} B = 0. \end{cases} \tag{1.1}$$

It is easy to verify that system (1.1) obtains the following energy conservation law:

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} (\|u\|^2_{L^2} + \varepsilon^2 \|j\|^2_{L^2} + \|E\|^2_{L^2} + \|B\|^2_{L^2}) + \mu (\|\nabla u\|^2_{L^2} + \varepsilon^2 \|\nabla j\|^2_{L^2}) + \frac{1}{\sigma} \|j\|^2_{L^2} = 0. \tag{1.2}$$

Hence, taking formally $\varepsilon \to 0$, one can obtain the following incompressible Navier–Stokes–Maxwell system with solenoidal Ohm's law:

$$\begin{cases} \partial_t u + u \cdot \nabla u - \mu \Delta u = -\nabla p + j \times B, \\ j = \sigma(-\nabla \tilde{p} + cE + u \times B), \\ \dfrac{1}{c} \partial_t E - \nabla \times B = -j, \\ \dfrac{1}{c} \partial_t B + \nabla \times E = 0, \\ \mathrm{div} u = 0, \ \mathrm{div} j = 0, \ \mathrm{div} E = 0, \ \mathrm{div} B = 0. \end{cases} \tag{NSMO}$$

The aim of this paper is to establish rigorously this asymptotic process and to show that system (NSMO) can be derived from system (NSM).

Before stating our main results, let us recall some known results. For system (NSM), Giga et al. [**5**] showed the existences of global weak solutions for any initial data $(u^{+,\mathrm{in}}, u^{-,\mathrm{in}}, E^{\mathrm{in}}, B^{\mathrm{in}}) \in (L^2(\mathbb{R}^3))^4$. The authors also established in [**5**] the local Kato-type solutions by using the fixed point argument for $(u^{+,\mathrm{in}}, u^{-,\mathrm{in}}, E^{\mathrm{in}}, B^{\mathrm{in}}) \in H^{1/2}(\mathbb{R}^3) \times H^{1/2}(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$, and proved these solutions to be global if the initial data are sufficiently small.

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For system (NSMO), though it enjoys the energy conservation law

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|u\|_{L^2}^2 + \|E\|_{L^2}^2 + \|B\|_{L^2}^2) + \mu\|\nabla u\|_{L^2}^2 + \frac{1}{\sigma}\|j\|_{L^2}^2 = 0,$$

due to the hyperbolic nature of the Maxwell equations, how to pass the limit in the nonlinear term $j \times B$ is unknown. Therefore, the global existence of Leray-type weak solution remains a very challenging problem. As a consequence, the convergence from system (NSM) to system (NSMO) as ε tends to zero in the context of weak solutions is also completely open. Nevertheless, if ε tends to zero and the speed c tends to infinity simultaneously by using the spectral properties of Maxwell's operator, Arsénio, Ibrahim and Masmoudi [**1**] established a rigorous asymptotic result from the weak solutions of system (NSM) to the weak solutions of the viscous incompressible magnetohydrodynamic system

$$\begin{cases} \partial_t u + u \cdot \nabla u - \mu\Delta u = -\nabla p + (\nabla \times B) \times B, \\ \partial_t B - \dfrac{1}{\sigma}\Delta B = \nabla \times (u \times B), \\ \mathrm{div} u = 0,\ \mathrm{div} B = 0. \end{cases}$$

We find that there is another system, which is mathematically very much similar to system (NSMO) and is also called an incompressible Navier–Stokes–Maxwell system with solenoidal Ohm's law in some literature. For its analytical studies, the reader may refer to Masmoudi [**9**] for global strong solutions in two dimension, Ibrahim and Keraani [**7**] for global mild solutions in two and three dimensions with small initial data, and Greman, Ibrahim and Masmoudi [**3**] for a refined result of globally small mild solutions. It is worthy to mention that the method used in [**3**, **7**, **9**] can be directly applied to solve system (NSMO). Very recently, Jiang and Luo [**8**] showed that system (NSMO) allows a unique global in time classical solution for small initial data $u^{\mathrm{in}} \in H^s(\mathbb{R}^3)$, E^{in}, $B^{\mathrm{in}} \in H^{s+1}(\mathbb{R}^3)$ with $s \geqslant 2$, by employing the dissipation and decay properties of the electric field E and magnetic field B.

In this paper, we provide an rigorous asymptotic result from the two-fluid system (NSM) to system (NSMO) as ε tends to zero in the context of strong solutions. The trick in the proof is to derive enough uniform estimates with respect to ε by applying the decay and dissipation properties of the electromagnetic field. More specifically, motivated by the work of Jiang and Luo [**8**], plugging the second equation into the Ampère–Maxwell equation in system (1.1) yields

$$\frac{1}{c}\partial_t E - \nabla \times B + \sigma c E = \sigma\nabla\tilde{p} - \sigma u \times B + \sigma J,$$

with a decay term $\sigma c E$, and here $J = \varepsilon^2\partial_t j - \varepsilon^2\mu\Delta j + \varepsilon^2(u \cdot \nabla j + j \cdot \nabla u)$. Taking derivative with respect to t in the Faraday equation, one derives the following wave equation:

$$\frac{1}{c^2}\partial_{tt} B - \Delta B + \sigma\partial_t B = \sigma\nabla \times (u \times B) - \sigma\nabla \times J$$

with a damping term $\sigma\partial_t B$. We then establish uniform bounds with respect to ε for terms E, $\partial_t B$, ∇B, and especially $\varepsilon\partial_t j$.

Now we state our main result as follows:

THEOREM 1.1 (Global well-posedness for (NSM)). *Let $s \geqslant 3$ and $0 < \varepsilon \leqslant 1$. Assume the initial data*

$$(u_\varepsilon^{+,\mathrm{in}}(x),\ u_\varepsilon^{-,\mathrm{in}}(x),\ E_\varepsilon^{\mathrm{in}}(x),\ B_\varepsilon^{\mathrm{in}}(x)) \in (H^s(\mathbb{R}^3))^4$$

is divergence-free. Furthermore, if the initial data satisfy

$$\|u_\varepsilon^{+,\mathrm{in}}\|^2_{H^s(\mathbb{R}^3)} + \|u_\varepsilon^{-,\mathrm{in}}\|^2_{H^s(\mathbb{R}^3)} + \|E_\varepsilon^{\mathrm{in}}\|^2_{H^s(\mathbb{R}^3)} + \|B_\varepsilon^{\mathrm{in}}\|^2_{H^s(\mathbb{R}^3)} \leqslant \kappa_0,$$

where $\kappa_0 > 0$ is a small constant independent of ε. Then system (NSM) allows a unique global in time classical solution $(u_\varepsilon^+(t,x),\ u_\varepsilon^-(t,x),\ E_\varepsilon(t,x),\ B_\varepsilon(t,x))$ such that

$$\begin{aligned}
&u_\varepsilon^+,\ u_\varepsilon^- \in L^\infty(\mathbb{R}^+;\ H^s(\mathbb{R}^3)) \cap L^2(\mathbb{R}^+;\ \dot{H}^{s+1}(\mathbb{R}^3)),\ \textit{and}\\
&E_\varepsilon,\ B_\varepsilon \in L^\infty(\mathbb{R}^+; H^s(\mathbb{R}^3)).
\end{aligned}$$

Moreover, the energy inequality

$$\begin{aligned}
&\sup_{t\geqslant 0}\left(\left\|\frac{u_\varepsilon^{+,\mathrm{in}} + u_\varepsilon^{-,\mathrm{in}}}{2}\right\|^2_{H^s} + \left\|\frac{u_\varepsilon^{+,\mathrm{in}} - u_\varepsilon^{-,\mathrm{in}}}{2}\right\|^2_{H^s} + \|E_\varepsilon(t)\|^2_{H^s} + \|B_\varepsilon(t)\|^2_{H^s}\right)\\
&\quad + \int_0^\infty \left(\mu\left\|\nabla\frac{u_\varepsilon^{+,\mathrm{in}} + u_\varepsilon^{-,\mathrm{in}}}{2}(t)\right\|^2_{H^s} + \mu\left\|\nabla\frac{u_\varepsilon^{+,\mathrm{in}} - u_\varepsilon^{-,\mathrm{in}}}{2}(t)\right\|^2_{H^s}\right.\\
&\quad \left. + \frac{1}{\sigma}\left\|\frac{u_\varepsilon^{+} - u_\varepsilon^{-}}{2\varepsilon}\right\|^2_{H^s}\right)\mathrm{d}t\\
&\qquad \leqslant \left\|\frac{u_\varepsilon^{+,\mathrm{in}} + u_\varepsilon^{-,\mathrm{in}}}{2}\right\|^2_{H^s} + \left\|\frac{u_\varepsilon^{+,\mathrm{in}} - u_\varepsilon^{-,\mathrm{in}}}{2}\right\|^2_{H^s} + \|E_\varepsilon^{\mathrm{in}}\|^2_{H^s} + \|B_\varepsilon^{\mathrm{in}}\|^2_{H^s}
\end{aligned}$$

holds.

REMARK 1.2. The main focus of theorem (1.1) is to obtain uniform H^s energy bounds with respect to ε. The other properties of solutions such as long time behaviour can be found in [4].

THEOREM 1.3 (Strong convergence from (NSM) to (NSMO)). *Let $s \geqslant 3$, $0 < \varepsilon \leqslant 1$ and $(u_\varepsilon^+,\ u_\varepsilon^-,\ E_\varepsilon,\ B_\varepsilon)$ the global classical solution of the two-fluid system (NSM) with initial data $(u_\varepsilon^{+,\mathrm{in}}(x),\ u_\varepsilon^{-,\mathrm{in}}(x),\ E_\varepsilon^{\mathrm{in}}(x),\ B_\varepsilon^{\mathrm{in}}(x))$ satisfies the assumptions in theorem 1.1. Moreover, if*

$$\begin{aligned}
&\left(\frac{u_\varepsilon^{+,\mathrm{in}} + u_\varepsilon^{-,\mathrm{in}}}{2},\ E_\varepsilon^{\mathrm{in}},\ B_\varepsilon^{\mathrm{in}}\right) \longrightarrow (u^{\mathrm{in}},\ E^{\mathrm{in}},\ B^{\mathrm{in}})\ \textit{in}\ (H^{s-2}(\mathbb{R}^3))^3, \quad \textit{as}\ \varepsilon \to 0,\\
&\left\|\frac{u_\varepsilon^{+,\mathrm{in}} - u_\varepsilon^{-,\mathrm{in}}}{2\varepsilon}\right\|_{H^{s-2}(\mathbb{R}^3)} \leqslant C, \quad \textit{for some constant}\ C > 0\ \textit{independent of}\ \ \varepsilon.
\end{aligned}$$

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Then, as $\varepsilon \to 0$,

$$\left(\frac{u_\varepsilon^+ + u_\varepsilon^-}{2},\ E_\varepsilon,\ B_\varepsilon\right) \longrightarrow (u,\ E,\ B) \text{ in } (L^\infty(\mathbb{R}^+; H^{s-2}) \cap L^2(\mathbb{R}^+; \dot{H}^{s-1})) \times (L^\infty(\mathbb{R}^+; H^{s-2})^2,\ \frac{u_\varepsilon^+ - u_\varepsilon^-}{2\varepsilon} \longrightarrow j \text{ in } L^2(\mathbb{R}^+; H^{s-2}),$$

where (u, E, B, j) *is the global classical solution to system* (NSMO) *with initial data* $(u^{in},\ E^{in},\ B^{in})$.

The organisation of this paper is as follows: in the next section, by utilising basic energy estimates we establish the existence of global classical solutions to system (NSM) with small initial data assumptions (see theorem 1.1). In § 3, as ε tends to zero, we prove the strong convergence of solutions $(u_\varepsilon,\ j_\varepsilon,\ E_\varepsilon,\ B_\varepsilon)$ of system (NSM) to some (u, j, E, B), which is exactly the global classical solution to the limit system (NSMO) (see theorem 1.3).

At the end of this section, we introduce some notations which will be frequently used in the following text. We denote by $\|\cdot\|_{L^p}$ the norm of the Lebesgue space $L^p(\mathbb{R}^3)$ for simplicity, and denote by $\|\cdot\|_{H^s}$ the norm of usual Sobolev spaces $H^s(\mathbb{R}^3)$ $(s \in \mathbb{N})$. We also define $\langle \cdot, \cdot \rangle$ as the usual L^2-inner product, and $\langle \cdot, \cdot \rangle_{H^s}$ as the H^s-inner product. The notation $\alpha = (\alpha_1,\ \alpha_2,\ \alpha_3) \in \mathbb{N}^3$ stands for a multi-index with its length defined as $|\alpha| = \sum_{i=1}^3 \alpha_i$. Denote the multi-derivative operator $\nabla_x^\alpha = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3}$, and we use the notation ∇^k for $|\alpha| = k$. We will use $A \lesssim B$ to denote the relation $A \leqslant CB$ for some positive constant $C > 0$, which may be different on different lines.

2. Proof of theorem 1.1

As mentioned in § 1, by defining the bulk velocity $u = (u^+ + u^-)/2$ and electrical current $j = (u^+ - u^-)/2\varepsilon$, system (NSM) is equivalent to system (1.1). In the following text, we make estimates directly on system (1.1). For the sake of simplicity, we drop the index ε of the solution $(u_\varepsilon,\ j_\varepsilon,\ B_\varepsilon,\ E_\varepsilon)$. The proof of theorem 1.1 can be divided into three steps.

2.1. Step 1: a priori estimates for smooth solutions

Proposition 2.1. *Assume that* (u, j, E, B) *is a sufficiently smooth solution to system* (1.1) *on the time interval* $[0, T]$ *for some* $T > 0$. *Denote the energy function and energy dissipation function as follows:*

$$F(t) = \|u\|_{H^s}^2 + \|\varepsilon j\|_{H^s}^2 + \|E\|_{H^s}^2 + \|B\|_{H^s}^2, \tag{2.1}$$

$$G(t) = \mu\|\nabla u\|_{H^s}^2 + \mu\|\varepsilon\nabla j\|_{H^s}^2 + \frac{1}{\sigma}\|j\|_{H^s}^2. \tag{2.2}$$

Then there exists a positive constant $C = C(s, \mu, c, \sigma) > 0$, *such that*

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}F(t) + G(t) \leqslant CF(t)^{1/2}G(t). \tag{2.3}$$

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Proof. Substituting the H^s $(s \geqslant 3)$ inner product in system (1.1) with (u, j, E, B), we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{H^s}^2 + \mu\|\nabla u\|_{H^s}^2 = -\langle u\cdot\nabla u, u\rangle_{H^s} - \varepsilon^2\langle j\cdot\nabla j, u\rangle_{H^s} + \langle j\times B, u\rangle_{H^s},$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varepsilon j\|_{H^s}^2 + \mu\|\varepsilon\nabla j\|_{H^s}^2 + \frac{1}{\sigma}\|j\|_{H^s}^2 = -\varepsilon^2\langle u\cdot\nabla j, j\rangle_{H^s}$$
$$- \varepsilon^2\langle j\cdot\nabla u, j\rangle_{H^s} + c\langle E, j\rangle_{H^s} + \langle u\times B, j\rangle_{H^s},$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|E\|_{H^s}^2 = c\langle\nabla\times B, E\rangle_{H^s} - c\langle j, E\rangle_{H^s},$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|B\|_{H^s}^2 = -c\langle\nabla\times E, B\rangle_{H^s}.$$

Combining the above inequalities and using the cancellations

$$c\langle\nabla\times B, E\rangle_{H^s} - c\langle\nabla\times E, B\rangle_{H^s} = 0, \text{ and } c\langle E, j\rangle_{H^s} - c\langle j, E\rangle_{H^s} = 0,$$

we conclude that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|u\|_{H^s}^2 + \|\varepsilon j\|_{H^s}^2 + \|E\|_{H^s}^2 + \|B\|_{H^s}^2) + \mu\|\nabla u\|_{H^s}^2 + \mu\|\varepsilon\nabla j\|_{H^s}^2 + \frac{1}{\sigma}\|j\|_{H^s}^2 \\
&\quad = -\langle u\cdot\nabla u, u\rangle_{H^s} - \varepsilon^2\langle j\cdot\nabla j, u\rangle_{H^s} + \langle j\times B, u\rangle_{H^s} \\
&\qquad - \varepsilon^2\langle u\cdot\nabla j, j\rangle_{H^s} - \varepsilon^2\langle j\cdot\nabla u, j\rangle_{H^s} + \langle u\times B, j\rangle_{H^s} \\
&\quad = \sum_{i=1}^{6} I_i.
\end{aligned} \tag{2.4}$$

For the term I_1, due to the divergence-free property of velocity field u, the Hölder inequality, the Sobolev inequalities $\|f\|_{L^3(\mathbb{R}^3)} \lesssim \|f\|_{H^1(\mathbb{R}^3)}$, and $\|f\|_{L^6(\mathbb{R}^3)} \lesssim \|\nabla f\|_{L^2(\mathbb{R}^3)}$, we can obtain

$$\begin{aligned}
I_1 &= -\sum_{k=0}^{s}\langle\nabla^k(u\cdot\nabla u) - u\nabla\nabla^k u, \nabla^k u\rangle \\
&= -\sum_{k=1}^{s}\sum_{a+b=k, a\geqslant 1}\langle\nabla^a u\nabla^b\nabla u, \nabla^k u\rangle \\
&\leqslant \sum_{k=1}^{s}\sum_{a+b=k, a\geqslant 1}\|\nabla^a u\|_{L^6}\|\nabla^b\nabla u\|_{L^3}\|\nabla^k u\|_{L^2} \\
&\lesssim \sum_{k=1}^{s}\sum_{a+b=k, a\geqslant 1}\|\nabla u\|_{H^a}\|\nabla u\|_{H^{b+1}}\|\nabla u\|_{H^{k-1}} \\
&\lesssim (\|\nabla u\|_{H^{s-1}}\|\nabla u\|_{H^s} + \|\nabla u\|_{H^s}\|\nabla u\|_{H^{s-1}})\|\nabla u\|_{H^{s-1}} \\
&\lesssim \|\nabla u\|_{H^{s-1}}\|\nabla u\|_{H^s}\|\nabla u\|_{H^{s-1}}.
\end{aligned}$$

Similar arguments lead to

$$I_4 \lesssim \varepsilon^2(\|\nabla u\|_{H^{s-1}}\|\nabla j\|_{H^s} + \|\nabla u\|_{H^s}\|\nabla j\|_{H^{s-1}})\|\nabla j\|_{H^{s-1}},$$

and

$$I_2 + I_5 = -\varepsilon^2 \sum_{k=0}^{s} (\langle \nabla^k (j\cdot\nabla j) - j\cdot\nabla\nabla^k j, \nabla^k u\rangle + \langle \nabla^k (j\cdot\nabla u) - j\cdot\nabla\nabla^k u, \nabla^k j\rangle)$$
$$\lesssim \varepsilon^2 (\|\nabla j\|_{H^{s-1}}\|\nabla j\|_{H^s}\|\nabla u\|_{H^{s-1}} + \|\nabla j\|_{H^{s-1}}\|\nabla u\|_{H^s}\|\nabla j\|_{H^{s-1}}).$$

The above I_2 and I_5 are estimated together through the summation $I_2 + I_5$ because of the simple cancellation $\langle j\cdot\nabla\nabla^k j, \nabla^k u\rangle + \langle j\cdot\nabla\nabla^k u, \nabla^k j\rangle = 0$ (since $\text{div} j = 0$).

For the last two terms I_3 and I_6, from the cancellations

$$\langle \nabla^k j\times B, \nabla^k u\rangle + \langle \nabla^k u\times B, \nabla^k j\rangle = 0, \quad k = 0, 1, \ldots, s$$

we obtain

$$I_3 + I_6 = \sum_{k=0}^{s} \langle \nabla^k (j\times B) - \nabla^k j\times B, \nabla^k u\rangle + \langle \nabla^k (u\times B) - \nabla^k u\times B, \nabla^k j\rangle$$
$$= II_1 + II_2.$$

Applying the Hölder inequality and Sobolev embedding inequalities $H^1(\mathbb{R}^3) \hookrightarrow L^3(\mathbb{R}^3)$, $\dot{H}^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$, and $H^2(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$, we derive that

$$\begin{aligned}
II_1 &= \sum_{k=0}^{s} \langle \nabla^k (j\times B) - \nabla^k j\times B, \nabla^k u\rangle \\
&= \sum_{k=1}^{s} \left(\sum_{a+b=k, a,b\geqslant 1} \langle \nabla^a j\times\nabla^b B, \nabla^k u\rangle + j\times\langle \nabla^k B, \nabla^k u\rangle \right) \\
&\leqslant \sum_{k=1}^{s} \left(\sum_{a+b=k, a,b\geqslant 1} \|\nabla^a j\|_{L^3}\|\nabla^b B\|_{L^6}\|\nabla^k u\|_{L^2} + \|j\|_{L^\infty}\|\nabla^k B\|_{L^2}\|\nabla^k u\|_{L^2} \right) \\
&\lesssim \sum_{k=1}^{s} (\|\nabla j\|_{H^{k-1}} + \|j\|_{L^\infty})\|\nabla B\|_{H^{k-1}}\|\nabla u\|_{H^{k-1}} \\
&\lesssim \|j\|_{H^s}\|\nabla B\|_{H^{s-1}}\|\nabla u\|_{H^{s-1}}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
II_2 &= \sum_{k=1}^{s} \langle \nabla^k (u\times B) - \nabla^k u\times B, \nabla^k j\rangle \\
&\lesssim \sum_{k=1}^{s} (\|\nabla u\|_{H^{k-1}} + \|u\|_{L^\infty})\|\nabla B\|_{H^{k-1}}\|\nabla j\|_{H^{k-1}}.
\end{aligned}$$

By splitting u into lower and higher frequencies and using Bernstein's lemma (see [2], p. 308), we found that

$$\|u\|_{L^\infty} \lesssim \|u\|_{L^6} + \|\nabla u\|_{L^\infty} \lesssim \|\nabla u\|_{L^2} + \|\nabla u\|_{H^2} \lesssim \|\nabla u\|_{H^2}, \tag{2.5}$$

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from which it follows that

$$II_2 \lesssim \sum_{k=1}^{s}(\|\nabla u\|_{H^{k-1}} + \|u\|_{L^\infty})\|\nabla B\|_{H^{k-1}}\|\nabla j\|_{H^{k-1}}$$
$$\lesssim \|\nabla u\|_{H^s}\|B\|_{H^s}\|j\|_{H^s}.$$

Plugging all the above inequalities into (2.4), and recalling the definitions of energy and energy dissipation functions $F(t)$ and $G(t)$, we conclude that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}F(t) + G(t) \leqslant CF(t)^{1/2}G(t).$$

This completes the proof of proposition. □

2.2. Step 2: global existence for small data

Define the cut-off mollifier

$$J_k f = \mathcal{F}^{-1}(I_{B(0,k)}(\xi)\hat{f}(\xi))$$

with $\mathcal{F}^{-1}$ the usual Fourier inverse transform. We consider the approximate system of (1.1)

$$\begin{cases} \partial_t u^k + J_k(J_k u^k\cdot\nabla J_k u^k) + \varepsilon^2 J_k(J_k j^k\cdot\nabla J_k j^k) - \mu\Delta J_k u^k \\ \quad = -\nabla p^k + J_k(J_k j^k \times J_k B^k), \\ \varepsilon^2\partial_t j^k + \varepsilon^2 J_k(J_k u^k\cdot\nabla J_k j^k + J_k j^k\cdot\nabla J_k u^k) - \varepsilon^2\mu\Delta J_k j^k + \dfrac{1}{\sigma}J_k j^k \\ \quad = -\nabla\tilde{p}^k + cJ_k E^k + J_k(J_k u^k\times J_k B^k), \\ \dfrac{1}{c}\partial_t E^k - \nabla\times J_k B^k = -J_k j^k, \\ \dfrac{1}{c}\partial_t B^k + \nabla\times J_k E^k = 0, \\ \mathrm{div}u^k = 0,\ \mathrm{div}j^k = 0,\ \mathrm{div}E^k = 0,\ \mathrm{div}B^k = 0, \\ (u^{kin}, j^{kin}, E^{kin}, B^{kin}) = (J_k u^{\mathrm{in}}, J_k j^{\mathrm{in}}, J_k E^{\mathrm{in}}, J_k B^{\mathrm{in}}). \end{cases}$$

The theory of ordinary differential equation ensures that there a maximal $T^k > 0$ such that the approximate system has a unique solution $(u^k, j^k, E^k, B^k) \in C([0,T^k); H^s)^4$. Observe that $J_k^2 = J_k$. Then $(J_k u^k, J_k j^k, J_k E^k, J_k B^k)$ also solves this approximate system. Hence, uniqueness guarantees $(u^k, j^k, E^k, B^k) = (J_k u^k, J_k j^k, J_k E^k, J_k B^k)$ and (u^k, j^k, E^k, B^k) also solves

$$\begin{cases} \partial_t u^k + J_k(u^k\cdot\nabla u^k) + \varepsilon^2 J_k(j^k\cdot\nabla j^k) - \mu\Delta u^k = -\nabla p^k + J_k(j^k\times B^k), \\ \varepsilon^2\partial_t j^k + \varepsilon^2 J_k(u^k\cdot\nabla j^k + j^k\cdot\nabla u^k) - \varepsilon^2\mu\Delta j^k + \dfrac{1}{\sigma}j^k \\ \quad = -\nabla\tilde{p}^k + cE^k + J_k(u^k\times B^k), \\ \dfrac{1}{c}\partial_t E^k - \nabla\times B^k = -j^k, \\ \dfrac{1}{c}\partial_t B^k + \nabla\times E^k = 0. \end{cases}$$

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Following exactly the lines of reasoning which led to the a priori energy estimate (2.3), we find similar energy estimates for the above approximate system

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}F^k(t) + G^k(t) \leqslant CF^k(t)^{1/2}G^k(t),$$

where

$$\mathcal{E}^k(t) = \|u^k\|_{H^s}^2 + \|\varepsilon j^k\|_{H^s}^2 + \|E^k\|_{H^s}^2 + \|B^k\|_{H^s}^2,$$
$$\mathcal{D}^k(t) = \mu\|\nabla u^k\|_{H^s}^2 + \mu\|\varepsilon\nabla j^k\|_{H^s}^2 + \frac{1}{\sigma}\|j^k\|_{H^s}^2.$$

If the initial energy $F(0)$ is small enough, for sufficient large k, we claim that (u^k, j^k, E^k, B^k) exists globally in time. In fact, assume that $CF(0)^{1/2} \leqslant 1/8$. The relation $\lim_{k\to\infty}\|J_k \cdot\|_{H^s} = \|\cdot\|_{H^s}$ implies that there exists sufficiently large k_0, such that for all $k \geqslant k_0$, $CF^k(0)^{1/2} \leqslant 1/4$. Define

$$T_k = \sup\left\{t \geqslant 0;\ \sup_{\tau\in[0,t]} CF^k(t)^{1/2} \leqslant \frac{1}{2}\right\}.$$

If T_k is finite, then for all $t \in [0, T_k]$,

$$\frac{\mathrm{d}}{\mathrm{d}t}F^k(t) + G^k(t) \leqslant 0,$$

from which it follows:

$$\sup_{t\in[0,T_k]} F^k(t) + \int_0^{T_k} G^k(t)\,\mathrm{d}t \leqslant F^k(0).$$

This means

$$\sup_{t\in[0,T_k]} CF^k(t)^{1/2} \leqslant CF^k(0)^{1/2} \leqslant \frac{1}{4}.$$

By the continuity of $F^k(t)$, there exists a t^{k*}, such that

$$\sup_{\tau\in[0,T_k+t^{k*}]} CF^k(t)^{1/2} \leqslant \frac{1}{2},$$

which contradicts the definition of T_k. Hence, we derive that $T_k = \infty$. This implies

$$\sup_{t\geqslant 0} F^k(t) + \int_0^{\infty} G^k(t)\,\mathrm{d}t \leqslant F(0).$$

This uniform bound together with standard compactness arguments then finds the existence of a global solution $(u, j) \in L^\infty(\mathbb{R}^+; H^s(\mathbb{R}^3)) \cap L^2(\mathbb{R}^+; \dot{H}^{s+1}(\mathbb{R}^3))$, $(E, B) \in$

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$L^\infty(\mathbb{R}^+; H^s(\mathbb{R}^3))$ to system (1.1) satisfying

$$\begin{aligned}\sup_{t\geqslant 0}&\left(\|u\|_{H^s}^2+\|\varepsilon j\|_{H^s}^2+\|E\|_{H^s}^2+\|B\|_{H^s}^2\right)\\&+\int_0^\infty\left(\mu\|\nabla u\|_{H^s}^2+\mu\|\varepsilon\nabla j\|_{H^s}^2+\frac{1}{\sigma}\|j\|_{H^s}^2\right)\\&\leqslant\|u^{\text{in}}\|_{H^s}+\|\varepsilon j^{\text{in}}\|_{H^s}+\|E^{\text{in}}\|_{H^s}+\|B^{\text{in}}\|_{H^s}.\end{aligned}\tag{2.6}$$

Using system (1.1), we show that $(u_t, j_t, E_t, B_t) \in L^2(\mathbb{R}^3, H^{s-1}(\mathbb{R}^3))$. Consequently, we deduce that for all $T > 0$, $(u, j, E, B) \in C([0, T]; H^{s-1}(\mathbb{R}^3))$.

2.3. Step 3: uniqueness

Suppose that (u_i, j_i, E_i, B_i) $(i = 1, 2)$

$$(u_i, j_i, E_i, B_i) \in L^\infty([0, T]; H^s(\mathbb{R}^3)) \cap C([0, T]; H^{s-1}(\mathbb{R}^3)) \tag{2.7}$$

are two solutions of system (1.1) with the same initial data. It is obvious that the difference $(u_1 - u_2, j_1 - j_2, E_1 - E_2, B_1 - B_2)$ satisfies

$$\begin{cases}\partial_t(u_1-u_2)+u_1\cdot\nabla(u_1-u_2)+(u_1-u_2)\cdot\nabla u_2\\\qquad+\varepsilon^2(j_1\cdot\nabla(j_1-j_2)+(j_1-j_2)\cdot\nabla j_2)-\mu\Delta(u_1-u_2)\\\quad=-\nabla(p_1-p_2)+j_1\times(B_1-B_2)+(j_1-j_2)\times B_2,\\\varepsilon^2\partial_t(j_1-j_2)+\varepsilon^2(u_1\cdot\nabla(j_1-j_2)+(u_1-u_2)\cdot\nabla j_2+j_1\cdot\nabla(u_1-u_2)\\\qquad+(j_1-j_2)\cdot u_2)-\varepsilon^2\mu\Delta(j_1-j_2)+\frac{1}{\sigma}(j_1-j_2)\\\quad=-\nabla(\tilde{p}_1-\tilde{p}_2)+c(E_1-E_2)+u_1\times(B_1-B_2)+(u_1-u_2)\times B_2,\\\dfrac{1}{c}\partial_t(E_1-E_2)-\nabla\times(B_1-B_2)=-(j_1-j_2),\\\dfrac{1}{c}\partial_t(B_1-B_2)+\nabla\times(E_1-E_2)=0,\\\operatorname{div}(u_1-u_2)=\operatorname{div}(j_1-j_2)=\operatorname{div}(E_1-E_2)=\operatorname{div}(B_1-B_2)=0.\end{cases}\tag{2.8}$$

Define

$$\begin{aligned}\bar{F}(t)&=\|(u_1-u_2)\|_{L^2}^2+\|\varepsilon(j_1-j_2)\|_{L^2}^2+\|(E_1-E_2)\|_{L^2}^2+\|(B_1-B_2)\|_{L^2}^2,\\\bar{G}(t)&=\mu\|\nabla(u_1-u_2)\|_{L^2}^2+\mu\|\varepsilon\nabla(j_1-j_2)\|_{L^2}^2+\frac{1}{\sigma}\|j_1-j_2\|_{L^2}^2.\end{aligned}$$

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Substituting the L^2 inner product in system (2.8) with $(u_1 - u_2, j_1 - j_2, E_1 - E_2, B_1 - B_2)$, we derive

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\bar{F}(t) + \bar{G}(t)\\
&= -\langle u_1\cdot\nabla(u_1-u_2), u_1-u_2\rangle_{L^2} - \langle (u_1-u_2)\cdot\nabla u_2, u_1-u_2\rangle_{L^2}\\
&\quad -\varepsilon^2\langle j_1\cdot\nabla(j_1-j_2), u_1-u_2\rangle_{L^2} - \varepsilon^2\langle (j_2-j_1)\cdot\nabla j_2, u_1-u_2\rangle_{L^2}\\
&\quad +\langle j_1\times(B_1-B_2), u_1-u_2\rangle_{L^2} + \langle (j_1-j_2)\times B_2, u_1-u_2\rangle_{L^2}\\
&\quad -\varepsilon^2\langle u_1\cdot\nabla(j_1-j_2), j_1-j_2\rangle_{L^2} - \varepsilon^2\langle (u_1-u_2)\cdot\nabla j_2), j_1-j_2\rangle_{L^2}\\
&\quad -\varepsilon^2\langle j_1\cdot\nabla(u_1-u_2), j_1-j_2\rangle_{L^2} - \varepsilon^2\langle (j_1-j_2)\cdot\nabla u_2, j_1-j_2\rangle_{L^2}\\
&\quad +\langle u_1\times(B_1-B_2), j_1-j_2\rangle_{L^2} + \langle (u_1-u_2)\times B_2, j_1-j_2\rangle_{L^2}\\
&= \sum_{i=1}^{12} II_i. \qquad (2.9)
\end{aligned}$$

Due to the divergence-free property of u and j, we obtain $II_1 = II_7 = II_3 + II_9 = 0$. It is easy to check that $II_6 + II_{12} = 0$. Applying the Hölder inequality, we derive

$$\begin{aligned}
II_2 + II_4 + II_8 + II_{10} &\lesssim (\|\nabla u_2\|_{L^\infty} + \|\varepsilon\nabla j_2\|_{L^\infty})(\|u_1-u_2\|_{L^2}^2 + \|\varepsilon(j_1-j_2)\|_{L^2}^2)\\
&\lesssim (\|\nabla u_2\|_{L^\infty} + \|\varepsilon\nabla j_2\|_{L^\infty})\bar{F}(t),\\
II_5 &\lesssim \|j_1\|_{L^\infty}\|B_1-B_2\|_{L^2}\|u_1-u_2\|_{L^2} \lesssim \|j_1\|_{L^\infty}\bar{F}(t),\\
II_{11} &\lesssim \|u_1\|_{L^\infty}\|B_1-B_2\|_{L^2}\|j_1-j_2\|_{L^2}\\
&\lesssim \|u_1\|_{L^\infty}\bar{F}(t)^{1/2}\bar{G}(t)^{1/2},
\end{aligned}$$

Applying the Young inequality to control $\bar{G}(t)^{1/2}$ by $\bar{G}(t)$, we conclude that

$$\frac{d}{dt}\bar{F}(t) + \bar{G}(t) \lesssim M_{12}(t)\bar{F}(t)$$

with

$$M_{12}(t) = \|\nabla u_2\|_{L^\infty} + \|\varepsilon\nabla j_2\|_{L^\infty} + \|j_1\|_{L^\infty} + \|u_1\|_{L^\infty}^2.$$

Assumption (2.7) guarantees that $M_{12} \in L^1[0,T]$(recall $s \geqslant 3$, $H^{s-1} \hookrightarrow L^\infty$). By applying Gronwall's lemma, we derive

$$\sup_{0\leqslant t\leqslant T}\bar{F}(T) + \int_0^T G(\tau)d\tau \leqslant \bar{F}(0)e^{\int_0^T M_{12}(\tau)d\tau} = 0.$$

As a consequence, we obtain the uniqueness.

Going back to the relation $u = (u^+ + u^-)/2$ and $j = (u^+ - u^-)/2\varepsilon$, completes the whole proof of theorem 1.1.

3. Proof of theorem 1.3

The proof of theorem 1.3 can be divided into three steps.

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3.1. Step 1: uniform bounds

First, according to the above section, under the initial assumption of theorem 1.3, for every $0 < \varepsilon \leqslant 1$, system (1.1) allows a unique global classical solution $(u_\epsilon, j_\varepsilon, E_\varepsilon, B_\varepsilon)$, uniformly bounded with respect to ε as in (2.6). In the following, we obtain more uniform estimates.

3.1.1. *Further bounds for electric and magnetic fields* The divergence-free condition of the magnetic field B yields

$$\nabla \times (\nabla \times B) = -\Delta B.$$

we then derive from system (1.1) that

$$\begin{cases} \frac{1}{c}\partial_t E - \nabla \times B + \sigma c E = \sigma \nabla \tilde{p} - \sigma u \times B + \sigma J, \\ \frac{1}{c}\partial_t B + \nabla \times E = 0, \\ \frac{1}{c^2}\partial_{tt} B - \Delta B + \sigma \partial_t B = \sigma \nabla \times (u \times B) - \sigma \nabla \times J. \end{cases} \tag{3.1}$$

with $J = \varepsilon^2 \partial_t j - \varepsilon^2 \mu \Delta j + \varepsilon^2 (u \cdot \nabla j + j \cdot \nabla u)$. Substituting the H^{s-1} inner product in the first and second equations of system (3.1) with (E, B) and the H^{s-2} product in the third equation with $\partial_t B$, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\bigg(\frac{1}{c}\|E\|_{H^{s-1}}^2 + \frac{1}{c}\|B\|_{H^{s-1}}^2 + \frac{1}{c^2}\|\partial_t B\|_{H^{s-2}}^2 \\ &\qquad + \|\nabla B\|_{H^{s-2}}^2\bigg) + \sigma c\|E\|_{H^{s-1}}^2 + \sigma\|\partial_t B\|_{H^{s-2}}^2 \\ &\quad = \sigma\langle -(u \times B) + J, E\rangle_{H^{s-1}} + \sigma\langle \nabla \times (u \times B) - \nabla \times J, \partial_t B\rangle_{H^{s-2}} \\ &\quad \leqslant C(\|(u \times B)\|_{H^{s-1}}^2 + \|J\|_{H^{s-1}}^2) + \frac{1}{2}\sigma c\|E\|_{H^{s-1}}^2 + \frac{1}{2}\sigma\|\partial_t B\|_{H^{s-2}}^2. \end{aligned} \tag{3.2}$$

Next, substituting the H^{s-2}-inner product with B in the third equation of (3.1) yields

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{c^2}\langle \partial_t B, B\rangle_{H^{s-2}} - \frac{1}{c^2}\|\partial_t B\|_{H^{s-2}}^2 + \|\nabla B\|_{H^{s-2}}^2 + \frac{\sigma}{2}\frac{d}{dt}\|B\|_{H^{s-2}}^2 \\ &\quad = \sigma\langle \nabla \times (u \times B) - \nabla \times J, B\rangle_{H^{s-2}} \\ &\quad = \sigma\langle (u \times B) - J, \nabla \times B\rangle_{H^{s-2}} \\ &\quad \leqslant C(\|u \times B\|_{H^{s-1}}^2 + \|J\|_{H^{s-1}}^2) + \frac{1}{2}\|\nabla B\|_{H^{s-2}}^2. \end{aligned} \tag{3.3}$$

Notice that

$$\langle \partial_t B, B\rangle_{H^{s-2}} = \frac{1}{2}(\|\partial_t B + B\|_{H^{s-2}}^2 - \|\partial_t B\|_{H^{s-2}}^2 - \|B\|_{H^{s-2}}^2). \tag{3.4}$$

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The right-hand term $\|u\times B\|_{H^{s-1}}$ can be controlled as follows:

$$\begin{aligned}\|u\times B\|_{H^{s-1}} &\leqslant \sum_{k=0}^{s-1}\left(\sum_{a+b=k,a\geqslant 1}\|\nabla^a u\times\nabla^b B\|_{L^2}+\|u\times\nabla^k B\|_{L^2}\right)\\ &\leqslant \sum_{k=0}^{s-1}\left(\sum_{a+b=k,a\geqslant 1}\|\nabla^a u\|_{L^6}\|\nabla^b B\|_{L^3}+\|u\|_{L^\infty}\|\nabla^k B\|_{L^2}\right)\\ &\lesssim \|\nabla u\|_{H^{s-1}}\|B\|_{H^{s-1}},\end{aligned}\tag{3.5}$$

where we have used the Sobolev inequalities $\|f\|_{L^3(\mathbb{R}^3)}\lesssim\|f\|_{H^1(\mathbb{R}^3)}$ and $\|f\|_{L^6(\mathbb{R}^3)}\lesssim\|\nabla f\|_{L^2(\mathbb{R}^3)}$, and the inequality (2.5) with $s-1\geqslant 2$. Since H^{s-1} is a Banach algebra, the right-hand term $\|J\|_{H^{s-1}}$ is estimated by

$$\begin{aligned}\|J\|_{H^{s-1}} &\leqslant \|\varepsilon^2\partial_t j\|_{H^{s-1}}+\|\varepsilon^2\mu\Delta j\|_{H^{s-1}}+\|\varepsilon^2(u\cdot\nabla j+j\cdot\nabla u)\|_{H^{s-1}}\\ &\leqslant \|\varepsilon^2\partial_t j\|_{H^{s-1}}+C\varepsilon(\|\varepsilon\nabla j\|_{H^s}+\|u\|_{H^{s-1}}\|\varepsilon\nabla j\|_{H^{s-1}}\\ &\quad+\|\varepsilon j\|_{H^{s-1}}\|\nabla u\|_{H^{s-1}}).\end{aligned}\tag{3.6}$$

For some constant $\delta>0$, by plugging the relation (3.4) and inequalities (3.5) and (3.6) into (3.2)+$\delta\times$(3.3), we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{c}\|E\|_{H^{s-1}}^2+\frac{1}{c}\|B\|_{H^{s-1}}^2+\delta\left(\sigma-\frac{1}{c^2}\right)\|B\|_{H^{s-2}}^2+\frac{1-\delta}{c^2}\|\partial_t B\|_{H^{s-2}}^2\right.\\ &\quad\left.+\|\nabla B\|_{H^{s-2}}^2+\frac{\delta}{c^2}\|\partial_t B+B\|_{H^{s-2}}^2\right]\\ &\quad+\sigma c\|E\|_{H^{s-1}}^2+\left(\sigma-\frac{2\delta}{c^2}\right)\|\partial_t B\|_{H^{s-2}}^2+\delta\|\nabla B\|_{H^{s-2}}^2\\ &\leqslant \frac{C_1}{4}\|\varepsilon^2\partial_t j\|_{H^{s-1}}^2+Ch_1(t),\end{aligned}\tag{3.7}$$

where C_1 is a constant depending only on s, c, σ, and μ, the function

$$\begin{aligned}h_1(t)\triangleq{}&(\varepsilon\|\varepsilon\nabla j\|_{H^s}+\varepsilon\|u\|_{H^{s-1}}\|\varepsilon\nabla j\|_{H^{s-1}}+\varepsilon\|\varepsilon j\|_{H^{s-1}}\|\nabla u\|_{H^{s-1}}\\ &+\|\nabla u\|_{H^{s-1}}\|B\|_{H^{s-1}})^2\end{aligned}$$

is uniformly bounded in $L^1(\mathbb{R}^+)$ according to the energy bound (2.6). We can choose $\delta>0$ small enough such that $0<\delta<\frac{1}{2}\min\{\sigma c^2/2,c,1\}\in(0,1)$ to guarantee coefficients in (3.7) are positive.

Next, in order to bound the term $\varepsilon^2\partial_t j$, substituting the H^{s-1}-inner product of the second and third equations in system (1.1) with $\varepsilon^2\partial_t j$ and $\varepsilon^2 j$, respectively, we

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deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mu}{2}\|\varepsilon^2\nabla j\|_{H^{s-1}}^2+\frac{1}{2\sigma}\|\varepsilon j\|_{H^{s-1}}^2\right)-\langle cE,\varepsilon^2\partial_t j\rangle_{H^{s-1}}+\|\varepsilon^2\partial_t j\|_{H^{s-1}}^2 \tag{3.8}$$

$$=-\langle\varepsilon^2(u\cdot\nabla j+j\cdot\nabla u),\varepsilon^2\partial_t j\rangle_{H^{s-1}}+\langle u\times B,\varepsilon^2\partial_t j\rangle_{H^{s-1}}, \tag{3.9}$$

and

$$-\langle c\partial_t E,\varepsilon^2 j\rangle_{H^{s-1}}=-c^2\langle\nabla\times B,\varepsilon^2 j\rangle_{H^{s-1}}+c^2\varepsilon^2\|j\|_{H^{s-1}}^2. \tag{3.10}$$

We derive H^{s-1} $(s-1\geqslant 2)$ is a Banach algebra such that

$$\begin{aligned}&-\langle\varepsilon^2(u\cdot\nabla j+j\cdot\nabla u),\varepsilon^2\partial_t j\rangle_{H^{s-1}}\\ &\quad\lesssim(\varepsilon^2\|u\|_{H^{s-1}}\|\nabla j\|_{H^{s-1}}+\varepsilon^2\|j\|_{H^{s-1}}\|\nabla u\|_{H^{s-1}})\|\varepsilon^2\partial_t j\|_{H^{s-1}}\\ &\quad\leqslant C(\varepsilon^2\|u\|_{H^{s-1}}^2\|\varepsilon\nabla j\|_{H^{s-1}}^2+\varepsilon^2\|\varepsilon j\|_{H^{s-1}}^2\|\nabla u\|_{H^{s-1}}^2)+\frac{1}{4}\|\varepsilon^2\partial_t j\|_{H^{s-1}}^2.\end{aligned}$$

The inequality (3.5) yields

$$\langle u\times B,\varepsilon^2\partial_t j\rangle_{H^{s-1}}\leqslant C\|\nabla u\|_{H^{s-1}}^2\|B\|_{H^{s-1}}^2+\frac{1}{4}\|\varepsilon^2\partial_t j\|_{H^{s-1}}^2.$$

For the right-hand term of (3.10), direct computations yield

$$\begin{aligned}-c^2\varepsilon^2\langle\nabla\times B,j\rangle_{H^{s-1}}&=-c^2\varepsilon^2\left(\sum_{k=1}^{s-1}\langle\nabla^k B,\nabla^k\nabla\times j\rangle+\langle\nabla\times B,j\rangle\right)\\ &\leqslant\varepsilon^2\left(\sum_{k=1}^{s-1}\|\nabla^k B\|_{L^2}\|\nabla^k\nabla\times j\|_{L^2}+\|\nabla\times B\|_{L^2}\|j\|_{L^2}\right)\\ &\lesssim\varepsilon^2\|\nabla B\|_{H^{s-2}}(\|\nabla j\|_{H^{s-1}}+\|j\|_{H^{s-1}})\\ &\leqslant\frac{\delta}{2C_1}\|\nabla B\|_{H^{s-2}}^2+C\varepsilon^2\|\varepsilon\nabla j\|_{H^{s-1}}^2+C\varepsilon^4\|j\|_{H^{s-1}}^2.\end{aligned}$$

Combining all the above inequalities yields

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mu}{2}\|\varepsilon^2\nabla j\|_{H^{s-1}}^2+\frac{1}{2\sigma}\|\varepsilon j\|_{H^{s-1}}^2-\langle cE,\varepsilon^2 j\rangle_{H^{s-1}}\right)+\frac{1}{2}\|\varepsilon^2\partial_t j\|_{H^{s-1}}^2\\ &\quad\leqslant\frac{\delta}{2C_1}\|\nabla B\|_{H^{s-2}}^2+Ch_2(t),\end{aligned} \tag{3.11}$$

where

$$\begin{aligned}h_2(t)&=\varepsilon^2\|u\|_{H^{s-1}}^2\|\varepsilon\nabla j\|_{H^{s-1}}^2+\varepsilon^2\|\varepsilon j\|_{H^{s-1}}^2\|\nabla u\|_{H^{s-1}}^2+\|\nabla u\|_{H^{s-1}}^2\|B\|_{H^{s-1}}^2\\ &\quad+\varepsilon^2\|\varepsilon\nabla j\|_{H^{s-1}}^2+(\varepsilon^4+\varepsilon^2)\|j\|_{H^{s-1}}^2\end{aligned}$$

is uniformly bounded in $L^1(\mathbb{R}^+)$ according to the energy bound (2.6).

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Finally, (3.7)+(3.11)$\times C_1$ enables us to obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}H(t)+\sigma c\|E\|_{H^{s-1}}^2+\left(\sigma-\frac{2\delta}{c^2}\right)\|\partial_t B\|_{H^{s-2}}^2+\frac{\delta}{2}\|\nabla B\|_{H^{s-2}}^2\\ +\frac{C_1}{4}\|\varepsilon^2\partial_t j\|_{H^{s-1}}^2\leqslant Ch_1(t)+Ch_2(t),\end{aligned}\tag{3.12}$$

with

$$\begin{aligned}H(t)=\Bigg[&\frac{1}{c}\|E\|_{H^{s-1}}^2+\frac{1}{c}\|B\|_{H^{s-1}}^2+\delta\left(\sigma-\frac{1}{c^2}\right)\|B\|_{H^{s-2}}^2\\ &+\frac{1-\delta}{c^2}\|\partial_t B\|_{H^{s-2}}^2+\|\nabla B\|_{H^{s-2}}^2+\frac{\delta}{c^2}\|\partial_t B+B\|_{H^{s-2}}^2\\ &+\frac{C_1\mu}{2}\|\varepsilon^2\nabla j\|_{H^{s-1}}^2+\frac{C_1}{2\sigma}\|\varepsilon j\|_{H^{s-1}}^2-C_1\langle cE,\varepsilon^2 j\rangle_{H^{s-1}}\Bigg].\end{aligned}$$

If ε is small enough, $-C_1\langle cE,\varepsilon^2 j\rangle_{H^{s-1}}$ can be controlled by $(1/c)\|E\|_{H^{s-1}}^2+(C_1/2\sigma)\|\varepsilon j\|_{H^{s-1}}^2$, this implies $H(t)$ is a positive function. Integrating (3.12) in time, using system (1.1) and using the energy bound (2.6) again, we conclude that

$$\begin{aligned}&\int_0^\infty(\|E\|_{H^{s-1}}^2+\|\partial_t B\|_{H^{s-2}}^2+\|\nabla B\|_{H^{s-2}}^2+\|\varepsilon^2\partial_t j\|_{H^{s-1}}^2)\mathrm{d}t\\ &\quad\lesssim\int_0^\infty(h_1(t)+h_2(t))\,\mathrm{d}t+H(0)\\ &\quad\leqslant C(\|u^{\mathrm{in}}\|_{H^s},\|\varepsilon j^{\mathrm{in}}\|_{H^s},\|E^{\mathrm{in}}\|_{H^s},\|B^{\mathrm{in}}\|_{H^s}).\end{aligned}\tag{3.13}$$

3.1.2. *Uniform estimates for the time derivative of current* Note that

$$\begin{aligned}&\left\langle\frac{1}{\sigma}j-cE-u\times B,\frac{1}{\sigma}\partial_t j\right\rangle_{H^{s-2}}\\ &\quad=\left\langle\frac{1}{\sigma}j-cE-u\times B,\frac{1}{\sigma}\partial_t j-c\partial_t E-\partial_t u\times B-u\times\partial_t B\right\rangle_{H^{s-2}}\\ &\qquad+\left\langle\frac{1}{\sigma}j-cE-u\times B,c\partial_t E+\partial_t u\times B+u\times\partial_t B\right\rangle_{H^{s-2}}\\ &\quad=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\frac{1}{\sigma}j-cE-u\times B\right\|_{H^{s-2}}^2\\ &\qquad+\left\langle\frac{1}{\sigma}j-cE-u\times B,c\partial_t E+\partial_t u\times B+u\times\partial_t B\right\rangle_{H^{s-2}}.\end{aligned}$$

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Substituting the H^{s-2}-inner product in the second equation in system (1.1) with $(1/\sigma)\partial_t j$, and using the above equality, we deduce that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mu}{\sigma}\|\varepsilon\nabla j\|^2_{H^{s-2}}+\left\|\frac{1}{\sigma}j-cE-u\times B\right\|^2_{H^{s-2}}\right)+\frac{1}{\sigma}\|\varepsilon\partial_t j\|^2_{H^{s-2}}\\
&\quad=-\left\langle \varepsilon^2(u\cdot\nabla j+j\cdot\nabla u),\frac{1}{\sigma}\partial_t j\right\rangle_{H^{s-2}}\\
&\qquad-\left\langle \frac{1}{\sigma}j-cE-u\times B, c\partial_t E+\partial_t u\times B+u\times\partial_t B\right\rangle_{H^{s-2}}.
\end{aligned}$$

We now estimate the right-hand side terms by using the energy bounds (2.6) and (3.13). First, using again H^s as an algebra yields

$$\begin{aligned}
&-\langle \varepsilon^2(u\cdot\nabla j+j\cdot\nabla u),\frac{1}{\sigma}\partial_t j\rangle_{H^{s-2}}\leqslant \|(u\cdot\varepsilon\nabla j+\varepsilon j\cdot\nabla u)\|_{H^{s-2}}\|\frac{1}{\sigma}\varepsilon\partial_t j\|_{H^{s-2}}\\
&\quad\leqslant C(\|u\|^2_{H^s}\|\varepsilon\nabla j\|^2_{H^s}+\|\varepsilon j\|^2_{H^s}\|\nabla u\|^2_{H^s})+\frac{1}{2\sigma}\|\varepsilon\partial_t j\|^2_{H^{s-2}}\\
&\quad\triangleq Ch_{31}(t)+\frac{1}{2\sigma}\|\varepsilon\partial_t j\|^2_{H^{s-2}}
\end{aligned}$$

with $h_{31}(t)\in L^1(\mathbb{R}^+)$. By virtue of (3.5), we obtain

$$\begin{aligned}
\left\|\frac{1}{\sigma}j-cE-u\times B\right\|_{H^{s-2}}&\lesssim \|j\|_{H^{s-2}}+\|E\|_{H^{s-2}}\\
&\quad+\|\nabla u\|_{H^s}\|B\|_{H^s}\triangleq h_{32}(t)\in L^2(\mathbb{R}^+).
\end{aligned}$$

For $\|fg\|_{H^{s-2}}$, we calculate as follows:

$$\begin{aligned}
\|fg\|_{H^{s-2}}&\leqslant \sum_{k=0}^{s-2}\sum_{a+b=k}\|\nabla^a u\times\nabla^b B\|_{L^2}\leqslant \sum_{k=0}^{s-2}\sum_{a+b=k}\|\nabla^a u\|_{L^\infty}\|\nabla^b B\|_{L^2}\\
&\leqslant \sum_{k=0}^{s-2}\sum_{a+b=k}\|\nabla^a f\|_{H^2}\|\nabla^b g\|_{L^2}\lesssim \|f\|_{H^s}\|g\|_{H^{s-2}}.
\end{aligned}\tag{3.14}$$

Consequently, we derive that

$$\begin{aligned}
\|c\partial_t E+\partial_t u\times B+u\times\partial_t B\|_{H^{s-2}}&\lesssim \|\partial_t E\|_{H^{s-2}}+\|\partial_t u\|_{H^{s-2}}\|B\|_{H^s}\\
&\quad+\|u\|_{H^s}\|\partial_t B\|_{H^{s-2}}\triangleq h_{33}(t).
\end{aligned}$$

Note that

$$\begin{aligned}
&\|\partial_t E\|_{H^{s-2}}\lesssim \|\nabla\times B\|_{H^{s-2}}+\|j\|_{H^{s-2}}\lesssim \|\nabla B\|_{H^{s-2}}+\|j\|_{H^s},\\
&\|\partial_t u\|_{H^{s-2}}\lesssim \|u\|_{H^s}\|\nabla u\|_{H^s}+\|\varepsilon j\|_{H^s}\|\varepsilon\nabla j\|_{H^s}+\|\nabla u\|_{H^s}+\|j\|_{H^s}\|B\|_{H^s}.
\end{aligned}$$

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It follows that $h_{33}(t) \in L^2(\mathbb{R}^+)$. Finally, we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mu}{\sigma}\|\varepsilon\nabla j\|_{H^{s-2}}^2 + \left\|\frac{1}{\sigma}j - cE - u\times B\right\|_{H^{s-2}}^2\right) + \frac{1}{\sigma}\|\varepsilon\partial_t j\|_{H^{s-2}}^2 \lesssim h_{31}(t) + h_{32}(t)h_{33}(t)$$

with $h_{31}(t) + h_{32}(t)h_{33}(t)$ uniformly bounded in $L^1(\mathbb{R}^+)$. Hence, integrating in time yields

$$\begin{aligned}\int_0^\infty \|\varepsilon\partial_t j\|_{H^{s-2}}^2 \mathrm{d}t &\lesssim \int_0^\infty (h_{31}(t) + h_{32}(t)h_{33}(t))\,\mathrm{d}t \\ &\quad + (\|\varepsilon\nabla j^{\mathrm{in}}\|_{H^{s-2}}^2 + \|\frac{1}{\sigma}j^{\mathrm{in}} - cE^{\mathrm{in}} - u^{\mathrm{in}}\times B^{\mathrm{in}}\|_{H^{s-2}}^2 \\ &\leqslant C(\|u^{\mathrm{in}}\|_{H^s}, \|\varepsilon j^{\mathrm{in}}\|_{H^s}, \|E^{\mathrm{in}}\|_{H^s}, \|B^{\mathrm{in}}\|_{H^s}, \|j^{\mathrm{in}}\|_{H^{s-2}}). \quad (3.15)\end{aligned}$$

Hence, under the initial assumption for the electric current

$$\left\|j^{\mathrm{in}}\left(=\frac{u^{+,\mathrm{in}} - u^{-,\mathrm{in}}}{2\varepsilon}\right)\right\|_{H^{s-2}(\mathbb{R}^3)} \leqslant C$$

(for some constant $C > 0$ independent of ε) in theorem 1.3, $\int_0^\infty \|\varepsilon\partial_t j\|_{H^{s-2}}^2$ is also uniformly bounded.

In summary, under the assumption of theorem 1.3, for sufficiently small ε, system (1.1) allows a unique global classical solution $(u_\epsilon, j_\varepsilon, E_\varepsilon, B_\varepsilon)$, which is uniformly bounded with respect to ε in the following sense, by combining inequalities (2.6), (3.13), and (3.15):

$$\begin{aligned}&\sup_{t\geqslant 0}\left(\|u_\varepsilon\|_{H^s}^2 + \|\varepsilon j_\varepsilon\|_{H^s}^2 + \|E_\varepsilon\|_{H^s}^2 + \|B_\varepsilon\|_{H^s}^2\right) \\ &\quad + \int_0^\infty \big(\mu\|\nabla u_\varepsilon\|_{H^s}^2 + \mu\|\varepsilon\nabla j_\varepsilon\|_{H^s}^2 + \frac{1}{\sigma}\|j_\varepsilon\|_{H^s}^2 + \|E_\varepsilon\|_{H^{s-1}}^2 + \|\partial_t B_\varepsilon\|_{H^{s-2}}^2 \\ &\quad + \|\nabla B_\varepsilon\|_{H^{s-2}}^2 + \|\varepsilon^2\partial_t j_\varepsilon\|_{H^{s-1}}^2 + \|\varepsilon\partial_t j_\varepsilon\|_{H^{s-2}}^2\big)\mathrm{d}t \\ &\leqslant C(\|u_\varepsilon^{\mathrm{in}}\|_{H^s}, \|\varepsilon j_\varepsilon^{\mathrm{in}}\|_{H^{s-1}}, \|E_\varepsilon^{\mathrm{in}}\|_{H^s}, \|B_\varepsilon^{\mathrm{in}}\|_{H^s}, \|j_\varepsilon^{\mathrm{in}}\|_{H^{s-2}}). \quad (3.16)\end{aligned}$$

3.2. Step 2: convergence of the sequence

According to the uniform bound (3.16) obtained in Step 1, we have the following proposition.

PROPOSITION 3.1. *Under the assumption of theorem 1.3, as $\varepsilon \to 0$, $(u_\varepsilon, j_\varepsilon, E_\varepsilon, B_\varepsilon)$ is a Cauchy sequence (by extracting a subsequence arbitrarily) in the spaces*

$$(L^\infty(\mathbb{R}^+; H^{s-2}) \cap L^2(\mathbb{R}^+; \dot{H}^{s-1})) \times L^2(\mathbb{R}^+; H^{s-2}) \times (L^\infty(\mathbb{R}^+; H^{s-2}))^2.$$

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Proof. In fact, for $0 < \varepsilon, \eta \leqslant 1$, the difference between the two solutions $(u_\varepsilon - u_\eta, j_\varepsilon - j_\eta, E_\varepsilon - E_\eta, B_\varepsilon - B_\eta)$ satisfies

$$\begin{cases} \partial_t(u_\varepsilon - u_\eta) - \mu\Delta(u_\varepsilon - u_\eta) + (u_\varepsilon - u_\eta)\cdot\nabla u_\varepsilon + u_\eta\cdot\nabla(u_\varepsilon - u_\eta) \\ \quad = -\nabla(p_\varepsilon - p_\eta) + (j_\varepsilon - j_\eta)\times B_\varepsilon + j_\eta \times (B_\varepsilon - B_\eta) - \varepsilon^2 j_\varepsilon\cdot\nabla j_\varepsilon + \eta^2 j_\eta\cdot\nabla j_\eta, \\ \dfrac{1}{\sigma}(j_\varepsilon - j_\eta) = -\nabla(\tilde{p}_\varepsilon - \tilde{p}_\eta) + c(E_\varepsilon - E_\eta) + (u_\varepsilon - u_\eta)\times B_\varepsilon \\ \quad + u_\eta \times (B_\varepsilon - B_\eta) - J_\varepsilon + J_\eta, \\ \dfrac{1}{c}\partial_t(E_\varepsilon - E_\eta) - \nabla\times(B_\varepsilon - B_\eta) = -(j_\varepsilon - j_\eta), \\ \dfrac{1}{c}\partial_t(B_\varepsilon - B_\eta) + \nabla\times(E_\varepsilon - E_\eta) = 0, \\ \operatorname{div}(u_\varepsilon - u_\eta) = 0,\ \operatorname{div}(j_\varepsilon - j_\eta) = 0,\ \operatorname{div}(E_\varepsilon - E_\eta) = 0,\ \operatorname{div}(B_\varepsilon - B_\eta) = 0, \end{cases}$$

where $J_i = i^2\partial_t j_i - i^2\mu\Delta j_i + i^2(u_i\cdot\nabla j_i + j_i\cdot\nabla u_i),\ i = \varepsilon, \eta.$

Substituting the H^{s-2}-inner product with $(u_\varepsilon - u_\eta, j_\varepsilon - j_\eta, E_\varepsilon - E_\eta, B_\varepsilon - B_\eta)$, we have

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_\varepsilon - u_\eta\|_{H^{s-2}}^2 + \mu\|\nabla(u_\varepsilon - u_\eta)\|_{H^{s-2}}^2 \\ &\qquad = -\langle (u_\varepsilon - u_\eta)\cdot\nabla u_\varepsilon + u_\eta\cdot\nabla(u_\varepsilon - u_\eta), u_\varepsilon - u_\eta\rangle_{H^{s-2}} + \langle (j_\varepsilon - j_\eta)\times B_\varepsilon \\ &\qquad\quad + j_\eta\times(B_\varepsilon - B_\eta), u_\varepsilon - u_\eta\rangle_{H^{s-2}} + \langle -\varepsilon^2 j_\varepsilon\cdot\nabla j_\varepsilon + \eta^2 j_\eta\cdot\nabla j_\eta, u_\varepsilon - u_\eta\rangle_{H^{s-2}}, \end{aligned}$$

and

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|E_\varepsilon - E_\eta\|_{H^{s-2}}^2 + \|B_\varepsilon - B_\eta\|_{H^{s-2}}^2) + \frac{1}{\sigma}\|j_\varepsilon - j_\eta\|_{H^{s-2}}^2 \\ &\quad = \langle (u_\varepsilon - u_\eta)\times B_\varepsilon + u_\eta\times(B_\varepsilon - B_\eta), j_\varepsilon - j_\eta\rangle_{H^{s-2}} + \langle -J_\varepsilon + J_\eta, j_\varepsilon - j_\eta\rangle_{H^{s-2}}. \end{aligned}$$

where we have used cancellations

$$\begin{aligned} &c\langle E_\varepsilon - E_\eta, j_\varepsilon - j_\eta\rangle_{H^{s-2}} - c\langle j_\varepsilon - j_\eta, E_\varepsilon - E_\eta\rangle_{H^{s-2}} = 0, \\ &\langle \nabla\times(B_\varepsilon - B_\eta), E_\varepsilon - E_\eta\rangle_{H^{s-2}} - \langle \nabla\times(E_\varepsilon - E_\eta), B_\varepsilon - B_\eta\rangle_{H^{s-2}} = 0. \end{aligned}$$

Next, we bind the right-hand side terms. By applying the Hölder inequality and Sobolev inequalities $H^1(\mathbb{R}^3) \hookrightarrow L^3(\mathbb{R}^3)$ and $\dot{H}^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$, we have

$$\begin{aligned} &-\langle (u_\varepsilon - u_\eta)\cdot\nabla u_\varepsilon, u_\varepsilon - u_\eta\rangle_{H^{s-2}} \\ &\quad \leqslant \sum_{k=0}^{s-2}\sum_{a+b=k}\|\nabla^a(u_\varepsilon - u_\eta)\|_{L^6}\|\nabla^b(\nabla u_\varepsilon)\|_{L^3}\|\nabla^k(u_\varepsilon - u_\eta)\|_{L^2} \\ &\quad \lesssim \sum_{k=0}^{s-2}\sum_{a+b=k}\|\nabla(u_\varepsilon - u_\eta)\|_{H^a}\|\nabla u_\varepsilon\|_{H^{b+1}}\|\nabla^k(u_\varepsilon - u_\eta)\|_{L^2} \\ &\quad \lesssim \|\nabla(u_\varepsilon - u_\eta)\|_{H^{s-2}}\|\nabla u_\varepsilon\|_{H^{s-1}}\|u_\varepsilon - u_\eta\|_{H^{s-2}} \\ &\quad \leqslant C\|\nabla u_\varepsilon\|_{H^{s-1}}^2\|u_\varepsilon - u_\eta\|_{H^{s-2}}^2 + \frac{1}{10}\mu\|\nabla(u_\varepsilon - u_\eta)\|_{H^{s-2}}^2. \end{aligned}$$

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Similarly, we obtain

$$\begin{aligned}
&\langle j_\eta \times (B_\varepsilon - B_\eta), u_\varepsilon - u_\eta\rangle_{H^{s-2}} \\
&\quad \leqslant \sum_{k=0}^{s-2}\sum_{a+b=k} \|\nabla^a j_\eta\|_{L^3}\|\nabla^b(B_\varepsilon - B_\eta)\|_{L^2}\|\nabla^k(u_\varepsilon - u_\eta)\|_{L^6} \\
&\quad \lesssim \|j_\eta\|_{H^{s-1}}\|(B_\varepsilon - B_\eta)\|_{H^{s-2}}\|\nabla(u_\varepsilon - u_\eta)\|_{H^{s-2}} \\
&\quad \leqslant C\|j_\eta\|^2_{H^{s-1}}\|(B_\varepsilon - B_\eta)\|^2_{H^{s-2}} + \frac{1}{10}\mu\|\nabla(u_\varepsilon - u_\eta)\|^2_{H^{s-2}}, \\
&\langle -\varepsilon^2 j_\varepsilon\cdot\nabla j_\varepsilon, u_\varepsilon - u_\eta\rangle_{H^{s-2}} \\
&\quad \lesssim \varepsilon\|j_\varepsilon\|_{H^{s-1}}\|\varepsilon\nabla j_\varepsilon\|_{H^{s-2}}\|\nabla(u_\varepsilon - u_\eta)\|_{H^{s-2}} \\
&\quad \leqslant C\varepsilon^2\|j_\varepsilon\|^2_{H^{s-1}}\|\varepsilon\nabla j_\varepsilon\|^2_{H^{s-2}} + \frac{1}{10}\|\nabla(u_\varepsilon - u_\eta)\|^2_{H^{s-2}}, \\
&\langle \eta^2 j_\eta\cdot\nabla j_\eta, u_\varepsilon - u_\eta\rangle_{H^{s-2}} \\
&\quad \leqslant C\eta^2\|j_\eta\|^2_{H^{s-1}}\|\eta\nabla j_\eta\|^2_{H^{s-2}} + \frac{1}{10}\|\nabla(u_\varepsilon - u_\eta)\|^2_{H^{s-2}}.
\end{aligned}$$

Recall the equality (2.5) with $s-1 \geqslant 2$. By direct computations, we then derive that

$$\begin{aligned}
\langle u_\eta\times&(B_\varepsilon - B_\eta), j_\varepsilon - j_\eta\rangle_{H^{s-2}} = \langle u_\eta \times (B_\varepsilon - B_\eta), (j_\varepsilon - j_\eta)\rangle \\
&+ \sum_{k=1}^{s-2}\left\langle \sum_{a+b=k, b\neq k} \nabla^a u_\eta \times \nabla^b(B_\varepsilon - B_\eta) + u_\eta \times \nabla^k(B_\varepsilon - B_\eta), \nabla^k(j_\varepsilon - j_\eta)\right\rangle \\
&\leqslant \|u_\eta\|_{L^6}\|(B_\varepsilon - B_\eta)\|_{L^3}\|(j_\varepsilon - j_\eta)\|_{L^2} \\
&+ \sum_{k=1}^{s-2}\left(\sum_{a+b=1, b\neq k}\|\nabla^a u_\eta\|_{L^6}\|\nabla^b(B_\varepsilon - B_\eta)\|_{L^3} + \|u_\eta\|_{L^\infty}\|\nabla^k(B_\varepsilon - B_\eta)\|_{L^2}\right) \\
&\quad \times \|\nabla^k(j_\varepsilon - j_\eta)\|_{L^2} \\
&\lesssim \|\nabla u_\eta\|_{H^{s-1}}\|B_\varepsilon - B_\eta\|_{H^{s-2}}\|j_\varepsilon - j_\eta\|_{H^{s-2}} \\
&\leqslant C\|\nabla u_\eta\|^2_{H^{s-1}}\|B_\varepsilon - B_\eta\|^2_{H^{s-2}} + \frac{\sigma}{6}\|j_\varepsilon - j_\eta\|^2_{H^{s-2}}.
\end{aligned}$$

Similar arguments lead to

$$\begin{aligned}
&\langle u_\eta\cdot\nabla(u_\varepsilon - u_\eta), u_\varepsilon - u_\eta\rangle_{H^{s-2}} \\
&\quad \lesssim \|\nabla u_\eta\|_{H^{s-1}}\|\nabla(u_\varepsilon - u_\eta)\|_{H^{s-2}}\|u_\varepsilon - u_\eta\|_{H^{s-2}} \\
&\quad \leqslant C\|\nabla u_\eta\|^2_{H^{s-1}}\|u_\varepsilon - u_\eta\|^2_{H^{s-2}} + \frac{1}{10}\mu\|\nabla(u_\varepsilon - u_\eta)\|^2_{H^{s-2}}.
\end{aligned}$$

Next, from the cancellation

$$\begin{aligned}
&\langle\nabla^k(j_\varepsilon - j_\eta)\times B_\varepsilon, \nabla^k(u_\varepsilon - u_\eta)\rangle \\
&\quad + \langle\nabla^k(u_\varepsilon - u_\eta)\times B_\varepsilon, \nabla^k(j_\varepsilon - j_\eta)\rangle = 0, k = 0, 1\ldots, m-1,
\end{aligned}$$

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we derive that

$$
\begin{aligned}
&\langle (j_\varepsilon - j_\eta)\times B_\varepsilon, u_\varepsilon - u_\eta\rangle_{H^{s-2}} + \langle (u_\varepsilon - u_\eta)\times B_\varepsilon, j_\varepsilon - j_\eta\rangle_{H^{s-2}}\\
&\quad = \sum_{k=1}^{s-2}\sum_{a+b=k, b\geqslant 1} (\langle \nabla^a (j_\varepsilon - j_\eta)\times \nabla^b B_\varepsilon, \nabla^k (u_\varepsilon - u_\eta)\rangle\\
&\qquad + \langle \nabla^a (u_\varepsilon - u_\eta)\times \nabla^b B_\varepsilon, \nabla^k (j_\varepsilon - j_\eta)\rangle)\\
&\quad \leqslant \sum_{k=1}^{s-2}\sum_{a+b=k, b\geqslant 1} (\|\nabla^a (j_\varepsilon - j_\eta)\|_{L^3}\|\nabla^b B_\varepsilon\|_{L^6}\|\nabla^k (u_\varepsilon - u_\eta)\|_{L^2}\\
&\qquad + \|\nabla^a (u_\varepsilon - u_\eta)\|_{L^3}\|\nabla^b B_\varepsilon\|_{L^6}\|\nabla^k (j_\varepsilon - j_\eta)\|_{L^2})\\
&\quad \lesssim \|(u_\varepsilon - u_\eta)\|_{H^{s-2}}\|\nabla B_\varepsilon\|_{H^{s-2}}\|j_\varepsilon - j_\eta\|_{H^{s-2}}\\
&\quad \leqslant C\|\nabla B_\varepsilon\|^2_{H^{s-2}}\|u_\varepsilon - u_\eta\|^2_{H^{s-2}} + \frac{\sigma}{6}\|(j_\varepsilon - j_\eta)\|^2_{H^{s-2}}.
\end{aligned}
$$

Finally, the Hölder inequality leads to the fact that

$$
\langle -J_\varepsilon + J_\eta, j_\varepsilon - j\rangle_{H^{s-2}} \leqslant C(\|J_\varepsilon\|^2_{H^{s-2}} + \|J_\eta\|^2_{H^{s-2}}) + \frac{\sigma}{6}\|j_\varepsilon - j\|^2_{H^{s-2}}.
$$

Direct calculations yield

$$
\begin{aligned}
\|J_i\|_{H^{s-2}} &= \|i^2\partial_t j_i - i^2\mu\Delta j_i + i^2(u_i\cdot\nabla j_i + j_i\cdot\nabla u_i)\|_{H^{s-2}} \\
&\leqslant Ci(\|i\partial_t j_i\|_{H^{s-2}} + \|i\nabla j_i\|_{H^s} + \|u_i\|_{H^s}\|i\nabla j_i\|_{H^s}\\
&\quad + \|ij_i\|_{H^s}\|\nabla u_i\|_{H^s}), \quad i = \varepsilon, \eta.
\end{aligned} \tag{3.17}
$$

Combining all the above inequalities, we obtain

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\|u_\varepsilon - u_\eta\|^2_{H^{s-2}} + \|E_\varepsilon - E_\eta\|^2_{H^{s-2}} + \|B_\varepsilon - B_\eta\|^2_{H^{s-2}})\\
&\quad + \mu\|\nabla(u_\varepsilon - u_\eta)\|^2_{H^{s-2}} + \frac{1}{\sigma}\|j_\varepsilon - j_\eta\|^2_{H^{s-2}}\\
&\qquad \leqslant Cg_1(t)(\|u_\varepsilon - u\|^2_{H^{s-2}} + \|(B_\varepsilon - B_\eta)\|^2_{H^{s-2}}) + Cg_2(t),
\end{aligned}
$$

where

$$
\begin{aligned}
g_1(t) &= \|\nabla u_\varepsilon\|^2_{H^{s-1}} + \|j_\eta\|^2_{H^{s-1}} + \|\nabla u_\eta\|^2_{H^{s-1}} + \|\nabla B_\varepsilon\|^2_{H^{s-2}},\\
g_2(t) &= \sum_{i=\varepsilon,\eta} i^2(\|j_i\|^2_{H^{s-1}}\|i\nabla j_i\|^2_{H^{s-2}} + \|i\partial_t j_i\|^2_{H^{s-2}} + \|i\nabla j_i\|^2_{H^s}\\
&\quad + \|u_i\|^2_{H^s}\|i\nabla j_i\|^2_{H^s} + \|ij_i\|^2_{H^s}\|\nabla u_i\|^2_{H^s}).
\end{aligned}
$$

According to the uniform bounds (3.16), we show that

$$
\int_0^\infty g_1(t)\mathrm{d}t \leqslant C, \quad \text{and} \quad \int_0^\infty g_2(t)\mathrm{d}t \leqslant C(\varepsilon^2 + \eta^2).
$$

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The Grönwall inequality then yields

$$\sup_{t\geqslant 0}(\|u_\varepsilon-u_\eta\|^2_{H^{s-2}}+\|E_\varepsilon-E_\eta\|^2_{H^{s-2}}+\|B_\varepsilon-B_\eta\|^2_{H^{s-2}})$$
$$+\int_0^\infty \mu(\|\nabla(u_\varepsilon-u_\eta)\|^2_{H^{s-2}}+\frac{1}{\sigma}\|j_\varepsilon-j_\eta\|^2_{H^{s-2}})\mathrm{d}t$$
$$\leqslant C(\|u^{\mathrm{in}}_\varepsilon-u^{\mathrm{in}}_\eta\|^2_{H^{s-2}}+\|E^{\mathrm{in}}_\varepsilon-E^{\mathrm{in}}_\eta\|^2_{H^{s-2}}+\|B^{\mathrm{in}}_\varepsilon-B^{\mathrm{in}}_\eta\|^2_{H^{s-2}}+\varepsilon^2+\eta^2).$$

This completes the proof of proposition 3.1. □

3.3. Step 3: passing to the limit

Let (u,j,B,E) be the limit of Cauchy sequence $(u_\varepsilon,j_\varepsilon,B_\varepsilon,E_\varepsilon)$ in space

$$(L^\infty(\mathbb{R}^+;H^{s-2})\cap L^2(\mathbb{R}^+;\dot{H}^{s-1}))\times L^2(\mathbb{R}^+;H^{s-2})\times(L^\infty(\mathbb{R}^+;H^{s-2}))^2.$$

In system (1.1), we show terms with coefficient ε^2 converging to zero in some sense. In fact, as $\varepsilon\to 0$, according to uniform estimate (3.16)

$$\begin{aligned}
\varepsilon^2\|j_\varepsilon\cdot\nabla j_\varepsilon\|_{L^2(\mathbb{R}^+;H^{s-1})} &\lesssim \varepsilon\|j_\varepsilon\|_{L^2(\mathbb{R}^+;H^{s-1})}\|\varepsilon\nabla j_\varepsilon\|_{L^\infty(\mathbb{R}^+;H^{s-1})}\\
&\lesssim \varepsilon\|j_\varepsilon\|_{L^2(\mathbb{R}^+;H^{s})}\|\varepsilon j_\varepsilon\|_{L^\infty(\mathbb{R}^+;H^{s})}\to 0,\\
\varepsilon^2\|\partial_t j_\varepsilon\|_{L^2(\mathbb{R}^+;H^{s-2})} &\leqslant \varepsilon\|\varepsilon\partial_t j_\varepsilon\|_{L^2(\mathbb{R}^+;H^{s-2})}\to 0,\\
\varepsilon^2\|\mu\Delta j_\varepsilon\|_{L^2(\mathbb{R}^+;H^{s-1})} &\leqslant \varepsilon\mu\|\varepsilon\nabla j_\varepsilon\|_{L^2(\mathbb{R}^+;H^{s})}\to 0,\\
\varepsilon^2\|u_\varepsilon\cdot\nabla j_\varepsilon\|_{L^2(\mathbb{R}^+;H^{s})} &\lesssim \varepsilon\|u_\varepsilon\|_{L^\infty(\mathbb{R}^+;H^{s})}\|\varepsilon\nabla j_\varepsilon\|_{L^2(\mathbb{R}^+;H^{s})}\to 0,\\
\varepsilon^2\|j_\varepsilon\cdot\nabla u_\varepsilon\|_{L^2(\mathbb{R}^+;H^{s})} &\lesssim \varepsilon\|j_\varepsilon\|_{L^\infty(\mathbb{R}^+;H^{s})}\|\varepsilon\nabla u_\varepsilon\|_{L^2(\mathbb{R}^+;H^{s})}\to 0.
\end{aligned}$$

Next, in order to pass the limit in system (1.1) for the other nonlinear terms, we establish the following product law:

$$\begin{aligned}
\|fg\|_{H^{s-3}} &\leqslant \sum_{k=0}^{s-3}\sum_{a+b=k}\|\nabla^a u\times\nabla^b B\|_{L^2}\leqslant\sum_{k=0}^{s-3}\sum_{a+b=k}\|\nabla^a u\|_{L^3}\|\nabla^b B\|_{L^6}\\
&\leqslant \sum_{k=0}^{s-3}\sum_{a+b=k}\|\nabla^a f\|_{H^1}\|\nabla^b g\|_{H^1}\lesssim\|f\|_{H^{s-2}}\|g\|_{H^{s-2}}.
\end{aligned}\tag{3.18}$$

Together with the convergence result—proposition 3.1, we derive

$$\begin{aligned}
&\|j_\varepsilon\times B_\varepsilon-j\times B\|_{L^2(\mathbb{R}^+;H^{s-3})}\\
&\quad\leqslant\|(j_\varepsilon-j)\times B_\varepsilon\|_{L^2(\mathbb{R}^+;H^{s-3})}+\|j\times(B_\varepsilon-B)\|_{L^2(\mathbb{R}^+;H^{s-3})}\\
&\quad\lesssim\|j_\varepsilon-j\|_{L^2(\mathbb{R}^+;H^{s-2})}\|B_\varepsilon\|_{L^\infty(\mathbb{R}^+;H^{s-2})}+\|j\|_{L^2(\mathbb{R}^+;H^{s-2})}\|B_\varepsilon-B\|_{L^\infty(\mathbb{R}^+;H^{s-2})}\\
&\quad\lesssim\|j_\varepsilon-j\|_{L^2(\mathbb{R}^+;H^{s-2})}\|B_\varepsilon\|_{L^\infty(\mathbb{R}^+;H^{s})}+\|j\|_{L^2(\mathbb{R}^+;H^{s})}\|B_\varepsilon-B\|_{L^\infty(\mathbb{R}^+;H^{s-2})}\\
&\quad\to 0,\text{as }\varepsilon\to 0.
\end{aligned}$$

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Similarly

$$
\begin{aligned}
&\|u_\varepsilon \times B_\varepsilon - u \times B\|_{L^\infty(\mathbb{R}^+;H^{s-3})} \\
&\quad \lesssim \|u_\varepsilon - u\|_{L^\infty(\mathbb{R}^+;H^{s-2})}\|B_\varepsilon\|_{L^\infty(\mathbb{R}^+;H^{s-2})} \\
&\quad + \|u\|_{L^\infty(\mathbb{R}^+;H^{s-2})}\|B_\varepsilon - B\|_{L^\infty(\mathbb{R}^+;H^{s-2})} \\
&\quad \lesssim \|u_\varepsilon - u\|_{L^\infty(\mathbb{R}^+;H^{s-2})}\|B_\varepsilon\|_{L^\infty(\mathbb{R}^+;H^{s})} + \|u\|_{L^\infty(\mathbb{R}^+;H^{s})}\|B_\varepsilon - B\|_{L^\infty(\mathbb{R}^+;H^{s-2})} \\
&\quad \to 0, \text{as } \varepsilon \to 0, \\
&\|u_\varepsilon \cdot \nabla u_\varepsilon - u \cdot \nabla u\|_{L^2(\mathbb{R}^+;H^{s-1})} \\
&\quad \lesssim \|u_\varepsilon - u\|_{L^\infty(\mathbb{R}^+;H^{s-2})}\|u_\varepsilon\|_{L^2(\mathbb{R}^+;H^{s-2})} \\
&\quad + \|u\|_{L^\infty(\mathbb{R}^+;H^{s-2})}\|\nabla u_\varepsilon - \nabla u\|_{L^2(\mathbb{R}^+;H^{s-2})} \\
&\quad \lesssim \|u_\varepsilon - u\|_{L^\infty(\mathbb{R}^+;H^{s-2})}\|u_\varepsilon\|_{L^2(\mathbb{R}^+;H^{s})} + \|u\|_{L^\infty(\mathbb{R}^+;H^{s})}\|\nabla u_\varepsilon - \nabla u\|_{L^2(\mathbb{R}^+;H^{s-2})} \\
&\quad \to 0, \text{as } \varepsilon \to 0.
\end{aligned}
$$

Finally, on passing the limit in system (1.1), we conclude (u, j, B, E) is indeed a solution to (NSMO). This completes the whole proof of theorem 1.3.

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